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Elliptic problems involving the $p(x)$-Laplacian with competing nonlinearities

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ABSTRACT

By using the fibering method we study the existence of nonnegative solutions for the elliptic problem

$$\left.\begin{array}{l} -\Delta_{p(x)}u = -\lambda a(x)|u|^{p(x)-2}u + \mu b(x)|u|^{q(x)-2}u - \varepsilon c(x)|u|^{t(x)-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{array}\right\}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, $a(.)$, $b(.)$, $c(.)$ are essentially bounded functions and $p(.)$, $q(.)$, $t(.)$ are continuous on $\overline{\Omega}$.

1. Introduction

Suppose that Ω is a domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$. Consider the quasilinear elliptic problem

$$\left.\begin{array}{l} -\Delta_{p(x)}u = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{array}\right\}$$

where $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplace operator and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function. When p is a constant this operator appears in models which describe a variety of phenomena in nature including:

(i) Fluid dynamics. The shear stress $\vec{\tau}(x)$ and the gradient ∇u of the velocity of the fluid are related via the equation $\vec{\tau}(x) = r(x)|\nabla u(x)|^{p-2}\nabla u(x)$. When the fluid is Newtonian $p = 2$ while if it is pseudoplastic or dilatant, $p > 2$ or $p < 2$ respectively [9].
(ii) Flow through porous media (for instance in flow through rock filled dams), where $p = 3/2$, see [16].
(iii) Nonlinear elasticity, with $p \geqslant 2$, see [12,18].
(iv) Glaciology, $p \in (1, 4/3]$, see [13].
(v) Image restoration, $p \in [1, 2]$, see [5,7].

When the exponent $p(.)$ is not constant, the $p(x)$-Laplace operator appears in models for

(i) electrorheological fluids [3,15];
(ii) image restoration, where $p(x) \in [1, 2]$, see [8];
(iii) nonlinear Darcy's law in porous medium [4].

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In this paper we study the problem

$$\left.\begin{array}{l} -\Delta_{p(x)}u = -\lambda a(x)|u|^{p(x)-2}u + \mu b(x)|u|^{q(x)-2}u - \varepsilon c(x)|u|^{t(x)-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{array}\right\} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$, $p, q, t : \overline{\Omega} \to (1, +\infty)$ are continuous functions while λ, μ and ε are positive constants. We assume further that the function $b(.)$ changes sign while $a(.)$ and $c(.)$ remain nonnegative in Ω. Problem (1) was considered in [2] for the case $c \equiv 0$ with $p(x) < q(x)$ in $\overline{\Omega}$ where, via an application of the mountain pass theorem, the existence of an infinite number of solutions was proved. The case $\Omega = \mathbb{R}^N$ is studied in [1] where, under appropriate assumptions on the behavior of $q(.)$ at infinity, the existence of a solution is shown. Note that the sign of the solutions is not examined in [2], while none of the aforementioned papers examines the case where $q(x) < p(x)$ in $\overline{\Omega}$.

Our purpose in this work is to provide conditions on the data of (1) which guarantee the existence of a nonnegative solution and also examine the behavior of the solution and the energy functional as $\varepsilon \to 0$. To do this we employ Pohozaev's fibering method, see [11,14], which decomposes the Sobolev space $W_0^{1,p(.)}(\Omega)$ into rays and examines the behavior of the energy functional on them.

2. Mathematical background

In this section we recall some definitions and basic properties of the variable exponent spaces $L^{p(.)}(\Omega)$ and $W^{1,p(.)}(\Omega)$. For more details see [10].

Let

$$C_+(\overline{\Omega}) = \{p : \overline{\Omega} \to \mathbb{R}\text{: } p \text{ is continuous and } p(x) > 1 \text{ for every } x \in \overline{\Omega}\}.$$

If $s \in C_+(\overline{\Omega})$ we denote $s^+ := \sup_{x\in\overline{\Omega}} s(x)$ and $s^- := \inf_{x\in\overline{\Omega}} s(x)$.

Given $p \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space $L^{p(.)}(\Omega)$ is defined by

$$L^{p(.)}(\Omega) := \left\{u : \Omega \to \mathbb{R}\text{: } u \text{ is measurable and } \int_\Omega |u|^{p(x)}\,dx < \infty\right\}.$$

This space supplied with the so-called Luxemburg norm

$$\|u\|_{p(.)} = \inf\left\{\lambda > 0\text{: } \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \leqslant 1\right\}$$

becomes a Banach space and shares many of the properties of the classical Lebesgue spaces like separability, reflexivity and uniform convexity. It is easy to see that $\|u\|_{p(.)}$ satisfies the following inequalities:

$$\left(\int_\Omega |u|^{p(x)}\,dx\right)^{1/p^-} \leqslant \|u\|_{p(.)} \leqslant \left(\int_\Omega |u|^{p(x)}\,dx\right)^{1/p^+} \quad \text{if } \|u\|_{p(.)} < 1 \tag{2}$$

and

$$\left(\int_\Omega |u|^{p(x)}\,dx\right)^{1/p^+} \leqslant \|u\|_{p(.)} \leqslant \left(\int_\Omega |u|^{p(x)}\,dx\right)^{1/p^-} \quad \text{if } \|u\|_{p(.)} \geqslant 1. \tag{3}$$

Furthermore, if $p, s \in C_+(\overline{\Omega})$ with $p(x) < s(x)$ in $\overline{\Omega}$, then the embedding $L^{s(.)}(\Omega) \subseteq L^{p(.)}(\Omega)$ is continuous.

The variable exponent Sobolev space $W^{1,p(.)}(\Omega)$ is defined by

$$W^{1,p(.)}(\Omega) := \{u \in L^{p(.)}(\Omega)\text{: } |\nabla u| \in L^{p(.)}(\Omega)\}$$

and is equipped with the norm

$$\|u\|_{1,p(.)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}.$$

The closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$ is denoted by $W_0^{1,p(.)}(\Omega)$. The critical Sobolev exponent is defined by

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geqslant N. \end{cases}$$

The spaces $W^{1,p(.)}(\Omega)$ and $W_0^{1,p(.)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces. The analogue of Poincaré's inequality states that if $u \in W_0^{1,p(.)}(\Omega)$, then

$$\|u\|_{p(.)} \leqslant C\|\nabla u\|_{p(.)} \tag{4}$$

for some $C > 0$. Consequently, the norms $\|u\|_{1,p(.)}$ and $\|\nabla u\|_{p(.)}$ are equivalent on $W_0^{1,p(.)}(\Omega)$. Furthermore, if $q(x) < p^*(x)$ in $\overline{\Omega}$, then $W^{1,p(.)}(\Omega)$ is imbedded compactly in $L^{q(.)}(\Omega)$. Thus, if $u \in W_0^{1,p(.)}(\Omega)$, in view of (2), (3) and (4)

$$\int_\Omega |u|^{q(x)}\,dx \leqslant c\left(\int_\Omega |\nabla u|^{p(x)}\,dx\right)^\beta, \tag{5}$$

where $c > 0$ and

$$\beta := \begin{cases} \frac{q^-}{p^+} & \text{if } \|u\|_{q(.)} < 1 \text{ and } \|\nabla u\|_{p(.)} < 1, \\ \frac{q^+}{p^+} & \text{if } \|u\|_{q(.)} \geqslant 1 \text{ and } \|\nabla u\|_{p(.)} < 1, \\ \frac{q^+}{p^-} & \text{if } \|u\|_{q(.)} \geqslant 1 \text{ and } \|\nabla u\|_{p(.)} \geqslant 1, \\ \frac{q^-}{p^-} & \text{if } \|u\|_{q(.)} < 1 \text{ and } \|\nabla u\|_{p(.)} \geqslant 1. \end{cases} \tag{6}$$

3. Hypotheses and main results

We make the following assumptions concerning the data of problem (1):

$H(1)$ $p, q, t \in C_+(\overline{\Omega})$ with $p(x) < N$ and $q(x) < p^* < t(x)$ for every $x \in \overline{\Omega}$.
$H(2)$ $a, b, c \in L^\infty(\Omega)$ with $a, c \geqslant 0$ a.e. in Ω and $m\{x \in \Omega\colon\ b(x) > 0\} > 0$.

The energy functional of problem (1) is defined on the space $E := W_0^{1,p(.)}(\Omega) \cap L^{t(.)}(\Omega)$ which is supplied with the norm

$$\|u\|_E =: \|u\|_{1,p(.)} + \|u\|_{t(.)},$$

and is given by

$$\Phi_\varepsilon(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \lambda\int_\Omega \frac{a(x)}{p(x)}|u|^{p(x)}\,dx - \mu\int_\Omega \frac{b(x)}{q(x)}|u|^{q(x)}\,dx + \varepsilon\int_\Omega \frac{c(x)}{t(x)}|u|^{t(x)}\,dx. \tag{7}$$

We define the extended functional $\mathcal{F} : \mathbb{R} \times E \to \mathbb{R}$ by setting for any $r > 0$ and $v \in E$

$$\begin{aligned}\mathcal{F}(r, v) = {} & \int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)} r^{p(x)}\,dx + \lambda\int_\Omega \frac{a(x)}{p(x)}|v|^{p(x)} r^{p(x)}\,dx \\ & - \mu\int_\Omega \frac{b(x)}{q(x)}|v|^{q(x)} r^{q(x)}\,dx + \varepsilon\int_\Omega \frac{c(x)}{t(x)}|v|^{t(x)} r^{t(x)}\,dx.\end{aligned} \tag{8}$$

If $u = rv$ is a non-trivial critical point of $\Phi_\varepsilon(.)$, then

$$\mathcal{F}_r(r, v) = 0. \tag{9}$$

Assume that $r = r(v) > 0$ satisfies (9) for every v in $E\backslash\{0\}$ and $r(.) \in C^1(E\backslash\{0\})$. Then the reduced functional

$$\widehat{\Phi}_\varepsilon(v) := \Phi_\varepsilon\big(r(v)v\big) \tag{10}$$

is well defined and it is continuously differentiable on E. We will study $\widehat{\Phi}_\varepsilon(.)$ subject to the constraint

$$H(v) = 1,$$

where $H : E \to \mathbb{R}$ is defined by

$$H(v) := \int_\Omega \big[|\nabla v|^{p(x)} + \lambda a(x)|v|^{p(x)}\big]\,dx + \varepsilon\int_\Omega c(x)|u|^{t(x)}\,dx. \tag{11}$$

The main tool we will use is the fibering method which is based on the following fact:

Lemma 1. *(See [14, Theorem 1.2.2].) Let $H : E \to \mathbb{R}$ be a functional which is continuously Fréchet-differentiable in E and satisfies the conditions*:

$$H(0) = 0 \quad \text{and} \quad \langle H'(v), v\rangle \neq 0 \quad \text{if } H(v) = 1. \tag{12}$$

If $v \neq 0$ is a conditional critical point of $\widehat{\Phi}_\varepsilon(.)$ under the constraint $H(v) = 1$, then $u := r(v)v$ is a nonzero critical point of $\Phi_\varepsilon(.)$.

It is clear that the functional H, as defined in (11), satisfies (12).
We distinguish the following cases:

Case 1. $q^+ < p^-$.

Theorem 2. *Suppose that $H(1)$ and $H(2)$ are satisfied, $\lambda, \mu > 0$ and $\varepsilon \geqslant 0$. Then (1) admits a nonnegative solution.*

Proof. Let

$$S^1 := \{v \in E\colon\ H(v) = 1\} \tag{13}$$

and

$$B := \left\{v \in E\colon \int_\Omega b(x)|v|^{q(x)}\,dx > 0\right\}.$$

Relation (9) is equivalent to

$$\int_\Omega \left[|\nabla v|^{p(x)} + \lambda a(x)|v|^{p(x)}\right] r^{p(x)}\,dx + \varepsilon \int_\Omega c(x)|v|^{t(x)} r^{t(x)}\,dx = \mu \int_\Omega b(x)|v|^{q(x)} r^{q(x)}\,dx, \tag{14}$$

which, in view of $H(2)$, has a unique positive solution $r := r(v)$ for every $v \in B$. By the implicit function theorem, see Theorem 4.B, p. 150 in [17], $r(.) \in C^1(E\backslash\{0\})$. If $v \in S^1 \cap B$ and $r(v) \geqslant 1$, then by $H(1)$ and (14)

$$r^{p^-}\left\{\int_\Omega \left[|\nabla v|^{p(x)} + \lambda a(x)|v|^{p(x)}\right] dx + \varepsilon \int_\Omega c(x)|v|^{t(x)}\,dx\right\} = r^{p^-} \leqslant \mu \int_\Omega b(x)|v|^{q(x)} r^{q(x)}\,dx \leqslant r^{q^+} \mu \int_\Omega b|v|^{p(x)}\,dx,$$

and so

$$r^{p^- - q^+} \leqslant \mu \int_\Omega b|v|^{p(x)}\,dx.$$

Therefore, $r(.)$ is bounded on $S^1 \cap B$. On the other hand, if $v \in S^1 \cap B$, by $H(1)$, (8) and (14),

$$\widehat{\Phi}_\varepsilon(v) < \left(\frac{1}{p^-} - \frac{1}{q^+}\right) \int_\Omega |\nabla v|^{p(x)} r^{p(x)}\,dx + \lambda\left(\frac{1}{p^-} - \frac{1}{q^+}\right) \int_\Omega a|v|^{p(x)} r^{p(x)}\,dx + \varepsilon\left(\frac{1}{t^-} - \frac{1}{q^+}\right) \int_\Omega c|v|^{t(x)} r^{p(x)}\,dx,$$

and so $\widehat{\Phi}_\varepsilon(v) < 0$. Consequently,

$$M := \inf\{\widehat{\Phi}_\varepsilon(v)\colon\ v \in S^1 \cap B\} < 0.$$

We will show that the infimum is attained at a point in $S^1 \cap B$. To show this we let $v_n \in S^1$, $n \in \mathbb{N}$, be a sequence such that $\widehat{\Phi}_\varepsilon(v_n) \to M$. Since v_n is bounded in E, there exists a subsequence of v_n, still denoted by v_n, such that $v_n \to v_0$ weakly in $W_0^{1,p(.)}(\Omega)$ and $L^{t(.)}(\Omega)$ and strongly in $L^{p(.)}(\Omega)$ and $L^{q(.)}(\Omega)$. Furthermore, since $r(.)$ is bounded on $S^1 \cap B$, we may also assume that $r(v_n) \to r_0$. By the lower semicontinuity of the norms $\|.\|_{1,p(.)}$ and $\|.\|_{t(.)}$ we see that

$$\widehat{\Phi}_\varepsilon(r_0 v_0) \leqslant \liminf_{n\to+\infty} \widehat{\Phi}_\varepsilon(v_n) = M, \tag{15}$$

which shows that $r_0 > 0$ and $v_0 \neq 0$. Also, by applying (14) to the sequence v_n and allowing $n \to +\infty$, we get

$$\int_\Omega |\nabla v_0|^{p(x)} r_0^{p(x)}\,dx + \lambda \int_\Omega a|v_0|^{p(x)} r_0^{p(x)}\,dx + \varepsilon \int_\Omega c|v_0|^{t(x)} r_0^{t(x)}\,dx$$
$$\leqslant \mu \int_\Omega b|v_0|^{q(x)} r_0^{q(x)}\,dx,$$

which yields $r_0 \leqslant r(v_0)$ and $r_0 v_0 \in B$. If we assume that $r_0 < r(v_0)$, then by exploiting the fact that the function $r \to \Phi_\varepsilon(rv)$ is strictly decreasing on the interval $[0, r(v_0)]$ we have

$$\widehat{\Phi}_\varepsilon(v_0) = \Phi_\varepsilon\big(r(v_0)v_0\big) < \widehat{\Phi}_\varepsilon(r_0 v_0) = M. \tag{16}$$

On the other hand, for $\rho > 0$, in view of (14), $r(\rho v_0)$ satisfies

$$\int_\Omega |\nabla \rho v_0|^{p(x)} r(\rho v_0)^{p(x)}\,dx + \lambda \int_\Omega a|\rho v_0|^{p(x)} r(\rho v_0)^{p(x)}\,dx - \mu \int_\Omega b|\rho v_0|^{q(x)} r(\rho v_0)^{q(x)}\,dx + \varepsilon \int_\Omega c|\rho v_0|^{t(x)} r(\rho v_0)^{t(x)}\,dx = 0.$$

Thus,

$$\int_\Omega |\nabla v_0|^{p(x)} \big[\rho r(\rho v_0)\big]^{p(x)}\,dx + \lambda \int_\Omega a|v_0|^{p(x)} \big[\rho r(\rho v_0)\big]^{p(x)}\,dx - \int_\Omega b|v_0|^{q(x)} \big[\rho r(\rho v_0)\big]^{q(x)}\,dx + \varepsilon \int_\Omega c|v_0|^{t(x)} \big[\rho r(\rho v_0)\big]^{t(x)}\,dx = 0,$$

which shows that

$$r(v_0) = \rho r(\rho v_0). \tag{17}$$

Let $s > 0$ be such that $s v_0 \in S^1$. Combining relations (16) and (17) we get

$$\widehat{\Phi}_\varepsilon(s v_0) = \Phi_\varepsilon\big(r(t v_0) s v_0\big) = \Phi_\varepsilon\big(r(v_0)v_0\big) = \widehat{\Phi}_\varepsilon(v_0) < M,$$

contradicting the definition of M. Thus, we must have $r_0 = r(v_0)$, and so (15) and (17) imply that $\widehat{\Phi}_\varepsilon(s v_0) = M$. By Lemma 1 we see that $u := r(v_0)v_0$ is a solution of (1). Since $|u|$ is also a minimizer, we may assume that $u \geqslant 0$. $\square$

Case 2. $p^+ < q^-$.

The following lemma demonstrates that in this case (1) may not have a non-trivial solution. It is based on a similar result in [6].

Lemma 3. *Assume that $H(1)$ and $H(2)$ hold with $b(x) \geqslant 0$ a.e. and $c(x) > \eta > 0$ a.e. in Ω. Then, for every $\lambda, \varepsilon > 0$ there exists $\mu^*(\lambda, \varepsilon) > 0$ such that (1) does not admit a non-trivial solution for $0 < \mu < \mu^*(\lambda, \varepsilon)$.*

Proof. Let u be a solution to (1). Then we have

$$\int_\Omega \big[|\nabla u|^{p(x)} + \lambda a(x)|u|^{p(x)}\big]\,dx + \varepsilon \int_\Omega c(x)|u|^{t(x)}\,dx = \mu \int_\Omega b(x)|u|^{q(x)}\,dx. \tag{18}$$

Young's inequality implies that

$$\mu \int_\Omega b(x)|u|^{q(x)}\,dx \leqslant \frac{\varepsilon q^+}{t^-} \int_\Omega c(x)|u|^{t(x)}\,dx + \frac{t^+ - q^-}{t^-} \mu^\gamma \|b\|_\infty^\delta \varepsilon^{-\zeta} \int_\Omega c(x)^{\frac{q(x)}{q(x)-t(x)}}\,dx,$$

where

$$\gamma := \begin{cases} \frac{t^+}{t^- - q^+} & \text{if } \mu \geqslant 1, \\ \frac{t^-}{t^+ - q^-} & \text{if } \mu < 1, \end{cases} \qquad \delta := \begin{cases} \frac{t^+}{t^- - q^+} & \text{if } \|b\|_\infty \geqslant 1, \\ \frac{t^-}{t^+ - q^-} & \text{if } \|b\|_\infty < 1, \end{cases}$$

$$\zeta := \begin{cases} \frac{q^-}{t^- - q^+} & \text{if } \varepsilon \geqslant 1, \\ \frac{q^+}{t^+ - q^-} & \text{if } \varepsilon < 1, \end{cases}$$

and so (18) yields

$$\int_\Omega \big[|\nabla u|^{p(x)} + \lambda a(x)|u|^{p(x)}\big]\,dx \leqslant \frac{t^+ - q^-}{t^-} \mu^\gamma \|b\|_\infty^\delta \varepsilon^{-\zeta} \int_\Omega c(x)^{\frac{q(x)}{q(x)-t(x)}}\,dx. \tag{19}$$

By (6) and (18) we get

$$\hat{c}\left(\int_\Omega b(x)|u|^{q(x)}\,dx\right)^{\frac{1}{\beta}} \leqslant \int_\Omega \left[|\nabla u|^{p(x)} + \lambda a(x)|u|^{p(x)}\right]dx \leqslant \mu \int_\Omega b(x)|u|^{q(x)}\,dx \tag{20}$$

where $\hat{c} > 0$ and β is defined by (6) with q in the place of s. Thus,

$$\left(\frac{\hat{c}^\beta}{\mu}\right)^{\frac{1}{\beta-1}} \leqslant \hat{c}\left(\int_\Omega b(x)|u|^{q(x)}\,dx\right)^{\frac{1}{\beta}}. \tag{21}$$

In view of (19)–(21) we get

$$\mu \geqslant \mu^*(\lambda,\varepsilon) := \left(\frac{t^-}{t^+ - q^-}\frac{\hat{c}^{\frac{\beta}{\beta-1}}\varepsilon^\zeta}{\|b\|_\infty^\delta \int_\Omega c(x)^{\frac{q(x)}{q(x)-t(x)}}\,dx}\right)^{\frac{\beta-1}{\gamma(\beta-1)+1}}. \qquad \square$$

Lemma 4. *Suppose that c_1, c_2 are positive constants, $\gamma : \Omega \to \mathbb{R}$ is a nonnegative essentially bounded function with $m\{x \in \Omega: \gamma(x) > 0\} > 0$ and $q : \overline{\Omega} \to \mathbb{R}$ is a continuous function such that $1 < p < q(x) < t$ for every $x \in \overline{\Omega}$, where $p, t \in \mathbb{R}$. Then, for large enough $\mu > 0$, the equation*

$$c_1 r^p + c_2 r^t - \mu \int_\Omega \gamma(x) r^{q(x)}\,dx = 0$$

admits two solutions $r_1, r_2 > 0$.

Proof. Let $g(r) := c_1 + c_2 r^{t-p} - \mu \int_\Omega \gamma(x) r^{q(x)-p}\,dx$, $r > 0$. Then $g(0) = c_1$ and $\lim_{r\to+\infty} g(r) = +\infty$. It is easy to see that $g'(r)$ has a unique positive zero. Since $\inf_{r>0} g(r) < 0$ for large enough $\mu > 0$, the result follows. $\square$

For the next existence result we make the following hypotheses concerning the function $c(.)$ and the exponents p, q and t:

$H(3)$ $supp(b(.)^+) \subseteq supp(c(.))$, where $b(x)^+ = \max\{b(x), 0\}$, $x \in \Omega$.
$H(4)$ $p^+(t^- - p^-) > q^-(t^- - q^+)$ and $t^+ \leqslant \min\{\frac{q^- p^+(q^+ - p^-)}{p^+(t^- - p^-) - q^-(t^- - q^+)}, \frac{q^-(t^- - p^-)}{q^+ - p^-}\}$.
$H(5)$ $q(.)$ is a constant or $p(.)$ and $t(.)$ are constants.

Note that if $p(.)$, $q(.)$ and $t(.)$ are constant functions, then $H(4)$ is satisfied if $t < p + q$.

Theorem 5. *Suppose that $H(1)$–$H(5)$ hold with $p^+ < q$. Then for every $\lambda, \varepsilon > 0$ there exists $\mu^*(\lambda,\varepsilon) > 0$, such that* (1) *admits a nonnegative solution for every $\mu > \mu^*(\lambda,\varepsilon)$.*

Proof. Assume first that $q(.)$ is constant. For $v \in B$ and $r > 0$ define

$$G_\varepsilon(r,v) := \frac{\int_\Omega [|\nabla v|^{p(x)} + \lambda a(x)|v|^{p(x)}] r^{p(x)-q}\,dx + \varepsilon \int_\Omega c(x)|v|^{t(x)} r^{t(x)-q}\,dx}{\int_\Omega b(x)|v|^q\,dx},$$

$$c_1(v) := \frac{\int_\Omega [|\nabla v|^{p(x)} + \lambda a(x)|v|^{p(x)}]\,dx}{\int_\Omega b(x)|v|^q\,dx}$$

and

$$c_2(v) := \frac{\varepsilon \int_\Omega c(x)|v|^{t(x)}\,dx}{\int_\Omega b(x)|v|^q\,dx}.$$

Then, if $r \geqslant 1$,

$$\begin{aligned} c_1(v) r^{p^- - q} + c_2(v) r^{t^- - q} &= g_l^1(r,v) \leqslant G_\varepsilon(r,v) \\ &\leqslant g_u^1(r,v) = c_1(v) r^{p^+ - q} + c_2(v) r^{t^+ - q} \end{aligned} \tag{22}$$

while, if $r < 1$,

$$\begin{aligned} c_1(v)r^{p^+-q} + c_2(v)r^{t^+-q} = g_l^2(r,v) &\leqslant G_\varepsilon(r,v) \\ &\leqslant g_u^2(r,v) = c_1(v)r^{p^--q} + c_2(v)r^{t^--q}. \end{aligned} \tag{23}$$

Thus,

$$\lim_{r\to 0^+} G_\varepsilon(r,v) = \lim_{r\to+\infty} G_\varepsilon(r,v) = +\infty.$$

Note that $\frac{\partial}{\partial r}G_\varepsilon(.\,,v) = 0$ iff

$$\int_\Omega \big(q-p(x)\big)\big[|\nabla v|^{p(x)} + \lambda a(x)|v|^{p(x)}\big]r^{p(x)}\,dx - \varepsilon\int_\Omega \big(t(x)-q\big)c(x)|v|^{t(x)}r^{t(x)}\,dx = 0. \tag{24}$$

It is easy to see that (24) has a unique zero $r_*(v)$ which is a point of global minimum for $G_\varepsilon(.\,,v)$. Consequently, for large $\mu > 0$, the equation $G_\varepsilon(r,v) = \mu$ has exactly two solutions $r_1(v)$ and $r_2(v)$ with $r_1(v) < r_2(v)$. Note that $r_1(v)$ and $r_2(v)$ are also the solutions of (14). We define $r(v) := r_2(v)$. It is easy to see that $r(v)$ increases as μ increases or ε decreases.

Let

$$B_0^\varepsilon(\mu) := \big\{u \in B\colon\ \mu > G_\varepsilon\big(r_*(u),u\big)\big\}.$$

It is clear that $B_0^\varepsilon(\mu) \neq \emptyset$ if μ is large enough. Next, we will find an upper bound for $r_*(v)$ when $v \in B_0^\varepsilon(\mu)$. By the definition of $B_0^\varepsilon(\mu)$

$$\begin{aligned} &\int_\Omega p(x)\big[|\nabla v|^{p(x)} + \lambda a(x)|v|^{p(x)}\big]r_*(v)^{p(x)}\,dx + \varepsilon\int_\Omega t(x)c(x)|v|^{t(x)}r_*(v)^{t(x)}\,dx \\ &\quad < \mu q r_*(v)^q \int_\Omega b(x)|v|^q\,dx, \end{aligned}$$

so, if $r_*(v) \geqslant 1$, then

$$r_*(v)^{t^-}\varepsilon t^- \int_\Omega c(x)|v|^{t(x)}\,dx < \mu q r_*(v)^q \int_\Omega b(x)|v|^q\,dx,$$

while if $r_*(v) < 1$, then

$$r_*(v)^{t^+}\varepsilon t^- \int_\Omega c(x)|v|^{t(x)}\,dx < \mu q r_*(v)^q \int_\Omega b(x)|v|^q\,dx.$$

Consequently,

$$r_*(v) < \left[\frac{\mu q \int_\Omega b(x)|v|^q\,dx}{\varepsilon t^- \int_\Omega c(x)|v|^{t(x)}\,dx}\right]^{1/(t^--q)}, \quad \text{if } r_*(v) \geqslant 1, \tag{25}$$

and

$$r_*(v) < \left[\frac{\mu q \int_\Omega b(x)|v|^q\,dx}{\varepsilon t^- \int_\Omega c(x)|v|^{t(x)}\,dx}\right]^{1/(t^+-q)}, \quad \text{if } r_*(v) < 1.$$

We shall show that if $v \in B_0^\varepsilon(\mu) \cap S^1$ then

$$1 < \int_\Omega c(x)|v|^{t(x)}\,dx + d\bigg(\int_\Omega c(x)|v|^{t(x)}\,dx\bigg)^\eta,$$

where $d > 0$ and

$$\eta := \begin{cases} \frac{q(t^--p^-)-t^+(q-p^-)}{t^+(t^--q)} & \text{if } \|v\|_q < 1 \text{ and } r_*(v) \geqslant 1, \\ \frac{p^+}{t^+} & \text{if } \|v\|_q < 1 \text{ and } r_*(v) < 1, \\ \frac{q(t^+-p^+)-t^+(q-p^+)}{t^+(t^+-q)} & \text{if } \|v\|_q \geqslant 1 \text{ and } r_*(v) < 1, \\ \frac{q(t^--p^-)-t^+(q-p^-)}{t^+(t^--q)} & \text{if } \|v\|_q \geqslant 1 \text{ and } r_*(v) \geqslant 1. \end{cases}$$

We will only present the case where $\|v\|_q < 1$ and $r_*(v) \geqslant 1$, the remaining three can be treated similarly. Since $G_\varepsilon(r_*(v), v) < \mu$, we have

$$\int_\Omega \left[|\nabla v|^{p(x)} + \lambda a(x)|v|^{p(x)}\right] r_*(v)^{p(x)}\,dx < \mu r_*(v)^q \int_\Omega b(x)|v|^q\,dx, \tag{26}$$

and so

$$r_*(v)^{p^-} \int_\Omega \left[|\nabla v|^{p(x)} + \lambda a(x)|v|^{p(x)}\right] dx < r_*(v)^q \mu \int_\Omega b(x)|v|^q\,dx,$$

which, in view of (11), yields

$$\left(1 - \int_\Omega c(x)|v|^{t(x)}\,dx\right) < r_*(v)^{q-p^-} \mu \int_\Omega b(x)|v|^q\,dx.$$

On combining this inequality with (25) we get

$$\begin{aligned}
&\left(\varepsilon \int_\Omega c(x)|v|^{t(x)}\,dx\right)^{\frac{q-p^-}{t^- - q}} \left(1 - \int_\Omega c(x)|v|^{t(x)}\,dx\right) \\
&\quad < \left(\frac{\mu q^+}{t^-}\right)^{\frac{q-p^-}{t^- - q}} \left(\mu \int_\Omega b(x)|v|^q\,dx\right)^{\frac{t^- - p^-}{t^- - q}}.
\end{aligned} \tag{27}$$

Hypotheses $H(3)$ and $H(4)$, (2) and (3) imply that

$$\int_\Omega b(x)|v|^q\,dx \leqslant \hat{d}\left(\int_\Omega c(x)|v|^{t(x)}\,dx\right)^{\frac{q}{t^+}}$$

for some $\hat{d} > 0$, which, in view of (27), gives

$$1 < \int_\Omega c(x)|v|^{t(x)}\,dx + d\left(\int_\Omega c(x)|v|^{t(x)}\,dx\right)^{\frac{q(t^- - p^-) - t^+(q - p^-)}{t^+(t^- - q)}},$$

where $d > 0$. Consequently, $\int_\Omega c(x)|v|^{t(x)}\,dx$, $v \in B_0^\varepsilon(\mu)$, is bounded away from 0. By (14)

$$\int_\Omega c(x)|v|^{t(x)} r^{t(x)}\,dx \leqslant \mu r^q \int_\Omega b(x)|v|^q\,dx,$$

which implies that, if $r \geqslant 1$,

$$r^{t^- - q} \leqslant \frac{\mu \int_\Omega b(x)|v|^q\,dx}{\varepsilon \int_\Omega c(x)|v|^{t(x)}\,dx} \tag{28}$$

while, if $r < 1$

$$r^{t^+ - q} \leqslant \frac{\mu \int_\Omega b(x)|v|^q\,dx}{\varepsilon \int_\Omega c(x)|v|^{t(x)}\,dx}. \tag{29}$$

Since $\int_\Omega c(x)|v|^{t(x)}\,dx$, $v \in B_0^\varepsilon(\mu)$, is bounded away from 0 we see that $r(v)$, $v \in B_0^\varepsilon(\mu)$, is bounded above. In view of (8), (10) and (14)

$$\begin{aligned}
\widehat{\Phi}_\varepsilon(v) &\leqslant \frac{1}{p} \int_\Omega \left[|\nabla v|^{p(x)} + \lambda a(x)|v|^{p(x)}\right] r^{p(x)}\,dx - \mu \int_\Omega \frac{b(x)}{q}|v|^q r^q\,dx + \varepsilon \int_\Omega \frac{c(x)}{t^-}|v|^{t(x)} r^{t(x)}\,dx \\
&= \left(\frac{1}{p} - \frac{1}{t^-}\right) \int_\Omega \left[|\nabla v|^{p(x)} + \lambda a(x)|v|^{p(x)}\right] r^{p(x)}\,dx - \mu\left(\frac{1}{q} - \frac{1}{t^-}\right) \int_\Omega b(x)|v|^q r^q\,dx.
\end{aligned} \tag{30}$$

Since $r(v)$ increases with μ and $p(x) < q$ in $\overline{\Omega}$, by taking μ large enough, say $\mu^*(\lambda, \varepsilon)$, we have that $\widehat{\Phi}_\varepsilon(v) < 0$ for at least one $v \in S^1 \cap B_0^\varepsilon(\mu^*(\lambda, \varepsilon))$. We claim that the infimum for $\widehat{\Phi}_\varepsilon(.)$ is attained at a point of $S^1 \cap B_0^\varepsilon(\mu^*(\lambda, \varepsilon))$. To show this, let $v_n \in S^1$, $n \in \mathbb{N}$, be a sequence such that $\widehat{\Phi}_\varepsilon(v_n) \to M$. Since v_n is bounded in E, there exists a subsequence of v_n, still denoted by v_n, such that $v_n \to v_0$ weakly in $W_0^{1,p(.)}(\Omega)$ and $L^{t(.)}(\Omega)$ and strongly in $L^{p(.)}(\Omega)$ and $L^q(\Omega)$. Furthermore, since $r(.)$ is bounded in $S^1 \cap B_0^\varepsilon(\mu^*(\lambda,\varepsilon))$, we may also assume that $v_0 \neq 0$ and $r(v_n) \to r_0 > 0$. Working as in case 1 and exploiting the fact that the function $r \to \Phi_\varepsilon(rv)$ has a global minimum at $r = r(v_0)$ we conclude that $r_0 = r(v_0)$. The sequence $r_*(u_n)$, $n \in \mathbb{N}$, is bounded so we may assume that $r_*(u_n) \to r_*^0$. In view of (22) and (23) we get

$$G_\varepsilon\big(r_*(v_n), v_n\big) \geqslant \begin{cases} \min_{r\geqslant 1} g_l^1(r, v_n) & \text{if } r_*(v_n) \geqslant 1, \\ \min_{0<r<1} g_l^2(r, v_n) & \text{if } r_*(v_n) < 1, \end{cases}$$

for every $n \in \mathbb{N}$. Since $\min_{r\geqslant 1} g_l^1(r, v_n) \geqslant \min_{r>0} g_l^1(r, v_n) = c_1(v_n)^{\frac{t^+-q}{t^+-p^+}} c_2(v_n)^{\frac{q-p^+}{t^+-p^+}}$, with a similar inequality holding for $\min_{0<r<1} g_l^1(r, v_n)$, we see that, in the limit, $G_\varepsilon(r_*^0, v_0) > 0$. Thus $r_*^0 > 0$. The lower semicontinuity of the norms yields $\mu \geqslant G_\varepsilon(r_*^0, v_0)$. Since the minimum of $r \to G_\varepsilon(r, v_0)$ occurs at $r = r_*(v_0)$, we also have $\mu \geqslant G_\varepsilon(r_*(v_0), v_0)$. We claim that $\mu > G_\varepsilon(r_*(v_0), v_0)$. Indeed, let us assume that $\mu = G_\varepsilon(r_*(v_0), v_0)$. Since $\mu = G_\varepsilon(r(v_n), v_n)$ for every $n \in \mathbb{N}$, we obtain

$$\int_\Omega \big[|\nabla v_n|^{p(x)} + \lambda a(x)|v_n|^{p(x)}\big] r(v_n)^{p(x)}\,dx + \varepsilon \int_\Omega c(x)|v_n|^{t(x)} r(v_n)^{t(x)}\,dx$$
$$= \mu r(v_n)^q \int_\Omega b(x)|v_n|^q\,dx$$

which in the limit gives

$$\int_\Omega \big[|\nabla v_0|^{p(x)} + \lambda a(x)|v_0|^{p(x)}\big] r(v_0)^{p(x)}\,dx + \varepsilon \int_\Omega c(x)|v_0|^{t(x)} r(v_0)^{t(x)}\,dx$$
$$\leqslant \mu r(v_0)^q \int_\Omega b(x)|v_0|^q\,dx,$$

that is, $\mu \geqslant G_\varepsilon(r(v_0), v_0)$. Thus

$$r_*(v_0) = r(v_0) = r_0. \tag{31}$$

On the other hand, $\frac{\partial}{\partial r} G_\varepsilon(r_*(v_n), v_n) = 0$, and so for every $n \in \mathbb{N}$

$$\bigg(\int_\Omega p(x)\big[|\nabla v_n|^{p(x)} + \lambda a(x)|v_n|^{p(x)}\big] r_*(v_n)^{p(x)}\,dx + \varepsilon \int_\Omega t(x)c(x)|v_n|^{t(x)} r_*(v_n)^{t(x)}\bigg)\bigg(\int_\Omega b(x)|v_n|^q r_*(v_n)^q\,dx\bigg)$$
$$= \bigg(\int_\Omega \big[|\nabla v_n|^{p(x)} + \lambda a(x)|v_n|^{p(x)}\big] r_*(v_n)^{p(x)}\,dx + \varepsilon \int_\Omega c(x)|v_n|^{t(x)} r_*(v_n)^{t(x)}\bigg)\bigg(\int_\Omega q(x)b(x)|v_n|^q r_*(v_n)^q\,dx\bigg),$$

which implies that

$$\big(q - p^-\big) \int_\Omega \big[|\nabla v_n|^{p(x)} + \lambda a(x)|v_n|^{p(x)}\big] r_*(v_n)^{p(x)}\,dx$$
$$\geqslant \varepsilon\big(t^- - q\big) \int_\Omega c(x)|v_n|^{t(x)} r_*(v_n)^{t(x)}\,dx. \tag{32}$$

In view of (8), (14), (31), (32)

$$M = \lim_{n\to+\infty} \Phi_\varepsilon\big(r(v_n)v_n\big)$$
$$\geqslant \lim_{n\to+\infty} \varepsilon\bigg(\frac{(q-p^+)(t^- - q)}{p^+(q-p^-)} - \frac{t^+ - q}{t^+}\bigg) \int_\Omega c(x)|v_n|^{t(x)} r_*(v_n)^{t(x)}\,dx \geqslant 0,$$

a contradiction. Thus $\mu > G_\varepsilon(r_*(v_0), v_0)$, that is $v_0 \in B_0^\varepsilon(\mu^*(\lambda,\varepsilon))$. Working as in the proof of Theorem 2 we conclude that $u := |r(v_0)v_0|$ is a solution to (1).

Assume next that $p(x) = p$ and $t(x) = t$ for every $x \in \overline{\Omega}$ while $q(.)$ varies with $x \in \overline{\Omega}$. By (14),

$$r^p \int_\Omega \big[|\nabla v|^{p(x)} + \lambda a(x)|v|^{p(x)}\big]\,dx + \varepsilon r^t \int_\Omega c(x)|v|^{t(x)}\,dx = \mu \int_\Omega b(x)|v|^{q(x)} r^{q(x)}\,dx, \tag{33}$$

and Lemma 4 implies that (33) admits exactly two solutions $r_1(v)$ and $r_2(v)$ with $r_1(v) < r_2(v)$. A nonnegative solution to (1) can be obtained by modifying the previous arguments to our current assumptions. $\square$

It is clear that if $c \equiv 0$ the previous theorem is not valid. In order to further study problem (1) we will examine the behavior of the solutions to (1) as $\varepsilon \to 0$. Let $\lambda > 0$, $\varepsilon_0 > 0$ and $\mu > \mu^*(\lambda, \varepsilon_0)$. Theorem 5 implies that (1) has a solution $u_0 = r_{\varepsilon_0}(v_{\varepsilon_0})v_{\varepsilon_0}$, $v_{\varepsilon_0} \in B_0^{\varepsilon}(\mu) \cap S^1$. Assume first that $q(.)$ is a constant. We claim that (1) admits a solution $u_\varepsilon = r_\varepsilon(v_\varepsilon)v_\varepsilon$ for every $\varepsilon \in (0, \varepsilon_0)$. Indeed, if $v \in B_0^{\varepsilon}(\mu)$ then $G_\varepsilon(r_*(v), v) \leqslant \max\{g_u^1(r_u^1, v), g_u^2(r_u^2, v)\}$, where r_u^1, r_u^2 are the points of global minimum for the functions $g_u^1(., v)$ and $g_u^2(., v)$. In view of (22) and (23),

$$g_u^1(r_u^1, v) = k_1 c_1(v)^{\frac{t^+-q}{t^+-p^+}} c_2(v)^{\frac{q-p^+}{t^+-p^+}} = \frac{k_1(\int_\Omega [|\nabla v|^{p(x)} + \lambda a(x)|v|^{p(x)}]\,dx)^{\frac{t^+-q}{t^+-p^+}} (\int_\Omega c(x)|v|^{t(x)}\,dx)^{\frac{q-p^+}{t^+-p^+}} \varepsilon^{\frac{q-p^+}{t^+-p^+}}}{\int_\Omega b(x)|v|^q\,dx}$$

and

$$g_u^2(r_u^2, v) = k_2 c_1(v)^{\frac{t^--q}{t^--p^-}} c_2(v)^{\frac{q-p^-}{t^--p^-}} = \frac{k_2(\int_\Omega [|\nabla v|^{p(x)} + \lambda a(x)|v|^{p(x)}]\,dx)^{\frac{t^--q^-}{t^--p^-}} (\int_\Omega c(x)|v|^{t(x)}\,dx)^{\frac{q-p^-}{t^--p^-}} \varepsilon^{\frac{q-p^-}{t^--p^-}}}{\int_\Omega b(x)|v|^q\,dx},$$

where k_1, k_2 are some positive constants independent of ε and v. Thus, $G_\varepsilon(r_*(v), v) \downarrow 0$ as $\varepsilon \downarrow 0$ and so $G_\varepsilon(r_*(v_{\varepsilon_0}), v_{\varepsilon_0}) < \mu$ for every $\varepsilon \in (0, \varepsilon_0)$. Consequently, $v_{\varepsilon_0} \in B_0^{\varepsilon}(\mu) \cap S^1$, and so $B_0^{\varepsilon}(\mu) \cap S^1 \neq \emptyset$. On the other hand, in view of (22) and (23), $r_\varepsilon(v_{\varepsilon_0}) \to +\infty$ as $\varepsilon \to 0$. By (30), $\widehat{\Phi}_\varepsilon(v_{\varepsilon_0}) < 0$ for every $\varepsilon \in (0, \varepsilon_0)$. Thus, $\inf\{\widehat{\Phi}_\varepsilon(v)\colon\ v \in S^1 \cap B_0^{\varepsilon}(\mu)\} < 0$ for every $\varepsilon \in (0, \varepsilon_0)$. By repeating the arguments in the proof of Theorem 2 we get that (1) has a solution $u_\varepsilon = r_\varepsilon(v_\varepsilon)v_\varepsilon$, $v_\varepsilon \in B_0^{\varepsilon}(\mu) \cap S^1$, for every $\varepsilon \in (0, \varepsilon_0)$. Since $r_\varepsilon(v_{\varepsilon_0}) \to +\infty$ as $\varepsilon \to 0$, (30) implies that $\widehat{\Phi}_\varepsilon(v_{\varepsilon_0}) \to -\infty$. Thus, $\Phi_\varepsilon(u_\varepsilon) = \widehat{\Phi}_\varepsilon(v_\varepsilon) \to -\infty$ as well. By (5) and (7) we get that $\|u_\varepsilon\|_E \to +\infty$ as $\varepsilon \to 0$. We are led to the same conclusions if we assume that $p(.)$ and $t(.)$ are constants while $q(.)$ varies with $x \in \overline{\Omega}$.

Therefore we have the following:

Theorem 6. *Suppose that hypotheses $H(1)$–$H(5)$ are satisfied with $p^+ < q^-$, $\lambda > 0$, $\varepsilon_0 > 0$ and $\mu > \mu^*(\lambda, \varepsilon_0)$. Then the problem (1) admits a solution u_ε for every $\varepsilon \in (0, \varepsilon_0)$ with $\|u_\varepsilon\|_E \to +\infty$ and $\Phi_\varepsilon(u_\varepsilon) \to -\infty$ as $\varepsilon \to 0$.*

Remark 7. The results presented here also hold if we assume that $t(.)$ is subcritical, that is $t(x) < \frac{Np(x)}{N-p(x)}$ in Ω, and in this case $E := W_0^{1,p(.)}(\Omega)$.

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