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Chaos among self-maps of the Cantor space ☆

Emma D'Aniello [a,*], Udayan B. Darji [b]

[a] Dipartimento di Matematica, Seconda Università degli Studi di Napoli, Via Vivaldi 43, 81100 Caserta, Italy
[b] Department of Mathematics, University of Louisville, Louisville, KY 40208-2772, USA

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ABSTRACT

Glasner and Weiss have shown that a generic homeomorphism of the Cantor space has zero topological entropy. Hochman has shown that a generic transitive homeomorphism of the Cantor space has the property that it is topologically conjugate to the universal odometer and hence far from being chaotic in any sense. We show that a generic self-map of the Cantor space has zero topological entropy. Moreover, we show that a generic self-map of the Cantor space has no periodic points and hence is not Devaney chaotic nor Devaney chaotic on any subsystem. However, we exhibit a dense subset of the space of all self-maps of the Cantor space each element of which has infinite topological entropy and is Devaney chaotic on some subsystem.

1. Introduction

In this note by a topological dynamical system we mean a pair (X, f) where X is a compact metric space and f is a continuous self-map. Many interesting properties of topological dynamical systems have been investigated through the years and one of them is that of chaos. There are numerous definitions of chaos but three notions most studied are positive topological entropy, chaos according to Devaney and chaos according to Li–Yorke. Topological entropy was the first among the three to be defined. It is a topological analogue of the important concept of entropy (in the measure theoretic sense). We leave the definition of entropy to a latter section as it is somewhat technical. The system (X, f) is chaotic according to Devaney if the mapping f is transitive and periodic points of f are dense in X. Often this definition is somewhat rigid so one only requires these properties on a subsystem, i.e., there is a compact set $Y \subseteq X$ such that $f|_Y$, the restriction of f to Y, is Devaney chaotic. Finally, f is chaotic according to Li–Yorke if there is an uncountable scrambled set S, i.e., for all distinct points $x, y \in S$ we have that $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$ and $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$. It is known that Devaney chaos implies Li–Yorke chaos [21,24]. It is also known that positive topological entropy implies Li–Yorke chaos [9]. In special cases such as when X is the unit interval we have that positive topological entropy implies Devaney chaos on a subsystem. However, in the general setting of compact metric spaces there are no implications between positive topological entropy and Devaney chaos on a subsystem.

Recently many authors have investigated the behavior of a generic dynamical system. More precisely, let us fix the compact space X and consider $\mathcal{H}(X)$, the set of all homeomorphisms of X, and $\mathcal{C}(X)$, the set of all continuous self-maps of X. Both of these spaces are Polish spaces, i.e., complete, separable and metric. Hence the Baire category theorem holds. We may ask what type of dynamical behavior is exhibited by a generic homeomorphism or a generic continuous map. Dynamical behavior of a generic homeomorphism is very thoroughly investigated in the monograph by Akin, Hurley and Kennedy [3].

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* Corresponding author.
E-mail addresses: emma.daniello@unina2.it (E. D'Aniello), ubdarj01@louisville.edu (U.B. Darji).

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The earliest result, to our knowledge, concerning chaotic behavior of generic maps is by Yano [26]. In 1980 he showed that a generic homeomorphism of a manifold of dimension at least 2 has the property that it has infinite topological entropy. He also showed that the same result holds for a generic self-map of a manifold of dimension at least 2. His idea was to show that a generic element of these spaces contains a horseshoe type structure. In 2001 Glasner and Weiss [18] defined the *topological Rohlin property* and showed that the group of homeomorphisms of S^d, $2 \leqslant d \leqslant \infty$, as well as the group of homeomorphisms of the Cantor space, has this property. Using this fact they concluded that the generic homeomorphism of S^d, $2 \leqslant d \leqslant \infty$, has infinite topological entropy. On the other hand, they showed that a generic homeomorphism of the Cantor space has zero topological entropy. In 2007 Kościelniak [23] showed that a generic homeomorphism of a manifold of dimension d, $2 \leqslant d < \infty$, has the property that some power of it is semi-conjugate to the shift map and has infinite topological entropy. Apparently, authors of [18] and [23] were not aware of the result of Yano as his paper is not cited in either article. In 2008 Hochman proved many interesting results concerning dynamical properties of a generic homeomorphism of the Cantor set [20]. Among these is the fact that a generic transitive homeomorphism of the Cantor space is conjugate to the universal adding machine and hence has zero topological entropy. In [15] it was shown that odometers appear in abundance as the ω-limit sets of generic maps on manifolds.

We now turn to Devaney chaos. Following the work of [1,22], Daalderop and Fokkink [14] showed that a generic measure-preserving homeomorphism on a compact d-dimensional manifold, $d \geqslant 2$, is chaotic in the sense of Devaney. Some of the results in [14] were generalized to σ-compact manifolds by Alpern and Prasad in [4]. Bernardes has obtained results similar in spirit (see [6–8]).

In this note we show that there is a dense set of the space of all continuous self-maps of the Cantor space each element of which has infinite topological entropy and is Devaney chaotic on a subsystem. However, a generic continuous map of the Cantor space is neither Devaney chaotic nor Devaney chaotic on any subsystem and has zero topological entropy.

We note that the situation on the interval is well understood. Of course, a homeomorphism of the interval is very simple from point of view of chaos: it is not chaotic in any sense. A generic continuous self-map of the interval has topological entropy infinite and hence it is chaotic on a Devaney subsystem and chaotic according to Li–Yorke [10].

2. Definitions and background material

A *topological dynamical system* for us is simply (X, f), where X is a compact metric space without isolated points and f is a continuous self-map of X. The system (X, f) is *transitive* if there is $x \in X$ such that $\{f^n(x)\colon n \in \mathbb{N}\}$, the forward orbit of x, is dense in X [13].

Throughout this article we use the two spaces X and $\mathcal{C}(X)$. For $x \in X$, $B_\epsilon(x)$ denotes the open ball of radius ϵ centered at x, that is $B_\epsilon(x) = \{y \in X\colon d(x, y) < \epsilon\}$. For f and h in $\mathcal{C}(X)$, we write $d(f, h) = \sup_{x \in X} d(f(x), h(x))$ and, hence, $B_\epsilon(f) = \{g \in \mathcal{C}(X)\colon d(f, g) < \epsilon\}$.

We say that the system (X, f) is *Devaney chaotic* if the following conditions hold:

(1) (X, f) is transitive;
(2) the set of periodic points of f is dense in X;
(3) f has a sensitive dependence on initial conditions, i.e., there is $\delta > 0$ such that for all $x \in X$ and $\epsilon > 0$, there exist $y \in X$ within ϵ of x and $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y))$, the distance between $f^n(x)$ and $f^n(y)$, is greater than δ.

It is well known that conditions (1) and (2) imply (3). For example, see [5], or [19], or [2]. Often, it is the case that (X, f) is not transitive but in some intuitive sense the system is still chaotic. To capture this idea, we will say that (X, f) *is Devaney chaotic on a subsystem* if there is a compact subset K of X such that $f(K) = K$ and $f|_K$, the restriction of f to K, is Devaney chaotic. We refer the reader to [16] for more information on Devaney chaos.

Another measure of chaos in topological dynamics is that of topological entropy. There is a significant difference between the dynamical systems which have positive topological entropy and the topological systems which have zero topological entropy. One often thinks of dynamical systems with positive topological entropy as chaotic. There are several equivalent definitions of topological entropy. We use the following due to Bowen [12] and Dinaburg [17].

Let (X, d) be a compact metric space and $f : X \to X$ be a continuous map. For each natural number n, a new metric d_n is defined on X by the formula

$$d_n(x, y) = \max\bigl\{d\bigl(f^i(x), f^i(y)\bigr)\colon 0 \leqslant i < n\bigr\}.$$

A subset E of X is said to be *(n, ϵ, f)-separated* if each pair of distinct points of E is at least ϵ apart in the metric d_n. Denote by $N(n, \epsilon, f)$ the maximum cardinality of an (n, ϵ, f)-separated set. The *topological entropy of the map f* is defined by

$$ent(f) = \lim_{\epsilon \to 0}\left(\limsup_{n \to \infty} \frac{1}{n} \log N(n, \epsilon, f)\right).$$

We recall [13] that in the above we may use lim inf, i.e.,

$$ent(f) = \lim_{\epsilon \to 0}\left(\liminf_{n \to \infty} \frac{1}{n} \log N(n, \epsilon, f)\right).$$

It is a basic fact that $ent(f)$ is a topological invariant, i.e., $ent(f)$ depends only on the topology of X and not on the metric which generates it. Another basic fact we will use is that if (X, f) and (Y, g) are topological dynamical systems, then $(X \times Y, f \times g)$ is a topological dynamical system which has the property that $ent(f \times g) = ent(f) + ent(g)$. Two maps $f, g : X \to X$ are *topologically conjugate* if there exists a homeomorphism k of X such that $f = k \circ g \circ k^{-1}$. It is also well known that for conjugate maps f and g, $ent(f) = ent(g)$. We also use the fact that $ent(f^n) = ent(f)^n$, for $n \in \mathbb{N}$. We refer the reader to [13] for more information on topological entropy.

We will say that a topological space is a *Cantor space* if it is homeomorphic to middle $\frac{1}{3}$-Cantor subset of the unit interval. It is well known that every non-empty, compact, perfect, zero-dimensional metric space is a Cantor space. We will use $2^{\mathbb{N}}$, the set of all infinite sequences of 0's and 1's, as our model of the Cantor space. We will use $\alpha, \beta, \ldots,$ etc. to denote elements of $2^{\mathbb{N}}$. The set $2^{<\mathbb{N}}$ will denote the set of all finite sequence of 0's and 1's. If $\alpha \in 2^{<\mathbb{N}}$, then

$$[\alpha] = \{\beta \in 2^{\mathbb{N}}\colon\ \alpha \text{ is an initial segment of } \beta\}.$$

We recall that $\{[a]\colon\ \alpha \in 2^{<\mathbb{N}}\}$ is a basis for the product topology on $2^{\mathbb{N}}$.

If X is any compact metric space, then we use $\mathcal{C}(X)$ to denote the set of all continuous self-map of X. We endow this set with the sup norm and hence it forms a complete, separable, metric space. Given any complete metric space Y, we say that *a generic element of Y has property P* if the set of elements of Y which have property P is *comeager in Y*, i.e., it contains a dense G_δ subset of Y, or, equivalently, the set of those elements of Y which do not satisfy property P is a *meager subset of Y*, i.e., it is the union of countably many nowhere dense sets. As Baire category theorem holds in every complete metric space, meager sets can be considered "topologically small" whereas comeager sets are "topologically large". See [25] for information on Baire category.

3. Main results

In this section we prove our main results.

3.1. Density of chaos

We first prove that the set of chaotic maps is dense in $\mathcal{C}(2^{\mathbb{N}})$.

The *shift map* $\Sigma : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is defined by

$$\Sigma\big(\alpha(1), \alpha(2), \ldots\big) = \big(\alpha(2), \alpha(3), \ldots\big).$$

It is well known that $ent(\Sigma) = \ln 2$. For example, see [13]. It is easy to see that the set of periodic sequence in $2^{\mathbb{N}}$ is dense in $2^{\mathbb{N}}$ and hence the set of periodic points of Σ is dense $2^{\mathbb{N}}$. Moreover, if $\alpha \in 2^{\mathbb{N}}$ is any element which contains all finite sequences of 0's and 1's as substrings, then the orbit of α under Σ is dense in $2^{\mathbb{N}}$. Hence Σ is transitive. Therefore, we have that Σ is Devaney chaotic.

We need the following lemma in order to prove the density of chaotic maps.

Lemma 3.1.1. *Let $n \in \mathbb{N}$. Then, there exists a map $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that $F : \{0, \ldots, n-1\} \times 2^{\mathbb{N}} \to \{0, \ldots, n-1\} \times 2^{\mathbb{N}}$ defined by $F(i, \alpha) = ((i+1) \bmod n, f(\alpha))$ has infinite topological entropy and is Devaney chaotic.*

Proof. As $(2^{\mathbb{N}})^{\mathbb{N}}$ is homeomorphic to $2^{\mathbb{N}}$, it suffices to construct $f : (2^{\mathbb{N}})^{\mathbb{N}} \to (2^{\mathbb{N}})^{\mathbb{N}}$ such that corresponding F defined on $\{0, \ldots, n-1\} \times (2^{\mathbb{N}})^{\mathbb{N}}$ has infinite topological entropy and is Devaney chaotic.

We define $f : (2^{\mathbb{N}})^{\mathbb{N}} \to (2^{\mathbb{N}})^{\mathbb{N}}$ by

$$f(\alpha_1, \alpha_2, \ldots) = \big(\Sigma(\alpha_1), \Sigma(\alpha_2), \ldots\big).$$

As f is simply the infinite product of Σ and $ent(\Sigma) = \ln 2$, it follows that $ent(f) = \infty$. Moreover, F^n restricted to $\{0\} \times (2^{\mathbb{N}})^{\mathbb{N}}$ is topologically conjugate to f^n and hence F^n has infinite topological entropy which, in turn, implies that F has infinite topological entropy.

We next show that F is Devaney chaotic. Let U be any open set in $\{0, \ldots, n-1\} \times (2^{\mathbb{N}})^{\mathbb{N}}$. We need to show that U contains a periodic point of F. We first observe that U has a subset of the form $\{i\} \times [\sigma_1] \times [\sigma_2] \times \cdots [\sigma_k] \times 2^{\mathbb{N}} \times 2^{\mathbb{N}} \times \cdots$ where $\sigma_1, \ldots, \sigma_k \in \{0, 1\}^j$ for some $j \in \mathbb{N}$. By extending j, if necessary, we may assume that j is a multiple of n. Now consider the point $(i, \beta_1, \beta_2, \ldots)$ where β_i is the periodic extension of σ_i for $1 \leqslant i \leqslant k$ and β_i the constant zero sequence for $i > k$. Then, $(i, \beta_1, \beta_2, \ldots) \in U$ and $F^j((i, \beta_1, \beta_2, \ldots)) = (i, \beta_1, \beta_2, \ldots)$.

We next show that F is transitive, i.e., there is a point in $\{0, \ldots, n-1\} \times (2^{\mathbb{N}})^{\mathbb{N}}$ whose orbit under F is dense in $\{0, \ldots, n-1\} \times (2^{\mathbb{N}})^{\mathbb{N}}$. To this end, enumerate the set of all matrices (of all possible dimensions) of 0's and 1's as $S_1, S_2, \ldots$. Consider the infinite matrix

$$A = \begin{pmatrix} S_1 & 0 & 0 & 0 \\ S_2 & 0 & 0 & 0 \\ S_3 & 0 & 0 & 0 \\ \dots & 0 & 0 & 0 \\ \vdots & . & . & . \end{pmatrix}.$$

Let us consider the point $(0, \alpha_1, \alpha_2, \dots) \in \{0, \dots, n-1\} \times (2^{\mathbb{N}})^{\mathbb{N}}$ so that α_i is the ith column of A. We claim that the orbit of this point under f is dense in $\{0, \dots, n-1\} \times (2^{\mathbb{N}})^{\mathbb{N}}$. Indeed, let $U = \{i\} \times [\sigma_1] \times [\sigma_2] \times \cdots \times [\sigma_k] \times 2^{\mathbb{N}} \times \cdots \times 2^{\mathbb{N}} \times \cdots$ be any open set where $\sigma_1, \dots, \sigma_k \in \{0,1\}^j$ for some j which is a multiple of n. Let S be the $j \times k$ matrix whose tth column is σ_t, i.e.,

$$S = (\sigma_1 \quad \sigma_2 \quad \dots \quad \sigma_k).$$

Now consider the following $(nj+n) \times k$ matrix T where

$$T = \begin{pmatrix} S \\ 0 \\ S \\ 0 \\ S \\ \vdots \\ 0 \end{pmatrix}.$$

We note that T appears in the infinite matrix A. Hence, there exits $l \in \mathbb{N}$ such that $F^l((0, \alpha_1, \alpha_2, \dots)) \in \{l \bmod n\} \times [\sigma_1] \times [\sigma_2] \times \cdots \times [\sigma_k] \times 2^{\mathbb{N}} \times \cdots$, and, moreover, $F^{l+mj+m}((0, \alpha_1, \alpha_2, \dots)) \in \{(l+m) \bmod n\} \times [\sigma_1] \times [\sigma_2] \times \cdots \times [\sigma_k] \times 2^{\mathbb{N}} \times \cdots$ for all $1 \leqslant m \leqslant n$. Therefore, there exits $1 \leqslant m_0 \leqslant n$ such that $F^{l+jm_0+m}((0, \alpha_1, \alpha_2, \dots)) \in \{i\} \times [\sigma_1] \times [\sigma_2] \times \cdots \times [\sigma_k] \times 2^{\mathbb{N}} \times \cdots$, as required. □

Theorem 3.1.2. *The collection*

$$\mathcal{E} = \{ f \in \mathcal{C}(2^{\mathbb{N}}) \colon \mathit{ent}(f) = +\infty \text{ and } f \text{ is Devaney chaotic on a subsystem} \}$$

is dense in $\mathcal{C}(2^{\mathbb{N}})$.

Proof. Let $g \in \mathcal{C}(2^{\mathbb{N}})$ and $\epsilon > 0$. We will show that $B_\epsilon(g) \cap \mathcal{E}$ is non-empty.

By Lemma 3.3 in [15] we may choose $g_1 \in B_{\frac{\epsilon}{3}}(g)$ such that the orbit of some point of $2^{\mathbb{N}}$ under g_1 is finite. Applying Lemma 3.4 of [15] to this g_1, we may choose $g_2 \in B_{\frac{\epsilon}{3}}(g_1)$ and pairwise disjoint non-empty clopen subsets $V_0, \dots, V_{n-1}$ of $2^{\mathbb{N}}$ such that for all $0 \leqslant i \leqslant n-1$, we have that

(1) $g_2(V_i) \subseteq V_{(i+1) \bmod n}$,
(2) $\mathit{diam}(V_i) < \frac{\epsilon}{3}$.

Let $F \colon \{0, \dots, n-1\} \times 2^{\mathbb{N}} \to \{0, \dots, n-1\} \times 2^{\mathbb{N}}$ be a function of the type described in the statement of Lemma 3.1.1. As each V_i is a Cantor space, we may find a homeomorphism $h \colon \bigcup_{i=0}^{n-1} V_i \to \{0, \dots, n-1\} \times 2^{\mathbb{N}}$ such that $h(V_i) = \{i\} \times 2^{\mathbb{N}}$ for all $0 \leqslant i \leqslant n-1$.

We now define g_3, a modification of g_2 on $\bigcup_{i=0}^{n-1} V_i$, by setting

$$g_3(x) = \begin{cases} g_2(x) & \text{if } x \in 2^{\mathbb{N}} \setminus \bigcup_{i=0}^{n-1} V_i, \\ h^{-1} \circ F \circ h(x) & \text{if } x \in \bigcup_{i=0}^{n-1} V_i. \end{cases}$$

We note that $g_3(V_i) \subseteq V_{(i+1) \bmod n}$ and hence $d(g_3, g_2) < \frac{\epsilon}{3}$. This, in turn, implies that $d(g_3, g) < \epsilon$. Furthermore, the restriction of g_3 to the union of $\bigcup_{i=0}^{n-1} V_i$ is topologically conjugate to F. Therefore, we have that the g_3 has infinite topological entropy and g_3 is Devaney chaotic on a subsystem, completing the proof of the theorem. □

3.2. Genericity of zero entropy maps

We now prove that a generic map in $\mathcal{C}(2^{\mathbb{N}})$ has zero topological entropy.

The following lemma was proved for homeomorphism in [18] using the covering definition of entropy. Their proof also holds for the space continuous self-maps of the Cantor space. However, we give a proof here using the alternate definition of entropy for the sake of completeness.

Lemma 3.2.1. *The collection $\mathcal{Z} = \{f \in \mathcal{C}(2^{\mathbb{N}}):\ ent(f) = 0\}$ is a G_δ subset of $\mathcal{C}(2^{\mathbb{N}})$.*

Proof. Fix $n \in \mathbb{N}$, $\epsilon > 0$ and $\eta > 0$. Note that

$$\mathcal{Z}(n, \epsilon, \eta) = \left\{ f \in \mathcal{C}\big(2^{\mathbb{N}}\big):\ \frac{\log N(n, \epsilon, f)}{n} < \eta \right\}$$

is an open subset of $\mathcal{C}(2^{\mathbb{N}})$. Let

$$\mathcal{Z}(\epsilon, \eta) = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \mathcal{Z}(n, \epsilon, \eta).$$

Clearly, $\mathcal{Z}(\epsilon, \eta)$ is a G_δ subset of $\mathcal{C}(2^{\mathbb{N}})$, and moreover, for each $f \in \mathcal{Z}(\epsilon, \eta)$ we have that

$$\liminf_{n\to\infty} \frac{1}{n} \log N(n, \epsilon, f) \leqslant \eta.$$

Hence, $\mathcal{Z}$ is a G_δ set as

$$\mathcal{Z} = \bigcap_{k=1}^{\infty} \bigcap_{j=1}^{\infty} \mathcal{Z}\left(\frac{1}{j}, \frac{1}{k}\right). \qquad \square$$

Lemma 3.2.2. *Let $K_1, \ldots, K_n$ be non-empty, pairwise disjoint, clopen subsets of $2^{\mathbb{N}}$. Then, there is a homeomorphism $h : \bigcup_{i=1}^{n} K_i \to \bigcup_{i=1}^{n} K_i$ such that $h^n = id$ and $h(K_i) = K_{(i+1) \bmod n}$.*

Proof. Recall that any two non-empty, clopen subsets of $2^{\mathbb{N}}$ are homeomorphic. Hence, we may choose homeomorphisms h_i from K_i onto K_{i+1}, $1 \leqslant i \leqslant n-1$. Let $h_n = h_1^{-1} \circ \cdots \circ h_{n-1}^{-1}$ and $h = h_i$ on K_i for $1 \leqslant i \leqslant n$. Then, h has the desired properties. $\square$

Lemma 3.2.3. *Let $f \in \mathcal{C}(2^{\mathbb{N}})$ and $M, p \in \mathbb{N}$ such that for all $\sigma \in 2^{\mathbb{N}}$ and $k \in \mathbb{N}$, we have that $f^{M+p+k}(\sigma) = f^{M+k}(\sigma)$. Then, $ent(f) = 0$.*

Proof. Note that for such an f we have that $d_n = d_{M+p}$ for all $n \geqslant M + p$ where, as earlier,

$$d_n(x, y) = \max\big\{d\big(f^i(x), f^i(y)\big):\ 0 \leqslant i < n\big\}.$$

From this, it follows that $N(n, \epsilon, f) = N(M + p, \epsilon, f)$ for all $n \geqslant M + p$ and $\epsilon > 0$. As

$$\limsup_{n\to\infty} \frac{1}{n} \log N(n, \epsilon, f) = \limsup_{n\to\infty} \frac{1}{n} \log N(M + p, \epsilon, f) = 0,$$

we have that $ent(f) = 0$. $\square$

Lemma 3.2.4. *The set $\{f \in \mathcal{C}(2^{\mathbb{N}}):\ ent(f) = 0\}$ is dense in $\mathcal{C}(2^{\mathbb{N}})$.*

Proof. Let $f \in \mathcal{C}(2^{\mathbb{N}})$ and $\eta > 0$. We will find a $g \in \mathcal{C}(2^{\mathbb{N}})$ within η of f such that $ent(g) = 0$.

Let $U_1, \ldots, U_t$ be pairwise disjoint clopen sets with diameter less than η such that

- $U_i \cap f(2^{\mathbb{N}}) \neq \emptyset$ for all $1 \leqslant i \leqslant t$, and
- $f(2^{\mathbb{N}}) \subseteq \bigcup_{i=1}^{t} U_i$.

Let $V_i = f^{-1}(U_i)$. It will suffice to a find a $g \in \mathcal{C}(2^{\mathbb{N}})$ with $g(V_i) \subseteq U_i$ such that $ent(g) = 0$.

For each $1 \leqslant i \leqslant t$, choose $\alpha(i)$ to be such that $U_i \cap V_{\alpha(i)} \neq \emptyset$. Let K_i be the non-empty clopen set $U_i \cap V_{\alpha(i)}$. Note that K_i has the property that $f(K_i) \subseteq U_{\alpha(i)}$. As α is a function from $\{1, \ldots, t\}$ into itself, there exists $i \in \{1, \ldots, t\}$ and p such that p is the least positive integer such that $\alpha^p(i) = i$, i.e., $i \to \alpha(i) \to \cdots \to \alpha^{p-1}(i) \to \alpha^p(i) = i$ is cycle. Let $i_1 \to \alpha(i_1) \to \cdots \to \alpha^{p_1-1}(i_1) \to \alpha^{p_1}(i_1) = i_1$ through $i_s \to \alpha(i_s) \to \cdots \to \alpha^{p_s-1}(i_s) \to \alpha^{p_s}(i_s) = i_s$ be an enumeration of all distinct such cycles. Note that as α is a function, no two distinct cycles have an element in common.

For each $1 \leqslant j \leqslant s$, using Lemma 3.2.2, we may choose a homeomorphism h_j defined on $\bigcup_{l=1}^{p_j} K_{\alpha^l(i_j)}$ such that $h_j(K_{\alpha^l(i_j)}) = K_{\alpha^{(l+1)}(i_j)}$, and moreover, $h_j^{p_j}$ is identity. Note that the domains of h_j's are disjoint as the corresponding cycles are disjoint. Hence, $h = \bigcup_{j=1}^{s} h_j$ is a well-defined homeomorphism defined on a clopen subset of $2^{\mathbb{N}}$ such that $h^p = id$ where $p = p_1 \ldots p_s$.

We now complete the definition of h. If $l \in \{1, \dots, t\}$ such that h is not yet defined on K_l, define h to be an arbitrarily homeomorphism so that $h(K_l) = K_{\alpha(l)}$. Finally, if $1 \leqslant i \leqslant t$ and $V_i \setminus (\bigcup_{l=1}^{t} K_l)$ is not empty, then define h to be any homeomorphism from this set onto K_i. Now, our definition of h is complete. It is clear that h is well defined and that h is a continuous mapping whose range is a subset of $\bigcup_{l=1}^{t} K_l$.

We next observe that h is within η of f. To this end, let $\sigma \in 2^{\mathbb{N}}$ and $1 \leqslant i \leqslant t$ be such that $\sigma \in V_i$. If $\sigma \in V_i \setminus (\bigcup_{l=1}^{t} K_l)$, then, by definition of h, $h(\sigma) \in K_i \subseteq U_i$ and hence the distance between $h(\sigma)$ and $f(\sigma)$ is less than η as $f(V_i) = U_i$. Now suppose that $\sigma \in K_j$ for some $1 \leqslant j \leqslant t$. Note that $K_j = U_j \cap V_l$ for some $1 \leqslant l \leqslant t$. As $\sigma \in K_j$ and $V_1, \dots, V_t$ are pairwise disjoint, we must have that $l = i$, i.e., $K_j = U_j \cap V_i$ and $\alpha(j) = i$. But recall that $h(K_j) = K_{\alpha(j)} = K_i \subseteq U_i$. Hence, again $h(\sigma) \in U_i$ and the distance between $h(\sigma)$ and $f(\sigma)$ is less than η.

Finally, let us proceed to show that $ent(h) = 0$. Let $M = t + 1$ and $p = p_1 \dots p_s$. It suffices to show that M, p and h satisfy the hypothesis of Lemma 3.2.3. To this end, let $\sigma \in 2^{\mathbb{N}}$. Let $1 \leqslant i \leqslant t$ be such that $h(\sigma) \in K_i$. Consider the sequence, $i, \alpha(i), \alpha^2(i), \dots, \alpha^t(i)$. Note that $h^l(\sigma) \in K_{\alpha^{l-1}(i)}$. As $i, \alpha(i), \alpha^2(i), \dots, \alpha^t(i)$ has length $t + 1$, it must intersect some cycle of α. Let $\alpha^j(i)$ be such an element and $p_{j'}$ be the length of the cycle of α to which $\alpha^j(i)$ belongs. By the construction of h, we have that $h^{p_{j'}}$ is identity on $K_{\alpha^j(i)}$. Since $p_{j'}$ divides p, we also have that h^p is identity on $K_{\alpha^j(i)}$. As $h^{j+1}(\sigma) \in K_{\alpha^j(i)}$, we have that $h^{p+j+1}(\sigma) = h^p(h^{j+1}(\sigma)) = h^{j+1}(\sigma)$. Now let $k \in \mathbb{N}$. Then,

$$h^{M+p+k}(\sigma) = h^{M-j-1+k}\big(h^{p+j+1}(\sigma)\big) = h^{M-j-1+k}\big(h^{j+1}(\sigma)\big) = h^{M+k}(\sigma),$$

completing the proof. □

Corollary 3.2.5. *A generic $f \in \mathcal{C}(2^{\mathbb{N}})$ has the property that $ent(f) = 0$.*

Proof. This simply follows from Lemmas 3.2.1 and 3.2.4. □

3.3. Genericity of non-chaotic Devaney maps

We prove that a generic map of the Cantor space has no fixed points and hence is not Devaney chaotic nor Devaney chaotic on any subsystem.

Let X be a compact metric space. For each $f \in \mathcal{C}(X)$ and $k \in \mathbb{N}$, we let $P_k(f) = \{x \in \mathcal{C}(X)\colon\ f^k(x) = x\}$ and $Per_k(f) = \{x \in \mathcal{C}(X)\colon\ x$ has period k under $f\}$. The set $P_k(f)$ is closed, however, $Per_k(f)$ need not be.

Lemma 3.3.1. *Suppose X is a compact metric space and $f \in \mathcal{C}(X)$, $k \in \mathbb{N}$ such that $P_k(f) = Per_k(f)$. Then, there is $\epsilon > 0$ such that for all $x \in Per_k(f)$ we have that $d(x, f^i(x)) > \epsilon$ for all $1 \leqslant i \leqslant k - 1$.*

Proof. To obtain a contradiction, assume that there is no such ϵ. Hence, we may find a sequence $\{x_n\}$ in X and $\{i_n\}$ in $\{1, \dots, k - 1\}$ such that $d(x_n, f^{i_n}(x_n)) < 2^{-n}$ for all $n \in \mathbb{N}$. Furthermore, since X is compact, we may pass through a subsequence and assume that $\{x_n\}$ converges to some point $x \in X$. As each $i_n \in \{1, \dots, k-1\}$, passing through a subsequence once again, we may assume that $i_n = i$ for all n, i.e., all i_n's are equal. By the continuity of f, we have that $f^i(x) = x$. Since $P_k(f)$ is closed and $\{x_n\}$ is in $P_k(f)$, we have that $x \in P_k(f)$. As $f^i(x) = x$ and $1 \leqslant i \leqslant k - 1$, we have that $x \notin Per_k(f)$, contradicting that $P_k(f) = Per_k(f)$. □

Lemma 3.3.2. *Let $V_0, \dots, V_{k-1}$ be non-empty pairwise disjoint clopen subsets of $2^{\mathbb{N}}$. Then, there is a homeomorphism f defined on $\bigcup_{i=0}^{k-1} V_i$ such that $f(V_i) = V_{(i+1) \bmod k}$ and f has no periodic point.*

Proof. Consider the odometer $\mathbb{Z}_k^{\mathbb{N}}$ with g being the "+1" map. See Ref. [11] for more details. This map g has the property that g has no periodic points and $g(\{i\} \times \mathbb{Z}_k^{\mathbb{N}}) = \{(i+1) \bmod k\} \times \mathbb{Z}_k^{\mathbb{N}}$. Let $h_i : V_i \to \{i\} \times \mathbb{Z}_k^{\mathbb{N}}$ be a homeomorphism and define $f = h_{i+1}^{-1} \circ g \circ h_i$ on each V_i. Then, we have that $f(V_i) = V_{i+1}$ and as f is conjugate to g, f has no periodic points. □

Lemma 3.3.3. *Let $f \in \mathcal{C}(2^{\mathbb{N}})$, $k \in \mathbb{N}$ such that $P_k(f) = Per_k(f)$. Then, for every $\delta > 0$, there exists $g \in B_\delta(f)$ such that $P_k(g) = \emptyset$.*

Proof. Let ϵ be as in the proof of Lemma 3.3.1 and let $0 < \eta < \min\{\frac{\delta}{4}, \frac{\epsilon}{2}\}$ be such that if $x, y \in 2^{\mathbb{N}}$ and $d(x, y) < \eta$, then $d(f(x), f(y)) < \frac{\delta}{4}$. Let $U_1, \dots U_m$ be pairwise disjoint clopen subsets of $2^{\mathbb{N}}$ such that $diam(U_i) < \eta$, $U_i \cap P_k(f) \neq \emptyset$ for all $1 \leqslant i \leqslant t$ and $P_k(f) \subseteq \bigcup_{i=1}^{m} U_i$.

We will describe a sequence of modifications of f which will yield the desired g. Here is how we proceed at the first step. Let $x \in U_1 \cap Per_k(f)$. Choose $r_0^1, \dots, r_{k-1}^1$ so that $f^i(x) \in U_{r_i^1}$ for all $0 \leqslant i \leqslant k - 1$. By our choice of ϵ, we have that $U_{r_i^1} \cap U_{r_j^1} = \emptyset$ if $i \neq j$. Now using Lemma 3.3.2 we choose a continuous function g_1 defined on $\bigcup_{i=0}^{k-1} U_{r_i^1}$ such that

$g_1(U_{r_i^1}) = U_{r^1_{(i+1) \bmod k}}$ and g_1 has no periodic point. Let $g_1 = f$ on the rest of $2^{\mathbb{N}}$. We first note that $g_1 \in B_\delta(f)$. Indeed, let $t \in U_{r_i^1}$ for some $0 \leqslant i \leqslant k-1$. For the sake of notational convenience, let $x_i = f^i(x)$ and note that $x_i \in U_{r_i^1}$. Then,

$$d\big(f(t), g(t)\big) \leqslant d\big(f(t), f(x_i)\big) + d\big(f(x_i), g(x_i)\big) + d\big(g(x_i), g(t)\big) \leqslant \frac{\delta}{4} + \frac{\delta}{4} + \frac{\delta}{4} < \delta.$$

We note that if $t \in 2^{\mathbb{N}}$ such that $g_1^j(t) \in \bigcup_{i=0}^{k-1} U_{r_i^1}$ for some $j \in \mathbb{N}$ then t is not a periodic point of g_1. This observation implies that if t is a periodic point of g_1, then the orbit of t under g_1 is exactly the orbit of t under f. Hence, $P_k(g_1) \subseteq P_k(f)$ and $P_k(g_1) = Per_k(g_1)$. At the end of the first step, we have a function $g_1 \in B_\delta(f)$ which has no periodic point in U_1 and $P_k(g_1) \subseteq \bigcup_{i=1}^m U_i$. We end the first stage by setting $V_1 = \bigcup_{i=0}^{k-1} U_{r_i^1}$. The successive functions will leave g_1 unchanged on V_1. Thus we have removed a portion of $P_k(f)$ without introducing new periodic points.

We now show how to proceed at the second stage. The rest of the construction will be analogous to this stage. Choose the least integer n such that U_n contains a point of $P_k(g_1)$. Note that if there is no such n, i.e., $P_k(g_1) = \emptyset$, then $g = g_1$ is the desired final function. Otherwise, label this n as r_0^2 and choose a point $x \in U_{r_0^2} \cap P_k(g_1)$. As before we choose $r_0^2, \dots, r_{k-1}^2$ so that $g_1^i(x) = f^i(x) \in U_{r_i^2}$ for all $0 \leqslant i \leqslant k-1$. By our choice of ϵ, we have that $U_{r_i^2} \cap U_{r_j^2} = \emptyset$ if $i \neq j$. Let $U'_{r_i^2} = U_{r_i^2} \setminus V_1$. As the orbit of x under g_1 does not intersect V_1, we have that each of $U'_{r_i^2}$ is non-empty. Now using Lemma 3.3.2 we choose a continuous function g_2 defined on $\bigcup_{i=0}^{k-1} U'_{r_i^2}$ such that $g_2(U'_{r_i^2}) = U'_{r^2_{(i+1) \bmod k}}$ and g_2 has no periodic point. Let $g_2 = g_1$ on the rest of $2^{\mathbb{N}}$. Arguing as before, we have that $g_2 \in B_\delta(f)$. We let $V_2 = V_1 \cup (\bigcup_{i=0}^{k-1} U'_{r_i^2})$. We note that g_2 has no periodic point in V_2, $U_1 \cup U_2 \subseteq V_2$ and if t is a periodic point of g_2, then the orbit of t under g_2 is exactly the orbit of t under f. Hence at the end of the second stage, we have constructed g_2, an approximation of f, which has no periodic point in $U_1 \cup U_2$ and $P_k(g_2) \subseteq \bigcup_{i=1}^m U_i$.

This process has to terminate at some step $l \leqslant m$. Setting $g = g_l$ we obtain the desired function. □

Lemma 3.3.4. *Let X be a compact metric space, $k \in \mathbb{N}$ and $\mathcal{A}_k = \{f \in \mathcal{C}(X)\colon\ P_k(f) \neq \emptyset\}$. Then, $\mathcal{A}_k$ is closed in $\mathcal{C}(X)$.*

Proof. This simply follows from the fact that the topology on $\mathcal{C}(X)$ is that of uniform convergence and X is compact. □

Theorem 3.3.5. *A generic $f \in \mathcal{C}(2^{\mathbb{N}})$ has the property that f has no periodic point.*

Proof. It suffices to prove that for each $k \in \mathbb{N}$ set $\mathcal{A}_k$ defined in Lemma 3.3.4 is nowhere dense. To this end, fix $k \in \mathbb{N}$, $f \in \mathcal{A}_k$ and $\epsilon > 0$. We will find $g \in B_\epsilon(f)$ such that $P_k(g) = \emptyset$. We construct such a g in the following fashion. First observe that $P_1(f) = Per_1(f)$. Apply Lemma 3.3.3, we may choose $g_1 \in B_{\frac{\epsilon}{k}}(f)$ such that $P_1(g_1) = \emptyset$. Now suppose for $1 < i < k$, g_i has been constructed so that $g_i \in B_{\frac{\epsilon}{k}}(g_{i-1})$ and $P_1(g_i) = \cdots = P_i(g_i) = \emptyset$. We note that $P_{i+1}(g_i) = Per_{i+1}(g_i)$ as $P_i(g_i) = \emptyset$. Applying Lemma 3.3.3 again, we may find g_{i+1} so that $g_{i+1} \in B_{\frac{\epsilon}{k}}(g_i)$ and $P_{i+1}(g_{i+1}) = \emptyset$. Setting $g = g_k$, the proof of the theorem is concluded. □

Corollary 3.3.6. *A generic $f \in \mathcal{C}(2^{\mathbb{N}})$ is not Devaney chaotic nor Devaney chaotic on any subsystem.*

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