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On the semiclassical approximation to the eigenvalue gap of Schrödinger operators

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ABSTRACT

We consider two types of Schrödinger operators $H(t) = -d^2/dx^2 + q(x) + t\cos x$ and $H(t) = -d^2/dx^2 + q(x) + A\cos(tx)$ defined on $L^2(\mathbb{R})$, where q is an even potential that is bounded from below, A is a constant, and $t > 0$ is a parameter. We assume that $H(t)$ has at least two eigenvalues below its essential spectrum; and we denote by $\lambda_1(t)$ and $\lambda_2(t)$ the lowest eigenvalue and the second one, respectively. The purpose of this paper is to study the asymptotics of the gap $\Gamma(t) = \lambda_2(t) - \lambda_1(t)$ in the limit as $t \to \infty$.

1. Introduction

Consider the Schrödinger operator

$$H(t) = -d^2/dx^2 + tV(x)$$

acting on $L^2(\mathbb{R})$, where $t > 0$ and V is a suitable potential such that $H(t)$ is selfadjoint. Suppose that V is bounded from below, and that $H(t)$ has eigenvalues $\lambda_1(t) < \lambda_2(t) < \cdots$, below its essential spectrum. The problem of the semiclassical limit is to determine the behavior of the eigenvalues $\lambda_n(t)$ as t tends to infinity. There have been many studies on the semiclassical limit of eigenvalues for double- and multiple-well potentials. The most common situation occurs when the potential is nonnegative and has several nondegenerate zeros. We mention the work of Harrell [2], Kirsch and Simon [4], Nakamura [6] and Simon [8]. For related literature on this subject, see Hislop and Sigal [3] and references therein.

The purpose of this paper is to study the semiclassical approximation to the eigenvalue gap of Schrödinger operators with periodic potentials. Let q be an even potential that is bounded from below. We will actually study two types of operators, that is,

$$H(t) = -d^2/dx^2 + q(x) + t\cos x \tag{1.1}$$

and

$$H(t) = -d^2/dx^2 + q(x) + A\cos(tx) \tag{1.2}$$

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defined on $L^2(\mathbb{R})$, where A is a constant. Here we assume that $H = -d^2/dx^2 + q(x)$ is selfadjoint. Also, $H(t)$ has at least two eigenvalues below its essential spectrum; and we denote by $\lambda_1(t)$ and $\lambda_2(t)$ the lowest eigenvalue and the second one, respectively. Our main goal in this paper will be to investigate the asymptotics of the gap

$$\Gamma(t) = \lambda_2(t) - \lambda_1(t)$$

in the limit as $t \to \infty$. We now state our results. The proofs will be given in Sections 2 and 3.

Theorem 1.1. *Under the above hypotheses, suppose in addition that*

$$\int_{-\infty}^{\infty} q(x)x^2 e^{-x^2}\,dx < \infty. \tag{1.3}$$

Then, for sufficiently large t, the eigenvalue gap $\Gamma(t)$ of (1.1) *satisfies*

$$\Gamma(t) \leqslant C\exp(-\alpha\sqrt{t}\,),$$

where C and α are positive constants both independent of t.

Theorem 1.2. *Under the above hypotheses, suppose in addition that $q \in K^1_{\mathrm{loc}}$ and $q_- \in K_1$. Then the first two eigenvalues of* (1.2) *satisfy*

$$\lim_{t\to\infty} \lambda_n(t) = \lambda_n, \quad n = 1, 2,$$

where λ_n is the nth eigenvalue of $H = -d^2/dx^2 + q(x)$. In particular,

$$\lim_{t\to\infty} \Gamma(t) = \lambda_2 - \lambda_1.$$

We remark that a potential $V \in K_1$ if and only if $\sup_{x\in\mathbb{R}} \int_{x-1}^{x+1} |V(y)|\,dy < \infty$, and $V \in K^1_{\mathrm{loc}}$ if and only if $V\varphi \in K_1$ for any $\varphi \in C_0^\infty(\mathbb{R})$. For a discussion of the class K_v, we refer to the book by Cycon, Froese, Kirsch and Simon [1]. In the proof of Theorem 1.2, we shall use the facts that if $q \in K^1_{\mathrm{loc}}$ and $q_- \in K_1$, then for any eigenfunction u of H, (i) $u(x) \to 0$ as $|x| \to \infty$ (see Theorem 2.4 of [1]), (ii) $u(x)u'(x) \to 0$ as $|x| \to \infty$. Here is a proof for (ii), using integration by parts, we have

$$u(x)u'(x) - u(0)u'(0) = \int_0^x u'(y)^2\,dy + \int_0^x u(y)u''(y)\,dy.$$

Since $u, u', u'' \in L^2(\mathbb{R})$, we see that $\lim_{x\to\infty} u(x)u'(x)$ exists. Suppose on the contrary that

$$\lim_{x\to\infty} u(x)u'(x) = \alpha \neq 0.$$

Then, for sufficiently large x,

$$\left|u'(x)\right| > \frac{|\alpha|}{2|u(x)|}.$$

Since $u(x) \to 0$ as $x \to \infty$, it follows that $|u'(x)| \to \infty$ as $x \to \infty$. However, this is inconsistent with u' being in $L^2(\mathbb{R})$ and hence α must be zero. The result that $\lim_{x\to-\infty} u(x)u'(x) = 0$ is proved in the same way.

2. Proof of Theorem 1.1

Lemma 2.1. *There are constants a and b, where $b < 0$, such that*

$$\lambda_2(t) \leqslant a + bt \quad \text{for all } t.$$

Proof. To prove the lemma, we shall use the Rayleigh–Ritz principle [7]:

$$\lambda_2(t) = \inf_{\varphi \in Q(H(t))\setminus\{0\},\ \varphi \perp \varphi_1(t,\cdot)} \frac{\langle \varphi, H(t)\varphi\rangle}{\langle \varphi, \varphi\rangle}, \tag{2.1}$$

where $Q(H(t))$ is the form domain of $H(t)$, and $\varphi_1(t,x)$ is the first eigenfunction of $H(t)$ associated to $\lambda_1(t)$. Since the potential $q(x) + t\cos x$ is symmetric, $\varphi_1(t,\cdot)$ is symmetric, so any antisymmetric function will be orthogonal to $\varphi_1(t,\cdot)$, and will therefore be a suitable trial function in (2.1). As a trial function for estimating $\lambda_2(t)$, let us take $\varphi(x) = (x - \sin x)e^{-x^2/2}$, so

$$\lambda_2(t) \leqslant a + bt,$$

where

$$a = \frac{1}{\langle\varphi, \varphi\rangle} \int_{-\infty}^{\infty} \left[\varphi'(x)^2 + q(x)\varphi^2(x)\right] dx$$

and

$$b = \frac{1}{\langle\varphi, \varphi\rangle} \int_{-\infty}^{\infty} \varphi^2(x) \cos x \, dx. \tag{2.2}$$

By hypothesis (1.3), we see that a is finite. It remains to show that $b < 0$. To compute the integral in (2.2), we use the basic identities:

$$\sin^2 x \cos x = \frac{1}{4}(\cos x - \cos 3x)$$

and

$$C(\alpha) \equiv \int_{-\infty}^{\infty} \cos(\alpha x) e^{-x^2} \, dx = \sqrt{\pi} e^{-\alpha^2/4} \quad \text{for } \alpha \in \mathbb{R}.$$

An elementary calculation then gives that

$$\begin{aligned}
\int_{-\infty}^{\infty} \varphi^2(x) \cos x \, dx &= \int_{-\infty}^{\infty} (x - \sin x)^2 \cos x e^{-x^2} \, dx \\
&= \int_{-\infty}^{\infty} \left(x^2 \cos x - x \sin 2x\right) e^{-x^2} \, dx + \frac{1}{4}\left[C(1) - C(3)\right] \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{4} \cos x - \cos 2x\right) e^{-x^2} \, dx + \frac{1}{4}\left[C(1) - C(3)\right] \\
&= \frac{1}{2} C(1) - C(2) - \frac{1}{4} C(3) \\
&= \sqrt{\pi} \left(\frac{1}{2} e^{-1/4} - e^{-1} - \frac{1}{4} e^{-9/4}\right) \\
&\approx \sqrt{\pi}(-0.004828) < 0
\end{aligned}$$

where we have used integration by parts twice in the third step. This proves that $b < 0$ and thus the lemma. □

Note that for each fixed $k > 0$, λ is an eigenvalue for the potential $q(x) + t \cos x$ if and only if λ/k^2 is an eigenvalue for the potential

$$V_{t,k}(x) \equiv \frac{1}{k^2}\left[q\left(\frac{x}{k}\right) + t \cos\left(\frac{x}{k}\right)\right].$$

Thus, if we denote by $\Gamma[V_{t,k}]$ the gap between the two lowest eigenvalues of the Schrödinger operator $-d^2/dx^2 + V_{t,k}(x)$ on $L^2(\mathbb{R})$, then

$$\Gamma[V_{t,k}] = \frac{1}{k^2} \Gamma(t). \tag{2.3}$$

To get estimates on $\Gamma(t)$ for t large, we will consider the specific operator

$$-d^2/dx^2 + V_{t,\delta t^{1/4}}(x), \tag{2.4}$$

where $\delta > 0$. Let $\psi_{t,\delta}(x)$ be the first normalized eigenfunction of (2.4). As a preliminary to a proof of Theorem 1.1, we need:

Lemma 2.2. *Let $0 < \beta < \sqrt{-2b}/\delta^2\pi$, where b is given by (2.2). Then, for sufficiently large t,*

$$\int_{-\infty}^{\infty} \psi_{t,\delta}^2(x) e^{-\beta x^2}\, dx \leqslant A(\beta,\delta) e^{-(\beta\delta^2\pi^2/4)\sqrt{t}},$$

where $A(\beta,\delta) > 0$ is a number independent of t.

Proof. For simplicity of notations, we write $V(x) = V_{t,\delta t^{1/4}}(x)$, $\psi(x) = \psi_{t,\delta}(x)$, and let μ be the first eigenvalue of (2.4) associated to $\psi(x)$. Then $-\psi'' + V\psi = \mu\psi$ so that

$$-\mu \int_{-\infty}^{\infty} \psi^2(x) e^{-\beta x^2}\, dx = \int_{-\infty}^{\infty} \psi''(x)\psi(x) e^{-\beta x^2}\, dx - \int_{-\infty}^{\infty} V(x)\psi^2(x) e^{-\beta x^2}\, dx.$$

On integrating by parts twice, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \psi''(x)\psi(x) e^{-\beta x^2}\, dx &= \int_{-\infty}^{\infty} \big[2\beta x\psi(x) - \psi'(x)\big]\psi'(x) e^{-\beta x^2}\, dx \\ &\leqslant 2\beta \int_{-\infty}^{\infty} x\psi(x)\psi'(x) e^{-\beta x^2}\, dx \\ &= \beta \int_{-\infty}^{\infty} \big(2\beta x^2 - 1\big)\psi^2(x) e^{-\beta x^2}\, dx \\ &\leqslant 2\beta^2 \int_{-\infty}^{\infty} x^2\psi^2(x) e^{-\beta x^2}\, dx.\end{aligned}$$

Hence

$$-\mu \int_{-\infty}^{\infty} \psi^2(x) e^{-\beta x^2}\, dx \leqslant \int_{-\infty}^{\infty} \big[2\beta^2 x^2 - V(x)\big]\psi^2(x) e^{-\beta x^2}\, dx. \tag{2.5}$$

Now let $m_q = \inf\{q(x)/x \in \mathbb{R}\}$. Then

$$V(x) \geqslant \frac{1}{\delta^2\sqrt{t}}\left[m_q + t\cos\left(\frac{x}{\delta t^{1/4}}\right)\right]$$

so that, by (2.5),

$$\begin{aligned}\left[-\mu - \frac{(\beta\delta\pi)^2}{2}\sqrt{t} + \frac{m_q}{\delta^2\sqrt{t}}\right]\int_{-\infty}^{\infty} \psi^2(x) e^{-\beta x^2}\, dx &\leqslant \int_{-\infty}^{\infty} \left[2\beta^2 x^2 - \frac{(\beta\delta\pi)^2}{2}\sqrt{t} - \frac{\sqrt{t}}{\delta^2}\cos\left(\frac{x}{\delta t^{1/4}}\right)\right]\psi^2(x) e^{-\beta x^2}\, dx \\ &= \int_{|x| \leqslant (\delta t^{1/4}\pi)/2} + \int_{|x| > (\delta t^{1/4}\pi)/2}.\end{aligned} \tag{2.6}$$

The first integral on the right of (2.6) is nonpositive, since

$$2\beta^2 x^2 \leqslant \frac{(\beta\delta\pi)^2}{2}\sqrt{t} \quad \text{and} \quad \cos\left(\frac{x}{\delta t^{1/4}}\right) \geqslant 0 \quad \text{for } |x| \leqslant \big(\delta t^{1/4}\pi\big)/2.$$

It follows that

$$\left[-\mu - \frac{(\beta\delta\pi)^2}{2}\sqrt{t} + \frac{m_q}{\delta^2\sqrt{t}}\right]\int_{-\infty}^{\infty} \psi^2(x) e^{-\beta x^2}\, dx \leqslant \int_{|x| > (\delta t^{1/4}\pi)/2} \left[2\beta^2 x^2 + \frac{\sqrt{t}}{\delta^2}\right]\psi^2(x) e^{-\beta x^2}\, dx. \tag{2.7}$$

We now let t be such that $(\delta t^{1/4}\pi)/2 > 1/\sqrt{\beta}$. Then, since the function $g(x) = 2\beta^2 x^2 e^{-\beta x^2}$ is monotone decreasing on the interval $[1/\sqrt{\beta}, \infty)$, the integral on the right of (2.7) is less than

$$\int_{|x|>(\delta t^{1/4}\pi)/2} \left[g\left(\frac{\delta t^{1/4}\pi}{2}\right) + \frac{\sqrt{t}}{\delta^2} e^{-\beta(\delta t^{1/4}\pi/2)^2} \right] \psi^2(x)\,dx = \left[\frac{(\beta\delta\pi)^2}{2} + \frac{1}{\delta^2}\right] \sqrt{t} e^{-(\beta\delta^2\pi^2/4)\sqrt{t}} \int_{|x|>(\delta t^{1/4}\pi)/2} \psi^2(x)\,dx.$$

Since $\psi(x)$ is normalized, it follows from (2.7) that

$$\left[-\mu - \frac{(\beta\delta\pi)^2}{2}\sqrt{t} + \frac{m_q}{\delta^2\sqrt{t}}\right] \int_{-\infty}^{\infty} \psi^2(x) e^{-\beta x^2}\,dx \leqslant \left[\frac{(\beta\delta\pi)^2}{2} + \frac{1}{\delta^2}\right] \sqrt{t} e^{-(\beta\delta^2\pi^2/4)\sqrt{t}}. \tag{2.8}$$

Recall that μ is the first eigenvalue of the operator (2.4). Hence

$$\mu = \frac{\lambda_1(t)}{\delta^2\sqrt{t}} < \frac{\lambda_2(t)}{\delta^2\sqrt{t}} \leqslant \frac{a+bt}{\delta^2\sqrt{t}}$$

by Lemma 2.1, and so

$$-\mu - \frac{(\beta\delta\pi)^2}{2}\sqrt{t} + \frac{m_q}{\delta^2\sqrt{t}} > \frac{m_q - a}{\delta^2\sqrt{t}} + \left[\frac{-b}{\delta^2} - \frac{(\beta\delta\pi)^2}{2}\right]\sqrt{t}.$$

Since $0 < \beta < \sqrt{-2b}/\delta^2\pi$, the right-hand side here is positive if t is large enough. Hence, from (2.8), we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} \psi^2(x) e^{-\beta x^2}\,dx &\leqslant \frac{2 + (\beta\delta^2\pi)^2}{[2(m_q - a)/t] + [-2b - (\beta\delta^2\pi)^2]} e^{-(\beta\delta^2\pi^2/4)\sqrt{t}} \\ &\leqslant A(\beta,\delta) e^{-(\beta\delta^2\pi^2/4)\sqrt{t}} \end{aligned}$$

if t is large enough, $A(\beta,\delta)$ being a positive number independent of t. □

We are now able to complete the proof of Theorem 1.1.

Proof. As in the proof of Lemma 2.2, we set $\psi(x) = \psi_{t,\delta}(x)$, the first normalized eigenfunction of the operator (2.4), and $V(x) = V_{t,\delta t^{1/4}}(x)$, where $\delta > 0$ will be specified in a moment. Then, by (2.3),

$$\Gamma(t) = \delta^2\sqrt{t}\,\Gamma[V]. \tag{2.9}$$

To get estimates on $\Gamma(t)$ for t large, we shall use the following variational principle, which was first exploited by Thompson and Kac [9] (see also Kirsch and Simon [5]).

$$\Gamma[V] = \inf\left\{ \frac{\int_{-\infty}^{\infty} f'(x)^2\psi^2(x)\,dx}{\int_{-\infty}^{\infty} f^2(x)\psi^2(x)\,dx} \;\middle|\; \int_{-\infty}^{\infty} f(x)\psi^2(x)\,dx = 0 \right\}.$$

As a trial function for estimating $\Gamma[V]$, we take

$$f(x) = \int_0^x e^{-y^2/2}\,dy.$$

Since the potential V is symmetric, the same is true for $\psi(x)$, so $\int_{-\infty}^{\infty} f(x)\psi^2(x)\,dx = 0$. It follows that

$$\Gamma[V] \leqslant \frac{\int_{-\infty}^{\infty} e^{-x^2}\psi^2(x)\,dx}{\int_{-\infty}^{\infty} f^2(x)\psi^2(x)\,dx}. \tag{2.10}$$

Now from the inequality

$$\int_x^{\infty} e^{-y^2/2}\,dy \leqslant 2e^{-x^2/2} \quad \text{for } x \geqslant 0,$$

we obtain

$$f(x) + 2e^{-x^2/2} \geqslant \int_0^{\infty} e^{-y^2/2}\,dy = \sqrt{\frac{\pi}{2}} \quad \text{for } x \geqslant 0$$

so that

$$f^2(x) + 4e^{-x^2/2}\Big[\max_{x\in\mathbb{R}} f(x)\Big] + 4e^{-x^2} \geqslant \frac{\pi}{2} \quad \text{for all } x \in \mathbb{R}.$$

Since $\psi(x)$ is normalized, this gives

$$\int_{-\infty}^{\infty} f^2(x)\psi^2(x)\,dx + 4\sqrt{\frac{\pi}{2}}\int_{-\infty}^{\infty} \psi^2(x)e^{-x^2/2}\,dx + 4\int_{-\infty}^{\infty} \psi^2(x)e^{-x^2}\,dx \geqslant \frac{\pi}{2}. \tag{2.11}$$

To apply Lemma 2.2 with $\beta = 1/2$ and $\beta = 1$, we now choose δ so small that

$$0 < \delta < \left(-2b/\pi^2\right)^{1/4}.$$

Then, for sufficiently large t,

$$\int_{-\infty}^{\infty} \psi^2(x)e^{-x^2/2}\,dx \leqslant C_1 e^{-(\delta^2\pi^2/8)\sqrt{t}}$$

and

$$\int_{-\infty}^{\infty} \psi^2(x)e^{-x^2}\,dx \leqslant C_2 e^{-(\delta^2\pi^2/4)\sqrt{t}}, \tag{2.12}$$

C_1 and C_2 being positive constants independent of t. These together with (2.11) imply that there is a constant C_3 independent of t such that

$$\int_{-\infty}^{\infty} f^2(x)\psi^2(x)\,dx \geqslant C_3 > 0,$$

if t is sufficiently large. It follows from this and (2.10) and (2.12) that

$$\Gamma[V] \leqslant \frac{C_2}{C_3} e^{-(\delta^2\pi^2/4)\sqrt{t}}$$

and so, by (2.9),

$$\Gamma(t) \leqslant \frac{C_2\delta^2}{C_3}\sqrt{t}e^{-(\delta^2\pi^2/4)\sqrt{t}} \leqslant Ce^{-\alpha\sqrt{t}}$$

provided that t is sufficiently large, where C and α are positive constants both independent of t. All this proves Theorem 1.1. $\square$

3. Proof of Theorem 1.2

We denote by $\varphi_n(x)$ and $\varphi_n(t,x)$, respectively, the normalized eigenfunctions corresponding to λ_n and $\lambda_n(t)$. By the Rayleigh–Ritz principle [7]:

$$\lambda_1(t) = \inf_{\varphi\in Q(H(t))\setminus\{0\}} \frac{\langle\varphi, H(t)\varphi\rangle}{\langle\varphi,\varphi\rangle}, \qquad \lambda_1 = \inf_{\varphi\in Q(H)\setminus\{0\}} \frac{\langle\varphi, H\varphi\rangle}{\langle\varphi,\varphi\rangle}.$$

As a trial function for estimating $\lambda_1(t)$, we take $\varphi = \varphi_1$; and for estimating λ_1, we take $\varphi = \varphi_1(t,\cdot)$. It then follows that

$$A\int_{-\infty}^{\infty} \varphi_1^2(t,x)\cos(tx)\,dx \leqslant \lambda_1(t) - \lambda_1 \leqslant A\int_{-\infty}^{\infty} \varphi_1^2(x)\cos(tx)\,dx. \tag{3.1}$$

But

$$\int_{-\infty}^{\infty} \varphi_1^2(x)\cos(tx)\,dx = \frac{-2}{t}\int_{-\infty}^{\infty} \varphi_1(x)\varphi_1'(x)\sin(tx)\,dx$$

on integrating by parts and using the fact that $\varphi_1(x) \to 0$ as $|x| \to \infty$. Hence

$$\left|\int_{-\infty}^{\infty} \varphi_1^2(x)\cos(tx)\,dx\right| \leqslant \frac{2}{t}\int_{-\infty}^{\infty} \left|\varphi_1(x)\varphi_1'(x)\right| dx \to 0 \tag{3.2}$$

as $t \to \infty$, since φ_1 and φ_1' are in $L^2(\mathbb{R})$. On the other hand, we have

$$\int_{-\infty}^{\infty} \varphi_1^2(t,x)\cos(tx)\,dx = \frac{-2}{t}\int_{-\infty}^{\infty} \varphi_1(t,x)\varphi_1'(t,x)\sin(tx)\,dx,$$

where φ_1' means the x-derivative of φ_1. Hence

$$\begin{aligned}\left|\int_{-\infty}^{\infty} \varphi_1^2(t,x)\cos(tx)\,dx\right| &\leqslant \frac{2}{t}\int_{-\infty}^{\infty} \left|\varphi_1(t,x)\varphi_1'(t,x)\right| dx\\ &\leqslant \frac{1}{t}\left[1+\int_{-\infty}^{\infty} \varphi_1'(t,x)^2\,dx\right] \end{aligned}\tag{3.3}$$

since $\varphi_1(t,\cdot)$ is normalized. Now let $m_q = \inf\{q(x)/x \in \mathbb{R}\}$, and note that $q + A\cos(tx) \in K_{\mathrm{loc}}^1$ and $[q + A\cos(tx)]_- \in K_1$. On integrating by parts and using the fact that $\varphi_1(t,x)\varphi_1'(t,x) \to 0$ as $|x| \to \infty$, the differential equation and the right-hand inequality of (3.1), we find

$$\begin{aligned}\int_{-\infty}^{\infty} \varphi_1'(t,x)^2\,dx &= -\int_{-\infty}^{\infty} \varphi_1(t,x)\varphi_1''(t,x)\,dx\\ &= \lambda_1(t) - \int_{-\infty}^{\infty} q(x)\varphi_1^2(t,x)\,dx - A\int_{-\infty}^{\infty} \varphi_1^2(t,x)\cos(tx)\,dx\\ &\leqslant \lambda_1(t) - m_q + |A|\\ &\leqslant \lambda_1 + 2|A| - m_q.\end{aligned}$$

Hence (3.3) gives

$$\int_{-\infty}^{\infty} \varphi_1^2(t,x)\cos(tx)\,dx \to 0$$

as $t \to \infty$. This together with (3.1) and (3.2) shows that

$$\lim_{t\to\infty} \lambda_1(t) = \lambda_1.$$

The proof of the corresponding result for $\lambda_2(t)$ is similar. By the Rayleigh–Ritz principle,

$$\lambda_2(t) = \inf_{\varphi \in Q(H(t))\setminus\{0\},\, \varphi \perp \varphi_1(t,\cdot)} \frac{\langle \varphi, H(t)\varphi\rangle}{\langle \varphi, \varphi\rangle},$$
$$\lambda_2 = \inf_{\varphi \in Q(H)\setminus\{0\},\, \varphi \perp \varphi_1} \frac{\langle \varphi, H\varphi\rangle}{\langle \varphi, \varphi\rangle}.$$

Since the potentials $q(x)$ and $q(x) + A\cos(tx)$ are symmetric, the first eigenfunctions φ_1 and $\varphi_1(t,\cdot)$ are symmetric; while the second eigenfunctions φ_2 and $\varphi_2(t,\cdot)$ are antisymmetric. We take $\varphi = \varphi_2$ in the Rayleigh–Ritz principle as applied to $\lambda_2(t)$, and take $\varphi = \varphi_2(t,\cdot)$ as applied to λ_2. It then follows that

$$A\int_{-\infty}^{\infty} \varphi_2^2(t,x)\cos(tx)\,dx \leqslant \lambda_2(t) - \lambda_2 \leqslant A\int_{-\infty}^{\infty} \varphi_2^2(x)\cos(tx)\,dx. \tag{3.4}$$

Again using integration by parts and the facts that $\varphi_2(x) \to 0$ and $\varphi_2(t,x)\varphi_2'(t,x) \to 0$ as $|x| \to \infty$, we obtain

$$\lim_{t\to\infty}\int_{-\infty}^{\infty} \varphi_2^2(t,x)\cos(tx)\,dx = \lim_{t\to\infty}\int_{-\infty}^{\infty} \varphi_2^2(x)\cos(tx)\,dx = 0.$$

This together with (3.4) implies that

$$\lim_{t\to\infty} \lambda_2(t) = \lambda_2.$$

The proof of Theorem 1.2 is now complete.

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