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Nonstandard principles for generalized functions

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ABSTRACT

We show that principles from nonstandard analysis hold to some extent for nonlinear generalized functions. The generalized functions under consideration are constructed as families of functions modulo a free filter, as it is usually done in applied analysis. In contrast with models of nonstandard analysis, we do not require the filter to be an ultrafilter. The principles are intended to be used as a tool for proving theorems, which we illustrate by means of an automatic continuity result that was not suspected by experts in the field.

1. Introduction

During the past decades, algebras of nonlinear generalized functions have been developed as a framework for modeling and understanding nonlinear partial differential equations and differential geometry with singular data [3,4,6,13]. As a rule, nonlinear generalized functions are constructed as equivalence classes of families of smooth functions. In contrast with distribution theory, such generalized functions can be viewed as pointwise functions acting on generalized points. The similarity with the generalized objects in nonstandard analysis has been observed in an early stage [11,13]. More recently, a number of fundamental tools for nonlinear generalized functions like *internal sets* and a *saturation principle* have been developed in a publication in this journal [16]. Unlike the objects in nonstandard analysis [17], nonlinear generalized functions are usually not constructed as families of smooth functions modulo a free ultrafilter. One can however view them naturally as families of smooth functions modulo a free filter, usually with a further identification, e.g. by means of certain growth conditions.

It is the goal of this paper to develop a number of principles known from nonstandard analysis (transfer, internal definition, countable saturation, spilling principles) in the more general setting of families modulo a free filter, relevant in practice for the theory of nonlinear generalized functions. Because of the more general setting, some of the principles only hold in a restricted form, but, contrary to what one could perhaps expect, transfer (e.g.) does *not* break down to the extent that it would become useless. We illustrate this by showing a result that came as a surprise to experts in the nonlinear theory of generalized functions (Theorem 7.5).

In fact, our setting is the same as Schmieden and Laugwitz's [18], in which such principles to our knowledge have not been investigated. The reason for this probably is the success of the corresponding theory using ultrafilters (i.e., nonstandard analysis), in which stronger versions of the principles hold, giving rise to a more elegant theory. In this context, we want to emphasize that the current paper does not intend to advocate the use of free filters instead of free ultrafilters. On the contrary, we hope that this paper will increase the awareness amongst researchers in the theory of nonlinear generalized functions of the usefulness of nonstandard ideas and the potential that nonstandard theories [15,19] may have to offer. We should also remark that the status of the generalized objects in nonstandard analysis often is one of idealized, 'auxiliary'

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objects, used to facilitate proofs about nongeneralized objects, whereas the nonlinear generalized functions are 'legitimate' objects in themselves, used as models for real world phenomena. As a result, it may be considered natural that, if a family $(f_\varepsilon)_{\varepsilon\in(0,1]}$ of functions represents the generalized function 0, at least $f_\varepsilon \to 0$ should hold as $\varepsilon \to 0$, a property that can be easily achieved modulo a free filter, but not modulo a free ultrafilter. Also, sometimes properties can more easily be shown modulo certain free filters than modulo a free ultrafilter (Theorem 7.9).

2. Generalized objects

For the definition of a filter, we refer to books on set theory or topology (e.g. [7–9]). A filter $\mathcal{F}$ on a set I is free if $\bigcap_{S\in\mathcal{F}} S = \emptyset$. A formula P_ε depending on $\varepsilon \in I$ holds a.e. iff $\{\varepsilon \in I\colon\ P_\varepsilon\}$ belongs to $\mathcal{F}$.

Throughout this paper, we fix an infinite index set I and a free filter $\mathcal{F}$ on I.

In particular, for applications to nonlinear generalized function theory, one can keep in mind the choice

$$I = \mathbb{N} \quad \text{with } \mathcal{F} = \{S \subseteq \mathbb{N}\colon\ \mathbb{N} \setminus S \text{ is finite}\} \tag{1}$$

($\mathcal{F}$ is the so-called Fréchet-filter) or

$$I = (0,1] \quad \text{with } \mathcal{F} = \bigl\{S \subseteq (0,1]\colon\ \bigl(\exists \eta \in (0,1]\bigr)\ (0,\eta) \subseteq S\bigr\}. \tag{2}$$

Then a property P_ε depending on $\varepsilon \in \mathbb{N}$ (resp. $\varepsilon \in (0,1]$) holds a.e. iff P_ε holds for sufficiently large $\varepsilon \in \mathbb{N}$ (resp. for sufficiently small $\varepsilon \in (0,1]$).

For the sake of generality, we develop the theory for any free filter $\mathcal{F}$ on any infinite index set (hence also including the case of a free ultrafilter $\mathcal{F}$; only starting from Section 5, we will impose extra conditions on $\mathcal{F}$).[1]

As in nonstandard analysis (i.e., the case in which $\mathcal{F}$ is an ultrafilter), we define generalized real numbers as elements of ${}^*\mathbb{R} := \mathbb{R}^I/\mathcal{F}$: families $(a_\varepsilon)_{\varepsilon\in I}$ of real numbers modulo $\mathcal{F}$. Hence by definition, for the equivalence classes $[a_\varepsilon]$, $[b_\varepsilon]$, we have

$$[a_\varepsilon] = [b_\varepsilon] \quad \iff \quad a_\varepsilon = b_\varepsilon \text{ a.e.}$$

Further, we inductively define so-called *internal* objects:

1. By definition, elements of ${}^*\mathbb{R}$ are internal objects.
2. Let $m \in \mathbb{N}$. If $[a_{1,\varepsilon}], \dots, [a_{m,\varepsilon}]$ are internal objects, then
$$\bigl[(a_{1,\varepsilon}, \dots, a_{m,\varepsilon})\bigr] := \bigl([a_{1,\varepsilon}], \dots, [a_{m,\varepsilon}]\bigr)$$
is an internal object.
3. If A_ε are nonempty sets (for each $\varepsilon \in I$) such that for each choice of $a_\varepsilon \in A_\varepsilon$, $[a_\varepsilon]$ is an internal object, then
$$[A_\varepsilon] := \bigl\{[a_\varepsilon]\colon\ a_\varepsilon \in A_\varepsilon \text{ a.e.}\bigr\}$$
is an internal object.

Any internal object is defined by applying these rules finitely many times.

In accordance with mathematical practice in analysis, we do not consider tuples to be sets (set-theorists will e.g. use the Kuratowski definition $(a,b) := \{\{a\},\{a,b\}\}$). Hence $[(a_\varepsilon, b_\varepsilon)]$ cannot be mistaken for an internal object defined by a family of sets (which, if well defined, yields another definition, unless $\mathcal{F}$ is an ultrafilter).

As in nonstandard analysis, we also define ${}^*a := [a]$ (the internal object corresponding to the constant family $(a)_\varepsilon$). In this text, a *nongeneralized object* is (by definition) an object a for which ${}^*a$ is well defined.

Remark 2.1. The map $*$ in this paper is a restriction of the map $*$ defined in nonstandard analysis. We can see this more explicitly as follows. Let $\mathcal{P}_\emptyset(A) := \{B \subseteq A\colon\ B \neq \emptyset\}$. Given a set X, let $\mathbb{S}$ be the smallest set satisfying

1. $X \in \mathbb{S}$,
2. if $Y \in \mathbb{S}$, then also $\mathcal{P}_\emptyset(Y) \in \mathbb{S}$,
3. if $Y_1, \dots, Y_n \in \mathbb{S}$ (for some $n \in \mathbb{N}$), then also $Y_1 \times \cdots \times Y_n \in \mathbb{S}$.

Then the *restricted superstructure* of X is the set $\widehat{X} := \bigcup_{Y\in\mathbb{S}} Y$, i.e.,

$$\widehat{X} = X \cup \mathcal{P}_\emptyset(X) \cup \bigcup_{n\in\mathbb{N}} X^n \cup \mathcal{P}_\emptyset\bigl(\mathcal{P}_\emptyset(X)\bigr) \cup \bigcup_{n\in\mathbb{N}} \mathcal{P}_\emptyset\bigl(X^n\bigr) \cup \bigcup_{n,m\in\mathbb{N}} X^n \times \mathcal{P}_\emptyset\bigl(X^m\bigr) \cup \cdots.$$

[1] A tutorial text for the use of nonstandard principles in generalized function theory intended for researchers in the nonlinear theory of generalized functions, focusing on the filter (2) and with additional examples can be found on `http://arxiv.org/abs/1101.6075`. For comparison, tutorial texts on nonstandard analysis are e.g. [2,12].

The set $\widehat{\mathbb{R}}$ is the set of nongeneralized objects. The internal objects in this paper form a subset of $\widehat{{}^*\mathbb{R}}$. (If $\mathcal{F}$ is an ultrafilter, the internal objects in this paper are exactly those internal objects from nonstandard analysis that belong to $\widehat{{}^*\mathbb{R}}$.) The map $*$ is a (non-surjective) map $\widehat{\mathbb{R}} \to \widehat{{}^*\mathbb{R}}$.

Hence (in contrast with the superstructure from nonstandard analysis), informally speaking, a set $A \in \widehat{X}$ can only contain elements 'of the same type': A cannot contain both elements of X and subsets of X, nor can A contain both subsets of X and functions $X \to X, \ldots$.

Lemma 2.2. *Let $[a_\varepsilon]$, $[b_\varepsilon]$ be internal objects. Then:*

1. $[a_\varepsilon] = [b_\varepsilon]$ *iff* $a_\varepsilon = b_\varepsilon$ *a.e.*
2. $[a_\varepsilon] \in [b_\varepsilon]$ *iff* $a_\varepsilon \in b_\varepsilon$ *a.e.*

Proof. 1. We proceed by induction. Equality in ${}^*\mathbb{R}$ is by definition equality a.e. on representatives. For m-tuples, we find by induction

$$\begin{aligned}
&\big[(a_{1,\varepsilon}, \ldots, a_{m,\varepsilon})\big] = \big[(b_{1,\varepsilon}, \ldots, b_{m,\varepsilon})\big] \\
&\quad\iff\quad [a_{j,\varepsilon}] = [b_{j,\varepsilon}], \quad \text{for } j = 1, \ldots, m \\
&\quad\iff\quad a_{j,\varepsilon} = b_{j,\varepsilon} \text{ a.e.}, \quad \text{for } j = 1, \ldots, m \\
&\quad\iff\quad (a_{1,\varepsilon}, \ldots, a_{m,\varepsilon}) = (b_{1,\varepsilon}, \ldots, b_{m,\varepsilon}) \text{ a.e.}
\end{aligned}$$

For nonempty sets A_ε, B_ε, if $A_\varepsilon = B_\varepsilon$ a.e., then by definition $[A_\varepsilon] = [B_\varepsilon]$. The converse statement follows if we show that $[A_\varepsilon] \subseteq [B_\varepsilon]$ implies that $A_\varepsilon \subseteq B_\varepsilon$ a.e.:

Choose $x_\varepsilon \in A_\varepsilon \setminus B_\varepsilon$, if $A_\varepsilon \nsubseteq B_\varepsilon$, and $x_\varepsilon \in A_\varepsilon$, if $A_\varepsilon \subseteq B_\varepsilon$. Then $[x_\varepsilon] \in [A_\varepsilon] \subseteq [B_\varepsilon]$, so $x_\varepsilon \in B_\varepsilon$ a.e. By the choice of x_ε, this implies that $A_\varepsilon \subseteq B_\varepsilon$ a.e.

2. By the definition of an internal set (rule 3). □

Remark 2.3. If we would allow $\varnothing$ as an internal object, the previous lemma would not hold. This motivates our choice to exclude $\varnothing$ from the restricted superstructure $\widehat{{}^*\mathbb{R}}$.

In order to incorporate m-ary relations R with domain D, we identify (as usual in set theory and nonstandard analysis) R with its graph $G_R = \{(x_1, \ldots, x_m) \in D\colon\ R(x_1, \ldots, x_m)\}$. For a family $(R_\varepsilon)_{\varepsilon \in I}$ of relations with domains $A_{1,\varepsilon} \times \cdots \times A_{m,\varepsilon}$ (with $[A_{j,\varepsilon}]$ internal sets, $j = 1, \ldots, m$), we therefore have

$$[G_{R_\varepsilon}] = \big\{\big[(x_{1,\varepsilon}, \ldots, x_{m,\varepsilon})\big]\colon\ R_\varepsilon(x_{1,\varepsilon}, \ldots, x_{m,\varepsilon}) \text{ a.e.}\big\}$$

which is the graph of a relation, denoted by $[R_\varepsilon]$, with domain $[A_{1,\varepsilon}] \times \cdots \times [A_{m,\varepsilon}]$ and

$$[R_\varepsilon]\big([x_{1,\varepsilon}], \ldots, [x_{m,\varepsilon}]\big) \quad\iff\quad R_\varepsilon(x_{1,\varepsilon}, \ldots, x_{m,\varepsilon}) \text{ a.e.}$$

Similarly, we consider a map $f: A \to B$ as a particular binary relation: $R(a, b)$ iff $f(a) = b$. The map f is thus identified with its graph $G_f = \{(x, f(x))\colon\ x \in A\}$. A family $(f_\varepsilon)_\varepsilon$ of maps $f_\varepsilon: A_\varepsilon \to B_\varepsilon$ (with $[A_\varepsilon]$, $[B_\varepsilon]$ internal sets) defines therefore a map $[f_\varepsilon]: [A_\varepsilon] \to [B_\varepsilon]$ with

$$[f_\varepsilon]\big([x_\varepsilon]\big) = \big[f_\varepsilon(x_\varepsilon)\big].$$

Remark 2.4. The internal subsets of a given internal set X, together with $\varnothing$, form a Boolean algebra under the operations $A \wedge B := A \cap B$, $A \vee B := [A_\varepsilon \cup B_\varepsilon]$ and $A' := [X_\varepsilon \setminus A_\varepsilon]$ (with $X = [X_\varepsilon]$, $A = [A_\varepsilon]$, $B = [B_\varepsilon]$). Notice that $A \cup B \subseteq A \vee B$ and $A' \subseteq X \setminus A$, but that $A \cup B$ and $X \setminus A$ are in general not internal, unless $\mathcal{F}$ is an ultrafilter.

3. Transfer

As in nonstandard analysis, we will proceed to show a transfer principle, i.e., for certain statements $P(a_1, \ldots, a_m)$ involving (nongeneralized) objects a_j, we generally have that $P(a_1, \ldots, a_m)$ is true iff $P({}^*a_1, \ldots, {}^*a_m)$ is true.

First, we define the formal language containing the statements that we will consider.

The language contains *variables* and *function variables*.

Inductively, *terms* are defined by the following rules:

- T1. A variable is a term.
- T2. If $t_1, \ldots, t_m$ are terms ($m > 1$), then also the m-tuple $(t_1, \ldots, t_m)$ is a term.
- T3. If t is a term and f is a function variable, then also $f(t)$ is a term.

Inductively, *formulas* are defined by the following rules:

F1. (Atomic formulas) If t_1, t_2 are terms, then $t_1 = t_2$, and $t_1 \in t_2$ are formulas.
F2. If P, Q are formulas, then $P \wedge Q$ is a formula.
F3. If P is a formula, x is a variable free in P and t is a term in which x does not occur, then $(\exists x \in t)\ P$ is a formula.
F4. If P is a formula, x is a variable free in P and t is a term in which x does not occur, then $(\forall x \in t)\ P$ is a formula.
F5. If P, Q are formulas, then $P \Longrightarrow Q$ is a formula.
F6. If P is a formula, then $\neg P$ is a formula.
F7. If P, Q are formulas, then $P \vee Q$ is a formula.

A *sentence* is a formula in which all occurring free variables are substituted by objects, which we call the *constants* or *parameters* of the sentence. The meaning associated to a sentence is given by the natural semantics. We introduce brackets in formulas to make clear the precedence of the operations. It is silently understood that function variables are substituted in such a way that the objects to which a substituted function f is applied are within the domain of f.

Notation. We denote $t(x_1, \dots, x_m)$ (or shortly $t(x_j)$) for a term t in which the only occurring variables are $x_1, \dots, x_m$. We denote by $t(c_1, \dots, c_m)$ (or shortly $t(c_j)$) the term t in which the variable x_j has been substituted by the object c_j (for $j = 1, \dots, m$).

Similarly, we denote $P(x_1, \dots, x_m)$ (or shortly $P(x_j)$) for a formula P in which the only occurring *free* variables are $x_1, \dots, x_m$. We denote by $P(c_1, \dots, c_m)$ (or shortly $P(c_j)$) the formula P in which the variable x_j has been substituted by the object c_j (for $j = 1, \dots, m$).

In order for transfer to be valid, we do not consider, in accordance with mathematical practice in analysis, real numbers as sets (equivalences of Cauchy sequences of rational numbers, e.g.). Similarly, we do not consider generalized real numbers as sets.[2] This avoids that sentences involving elements of real numbers (in which one is anyway not interested in analysis) like $(\exists x \in \mathbb{Q}^{\mathbb{N}})\ (x \in 1)$ would complicate the transfer principle. With this convention, internal sets contain only internal elements. We will also identify ${}^*a \in {}^*\mathbb{R}$ with $a \in \mathbb{R}$ and hence consider $\mathbb{R} \subseteq {}^*\mathbb{R}$.

Definition 3.1. A formula $P(x_j)$ is called *transferrable* if for all internal objects $[c_{j,\varepsilon}]$,

$$P(c_{j,\varepsilon}) \text{ is true a.e.}$$

is equivalent with

$$P\big([c_{j,\varepsilon}]\big) \text{ is true.}$$

Lemma 3.2. *Let $t(x_j)$ be a term formed by rules* T1–T3. *For internal objects $[c_{j,\varepsilon}]$,*

$$\big[t(c_{j,\varepsilon})\big] = t\big([c_{j,\varepsilon}]\big).$$

Proof. T1. If t is a variable, this is clear.

T2. Let $t_1, \dots, t_m$ be terms. For the term $(t_1, \dots, t_m)$, we find inductively,

$$\begin{aligned}\big[(t_1, \dots, t_m)(c_{j,\varepsilon})\big] &= \big[\big(t_1(c_{j,\varepsilon}), \dots, t_m(c_{j,\varepsilon})\big)\big] = \big(\big[t_1(c_{j,\varepsilon})\big], \dots, \big[t_m(c_{j,\varepsilon})\big]\big) \\ &= \big(t_1\big([c_{j,\varepsilon}]\big), \dots, t_m\big([c_{j,\varepsilon}]\big)\big) = (t_1, \dots, t_m)\big([c_{j,\varepsilon}]\big).\end{aligned}$$

T3. Let $t(x_j)$ be a term and f a function variable. For the term $f(t)$, we find inductively (with $[\phi_\varepsilon]$ an internal function),

$$\big[f(t)(\phi_\varepsilon, c_{j,\varepsilon})\big] = \big[\phi_\varepsilon\big(t(c_{j,\varepsilon})\big)\big] = [\phi_\varepsilon]\big(\big[t(c_{j,\varepsilon})\big]\big) = [\phi_\varepsilon]\big(t\big([c_{j,\varepsilon}]\big)\big) = f(t)\big([\phi_\varepsilon], [c_{j,\varepsilon}]\big). \qquad \square$$

Proposition 3.3. *Let $P(x_j)$ be a formula formed by applying rules* T1–T3, F1–F4 *only. Then $P(x_j)$ is transferrable.*

Proof. F1. For atomic formulas, this follows immediately from Lemmas 2.2 and 3.2.

We proceed by induction for more general formulas. We put $c_j := [c_{j,\varepsilon}]$.

F2. For a formula of the form $P(x_j) \wedge Q(x_j)$, we find inductively,

$$\begin{aligned}&P(c_j) \wedge Q(c_j) \text{ is true} \\ &\quad\Longleftrightarrow \quad P(c_{j,\varepsilon}) \text{ is true a.e., and } Q(c_{j,\varepsilon}) \text{ is true a.e.} \\ &\quad\Longleftrightarrow \quad P(c_{j,\varepsilon}) \wedge Q(c_{j,\varepsilon}) \text{ is true a.e.}\end{aligned}$$

[2] In order to realize this within set theory, one identifies $\mathbb{R}$ and ${}^*\mathbb{R}$ with sets of atoms [8].

F3. For a formula of the form $(\exists x \in t(x_j))\ P(x, x_j)$, we find inductively,

$$\begin{aligned} &\bigl(\exists x \in t(c_j)\bigr)\ P(x, c_j) \text{ is true} \\ &\quad\Longleftrightarrow\quad \text{there exists } c \in t(c_j) \text{ such that } P(c, c_j) \text{ is true} \\ &\quad\Longleftrightarrow\quad \text{there exists } (c_\varepsilon)_\varepsilon \text{ with } c_\varepsilon \in t(c_{j,\varepsilon}) \text{ a.e. such that } P(c_\varepsilon, c_{j,\varepsilon}) \text{ is true a.e.} \\ &\quad\Longleftrightarrow\quad \bigl(\exists x \in t(c_{j,\varepsilon})\bigr)\ P(x, c_{j,\varepsilon}) \text{ is true a.e.} \end{aligned}$$

F4. For a formula of the form $(\forall x \in t(x_j))\ P(x, x_j)$, we find inductively,

$$\begin{aligned} &\bigl(\forall x \in t(c_j)\bigr)\ P(x, c_j) \text{ is true} \\ &\quad\Longleftrightarrow\quad \text{for each } [c_\varepsilon] \text{ with } c_\varepsilon \in t(c_{j,\varepsilon}) \text{ a.e., } P\bigl([c_\varepsilon], c_j\bigr) \text{ is true} \\ &\quad\Longleftrightarrow\quad \text{if } c_\varepsilon \in t(c_{j,\varepsilon}) \text{ a.e., then } P(c_\varepsilon, c_{j,\varepsilon}) \text{ is true a.e.} \end{aligned}$$

We show that this is still equivalent with: $(\forall x \in t(c_{j,\varepsilon}))\ P(x, c_{j,\varepsilon})$ is true a.e.
$\Longrightarrow$: Choose

$$\begin{aligned} c_\varepsilon \in t(c_{j,\varepsilon}) \text{ with } \neg P(c_\varepsilon, c_{j,\varepsilon}), &\quad \text{if } \neg\bigl(\forall x \in t(c_{j,\varepsilon})\bigr)\ P(x, c_{j,\varepsilon}), \\ c_\varepsilon \in t(c_{j,\varepsilon}), &\quad \text{if } \bigl(\forall x \in t(c_{j,\varepsilon})\bigr)\ P(x, c_{j,\varepsilon}). \end{aligned}$$

(Since $t(c_j)$ is internal, $t(c_j) \neq \varnothing$, so w.l.o.g. $t(c_{j,\varepsilon}) \neq \varnothing$, $\forall \varepsilon$.) Then by assumption, $P(c_\varepsilon, c_{j,\varepsilon})$ is true a.e. By the choice of c_ε, this implies that $(\forall x \in t(c_{j,\varepsilon}))\ P(x, c_{j,\varepsilon})$ is true a.e.
$\Longleftarrow$: Let $c_\varepsilon \in t(c_{j,\varepsilon})$ a.e. Then by assumption, $P(c_\varepsilon, c_{j,\varepsilon})$ is true a.e. $\square$

Theorem 3.4 *(Transfer principle, restricted). Let $P(a_1, \ldots, a_m)$ be a sentence formed by applying rules* T1–T3, F1–F4 *only, in which the constants a_j are nongeneralized objects. Then $P(a_1, \ldots, a_m)$ is true iff $P({}^*a_1, \ldots, {}^*a_m)$ is true.*

Proof. This is a special case of Proposition 3.3. $\square$

Remark 3.5. If $\mathcal{F}$ is a nonmaximal free filter, the full transfer principle (i.e., including rules F5–F7) cannot hold. E.g., in that case, ${}^*\mathbb{R}$ is partially, but not totally ordered. Hence transfer cannot apply to the statement (containing $\vee$)

$$(\forall x, y \in \mathbb{R}) \quad (x \leqslant y \vee y \leqslant x).$$

Similarly, in that case, ${}^*\mathbb{R}$ is a ring, but not a field. Hence transfer cannot apply to the statement (containing $\neg$)

$$(\forall x, y \in \mathbb{R}) \quad \bigl(\neg(x = 0) \implies (\exists y \in \mathbb{R})\ (xy = 1)\bigr).$$

4. Internal definition and transfer (extended)

Theorem 4.1 *(Internal definition principle, I.D.P.). Let $P(x, x_j)$ be a transferrable formula. Let A, a_j be internal objects. Let $\{x \in A\colon P(x, a_j)\} \neq \varnothing$. Then $\{x \in A\colon P(x, a_j)\}$ is internal.*
Explicitly, if $A = [A_\varepsilon]$ and $a_j = [a_{j,\varepsilon}]$, then $\{x \in A\colon P(x, a_j)\} = [\{x \in A_\varepsilon\colon P(x, a_{j,\varepsilon})\}]$.

Proof. Let $\{x \in A\colon P(x, a_j)\} \neq \varnothing$, i.e., $(\exists x \in A)\ P(x, a_j)$. By transfer, $(\exists x \in A_\varepsilon)\ P(x, a_{j,\varepsilon})$ holds a.e. For an internal object $c = [c_\varepsilon]$, we have by transfer,

$$\begin{aligned} c \in \bigl\{x \in A\colon P(x, a_j)\bigr\} \quad &\Longleftrightarrow\quad c \in A \text{ and } P(c, a_j) \\ &\Longleftrightarrow\quad c_\varepsilon \in A_\varepsilon \text{ and } P(c_\varepsilon, a_{j,\varepsilon}) \text{ a.e.} \\ &\Longleftrightarrow\quad c_\varepsilon \in \bigl\{x \in A_\varepsilon\colon P(x, a_{j,\varepsilon})\bigr\} \text{ a.e.} \\ &\Longleftrightarrow\quad c \in \bigl[\bigl\{x \in A_\varepsilon\colon P(x, a_{j,\varepsilon})\bigr\}\bigr], \end{aligned}$$

where the latter internal set is well defined since the corresponding family is a family of nonempty sets (a.e.). Further, as A is internal, A has only internal elements. Hence $\{x \in A\colon P(x, a_j)\} = [\{x \in A_\varepsilon\colon P(x, a_{j,\varepsilon})\}]$ is internal. $\square$

Identifying $\mathbb{R}$ with a subset of ${}^*\mathbb{R}$, for a map $f: \mathbb{R}^n \to \mathbb{R}^m$, we have that ${}^*f: {}^*\mathbb{R}^n \to {}^*\mathbb{R}^m$ is an extension of f (in view of $[f_\varepsilon]([x_\varepsilon]) = [f_\varepsilon(x_\varepsilon)]$). We will therefore denote ${}^*f$ by f (as usual in nonstandard analysis). In the case of relations on $\mathbb{R}$, some confusion may arise in dropping the stars. E.g., for $a, b \in {}^*\mathbb{R}$, $a({}^*\neq)b$ is not equivalent with $\neg(a = b)$. We will drop the stars for $\leqslant$; on the other hand, we will use $a \neq b$ for $\neg(a = b)$, $a \nleqslant b$ for $\neg(a \leqslant b)$, and $a < b$ for $a \leqslant b \wedge a \neq b$. By transfer, $({}^*\mathbb{R}, +, \cdot, \leqslant)$ is a partially ordered commutative ring.

Example 4.2. 1. The archimedean property of $\mathbb{R}$

$$(\forall x \in \mathbb{R})\ (\exists n \in \mathbb{N}) \quad \bigl(n \geqslant |x|\bigr)$$

yields, by transfer,

$$\bigl(\forall x \in {}^*\mathbb{R}\bigr)\ \bigl(\exists n \in {}^*\mathbb{N}\bigr) \quad \bigl(n \geqslant |x|\bigr).$$

2. For $a \in \mathbb{R}$ and $R \in \mathbb{R}$ ($R > 0$), let $B(a, R) = \{x \in \mathbb{R}\colon\ |a| \leqslant R\}$. Then by transfer (or by I.D.P.),

$${}^*\bigl(B(a,R)\bigr) = \bigl\{x \in {}^*\mathbb{R}\colon\ |a| \leqslant R\bigr\}.$$

The version of transfer obtained so far is too weak for practical use. We will therefore allow another rule in the formation of terms:

T4. If P is a formula, x is a variable free in P and t is a term in which x does not occur, then $\{x \in t\colon\ P\}$ is a term.

We will call such a term a *set term*. We call the variable x *bound* (by the set term). We will denote by $t(x_j)$ a term t with x_j as its only *free* variables.

We define a unary predicate N ('is recursively nonempty') with the following semantics:

1. If a is not a tuple, then $N(a)$ iff $a \neq \varnothing$.
2. If $a = (a_1, \ldots, a_m)$, then $N(a)$ iff $N(a_1) \wedge \cdots \wedge N(a_m)$.

Lemma 4.3. *Let $t(x_j)$ be a term formed by rules* T1–T4*, in which all occurring formulas are transferrable. Let $[c_{j,\varepsilon}]$ be internal objects. If $N(t([c_{j,\varepsilon}]))$, then*

$$\bigl[t(c_{j,\varepsilon})\bigr] = t\bigl([c_{j,\varepsilon}]\bigr).$$

Proof. T1. Clear.

T2. If $t = (t_1, \ldots, t_m)$, then $N(t([c_{j,\varepsilon}]))$ iff $N(t_i([c_{j,\varepsilon}]))$ for $i = 1, \ldots, m$. Hence the claim follows by induction (as in Lemma 3.2).

T3. If $t = f(s)$, then $t([\phi_\varepsilon], [c_{j,\varepsilon}]) = [\phi_\varepsilon](s([c_{j,\varepsilon}]))$ is assumed to be well defined. In particular, $s([c_{j,\varepsilon}])$ is internal, and therefore $N(s([c_{j,\varepsilon}]))$. Hence the claim follows by induction (as in Lemma 3.2).

T4. Let P be a transferrable formula in which x is free and let $t(x_j)$ be a term in which x does not occur. For the term $\{x \in t(x_j)\colon\ P(x, x_j)\}$ with (by assumption) $\{x \in t([c_{j,\varepsilon}])\colon\ P(x, [c_{j,\varepsilon}])\} \neq \varnothing$, we have that also $t([c_{j,\varepsilon}])$ is a nonempty set, and thus $N(t([c_{j,\varepsilon}]))$. Further, we find inductively by Theorem 4.1

$$\bigl[\bigl\{x \in t(c_{j,\varepsilon})\colon\ P(x, c_{j,\varepsilon})\bigr\}\bigr] = \bigl\{x \in \bigl[t(c_{j,\varepsilon})\bigr]\colon\ P\bigl(x, [c_{j,\varepsilon}]\bigr)\bigr\} = \bigl\{x \in t\bigl([c_{j,\varepsilon}]\bigr)\colon\ P\bigl(x, [c_{j,\varepsilon}]\bigr)\bigr\}. \qquad \square$$

The condition that set terms cannot be recursively empty leads us to the adapted rules:

F1′. If t_1, t_2 are terms, then $t_1 = t_2 \wedge N(t_1) \wedge N(t_2)$ and $t_1 \in t_2 \wedge N(t_1) \wedge N(t_2)$ are formulas.
F4′. If P is a formula, x is a variable free in P and t is a term in which x does not occur, then $[(\forall x \in t)P] \wedge N(t)$ is a formula.

Notice that for terms $t(x_j)$ formed by rules T1–T3 and internal objects c_j, the side condition $N(t(c_j)) \neq \varnothing$ is always satisfied (and hence can be omitted from the formula).

Proposition 4.4. *Let $P(x_j)$ be a formula formed by applying rules* T1–T4, F1′, F2, F3, F4′. *Then $P(x_j)$ is transferrable.*

Proof. The induction of Proposition 3.3 goes through, provided that we can at any moment in the proof write $[t(c_{j,\varepsilon})] = t([c_{j,\varepsilon}])$. This is exactly what the side condition $N(t)$ accomplishes: if $N(t([c_{j,\varepsilon}]))$, then by induction, Lemma 4.3 can be applied.

Notice that also the side condition is transferrable: if the last rule applied in the formation of t is T1 or T3, then $N(t(\ldots))$ is always true. If $t = (t_1, \ldots, t_m)$, then inductively,

$$\begin{aligned} N\bigl(t\bigl([c_{j,\varepsilon}]\bigr)\bigr) \quad &\Longleftrightarrow \quad N\bigl(t_i\bigl([c_{j,\varepsilon}]\bigr)\bigr) \quad \text{for } i = 1, \ldots, m \\ &\Longleftrightarrow \quad N\bigl(t_i(c_{j,\varepsilon})\bigr) \text{ a.e.} \quad \text{for } i = 1, \ldots, m \quad \Longleftrightarrow \quad N\bigl(t(c_{j,\varepsilon})\bigr) \text{ a.e.} \end{aligned}$$

Finally, if $t = \{x \in s\colon\ P\}$, then inductively,

$$\begin{aligned} N\bigl(\bigl\{x \in s\bigl([c_{j,\varepsilon}]\bigr)\colon\ P\bigl(x, [c_{j,\varepsilon}]\bigr)\bigr\}\bigr) \quad &\Longleftrightarrow \quad \bigl(\exists x \in s\bigl([c_{j,\varepsilon}]\bigr)\bigr)\ P\bigl(x, [c_{j,\varepsilon}]\bigr) \\ &\Longleftrightarrow \quad \bigl(\exists x \in s(c_{j,\varepsilon})\bigr)\ P(x, c_{j,\varepsilon}) \text{ a.e.} \\ &\Longleftrightarrow \quad N\bigl(\bigl\{x \in s(c_{j,\varepsilon})\colon\ P(x, c_{j,\varepsilon})\bigr\}\bigr) \text{ a.e.} \end{aligned}$$

Hence the side condition also transfers properly. □

Theorem 4.5 *(Transfer principle, extended). Let $P(a_1, \dots, a_m)$ be a sentence formed by applying rules* T1–T4, F1′, F2, F3, F4′, *in which the constants a_j are nongeneralized objects. Then $P(a_1, \dots, a_m)$ is true iff $P({}^*a_1, \dots, {}^*a_m)$ is true.*

Proof. This is a special case of Proposition 4.4. □

In practice, an important corollary is that transfer can be applied to formulas that also contain '$\Longrightarrow$', under a constraint that is almost always fulfilled in practice:

F5′. If P, Q are formulas, x is a variable free in P and Q, and t is a term in which x does not occur, then $[(\forall x \in t)\ (P \Longrightarrow Q)] \wedge [(\exists x \in t)\ P]$ is a formula.

Corollary 4.6. *Let $P(x_j)$ be a formula formed by applying rules* T1–T3, F1–F4, F5′. *Then $P(x_j)$ is transferrable.*

Proof. The formula F5′ is equivalent with $[(\forall x \in \{x' \in t\colon\ P\})\ Q] \wedge N(\{x' \in t\colon\ P\})$, which is transferrable by Proposition 4.4. □

In practice, we will apply transfer to the formula $(\forall x \in t)\ (P \Longrightarrow Q)$, silently checking that the side condition $(\exists x \in t)\ P$ is fulfilled.

Remarks.

1. The formula $(\forall x \in t)\ [(P \Longrightarrow Q) \wedge R]$ can be treated similarly. In fact, it is equivalent with $[(\forall x \in t)\ (P \Longrightarrow Q)] \wedge [(\forall x \in t)\ R]$. Similarly, $(\forall x \in t)\ (\exists y \in s)\ (P \Longrightarrow Q)$ is equivalent with $(\forall x \in t)\ ([(\forall y \in s)\ P] \Longrightarrow [(\exists y \in s)\ Q])$.
2. A sentence containing the connective $\vee$ can sometimes be transferred using idempotent elements in ${}^*\mathbb{R}$. E.g., the sentence $(\forall x, y \in \mathbb{R})\ (x \leqslant y \vee y \leqslant x)$ is equivalent with $(\forall x, y \in \mathbb{R})\ (\exists e \in \mathbb{R})\ (e^2 = e \wedge xe \leqslant ye \wedge x(1-e) \leqslant y(1-e))$, which is transferrable.
3. A sentence containing the connective $\neg$ can sometimes be transferred if it can be pulled through to an atomic formula. E.g., $(\forall x \in \mathbb{R})\ (x \neq 0 \implies (\exists y \in \mathbb{R})\ (xy = 1))$ can be transferred as $(\forall x \in {}^*\mathbb{R})\ (x({}^*{\neq})0 \implies (\exists y \in {}^*\mathbb{R})\ (xy = 1))$. Notice that for $x \in {}^*\mathbb{R}$, $x({}^*{\neq})0$ is a stronger condition than $x \neq 0$ (unless $\mathcal{F}$ is an ultrafilter).

5. Saturation

Definition 5.1. We call a free filter $\mathcal{F}$ on I *selective* if for each sequence $(S_n)_{n\in\mathbb{N}}$ with $S_n^c := I \setminus S_n \notin \mathcal{F}$, there exist $\varepsilon_n \in S_n$ such that $\{\varepsilon_n\colon\ n \in \mathbb{N}\}^c \notin \mathcal{F}$.

We call a free filter $\mathcal{F}$ on I *blocked* if for each $S_j \subseteq I$ with $S_j^c \notin \mathcal{F}$ $(j = 1, 2)$, there exist disjoint $T_j \subseteq S_j$ with $T_j^c \notin \mathcal{F}$ $(j = 1, 2)$.[3]

Similarly, we call a free filter $\mathcal{F}$ on I *σ-blocked* if for each $S_j \subseteq I$ with $S_j^c \notin \mathcal{F}$ $(j \in \mathbb{N})$, there exist mutually disjoint $T_j \subseteq S_j$ with $T_j^c \notin \mathcal{F}$ $(j \in \mathbb{N})$.

A free filter $\mathcal{F}$ on I is called *$\aleph_1$-regular* (resp. *$\aleph_1$-incomplete*, also called σ-incomplete or δ-incomplete) [7] if there exist $S_n \in \mathcal{F}$ such that $\bigcap_{n\in\mathbb{N}} S_n = \varnothing$ (resp. $\bigcap_{n\in\mathbb{N}} S_n \notin \mathcal{F}$). For an ultrafilter, $\aleph_1$-regular is equivalent with $\aleph_1$-incomplete.

We call a filter $\mathcal{F}$ *common* if $\mathcal{F}$ is an $\aleph_1$-regular selective blocked free filter.

A filter $\mathcal{F}$ is called *Ramsey* [1] if for each decreasing sequence $(S_n)_{n\in\mathbb{N}}$ with $S_n \in \mathcal{F}$, there exist $\varepsilon_n \in S_n$ such that $\{\varepsilon_n\colon\ n \in \mathbb{N}\} \in \mathcal{F}$.

Lemma 5.2.

1. *Let $\mathcal{F}$ be a free ultrafilter. Then $\mathcal{F}$ is selective iff $\mathcal{F}$ is Ramsey.*
2. *If $\mathcal{F}$ is a selective free filter, then $\mathcal{F}$ is $\aleph_1$-incomplete.*

[3] The name *blocked* stems from the fact that this property is an obstruction for the filter to be an ultrafilter.

Proof. 1. For an ultrafilter, $S^c \notin \mathcal{F}$ iff $S \in \mathcal{F}$. Replacing $S_n \in \mathcal{F}$ by $S_1 \cap \cdots \cap S_n \in \mathcal{F}$, we may restrict ourselves to decreasing sequences only.

2. As $\mathcal{F}$ is selective, there exist $\varepsilon_n \in I$ such that $\{\varepsilon_n\colon n \in \mathbb{N}\}^c \notin \mathcal{F}$. As $\mathcal{F}$ is free, $S_n := \{\varepsilon_n\}^c \in \mathcal{F}$ for each $n \in \mathbb{N}$, but $\bigcap_{n\in\mathbb{N}} S_n \notin \mathcal{F}$. $\square$

Part 1 of Lemma 5.2 shows that our definition of a selective free filter is consistent with the fact that a Ramsey ultrafilter is also called a selective ultrafilter [10].

Examples 5.3.

1. The filters (1) and (2) are common. E.g., to see that they are selective, let $(S_n)_{n\in\mathbb{N}}$ be a sequence with $S_n^c \notin \mathcal{F}$, for each n. Then we can construct an increasing (resp. decreasing) sequence of elements $\varepsilon_n \in S_n$ such that $\varepsilon_n \to \infty$ (resp. $\varepsilon_n \to 0$). Then $\{\varepsilon_n\colon n \in \mathbb{N}\}^c \notin \mathcal{F}$.
   In particular, it is trivial to construct selective free filters (in contrast with selective free ultrafilters on $\mathbb{N}$, whose existence is not guaranteed under the ZFC axioms of set theory).
2. Let ω_1 be the first uncountable ordinal. Let
   $$I = \omega_1 \quad \text{with } \mathcal{F} = \bigl\{S \subseteq \omega_1\colon (\exists \eta \in \omega_1)\ (\eta, \omega_1) \subseteq S\bigr\}. \tag{3}$$
   Then $\mathcal{F}$ is σ-blocked: for each $n \in \mathbb{N}$, let $S_n \subseteq \omega_1$ with $S_n^c \notin \mathcal{F}$, i.e., for each $\eta \in \omega_1$, there exists $\varepsilon \geqslant \eta$ with $\varepsilon \in S_n$. Inductively choose for limit ordinals $\lambda \in \omega_1$ and $n \in \mathbb{N}$
   $$\varepsilon_\lambda := \sup_{\alpha<\lambda} \varepsilon_\alpha \in \omega_1,$$
   $$\varepsilon_{\lambda+n} \in S_n, \quad \varepsilon_{\lambda+n} > \varepsilon_{\lambda+n-1}.$$
   Then $T_n := \{\varepsilon_{\lambda+n}\colon \lambda \in \omega_1 \text{ is a limit ordinal}\} \subseteq S_n$ are mutually disjoint. Also $\varepsilon_\alpha \geqslant \alpha$ for each $\alpha \in \omega_1$. For any $\eta \in \omega_1$, there exists a limit ordinal $\lambda \in \omega_1$ with $\lambda \geqslant \eta$. Then $T_n \ni \varepsilon_{\lambda+n} \geqslant \lambda + n \geqslant \eta$. Hence $T_n^c \notin \mathcal{F}$.
   Further, $\mathcal{F}$ is $\aleph_1$-complete: if $S_n \in \mathcal{F}$, for each $n \in \mathbb{N}$, then there exist $\eta_n \in \omega_1$ such that $(\eta_n, \omega_1) \subseteq S_n$. Then $\mathcal{F} \ni (\sup_{n\in\mathbb{N}} \eta_n, \omega_1) \subseteq \bigcap_{n\in\mathbb{N}} S_n$.
   It follows that $\mathcal{F}$ is also not selective by Lemma 5.2.

Lemma 5.4. *Let $\mathcal{F}$ be a selective blocked free filter. Then $\mathcal{F}$ is σ-blocked.*

Proof. Let $S_j \subseteq I$ with $S_j^c \notin \mathcal{F}$, for each $j \in \mathbb{N}$. As $\mathcal{F}$ is blocked, we find $S_{j,2} \subseteq S_j$ with $S_{j,2}^c \notin \mathcal{F}$ and $S_{1,2} \cap S_{2,2} = \varnothing$ (as $\mathcal{F}$ is free, w.l.o.g. $S_{j,2} \subsetneqq S_j$, $\forall j$). For each $n \in \mathbb{N}$ ($n > 2$), we similarly find (repeatedly using the fact that $\mathcal{F}$ is blocked) $S_{j,n} \subsetneqq S_{j,n-1}$ with $S_{j,n}^c \notin \mathcal{F}$ and with mutually disjoint $S_{1,n}, \ldots, S_{n,n}$. As $\mathcal{F}$ is selective, we find $\varepsilon_{1,n} \in S_{1,n}$ such that $\{\varepsilon_{1,n}\colon n \in \mathbb{N}\}^c \notin \mathcal{F}$. Let $T_1 := \{\varepsilon_{1,n}\colon n \in \mathbb{N}\}$. Similarly, $\{\varepsilon_{2,n}\colon n \in \mathbb{N}\}^c \notin \mathcal{F}$ for some $\varepsilon_{2,n} \in S_{2,n}$, and we let $T_2 := \{\varepsilon_{2,n}\colon n \in \mathbb{N}\} \setminus T_1$. As $T_1 \cap T_2$ is finite, $T_2^c \notin \mathcal{F}$. And so on. $\square$

Theorem 5.5 *(Saturation principle). Let $\mathcal{F}$ be a common filter. Let X be an internal set. For each $n \in \mathbb{N}$, let $A_n \subseteq X$ be internal and $B_n \subseteq X$ such that $X \setminus B_n$ is internal or $B_n = X$. If $A_1 \cap \cdots \cap A_n \cap B_j \neq \varnothing$ for each $n, j \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} A_n \cap B_n \neq \varnothing$.*

Proof. Assume first that $B_n \neq X$, for some n. Then w.l.o.g. $B_n \neq X$ for each n. Let $X = [X_\varepsilon]$, $A_n = [A_{n,\varepsilon}]$ and $X \setminus B_n = [C_{n,\varepsilon}]$ for each n. As $C_{n,\varepsilon} \subseteq X_\varepsilon$ a.e., we may assume that $X \setminus B_n = [X_\varepsilon \setminus B_{n,\varepsilon}]$ for some $B_{n,\varepsilon} \subseteq X_\varepsilon$. Choose $x_{n,j} \in A_1 \cap \cdots \cap A_n \cap B_j$ for each $n, j \in \mathbb{N}$ with $j \leqslant n$. As X is internal, also $x_{n,j} \in X$ are internal. Let $x_{n,j} = [x_{n,j,\varepsilon}]$. Then $x_{n,j,\varepsilon} \in A_{1,\varepsilon} \cap \cdots \cap A_{n,\varepsilon}$ a.e. Hence

$$\widetilde{S}_n := \bigl\{\varepsilon \in I\colon (\forall j, k \leqslant n)\ (x_{n,j,\varepsilon} \in A_{k,\varepsilon})\bigr\} \in \mathcal{F}, \quad \forall n \in \mathbb{N}.$$

As $\mathcal{F}$ is $\aleph_1$-regular, we find $S_n \in \mathcal{F}$, $S_n \subseteq \widetilde{S}_n$, $(S_n)_n$ decreasing and $\bigcap_n S_n = \varnothing$.

Further, $x_{n,j} \notin [X_\varepsilon \setminus B_{j,\varepsilon}]$, i.e. $x_{n,j,\varepsilon} \notin B_{j,\varepsilon}$ does not hold a.e.

Let $T_{n,j} := \{\varepsilon \in I\colon x_{n,j,\varepsilon} \in B_{j,\varepsilon}\} \cap S_n$. As $S_n \in \mathcal{F}$, also $T_{n,j}^c \notin \mathcal{F}$, $\forall n, j \in \mathbb{N}$, $j \leqslant n$.

By Lemma 5.4, $\mathcal{F}$ is σ-blocked. So we find mutually disjoint $U_{n,j} \subseteq T_{n,j}$ with $U_{n,j}^c \notin \mathcal{F}$. As $\mathcal{F}$ is selective, there exist $\varepsilon_{n,j} \in U_{n,j}$ such that $\{\varepsilon_{n,j}\colon n, j \in \mathbb{N},\ j \leqslant n\}^c \notin \mathcal{F}$. Let

$$x_\varepsilon := \begin{cases} x_{n,j,\varepsilon}, & \varepsilon = \varepsilon_{n,j} \quad (j \leqslant n), \\ x_{n,1,\varepsilon}, & \varepsilon \in (S_n \setminus S_{n+1}) \setminus \{\varepsilon_{n,j}\colon n, j \in \mathbb{N},\ j \leqslant n\}. \end{cases}$$

As $\bigcap_{n\in\mathbb{N}} S_n = \varnothing$, this unambiguously defines x_ε for each $\varepsilon \in S_1$.

(1) Let $n \in \mathbb{N}$. We show that $S_n \setminus \{\varepsilon_{k,j}\colon j \leqslant k < n\} \subseteq \{\varepsilon \in I\colon x_\varepsilon \in A_{n,\varepsilon}\}$, whence $[x_\varepsilon] \in A_n$.

Let $\varepsilon \in S_n$. Then $\varepsilon \in S_m \setminus S_{m+1}$ for some $m \geqslant n$. If $\varepsilon \notin \{\varepsilon_{n,j}\colon n, j \in \mathbb{N},\ j \leqslant n\}$, then $x_\varepsilon = x_{m,1,\varepsilon} \in A_{n,\varepsilon}$ by definition of S_m. If $\varepsilon = \varepsilon_{k,j}$ for some $j \leqslant k$ with $k \geqslant n$, then $\varepsilon_{k,j} \in U_{k,j} \subseteq S_k$. Hence $x_\varepsilon = x_{k,j,\varepsilon} \in A_{n,\varepsilon}$ by definition of S_k.

(2) Let $j \in \mathbb{N}$. We show that $\{\varepsilon_{n,j}:\ n \in \mathbb{N},\ j \leqslant n\} \subseteq \{\varepsilon \in I:\ x_\varepsilon \in B_{j,\varepsilon}\}$, whence $[x_\varepsilon] \in B_j$.
If $\varepsilon = \varepsilon_{n,j}$ $(n \geqslant j)$, then $\varepsilon \in U_{n,j}$ and $x_\varepsilon = x_{n,j,\varepsilon}$; hence $x_\varepsilon \in B_{j,\varepsilon}$ by definition of $U_{n,j}$.
Finally, if $B_n = X$ for each n, then the proof is a simplified version of the previous argument (i.e., choosing $x_n \in A_1 \cap \cdots \cap A_n$, $\widetilde{S}_n := \{\varepsilon \in I:\ (\forall k \leqslant n)\ (x_{n,\varepsilon} \in A_{k,\varepsilon})\}$ and $x_\varepsilon := x_{n,\varepsilon}$ for $\varepsilon \in S_n \setminus S_{n+1}$). $\square$

The following special case resembles the classical countable saturation in nonstandard analysis more closely:

Corollary 5.6. *Let $\mathcal{F}$ be common filter. Let X be an internal set. For each $n \in \mathbb{N}$, let $A_n \subseteq X$ such that A_n or $X \setminus A_n$ is internal. If $(A_n)_{n\in\mathbb{N}}$ has the finite intersection property, then $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$.*

In contrast with nonstandard analysis (i.e., the case in which $\mathcal{F}$ is a free ultrafilter), a sequence of nonempty cointernal sets automatically has a nonempty intersection if $\mathcal{F}$ is a common filter.

Corollary 5.7 *(Quantifier switching, Q.S.). Let $\mathcal{F}$ be a common filter. Let X be an internal set. For each $n \in \mathbb{N}$, let $P_n(x, x_{n,j})$, $Q_n(x, y_{n,j})$ be transferrable formulas. Let $a_{n,j}$, $b_{n,j}$ be internal constants. If P_n gets stronger as n increases (i.e., for each $n \in \mathbb{N}$ and $x \in X$, $P_{n+1}(x, a_{n+1,j}) \implies P_n(x, a_{n,j})$) and if*

$$(\forall n, m \in \mathbb{N})\ (\exists x \in X) \quad \big(P_n(x, a_{n,j}) \wedge \neg Q_m(x, b_{m,j})\big),$$

then also

$$(\exists x \in X)\ (\forall n \in \mathbb{N}) \quad \big(P_n(x, a_{n,j}) \wedge \neg Q_n(x, b_{n,j})\big).$$

Proof. Let $A_n := \{x \in X:\ P_n(x, a_{n,j})\}$ and $B_n := \{x \in X:\ \neg Q_n(x, b_{n,j})\}$. By I.D.P., A_n, $X \setminus B_n$ are internal or empty. By assumption, A_n are not empty and $A_{n+1} \subseteq A_n$, $\forall n$. If $X \setminus B_n$ is empty, then $B_n = X$. By assumption, for each $n, m \in \mathbb{N}$, $A_1 \cap \cdots \cap A_n \cap B_m = A_n \cap B_m \neq \varnothing$. The result follows by the saturation principle. $\square$

6. Overspill and underspill

Definition 6.1. Let $a, b \in {}^*\mathbb{R}$. Then a is called *infinitely large* if $|a| \geqslant n$, for each $n \in \mathbb{N}$; a is called *finite* if $|a| \leqslant N$, for some $N \in \mathbb{N}$; a is called *infinitesimal* if $|a| \leqslant 1/n$, for each $n \in \mathbb{N}$. We denote $a \approx b$ iff $a - b$ is infinitesimal. We denote the set of finite elements of ${}^*\mathbb{R}$, resp. ${}^*\mathbb{N}$ by $\mathrm{Fin}({}^*\mathbb{R})$, resp. $\mathrm{Fin}({}^*\mathbb{N})$ and the set of infinitely large elements by ${}^*\mathbb{R}_\infty$, resp. ${}^*\mathbb{N}_\infty$.

Example 6.2. If $\mathcal{F}$ is not $\aleph_1$-regular, then ${}^*\mathbb{N}_\infty = \varnothing$: If $[n_\varepsilon] \in {}^*\mathbb{N}_\infty$, then $n_\varepsilon \in \mathbb{N}$ for each $\varepsilon \in I$ and for each $m \in \mathbb{N}$, there exists $X_m \in \mathcal{F}$ such that $n_\varepsilon \geqslant m$, for each $\varepsilon \in X_m$. As $\mathcal{F}$ is not $\aleph_1$-regular, there exists $\varepsilon_0 \in \bigcap_{m\in\mathbb{N}} X_m$. Then $n_{\varepsilon_0} \geqslant m$ for each $m \in \mathbb{N}$, a contradiction.

This example shows that the condition that $\mathcal{F}$ is $\aleph_1$-regular cannot be dropped in the statement of the saturation principle.

Lemma 6.3. *Let $\mathcal{F}$ be a common filter. Let $a \in {}^*\mathbb{R}$. If $|a| \leqslant m$ for each $m \in {}^*\mathbb{N}_\infty$, then a is finite.*

Proof. Suppose that a is not finite. Then $(\forall n \in \mathbb{N})\ (\exists m \in {}^*\mathbb{N})\ (m \geqslant n \wedge |a| \nleqslant m)$. By Q.S., there exists $m \in {}^*\mathbb{N}$ such that $|a| \nleqslant m$ and $m \geqslant n$, for each $n \in \mathbb{N}$, contradicting the hypotheses. $\square$

Recall that $\mathcal{P}_\emptyset(A) = \{X \subseteq A:\ X \neq \varnothing\}$.

Lemma 6.4. *Let $A \neq \varnothing$ be a nongeneralized set. Then ${}^*\mathcal{P}_\emptyset(A)$ is the set of all internal subsets of ${}^*A$.*

Proof. By definition, ${}^*\mathcal{P}_\emptyset(A)$ only has internal elements. Further,

$$[X_\varepsilon] \in {}^*\mathcal{P}_\emptyset(A) \iff X_\varepsilon \in \mathcal{P}_\emptyset(A) \text{ a.e.} \iff X_\varepsilon \subseteq A \text{ a.e.}$$

As in the proof of Lemma 2.2, this is still equivalent with $[X_\varepsilon] \subseteq {}^*A$. $\square$

Just like Q.S., the applications of saturation known as *overspill* and *underspill* are convenient for practical use.

Theorem 6.5 *(Spilling principles). Let $\mathcal{F}$ be a common filter. Let $A \subseteq {}^*\mathbb{N}$ be internal.*

1. *(Overspill) If A contains arbitrarily large finite elements (i.e., for each $n \in \mathbb{N}$, there exists $m \in A$ with $m \geqslant n$), then A contains an infinitely large element.*

2. (*Underspill*) *If A contains arbitrarily small infinitely large elements (i.e., for each $\omega \in {}^*\mathbb{N}_\infty$, there exists $a \in A$ with $a \leqslant \omega$), then A contains a finite element.*
3. (*Overspill*) *If $\mathbb{N} \subseteq A$, then there exists $\omega \in {}^*\mathbb{N}_\infty$ such that $\{n \in {}^*\mathbb{N}\colon\ n \leqslant \omega\} \subseteq A$.*
4. (*Underspill*) *If ${}^*\mathbb{N}_\infty \subseteq A$, then $A \cap \mathbb{N} \neq \varnothing$.*

Proof. 1. As $(\forall n \in \mathbb{N})\ (\exists m \in A)\ (m \geqslant n)$, $A \cap {}^*\mathbb{N}_\infty \neq \varnothing$ by Q.S.

2. By transfer on the sentence

$$\big(\forall X \in \mathcal{P}_\emptyset(\mathbb{N})\big)\ (\exists m \in X)\ (\forall n \in X) \quad (n \geqslant m),$$

every internal subset of ${}^*\mathbb{N}$ has a smallest element. Let n_{min} be the smallest element of A. Then $n_{min} \leqslant \omega$, for each $\omega \in {}^*\mathbb{N}_\infty$. By Lemma 6.3, n_{min} is finite.

3. First, let $n_0 \in \mathbb{N}$. By transfer on the sentence

$$\big(\forall X \in \mathcal{P}_\emptyset(\mathbb{N})\big) \quad \big[(1 \in X \wedge \cdots \wedge n_0 \in X) \implies (\forall m \in \mathbb{N})\ (m \leqslant n_0 \implies m \in X)\big]$$

(side conditions are trivially fulfilled), any internal subset of ${}^*\mathbb{N}$ that contains $\mathbb{N}$ also contains $\{m \in {}^*\mathbb{N}\colon\ m \leqslant n_0\}$, for any $n_0 \in \mathbb{N}$. Then

$$B = \big\{n \in {}^*\mathbb{N}\colon\ \big(\forall m \in {}^*\mathbb{N}\big)\ (m \leqslant n \implies m \in A)\big\}$$

is internal by I.D.P. (since the side condition is trivially fulfilled and $B \neq \varnothing$) and contains $\mathbb{N}$. By part 1, B contains an infinitely large ω. Hence $\{n \in {}^*\mathbb{N}\colon\ n \leqslant \omega\} \subseteq A$.

4. Let

$$B = \big\{n \in {}^*\mathbb{N}\colon\ \big(\forall m \in {}^*\mathbb{N}\big)\ (m \geqslant n \implies m \in A)\big\}.$$

By I.D.P., B is internal (since the side condition is trivially fulfilled and $B \neq \varnothing$). By part 2, B contains a finite element, i.e., there exists $n \in B$ and $N \in \mathbb{N}$ such that $n \leqslant N$. By definition of B, $N \in A$. □

7. Applications to generalized function theory

Remark 7.1. In generalized function theory, the spaces considered are usually not ${}^*\mathbb{R}$ or other spaces introduced so far, but they are rather quotients of these spaces modulo a further identification (e.g. by means of certain growth conditions). The reason why we nevertheless introduced them is that they can be used advantageously to prove statements about generalized functions, using the strong principles that were described in the previous sections. We briefly exemplify this in the following sections.

Similar principles for the objects obtained after a further identification often do not hold. In particular the transfer principle and the internal definition principle are then too restricted for practical use: e.g., with analogous definitions of internal objects, $\{x \in A\colon\ x \geqslant 0\}$ need no longer be internal if A is an internal set [16].

7.1. Automatic continuity

Assumption. In this section, we work with the filter (2) on $I = (0, 1]$.

We also denote $\rho := [\varepsilon] \in {}^*\mathbb{R}$.

Let E be a locally convex vector space (belonging to the nongeneralized objects) with its topology generated by a family of seminorms $(p_\lambda)_{\lambda \in \Lambda}$. The Colombeau module constructed on E [5] is defined as $\mathcal{G}_E := \mathcal{M}_E/\mathcal{N}_E$, where

$$\mathcal{M}_E = \big\{(u_\varepsilon)_\varepsilon \in E^I\colon\ (\forall \lambda \in \Lambda)\ (\exists N \in \mathbb{N})\ \big(p_\lambda(u_\varepsilon) \leqslant \varepsilon^{-N} \text{ a.e.}\big)\big\},$$
$$\mathcal{N}_E = \big\{(u_\varepsilon)_\varepsilon \in E^I\colon\ (\forall \lambda \in \Lambda)\ (\forall n \in \mathbb{N})\ \big(p_\lambda(u_\varepsilon) \leqslant \varepsilon^{n} \text{ a.e.}\big)\big\}.$$

For $E = \mathbb{R}$ (resp. $E = \mathbb{C}$), one denotes $\widetilde{\mathbb{R}} := \mathcal{G}_\mathbb{R}$ (resp. $\widetilde{\mathbb{C}} := \mathcal{G}_\mathbb{C}$).

If we define

$$\mathcal{M}_{^*E} = \big\{u \in {}^*E\colon\ (\forall \lambda \in \Lambda)\ (\exists N \in \mathbb{N})\ \big({}^*p_\lambda(u) \leqslant \rho^{-N}\big)\big\},$$
$$\mathcal{N}_{^*E} = \big\{u \in {}^*E\colon\ (\forall \lambda \in \Lambda)\ (\forall n \in \mathbb{N})\ \big({}^*p_\lambda(u) \leqslant \rho^{n}\big)\big\},$$

then, in view of Lemma 2.2, the identity map on representatives introduces isomorphisms $\mathcal{M}_E/\mathcal{F} \cong \mathcal{M}_{^*E}$ and $\mathcal{N}_E/\mathcal{F} \cong \mathcal{N}_{^*E}$. It follows that $\mathcal{G}_E \cong (\mathcal{M}_E/\mathcal{F})/(\mathcal{N}_E/\mathcal{F}) \cong \mathcal{M}_{^*E}/\mathcal{N}_{^*E}$, where the first isomorphism is also introduced by the identity on representatives (expressing that the identification up to $\mathcal{N}_E$ can equivalently performed in two steps, in the first step only identifying modulo $\mathcal{F}$).

Definition 7.2. We call *ρ-topology* on ${}^*E$ the translation invariant topology with finite intersections of sets $B_\lambda(0,\rho^m)$ ($\lambda \in \Lambda$, $m \in \mathbb{N}$) as a base of neighborhoods of 0. Here $B_\lambda(0,r) := \{u \in {}^*E\colon {}^*p_\lambda(u) < r\}$ for $r \in {}^*\mathbb{R}$, $r \geqslant 0$.

For $u, v \in {}^*E$, we write $u \approxeq v$ iff $u - v \in \mathcal{N}_{{}^*E}$.

Proposition 7.3 *(ρ-continuity). Let E, F be locally convex spaces and let the topology of E, resp. F be generated by an increasing sequence of seminorms $(p_n)_{n\in\mathbb{N}}$, resp. $(q_n)_{n\in\mathbb{N}}$. Let $T : {}^*E \to {}^*F$ be internal and $u \in {}^*E$. Then the following are equivalent:*

1. $(\forall m \in \mathbb{N})\ (\exists n \in \mathbb{N})\ (\forall v \in {}^*E)\ ({}^*p_n(v-u) \leqslant \rho^n \implies {}^*q_m(T(v)-T(u)) \leqslant \rho^m)$,
2. $(\forall v \in {}^*E)\ (v \approxeq u \implies T(v) \approxeq T(u))$.

Proof. $\Longrightarrow$: Let $v \in {}^*E$ with $v \approxeq u$. Hence ${}^*p_n(u-v) \leqslant \rho^n$, for each $n \in \mathbb{N}$. Let $m \in \mathbb{N}$. By assumption, ${}^*q_m(T(v)-T(u)) \leqslant \rho^m$. As $m \in \mathbb{N}$ is arbitrary, $T(v) \approxeq T(u)$.

$\Longleftarrow$: Let $m \in \mathbb{N}$. Consider the map $p : \mathbb{N} \times E \to \mathbb{R}$: $p(n,u) := p_n(u)$. By transfer, ${}^*p(n,u) = {}^*p_n(u)$ for each $n \in \mathbb{N}$ and $u \in {}^*E$. Also by transfer, as $(p_n)_{n\in\mathbb{N}}$ is increasing,

$$\left(\forall u \in {}^*E\right) \left(\forall n, m \in {}^*\mathbb{N}\right) \quad \left(n \leqslant m \implies {}^*p(n,u) \leqslant {}^*p(m,u)\right)$$

(the side condition for the implication is always fulfilled). Similarly for $q(n,v) := q_n(v)$. Define

$$A := \left\{n \in {}^*\mathbb{N}\colon \left(\forall v \in {}^*E\right) \left({}^*p(n, v-u) \leqslant \rho^n \implies {}^*q\left(m, T(v)-T(u)\right) \leqslant \rho^m\right)\right\}.$$

Let $n \in {}^*\mathbb{N}_\infty$. If ${}^*p(n, v-u) \leqslant \rho^n$, then $v \approxeq u$. Hence, by assumption, also $T(v) \approxeq T(u)$, and in particular ${}^*q(m, T(v) - T(u)) \leqslant \rho^m$.

So A contains all infinitely large $n \in {}^*\mathbb{N}$. By I.D.P., A is internal (the side condition for the implication is always fulfilled). By underspill, $A \cap \mathbb{N} \neq \emptyset$. $\square$

Definition 7.4. Let E, F be locally convex spaces. In analogy of [16], we call a map $T : \mathcal{G}_E \to \mathcal{G}_F$ internal if there exist $T_\varepsilon : E \to F$ (for each $\varepsilon \in I$) such that $T([u_\varepsilon]) = [T_\varepsilon(u_\varepsilon)]$ for each $[u_\varepsilon] \in \mathcal{G}_E$ (here [.] denotes the equivalence class modulo $\mathcal{N}_E$, resp. $\mathcal{N}_F$). This definition implies in particular that

$$(u_\varepsilon)_\varepsilon \in \mathcal{M}_E \quad\Longrightarrow\quad \left(T_\varepsilon(u_\varepsilon)\right)_\varepsilon \in \mathcal{M}_F, \tag{4}$$

$$(u_\varepsilon)_\varepsilon \in \mathcal{M}_E,\ (v_\varepsilon)_\varepsilon \in \mathcal{M}_E,\ (u_\varepsilon - v_\varepsilon)_\varepsilon \in \mathcal{N}_E \quad\Longrightarrow\quad \left(T_\varepsilon(u_\varepsilon) - T_\varepsilon(v_\varepsilon)\right)_\varepsilon \in \mathcal{N}_F. \tag{5}$$

The sharp topology on $\mathcal{G}_E$ is the topology induced by the ρ-topology on $\mathcal{M}_{{}^*E}$, i.e., with $p_\lambda : \mathcal{G}_E \to \widetilde{\mathbb{R}}$: $p_\lambda([u_\varepsilon]) := [p_\lambda(u_\varepsilon)]$, it is the translation invariant topology with finite intersections of sets

$$\left\{u \in \mathcal{G}_E\colon\ p_\lambda(u) < [\varepsilon]^m\right\} \quad (\lambda \in \Lambda,\ m \in \mathbb{N})$$

as a base of neighborhoods of 0 [5].

The following result came as a surprise to specialists in the theory of nonlinear generalized functions:

Theorem 7.5 *(Automatic continuity). Let E, F be metrizable locally convex spaces. Let $T : \mathcal{G}_E \to \mathcal{G}_F$ be internal. Then T is continuous for the sharp topology.*

Proof. If $T([u_\varepsilon]) = [T_\varepsilon(u_\varepsilon)]$, let $\bar{T}$ be the internal map $[T_\varepsilon] : {}^*E \to {}^*F$. By (4), $\bar{T}(\mathcal{M}_{{}^*E}) \subseteq \mathcal{M}_{{}^*F}$ and by (5), $u \approxeq v \implies \bar{T}u \approxeq \bar{T}v$, for each $u, v \in \mathcal{M}_{{}^*E}$. By Proposition 7.3, this means that $\bar{T}$ is ρ-continuous on $\mathcal{M}_{{}^*E}$. Hence T is continuous for the sharp topology on $\mathcal{G}_E$. $\square$

7.2. Pointwise regularity

For an open set $\Omega \subseteq \mathbb{R}^d$, the usual locally convex topology on $\mathcal{C}^\infty(\Omega)$ is described by the seminorms $p_m(u) := \sup_{x\in K_m,\, |\alpha|\leqslant m} |\partial^\alpha u(x)|$, where $(K_m)_m$ is a compact exhaustion of Ω ($m \in \mathbb{N}$). In this case, one usually denotes $\mathcal{G}(\Omega) := \mathcal{G}_{\mathcal{C}^\infty(\Omega)}$. We denote ${}^*\Omega_c := \bigcup_{K \Subset \Omega} {}^*K$.

Proposition 7.6. $\mathcal{M}_{{}^*\mathcal{C}^\infty(\Omega)} = \{u \in {}^*\mathcal{C}^\infty(\Omega)\colon (\forall \alpha \in \mathbb{N}^d)\ (\forall x \in {}^*\Omega_c)\ (\partial^\alpha u(x) \in \mathcal{M}_{{}^*\mathbb{R}})\}$.

Proof. For a finite set $A = \{a_1, \dots, a_n\}$, we denote $P_{a_1} \wedge \cdots \wedge P_{a_n}$ as $\bigwedge_{a\in A} P_a$. Let $m \in \mathbb{N}$. By transfer on

$$(\forall r \in \mathbb{R})\ \left(\forall u \in \mathcal{C}^\infty(\Omega)\right) \quad \left[p_m(u) \leqslant r \iff (\forall x \in K_m) \left(\bigwedge_{\alpha\in\mathbb{N}^d,\, |\alpha|\leqslant m} \left|\partial^\alpha u(x)\right| \leqslant r\right)\right],$$

we find for $u \in {}^*\mathcal{C}^\infty(\Omega)$ that

$$\begin{aligned}
&(\exists N\in\mathbb{N})\quad \bigl({}^*p_m(u)\leqslant\rho^{-N}\bigr)\\
&\quad\iff\quad (\exists N\in\mathbb{N})\ \bigl(\forall x\in{}^*K_m\bigr)\ \bigl(\forall\alpha\in\mathbb{N}^d,\ |\alpha|\leqslant m\bigr)\quad \bigl(\bigl|\partial^\alpha u(x)\bigr|\leqslant\rho^{-N}\bigr)\\
&\quad\iff\quad \bigl(\forall\alpha\in\mathbb{N}^d,\ |\alpha|\leqslant m\bigr)\ (\exists N\in\mathbb{N})\ \bigl(\forall x\in{}^*K_m\bigr)\quad \bigl(\bigl|\partial^\alpha u(x)\bigr|\leqslant\rho^{-N}\bigr).
\end{aligned}$$

By Q.S. on the negation of the latter formula, it is equivalent with

$$\bigl(\forall\alpha\in\mathbb{N}^d,\ |\alpha|\leqslant m\bigr)\ \bigl(\forall x\in{}^*K_m\bigr)\ (\exists N\in\mathbb{N})\quad \bigl(\bigl|\partial^\alpha u(x)\bigr|\leqslant\rho^{-N}\bigr).$$

Hence

$$\begin{aligned}
&u\in\mathcal{M}_{{}^*\mathcal{C}^\infty(\Omega)}\\
&\quad\iff\quad (\forall m\in\mathbb{N})\ \bigl(\forall\alpha\in\mathbb{N}^d,\ |\alpha|\leqslant m\bigr)\ \bigl(\forall x\in{}^*K_m\bigr)\ (\exists N\in\mathbb{N})\quad \bigl(\bigl|\partial^\alpha u(x)\bigr|\leqslant\rho^{-N}\bigr)\\
&\quad\iff\quad \bigl(\forall\alpha\in\mathbb{N}^d\bigr)\ \bigl(\forall x\in{}^*\Omega_c\bigr)\quad \bigl(\partial^\alpha u(x)\in\mathcal{M}_{{}^*\mathbb{R}}\bigr).\qquad\square
\end{aligned}$$

Similarly, $\mathcal{N}_{{}^*\mathcal{C}^\infty(\Omega)}=\{u\in{}^*\mathcal{C}^\infty(\Omega)\colon\ (\forall\alpha\in\mathbb{N}^d)\ (\forall x\in{}^*\Omega_c)\ (\partial^\alpha u(x)\in\mathcal{N}_{{}^*\mathbb{R}})\}$.

Definition 7.7. The subalgebra $\mathcal{G}^\infty(\Omega)$ of $\mathcal{G}^\infty$-regular Colombeau generalized functions on Ω is defined as

$$\Bigl\{(u_\varepsilon)_\varepsilon\in\mathcal{C}^\infty(\Omega)^I\colon\ (\forall K\Subset\Omega)\ (\exists N\in\mathbb{N})\ \bigl(\forall\alpha\in\mathbb{N}^d\bigr)\ \Bigl(\sup_{x\in K}\bigl|\partial^\alpha u_\varepsilon(x)\bigr|\leqslant\varepsilon^{-N}\text{a.e.}\Bigr)\Bigr\}/\mathcal{N}_{\mathcal{C}^\infty(\Omega)}.$$

As in Section 7.1 (and by transfer, as in the proof of Proposition 7.6),

$$\mathcal{G}^\infty(\Omega)\cong\bigl\{u\in{}^*\mathcal{C}^\infty(\Omega)\colon\ (\forall K\Subset\Omega)\ (\exists N\in\mathbb{N})\ \bigl(\forall\alpha\in\mathbb{N}^d\bigr)\ \bigl(\forall x\in{}^*K\bigr)\ \bigl(\bigl|\partial^\alpha u(x)\bigr|\leqslant\rho^{-N}\bigr)\bigr\}/\mathcal{N}_{{}^*\mathcal{C}^\infty(\Omega)}.$$

Definition 7.8. We say that $u\in{}^*\mathcal{C}^\infty(\Omega)$ is $\mathcal{G}^\infty$-regular at $x\in{}^*\Omega$ if there exists $N\in\mathbb{N}$ such that for each $\alpha\in\mathbb{N}^d$, $|\partial^\alpha u(x)|\leqslant\rho^{-N}$.

Theorem 7.9 *(Pointwise characterization of $\mathcal{G}^\infty(\Omega)$). Let $u\in{}^*\mathcal{C}^\infty(\Omega)$. The following are equivalent:*

1. $(\forall K\Subset\Omega)\ (\exists N\in\mathbb{N})\ (\forall\alpha\in\mathbb{N}^d)\ (\forall x\in{}^*K)\ (|\partial^\alpha u(x)|\leqslant\rho^{-N})$,
2. *u is $\mathcal{G}^\infty$-regular at each $x\in{}^*\Omega_c$.*

Proof. $\Longrightarrow$: Clear.

$\Longleftarrow$: Suppose that 1 does not hold. Then we find $K\Subset\Omega$ and $\alpha_n\in\mathbb{N}^d$, $\forall n\in\mathbb{N}$ such that $(\forall n\in\mathbb{N})$ $(\exists x\in{}^*K)$ $(|\partial^{\alpha_n}u(x)|\not\leqslant\rho^{-n})$. By Q.S., $(\exists x\in{}^*K)$ $(\forall n\in\mathbb{N})$ $(|\partial^{\alpha_n}u(x)|\not\leqslant\rho^{-n})$, contradicting the hypotheses. $\square$

Hence (cf. [14, Thm. 5.1]),

$$\mathcal{G}^\infty(\Omega)=\bigl\{u\in{}^*\mathcal{C}^\infty(\Omega)\colon\ u\ \text{is}\ \mathcal{G}^\infty\text{-regular at each}\ x\in{}^*\Omega_c\bigr\}/\mathcal{N}_{{}^*\mathcal{C}^\infty(\Omega)}.$$

If one works instead with an ultrafilter extending the common filter (2), as in [15], the proof of Theorem 7.9 breaks down, since cointernal sets then do not automatically have the finite intersection property.

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