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Approximation of extremal solution of non-Fourier moment problem and optimal control for non-homogeneous vibrating systems ☆

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ABSTRACT

Trigonometric non-Fourier moment problems arise as a result of various control problem study. In current paper, the extremal solution, i.e. the one with the least L^2-norm is searched for. Proposed is an algorithm that allows to change an infinite system of equations into the linear one with only a finite number of equations. The mentioned algorithm is based on the fact, that in the case of a Fourier moment problem, the extremal solution is periodic and easy to construct. The extremal solution of a non-Fourier moment problem close to a Fourier one is approximated by a sequence of solutions with periodicity disturbed in a finite number of equations. It is proved that this sequence of approximations converges to the desired extremal solution. The paper is concluded with the particular example whose consideration leads to a moment problem elaborated in the first part of the article.

1. Introduction

The non-Fourier moment problem arises very often, when we deal with the control related problems of non-homogeneous objects [15,16]. The results about existence of solutions of such moment problems are well-known (e.g. see [1,2] and references therein) and used by many authors. When the existence is proved, the next problem arising, is to give an effective algorithm for finding or approximating a particular solution. In many cases, the solution needed in applications is the optimal one, i.e. the one with the extremal norm. In the case of the Fourier moment problem, it is proved that the extremal solution, in the sense of L^2-norm, is a periodic function [7]. In the case of non-Fourier moment problem, the periodicity may be disturbed, what causes difficulties while dealing with the analytical solution of such problems. In this paper, the idea of periodicity is used to deal with a non-Fourier moment problem that is close to a Fourier one.

We propose an algorithm that approximates the extremal solution of non-Fourier moment problem stated in Hilbert space L^2. We use the fact that considered moment problem (1) is quadratically close to a Fourier one and the last has an easy to describe extremal solution $\widetilde{u}_o$ that is a periodic function (Theorem 2). We claim that the extremal solution of (1) can be approximated by a sequence (u_N) of solutions of the moment problems, where equations

$$\int_0^T u(t)\exp(i\omega_n t)\,dt = b_n$$

are replaced with

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$$\int_0^T u(t)\exp(int)\,dt = 0$$

for $n > N$. The above moment problem still is an infinite set of integral equations, but approximating its extremal solution with the use of the proposed algorithm leads to a finite set of linear equations. In fact, it is a Cramer system of equations.

The idea of replacing the infinite set of equations of some non-Fourier moment problem by a finite set is used, for example, by T. Strohmer and K. Gröchenig [17,6]. In those papers the problem of reconstruction of a function, given some sampling set is studied. The authors deal with some infinite set of equations, and their idea to approximate the solution is to truncate the problem to a finite set of equations. To deal with possible ill-posedness, they approximate the obtained system of equations by some set of trigonometric polynomials. In current paper, we use the idea of quadratically close sets [5] and a fact that the considered basis $(\exp(i\omega_n t))$ is quadratically close to a Fourier set $(\exp(int))$. This situation is characteristic for non-homogeneous objects. The finite set of equations is obtained as a consequence of periodicity of a function given by Fourier series.

The correctness of the proposed algorithm is checked in Section 3. We use the results of [5] to prove that the sequence (u_N) converges to the extremal solution of (1).

As an application, we consider the non-homogeneous string. In order for this paper to be self-contained, we state main result (Theorems 7) about location points of the eigenvalues of the operator connected with vibrations of this object. The proofs of facts stated in Section 4 use technics of mixed differential-integral equations and then the Neumann series described in [15] and in [16]. The different proofs of the mentioned theorems can be found in monographs [9,10,13,12] and others. The paper is concluded with numerical example in which, we obtain the approximation of the optimal control for a particular non-homogeneous string.

2. Extremal solution approximation

We consider the following non-Fourier moment problem.

$$\int_0^T u(t)\exp(i\omega_n t)\,dt = b_n, \quad n \in \mathbb{Z} \tag{1}$$

with $T > 0$, $b_n \in \mathbb{C}$, the ω_n's are different numbers and $\omega_n - n = O(1/n)$.

Remark 1. Assume $T \geqslant 2\pi$. The necessary and sufficient condition for the existence of a solution $u \in L^2(0,T)$ of the system (1) is the convergence of the series $\sum_{n=-\infty}^{\infty} b_n^2$, [11,18].

The function $u_o \in L^2(0,T)$ having the least L^2-norm and satisfying (1) is called the *extremal solution*. Our interest is to find exactly or, at least, to approximate such a function.

We consider the closure $\mathcal{L}$ of linear span over

$$\{\exp(i\omega_n t)\colon\ n \in \mathbb{Z},\ 0 \leqslant t \leqslant T\}.$$

Proposition 1. *The extremal solution of* (1) *belongs to* $\mathcal{L}$.

For the proof we notice that if u is any solution of (1), then it can be written in a unique way as $u_o + u_\perp$, where $u_o \in \mathcal{L}$ and $u_\perp$ belongs to the orthogonal complement of $\mathcal{L}$. Therefore u_o satisfies (1) and

$$\|u\|_2^2 = \|u_o\|_2^2 + \|u_\perp\|_2^2.$$

Consequently, $\|u\|_2 \geqslant \|u_o\|_2$ and the L^2-norm of u attains the least value if and only if $\|u_\perp\|_2 = 0$. The last means that the solution u of (1) with the least norm is equal to u_o, the member of $\mathcal{L}$.

Further on, we assume $T \geqslant 2\pi$ and use the transformation $f \mapsto \widehat{f}$ from $L^2(0,T)$ to $L^2(0,2\pi)$ defined as follows. Let m be a positive integer with property $2(m-1)\pi \leqslant T < 2m\pi$. We consider an extension of f onto $[0,2m\pi]$, where $f(t) = 0$ on $(T,2m\pi]$. With this in mind we put

$$\widehat{f}(t) = \sum_{k=0}^{m-1} f(t + 2k\pi), \quad 0 \leqslant t \leqslant 2\pi.$$

To simplify the notation we set

$$h_n(t) = \begin{cases} \exp(i\omega_n t) & \text{for } 0 \leqslant t \leqslant T \\ 0 & \text{for } T < t \leqslant 2m\pi. \end{cases}$$

Theorem 2. *Provided $\omega_n = n$ for all $n \in \mathbb{Z}$, the optimal solution u_o of (1) is a periodic function on $[0, T]$ (for $T \geqslant 2\pi$) with period 2π, i.e. if $t, t + 2\pi \in [0, T]$, then $u_o(t) = u_o(t + 2\pi)$.*

Considering the moment problem (1) for $\omega_n = n$ we have:

$$b_n = \int_0^T u(t) \exp(int)\, dt = \int_0^{2\pi} \widehat{u}(t) \exp(int)\, dt. \tag{2}$$

Because $\{\exp(int)\}_{n\in\mathbb{Z}}$ constitutes an orthonormal basis on $[0, 2\pi]$, there is a unique function $v \in L^2(0, 2\pi)$ satisfying (2). We extend v/m periodically onto $[0, T]$ to obtain function u_o. It is easy to show that u_o satisfies (2) and it is a member of $L^2(0, T)$ and, moreover, of $\mathcal{L}$. Thus u_o is extremal.

Our task is to approximate u_o in general case. For this purpose, given $N > 0$, we study the closure $\mathcal{L}_N$ of linear span over

$$\{h_n(t)\colon |n| \leqslant N,\ 0 \leqslant t \leqslant T\} \cup \{\exp(int)\colon |n| > N,\ 0 \leqslant t \leqslant T\}$$

and replace moment problem (1) with

$$\begin{aligned} &\int_0^T u(t) h_n(t)\, dt = b_n \quad \text{for } |n| \leqslant N \\ &\int_0^T u(t) \exp(int)\, dt = 0 \quad \text{for } |n| > N. \end{aligned} \tag{3}$$

We shall show (Section 3) that the extremal solution u_N of the moment problem (3) (in particular $u_N \in \mathcal{L}_N$) approximates u_o, in the sense that

$$\lim_{N\to\infty} u_N = u_o.$$

In a moment we shall prove that u_N can be found after solving the system of linear equations. Thus the infinite system (3) will be replaced with a finite one.

Remark 2. The systems that span $\mathcal{L}$ and $\mathcal{L}_N$ constitute Riesz bases in $[0, 2\pi]$. They still may be considered for $T > 2\pi$, but solutions of moment problems (1) or (3) are no longer unique. If $T < 2\pi$, mentioned moment problems may have no solution, as considered systems are not minimal on $L^2(0, T)$. Therefore we assume that $T \geqslant 2\pi$.

In order to find u_N, we propose the following algorithm.

Algorithm. We know that u_N belongs to $\mathcal{L}_N$ and write

$$u_N = \sum_{n=-N}^{N} \alpha_n h_n + g, \tag{4}$$

where g is periodical on $[0, T]$ with period 2π and

$$\int_0^{2\pi} g(t) \exp(int)\, dt = 0 \quad \text{for } |n| \leqslant N. \tag{5}$$

We need to find the function g and coefficients α_n for $|n| \leqslant N$. Assuming $|n| > N$ and using (4), (5), we obtain

$$\begin{aligned} 0 &= \int_0^T \left(\sum_{n=-N}^{N} \alpha_n h_n(t) + g(t) \right) \exp(int)\, dt \\ &= \sum_{k=0}^{m-1} \int_{2k\pi}^{2(k+1)\pi} \left(\sum_{n=-N}^{N} \alpha_n h_n(t) + g(t) \right) \exp(int)\, dt \\ &= \int_0^{2\pi} \left(\sum_{n=-N}^{N} \alpha_n \widehat{h}_n(t) + \widehat{g}(t) \right) \exp(int)\, dt. \end{aligned}$$

Therefore

$$\sum_{n=-N}^{N} \alpha_n \widehat{h}_n(t) + \widehat{g}(t) = \sum_{n=-N}^{N} d_n \exp(int)$$

for some d_n's. We notice that

$$\widehat{g}(t) = \begin{cases} mg(t) & \text{for } 0 \leqslant t \leqslant T - 2(m-1)\pi \\ (m-1)g(t) & \text{for } T - 2(m-1)\pi < t \leqslant 2\pi. \end{cases}$$

In particular, if we define the function $m : [0, T] \to \mathbb{R}$ by formula

$$m(t) = \begin{cases} -1/m & \text{for } 0 \leqslant t \leqslant T - 2(m-1)\pi \\ -1/(m-1) & \text{for } T - 2(m-1)\pi < t \leqslant 2\pi \end{cases}$$

then for $0 \leqslant t \leqslant 2\pi$,

$$g(t) = m(t) \sum_{n=-N}^{N} \alpha_n \widehat{h}_n(t) - m(t) \sum_{n=-N}^{N} d_n \exp(int)$$

and it extends periodically onto $[0, T]$.

Now, we use (5) and for $|n| \leqslant N$ obtain:

$$\begin{aligned} 0 &= \int_0^{2\pi} g(t) \exp(int)\, dt \\ &= \sum_{k=-N}^{N} \alpha_k \int_0^{2\pi} m(t) \widehat{h}_k(t) \exp(int) \\ &\quad - \sum_{k=-N}^{N} d_k \int_0^{2\pi} m(t) \exp\bigl(i(k+n)t\bigr)\, dt. \end{aligned} \tag{6}$$

What follows is the system of $2N+1$ equations with $4N+2$ unknown, the α_k's and d_k's. The second set of $2N+1$ equations we obtain from Eq. (4). Namely,

$$\begin{aligned} b_n &= \int_0^T u_N(t) h_n(t)\, dt \\ &= \sum_{k=-N}^{N} \alpha_k \left(\int_0^T h_k(t) h_n(t)\, dt + \int_0^{2\pi} m(t) \widehat{h}_k(t) \widehat{h}_n(t)\, dt \right) \\ &\quad - \sum_{k=-N}^{N} d_k \int_0^{2\pi} m(t) \widehat{h}_n(t) \exp(ikt)\, dt \end{aligned} \tag{7}$$

for $|n| \leqslant N$.

3. Correctness of the algorithm

We are going to proof that the function u_N obtained as a result of solution of the system of Eqs. (6)–(7) approximates the extremal solution of the moment problem (1). We shall use results and terminology of [5].

Let $\mathcal{H}$ be a (separable) Hilbert space. We say that two sequences (φ_ξ) and (φ'_ξ) are quadratically ε-close if

$$\sum_{\xi} \left\| \varphi_\xi - \varphi'_\xi \right\|^2 < \varepsilon, \quad \text{for some } \varepsilon > 0.$$

Lemma 3. *Assume (ψ_ξ) and (ψ'_ξ) are Riesz bases biorthogonal to (φ_ξ) and (φ'_ξ) respectively. If (φ_ξ) and (φ'_ξ) are quadratically ε-close then there exists a constant M, such that (ψ_ξ) and (ψ'_ξ) are quadratically $M\varepsilon$-close.*

For the proof, we recall that all (ψ_ξ), (ψ'_ξ), (φ_ξ) and (φ'_ξ) are Riesz bases and therefore there exists an orthonormal basis (e_ξ) of $\mathcal{H}$ and isomorphisms R, S with properties

$$Re_\xi = \varphi_\xi, \qquad Se_\xi = \varphi'_\xi, \qquad \left(R^*\right)^{-1} e_\xi = \psi_\xi, \qquad \left(S^*\right)^{-1} e_\xi = \psi'_\xi.$$

By hypothesis, $\sum_\xi \|\varphi_\xi - \varphi'_\xi\|^2 < \varepsilon$. We start with consideration of the terms of the last series.

$$\left\|\varphi_\xi - \varphi'_\xi\right\|^2 = \left\|(R-S)e_\xi\right\| = \left\|R\left(I - R^{-1}S\right)e_\xi\right\|^2 \geqslant \left\|R^{-1}\right\|^{-2}\left\|\left(I - R^{-1}S\right)e_\xi\right\|^2,$$

where I denotes the identity operator. We consider the expression $\|\psi_\xi - \psi'_\xi\|^2$ next. We have

$$\left\|\psi_\xi - \psi'_\xi\right\|^2 = \left\|\left(\left(R^*\right)^{-1} - \left(S^*\right)^{-1}\right)e_\xi\right\| = \left\|\left(S^*\right)^{-1}\left(S^*\left(R^*\right)^{-1} - I\right)e_\xi\right\|^2 \leqslant \left\|S^{-1}\right\|^2\left\|\left(I - R^{-1}S\right)^* e_\xi\right\|^2.$$

Now, let W be an operator with property $\sum_\xi \|We_\xi\|^2 < \infty$. Then

$$\sum_\xi \|We_\xi\|^2 = \sum_{\xi,\zeta}\left|\langle We_\xi, e_\zeta\rangle\right|^2 = \sum_{\xi,\zeta}\left|\langle e_\xi, W^* e_\zeta\rangle\right|^2 = \sum_\xi \left\|W^* e_\xi\right\|^2.$$

Because

$$\sum_\xi \left\|\left(I - R^{-1}S\right)e_\xi\right\|^2 < \left\|R^{-1}\right\|^2 \varepsilon < \infty$$

we may apply the above reasoning. Putting $M = \|S^{-1}\|^2\|R\|^2$, we obtain

$$\sum_\xi \left\|\psi_\xi - \psi'_\xi\right\|^2 \leqslant M \sum_\xi \left\|\varphi_\xi - \varphi'_\xi\right\|^2 < M\varepsilon.$$

Therefore (ψ_ξ) and (ψ'_ξ) are $M\varepsilon$-close.

Let us notice that our bases are "close" to orthogonal one: $(\exp(int))$. It may be shown that in this case the constant M is close to 1.

Let $\mathcal{H} = L^2(0,T)$. We denote

$$\varphi_n = \exp(i\omega_n t), \quad \varphi_n^N = \begin{cases} \exp(i\omega_n t), & \text{for } |n| \leqslant N \\ \exp(int), & \text{for } |n| > N \end{cases}$$

for integer n. Then $\mathcal{L}$ and $\mathcal{L}^N$ are closures of linear spans over $\{\varphi_n\colon n \in \mathbb{Z}\}$ and $\{\varphi_n^N\colon n \in \mathbb{Z}\}$ respectively.

Proposition 4. *Given $\varepsilon > 0$, there exists N_0 such that for $N \geqslant N_0$, sequences (φ_n) and (φ_n^N) are quadratically ε-close provided $\omega_n - n = O(1/n)$.*

In the beginning of the proof, we notice that $\omega_n - n = O(1/n)$ means the convergence of the series $\sum_{n=-\infty}^{\infty}(\omega_n - n)^2$. Let $\varepsilon > 0$ be given. We need to find N_0 such that for all $N \geqslant N_0$ the inequality

$$\sum_{|n|>N} \left\|e^{i\omega_n t} - e^{int}\right\|^2 < \varepsilon \tag{8}$$

holds. We consider $\|e^{i\omega_n t} - e^{int}\|^2$ to obtain

$$\begin{aligned} \left\|e^{i\omega_n t} - e^{int}\right\|^2 &= 2\int_0^T dt - \int_0^T e^{i(\omega_n - n)t}\,dt - \int_0^T e^{-i(\omega_n - n)t}\,dt \\ &= 2T - 2\frac{\sin((\omega_n - n)T)}{\omega_n - n} \\ &\leqslant \left(T^3/3\right)(\omega_n - n)^2. \end{aligned}$$

Therefore the series $\sum_{|n|>N}\|e^{i\omega_n t} - e^{int}\|^2$ converges and for sufficiently large N, Eq. (8) holds.

We consider the orthogonal complements $\mathcal{L}^\perp$, $\mathcal{L}_N^\perp$ of $\mathcal{L}$ and $\mathcal{L}_N$ respectively. Let (x_k) be the orthonormal basis of $\mathcal{L}^\perp$ and let $P : \mathcal{L}^\perp \to \mathcal{L}_N^\perp$, the projection. Then P is an isomorphism, so, in particular, (Px_k) forms a Riesz basis in $\mathcal{L}_N^\perp$.

Proposition 5. *There exists a constant C, such that for any sufficiently small ε the systems (x_k) and (Px_k) are quadratically $C\varepsilon$-close provided (φ_n) and (φ_n^N) are quadratically ε-close.*

Given $x \in \mathcal{L}^{\perp}$, we write $x = Px + Qx$, where $Q : \mathcal{L}^{\perp} \to \mathcal{L}_N$ is the projection. Then, because (φ_n^N) is a Riesz basis, there exists a constant C such that for any $y \in \mathcal{L}_N$ the inequality

$$\|y\|^2 \leqslant C \sum_n \left|\left\langle y, \varphi_n^N \right\rangle\right|^2$$

holds and because (φ_n^N) is quadratically ε-close to (φ_n), for sufficiently small ε, C can be chosen universally for all $N \geqslant N_0$ [18]. Then we have

$$\begin{aligned}\sum_k \|x_k - Px_k\|^2 = \sum_k \|Qx_k\|^2 &\leqslant C \sum_{k,n} \left|\left\langle Qx_k, \varphi_n^N \right\rangle\right|^2 \\ &= C \sum_{k,n} \left|\left\langle x_k, \varphi_n^N \right\rangle\right|^2 \\ &= C \sum_{k,n} \left|\left\langle x_k, \varphi_n^N - \varphi_n \right\rangle\right|^2.\end{aligned}$$

Because (x_k) constitutes the orthonormal set, the last is less or equal to [3]

$$C \sum_n \left\|\varphi_n^N - \varphi_n\right\|^2.$$

What follows is

$$\sum_k \|x_k - Px_k\|^2 \leqslant C \sum_n \left\|\varphi_n^N - \varphi_n\right\|^2 < C\varepsilon.$$

The proof is complete.

Now, let $(\psi_\xi) = (\psi_n) \cup (x_k)$ and $(\psi_\xi^N) = (\psi_n^N) \cup (Px_k)$.

By Lemma 3, we get that the systems (ψ_ξ^N), (ψ_ξ) biorthogonal to (φ_ξ^N) and (φ_ξ) respectively, are quadratically $M\varepsilon$-close.

Theorem 6. *The sequence (u_N) of functions obtained in Section 2 converges to the function u_o which is the extremal solution of the moment problem (1).*

Because (ψ_ξ) and (ψ_ξ^N) are Riesz bases in $\mathcal{H}$ we may write

$$u_o = \sum_\xi \langle u_o, \varphi_\xi \rangle \psi_\xi, \qquad u_N = \sum_\xi \left\langle u_N, \varphi_\xi^N \right\rangle \psi_\xi^N.$$

Let $\varepsilon > 0$ be given. Let $\sigma = \sum_{n\in\mathbb{Z}} |b_n|^2$. By Proposition 4, there exists N_1, such that for any $N > N_1$ inequality

$$\sum_{n\in\mathbb{Z}} \left\|\psi_n - \psi_n^N\right\|^2 < \frac{\varepsilon}{4\sigma}$$

holds. Further on, we have the existence of N_2 with property

$$\left\| \sum_{|n|>N} b_n \psi_n \right\| < \frac{\sqrt{\varepsilon}}{2} \quad \text{for } N > N_2.$$

Ultimately, we obtain for $N > \max\{N_1, N_2\}$:

$$\|u_o - u_N\| = \left\| \sum_{n\in\mathbb{Z}} \langle u_o, \varphi_n \rangle \psi_n - \sum_{n\in\mathbb{Z}} \left\langle u_N, \varphi_n^N \right\rangle \psi_n^N \right\| = \left\| \sum_{|n|\leqslant N} b_n\left(\psi_n - \psi_n^N\right) + \sum_{|n|>N} b_n \psi_n \right\|.$$

Because $(a+b)^2 \leqslant 2(a^2+b^2)$ for any positive numbers a and b, we get

$$\begin{aligned}\|u_o - u_N\|^2 &\leqslant 2\left(\left\| \sum_{|n|\leqslant N} b_n\left(\psi_n - \psi_n^N\right) \right\|^2 + \left\| \sum_{|n|>N} b_n \psi_n \right\|^2 \right) \\ &< 2\left(\sum_{|n|\leqslant N} |b_n|^2 \sum_{|n|\leqslant N} \left\|\psi_n - \psi_n^N\right\|^2 + \frac{\varepsilon}{4} \right) \\ &< \varepsilon.\end{aligned}$$

Therefore (u_N) converges to u_o.

We conclude this section with a following remark.

Remark 3. The major impact on the actual size of N is given by the speed the sequence (b_n) tends to 0 (while $n \to \pm\infty$). If that speed is slow, the sum σ takes large values and the constant N_1 must be also large. Here is the way how to deal with this inconvenience. We consider the moment problem

$$\int_0^T v_N(t)\exp(i\omega_n t)\,dt = b_n \quad \text{for } |n| \leqslant N$$

$$\int_0^T v_N(t)\exp(int)\,dt = b_n \quad \text{for } |n| > N. \tag{9}$$

Let us define the function $y(t) = (1/T)\sum_{n=-\infty}^{\infty} b_n \exp(-int)$ and replace v_N in (9) by $u_N + y$. As a result, we obtain the moment problem (3), but with coefficients $b_n - \int_0^T y(t)\exp(i\omega_n t)\,dt$ for $|n| \leqslant N$. Then we apply the Section 2 algorithm, to find the approximation u_N and immediately after it, v_N.

4. Example of application

The non-Fourier moment problem (1) considered in previous sections arises naturally while studying controllability of a non-homogeneous generalization of the wave equation

$$\ddot{w}(x,t) = \kappa w''(x,t), \tag{10}$$

where $\dot{w}$, $\ddot{w}$ mean the first and the second derivative with respect to variable t and prime denotes derivatives with respect to x.

Several authors (starting with pioneer work [13]) considered various generalizations of Eq. (10) for various purposes. Their study lead to the results similar to the ones outlined below. For our purposes, we consider a non-homogeneous, normalized (i.e. of length 1) string [14] given by the equation

$$\ddot{v}(x,t) = a(x)v''(x,t) \tag{11}$$

with Dirichlet with boundary conditions and one-side boundary control:

$$v(0,t) = 0, \qquad v(1,t) = u(t) \quad \text{for } t \geqslant 0.$$

The control problem for the object described by the above equations is to, given time $T > 0$, find an $L^2(0,T)$-function $u : [0,T] \to \mathbb{R}$ that, while imposed on the string, allows to control it from the initial state of rest

$$v(x,0) = \dot{v}(x,0) = 0 \quad (0 \leqslant x \leqslant 1) \tag{12}$$

to the final position

$$v(x,T) = y_T(x), \qquad \dot{v}(x,T) = \dot{y}_T(x) \quad (0 \leqslant x \leqslant 1) \tag{13}$$

where y_T, $\dot{y}_T$ are some functions defined on $[0,1]$.

The movement operator A given by formula $A\varphi = -a\varphi''$ is considered in Hilbert space H, whose underlying set is $L^2(0,1)$, but the inner product is defined by

$$\langle \varphi_1, \varphi_2 \rangle = \int_0^1 \frac{1}{a(x)}\varphi_1(x)\varphi_2(x)\,dx.$$

The domain $D(A)$ of the operator A is equal to

$$\{\varphi \in H^2(0,1)\colon \varphi(0) = \varphi(1) = 0\}.$$

It turns out that A is positive and self-adjoint, so its eigenvectors (v_n) form an orthogonal basis in H and the corresponding eigenvalues are simple and form an increasing sequence

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \to \infty.$$

Namely, following technics used (for example) in [15] one may prove the next theorem.

Theorem 7. *If λ_n is an eigenvalue of the operator A, then*

$$\lambda_n = \left(\int_0^1 \frac{ds}{\sqrt{a(s)}}\right)^{-2}\left(n\pi + O\left(\frac{1}{n}\right)\right)^2.$$

Basing on the ideas from the monograph [8], we obtain the following.

Theorem 8. *The unique weak solution of the initial value problem* (11), (12) *is*

$$y(x,t)=\sum_{n=1}^{\infty}\left((-1)^{n+1}\widehat{c}_n\int_0^t u(s)\sin\bigl(\sqrt{\lambda_n}(t-s)\bigr)\,ds\right)v_n(x), \tag{14}$$

where v_n is an eigenvector of the operator A with the corresponding eigenvalue λ_n and $\widehat{c}_n=(-1)^{n+1}\langle Id, v_n\rangle$ with $Id: x\mapsto x$.

Elementary calculations give the result $\widehat{c}_n=(-1)^n v_n'(1)/\lambda_n$. Therefore $\widehat{c}_n$ cannot be equal to zero or (by the uniqueness of the solution of the Cauchy problem) it would mean $v_n=0$ identically.

Now we can state the optimal control problem for the object considered by us.

Optimal Control Problem. Find the control function $u\in L^2(0,T)$ with the least norm, such that the weak solution (14) of the initial value problem (11), (12) satisfies end conditions (13).

To use the algorithm described in Section 2, we need to state the above problems as a moment problem, whose extremal solution will be the optimal control, i.e. a function that solves problems stated above. We mention that the method of reducing the controllability problem to an exponential moment problem is given in many papers, including [1,4,12]. We point out a couple of items to show the way how the coefficients in the moment problem arise from the parameters of the object itself and the final state functions y_T, $\dot{y}_T$.

For $n\geqslant 1$, let us define (real) numbers

$$\begin{aligned}\dot{c}_n&=\frac{(-1)^{n+1}}{\widehat{c}_n}\left(\langle y_T,v_n\rangle\sin(\sqrt{\lambda_n}T)+\frac{\langle\dot{y}_T,v_n\rangle}{\sqrt{\lambda_n}}\cos(\sqrt{\lambda_n}T)\right)\\ c_n&=\frac{(-1)^{n+1}}{\widehat{c}_n}\left(-\langle y_T,v_n\rangle\cos(\sqrt{\lambda_n}T)+\frac{\langle\dot{y}_T,v_n\rangle}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}T)\right).\end{aligned} \tag{15}$$

It follows from the Theorem 8 that the control u must satisfy the following infinite system of equations.

$$\begin{aligned}&\int_0^T u(t)\cos(\sqrt{\lambda_n}t)\,dt=\dot{c}_n\\ &\int_0^T u(t)\sin(\sqrt{\lambda_n}t)\,dt=c_n.\end{aligned} \tag{16}$$

It is the moment problem in the trigonometric form. The equivalent to (16) exponential moment problem in the form (1) we obtain after putting

$$J=\int_0^1\frac{ds}{\sqrt{a(s)}},\qquad \omega_n=\frac{J}{\pi}\sqrt{\lambda_n},\qquad \omega_{-n}=-\omega_n,$$

$$b_n=\frac{\pi}{J}(\dot{c}_n+ic_n),\qquad b_{-n}=\frac{\pi}{J}(\dot{c}_n-ic_n).$$

Let us put additionally $\omega_0=0$ and b_0 equal to some constant, say C, so the algorithm may be applied. Finally one may find the correct value of C using the Lagrange multiplier method.

As it was noticed (Remark 1), the solution of the moment problem (16) exists, provided $T\geqslant 2\pi$ and $\sum_{n=-\infty}^{\infty}|b_n|^2<\infty$. Therefore the algorithm described in Section 2 may be applied.

5. Numerical example

We consider the non-homogeneous string equation (11) with $a(x)=(x+1)^2$. The operator A ruling the movement of this string has eigenvalues (for $n\geqslant 1$) $\lambda_n=1/4+(n\pi/\ln 2)^2$ and the corresponding (normalized) eigenvectors are given by

$$v_n(x)=\sqrt{\frac{2}{\ln 2}}\sqrt{x+1}\sin\left(\frac{n\pi}{\ln 2}\ln(x+1)\right).$$

We apply the described algorithm for optimal control to reach the final state given by the sequence $b_n=(1/n^2+i/n^2)$ in time $T=6\pi$. The 5th approximation (i.e. $N=5$) of the optimal control function is given in Fig. 1.

The norm (see estimations at the end of Section 3) of u_o is approximated by $\|u_N\|$ up to 0.01. For comparison, we include the graph of the 10th approximation (see Fig. 2). The graphs were obtained using the Delphi program that is a realization of proposed algorithm.

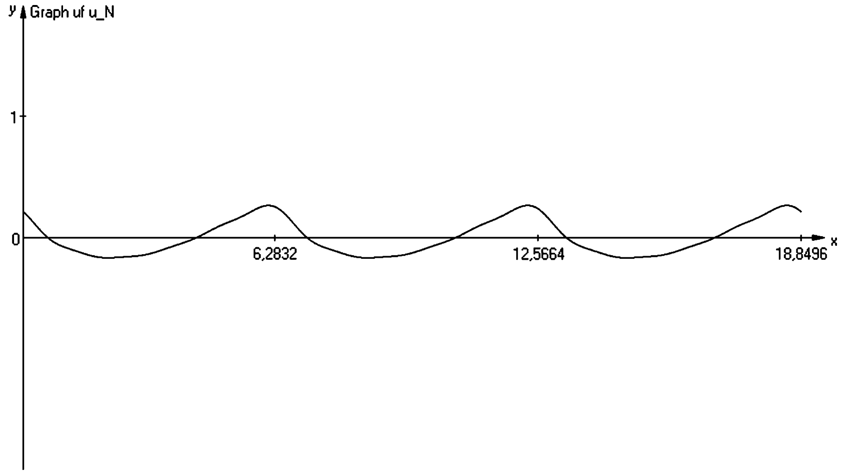

Fig. 1. The 5th approximation of the optimal control function.

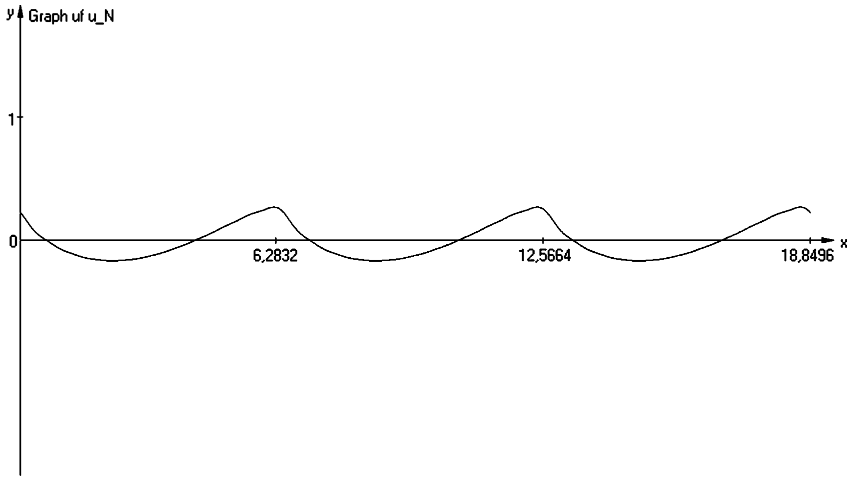

Fig. 2. The 10th approximation of the optimal control function.

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