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Extinction and non-extinction for a polytropic filtration equation with absorption and source

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ABSTRACT

This paper deals with extinction and non-extinction for the initial-boundary value problem of a polytropic filtration equation with absorption and source.

1. Introduction

In this paper, we consider the following polytropic filtration equation with absorption and source

$$\begin{cases} u_t = \operatorname{div}\left(\left|\nabla u^m\right|^{p-2}\nabla u^m\right) - \alpha u^q + \lambda u^r, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.1}$$

where $m > 0$, $0 < m(p-1) < 1$, $\lambda, r, \alpha > 0$, $q \leqslant 1$, $\Omega \subset \mathbb{R}^N$ $(N > p)$ is a bounded domain with smooth boundary, and the initial data $u_0(x)$ is a non-negative and bounded function with $u_0^m(x) \in W_0^{1,p}(\Omega)$.

Eq. (1.1) is the well-known non-Newtonian polytropic filtration equations, and this type of equations arises in various fields (see [9,11,25] and the references therein, where a more detailed physical background can be found). For example, in the mathematical model for a heat conduction process, the function $u(x,t)$ represents the temperature, the term $\operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m)$ represents the thermal diffusion, $-\alpha u^q$ is the absorption and λu^r is the source.

Because of the degeneracy and the singularity, Eq. (1.1) might not have classical solutions in general, and hence we introduce the definition of the weak solution. The definition of the weak solution of problem (1.1) reads as follows. For convenience, we define $\Omega_T = \Omega \times (0,T)$, $T > 0$.

Definition 1.1. A non-negative measurable function $u(x,t)$ defined in Ω_T is called a weak solution of problem (1.1), if

(1) for any $T > 0$,

$$u(x,t) \in X = \left\{L^{2q}(\Omega_T) \cap L^{2r}(\Omega_T), u_t(x,t) \in L^2(\Omega_T), \nabla u^m \in L^p(\Omega_T)\right\},$$

and $0 < \varphi(x,t) \in \overline{X} = \{L^2(\Omega_T), u_t(x,t) \in L^2(\Omega_T), \nabla u \in L^p(\Omega_T), u|_{\partial\Omega\times(0,T)} = 0\}$;

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(2) the integral equality

$$\int_{\Omega} u(x,T)\varphi(x,T)\,dx - \int_{\Omega} u_0(x)\varphi(x,0)\,dx = \int_0^T \int_{\Omega} \left(u\varphi_\tau - \left|\nabla u^m\right|^{p-2}\nabla u^m \nabla\varphi - \alpha u^q\varphi + \lambda u^r\varphi\right) dx\,d\tau \tag{1.2}$$

holds, and

$$u(x,0) = u_0(x), \quad x \in \Omega,$$
$$u(x,t) = 0, \quad x \in \partial\Omega \times (0,T).$$

Furthermore, we can define a subsolution (resp., supersolution) $\underline{u}(x,t)$ (resp., $\bar{u}(x,t)$). We need only to replace '=' in (1.2) by '$\leqslant$' (resp., '$\geqslant$') for any $\varphi(x,t) > 0$, $\underline{u}(x,t) \leqslant 0$ (resp., $\bar{u}(x,t) \geqslant 0$) in $\partial\Omega \times (0,T)$, and $\underline{u}(x,0) \leqslant u_0(x)$ (resp., $\bar{u}(x,0) \geqslant u_0(x)$) in Ω.

Under the some assumptions, the existence of a weak solution to problem (1.1) can be proved from the results of [5,9, 17,21,22].

The main purpose of this paper is to study the extinction property for the non-negative solution $u(x,t)$ of problem (1.1), i.e. there exists a finite time $T > 0$ such that $u(x,t) \equiv 0$ for all $(x,t) \in \Omega \times [T,+\infty)$. The first result concerning extinction of solutions for the general heat equation with absorption was established in [10]. In recent years, there have been many papers on extinction property for different kinds of evolution equations (see [2–4,6–16,19,20,23–25] and the references therein). For instance, in [11], the author considered the following p-Laplacian equation with absorption and source

$$\begin{cases} u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) - \beta u^q + \lambda u^r, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.3}$$

where $1 < p < 2$, $q \leqslant 1$, and $\beta, \lambda > 0$. He proved that if $q = 1$, then $r = p - 1$ is the critical extinction exponent for the weak solution, but if $q < 1$, extinction can always occur when $0 < q \leqslant r < 1$. In the absence of absorption (i.e. $\beta = 0$) for problem (1.3), Tian and Mu [20] proved that $r = p - 1$ is the critical extinction exponent for the weak solution of the problem (see also Yin and Jin in [24]). Moreover, in the absence of source (i.e. $\lambda = 0$) for problem (1.3), Gu [7] proved that if $p \in (1,2)$ or $q \in (0,1)$, then the solutions of the problem vanish in finite time. However, if $p \geqslant 2$ and $q \geqslant 1$, then the solutions of the problem cannot vanish in finite time.

In [9], the authors studied problem (1.1) without the absorption term (i.e. $\alpha = 0$) and showed that $r = m(p-1)$ is the critical extinction exponent for the weak solution of the problem (see also Zhou and Mu in [25]). Moreover, when the critical case $r = m(p-1)$ is concerned, they also proved that the parameter λ plays an important role. Namely, when λ belongs to different interval $(0,\lambda_1)$ or $(\lambda_1,+\infty)$, where λ_1 is the first eigenvalue of the p-Laplacian equation with homogeneous Dirichlet boundary-value condition, the weak solution has completely different properties.

To the best of our knowledge, no such result seems to be presented in the literature about problem (1.1). In this paper, by using an energy approach which was introduced in [1] and Lemma 1 in [18], Lemmas 3 and 4 in [11], we obtain our main results as follows.

Theorem 1.1. *If $q = 1$ and $r > m(p-1)$, then the weak solution of problem* (1.1) *vanishes in finite time provided that the initial data $u_0(x)$ is sufficiently small.*

Theorem 1.2. *If $q = 1$ and $r < m(p-1)$, then the weak solution of problem* (1.1) *cannot vanish in finite time for any non-negative initial data $u_0(x)$.*

Remark 1.1. For the case of $q = 1$, according to Theorems 1.1 and 1.2, we know that $r = m(p-1)$ is the critical extinction exponent of problem (1.1).

Let λ_1 be the first eigenvalue of the p-Laplacian equation with homogeneous Dirichlet boundary-value condition and $\phi(x)$ is the eigenfunction corresponding to the first eigenvalue λ_1. In this paper, we choose $\phi(x)$ that it satisfies $0 < \phi(x) \leqslant 1$ in Ω. For convenience, let $\|\cdot\|_p$ denote the $L^p(\Omega)$ norm, $1 \leqslant p \leqslant +\infty$.

Theorem 1.3. *For the case $q = 1$ and $r = m(p-1)$.*

(i) *If $\lambda < \lambda_1$, then the weak solution of problem* (1.1) *goes to zero in the sense of $\|u(\cdot,t)\|_{m+1}$ as $t \to +\infty$;*

(ii) *If* $\frac{Nm+N}{Nm+m+1} \leqslant p < 1 + \frac{1}{m}$ *and* $\lambda < \lambda_1$ *or* $1 < p < \frac{Nm+N}{Nm+m+1}$ *and* $\lambda < \frac{(Nm+N-Nmp-p)m^{p-1}p^{2p-1}}{[(N-p)(m+1-mp)]^p}\lambda_1$, *then the weak solution of problem* (1.1) *vanishes in finite time for any non-negative initial data* $u_0(x)$.

Remark 1.2. When $1 < p < \frac{Nm+N}{Nm+m+1}$, it is easy to show that $\frac{(Nm+N-Nmp-p)m^{p-1}p^{2p-1}}{[(N-p)(m+1-mp)]^p} < 1$.

Remark 1.3. For the case $1 < p < \frac{Nm+N}{Nm+m+1}$, when $\lambda \in [\frac{(Nm+N-Nmp-p)m^{p-1}p^{2p-1}}{[(N-p)(m+1-mp)]^p}\lambda_1, \lambda_1)$, we do not know whether the solution $u(x,t)$ of problem (1.1) possesses the property of extinction or non-extinction.

Theorem 1.4. *For the case* $q = 1$ *and* $r = m(p-1)$, *if one of the following cases holds*:

(i) $\lambda > \lambda_1$, *and for any non-negative initial data* $u_0(x)$;
(ii) $\lambda = \lambda_1$, *and for any identically positive initial data* $u_0(x)$,

then the weak solution of problem (1.1) *cannot vanish in finite time.*

Remark 1.4. For the case $q = 1$, according to our above results and [9] (see also [25]), it is not difficult to obtain that the critical extinction exponent of problem (1.1) is independent of the linear absorption. In fact, let $v(x,t) = u(x,t)e^{\alpha t}$, we obtain from (1.1)

$$\begin{cases} v_t = e^{(1-m(p-1))\alpha t}\operatorname{div}\left(\left|\nabla v^m\right|^{p-2}\nabla v^m\right) + \lambda e^{(1-r)\alpha t}v^r, \quad x \in \Omega,\ t > 0, \\ v(x,0) = u_0(x), \quad x \in \Omega, \\ v(x,t) = 0, \quad x \in \partial\Omega,\ t > 0. \end{cases} \tag{1.4}$$

Next, introduce the self-similar variables

$$w(x,\tau) = v(x,t), \quad x \in \Omega,\ t > 0, \text{ where } \tau = \frac{1}{(1-m(p-1))\alpha}e^{(1-m(p-1))\alpha t}.$$

Then it follows from (1.4) that

$$\begin{cases} w_\tau = \operatorname{div}\left(\left|\nabla w^m\right|^{p-2}\nabla w^m\right) + \lambda e^{(m(p-1)-r)\alpha t}w^r, \quad x \in \Omega,\ \tau > 0, \\ w(x,0) = u_0(x), \quad x \in \Omega, \\ w(x,\tau) = 0, \quad x \in \partial\Omega,\ \tau > 0. \end{cases} \tag{1.5}$$

Combining problem (1.5) and the results of [9] (see also [25]), it is easy to obtain Theorems 1.1–1.3 and 1.4(i). The proofs of Theorems 1.1–1.4 are rather lengthy and not essential to this paper, therefore, for the reader's convenience, the proofs are deferred to Appendix A.

Theorem 1.5. *For* $q < 1$, $m \geqslant 1$ *and* $r = m(p-1)$. *If* $\frac{Nm+N}{Nm+m+1} \leqslant p < 1 + \frac{1}{m}$ *and* $\lambda < \lambda_1$ *or* $1 < p < \frac{Nm+N}{Nm+m+1}$ *and* $\lambda < lm^{p-1}(\frac{p}{l+m(p-1)})^p\lambda_1$, *where* $l > (Nm+N-Nmp-p)/p$, *then the weak solution of problem* (1.1) *vanishes in finite time for any non-negative initial data.*

Theorem 1.6. *For* $q < 1$ *and* $m \geqslant 1$. *If* $\frac{Nm+N}{Nm+m+1} \leqslant p < 1 + \frac{1}{m}$ *and* $r > \frac{pq(m+1)+N(mp-m-q)}{p(m+1)+N(mp-m-q)}$ *or* $1 < p < \frac{Nm+N}{Nm+m+1}$ *and* $r > \frac{pq(l+1)+N(mp-m-q)}{p(l+1)+N(mp-m-q)}$, *where* $l > (Nm+N-Nmp-p)/p$, *then the weak solution of problem* (1.1) *vanishes in finite time for the initial data* $u_0(x)$ *is sufficiently small.*

Remark 1.5. When $q \geqslant m(p-1)$, the conditions of Theorem 1.6 imply $r > m(p-1)$.

Remark 1.6. Unfortunately, for $q < 1$ and $m \geqslant 1$, when $\frac{Nm+N}{Nm+m+1} \leqslant p < 1 + \frac{1}{m}$ and $r \leqslant \frac{pq(m+1)+N(mp-m-q)}{p(m+1)+N(mp-m-q)}$ or $1 < p < \frac{Nm+N}{Nm+m+1}$ and $r \leqslant \frac{pq(l+1)+N(mp-m-q)}{p(l+1)+N(mp-m-q)}$, where $l > (Nm+N-Nmp-p)/p$, we have to leave open the question whether the solution of problem (1.1) possesses the property of extinction or non-extinction.

This paper is organized as follows. In the next section, we consider the case $q < 1$ and prove Theorems 1.5 and 1.6. The proofs of Theorems 1.1–1.4 are given in Appendix A.

2. Proofs of the main results

In this section, we consider the case $q<1$ and give the proofs of Theorems 1.5 and 1.6.

Proof of Theorem 1.5. For $q<1$ and $r=m(p-1)$. We consider the first case $\frac{Nm+N}{Nm+m+1} \leqslant p < 1+\frac{1}{m}$ and $\lambda<\lambda_1$. Multiplying both sides of (1.1) by u^m and integrating over Ω, we arrive at the equality

$$\frac{1}{m+1}\frac{d}{dt}\int_\Omega u^{m+1}\,dx + \int_\Omega \left|\nabla u^m\right|^p dx + \alpha\int_\Omega u^{m+q}\,dx = \lambda\int_\Omega u^{m+r}\,dx. \tag{2.1}$$

Note that $\lambda_1\int_\Omega u^{mp}\,dx \leqslant \int_\Omega |\nabla u^m|^p\,dx$ and $r=m(p-1)$ and $\lambda<\lambda_1$, then we have

$$\frac{1}{m+1}\frac{d}{dt}\int_\Omega u^{m+1}\,dx + \left(1-\frac{\lambda}{\lambda_1}\right)\int_\Omega \left|\nabla u^m\right|^p dx + \alpha\int_\Omega u^{m+q}\,dx \leqslant 0. \tag{2.2}$$

According to Lemma 1 of [18] and $m\geqslant 1$, we obtain

$$\|u\|_{m+1} \leqslant \kappa_1 \left\|\nabla u^m\right\|_p^{\frac{\nu_1}{m}} \|u\|_{m+q}^{1-\nu_1}, \tag{2.3}$$

where

$$\nu_1 = m\left(\frac{1}{m+q}-\frac{1}{m+1}\right)\left(\frac{1}{N}-\frac{1}{p}+\frac{m}{m+q}\right)^{-1} = \frac{mNp(1-q)}{(m+1)[(p-N)(m+q)+Nmp]}$$

and κ_1 is a positive constant independent of q and ν_1.

Since $\frac{Nm+N}{Nm+m+1} \leqslant p < 1+\frac{1}{m}$ and $q<1$, it is not difficult to show that $\nu_1\in(0,1)$. Then using Young's inequality with ε, from (2.3) we derive

$$\begin{aligned} \|u\|_{m+1}^{h_1} &\leqslant \kappa_1^{h_1}\left\|\nabla u^m\right\|_p^{\frac{h_1\nu_1}{m}} \|u\|_{m+q}^{h_1(1-\nu_1)} \\ &\leqslant \kappa_1^{h_1}\left(\varepsilon_1\left\|\nabla u^m\right\|_p^p + C(\varepsilon_1)\|u\|_{m+q}^{\frac{h_1 mp(1-\nu_1)}{mp-h_1\nu_1}}\right), \end{aligned} \tag{2.4}$$

where $\varepsilon_1>0$ and $h_1>0$ are to be determined later. We choose

$$h_1 = \frac{(m+q)mp}{mp(1-\nu_1)+(m+q)\nu_1} = \frac{(m+1)[p(m+q)+N(mp-m-q)]}{(m+1)p+N(mp-m-q)}.$$

After a simple computation, we get $h_1\in(m,m+1)$ and $\frac{h_1 mp(1-\nu_1)}{mp-h_1\nu_1}=m+q$. Thus, from (2.4) we obtain

$$\frac{\kappa_1^{-h_1}}{C(\varepsilon_1)}\|u\|_{m+1}^{h_1} \leqslant \frac{\varepsilon_1}{C(\varepsilon_1)}\left\|\nabla u^m\right\|_p^p + \|u\|_{m+q}^{m+q}. \tag{2.5}$$

Combining (2.2) and (2.5), we derive

$$\frac{1}{m+1}\frac{d}{dt}\int_\Omega u^{m+1}\,dx + \left(1-\frac{\lambda}{\lambda_1}-\frac{\varepsilon_1\alpha}{C(\varepsilon_1)}\right)\int_\Omega \left|\nabla u^m\right|^p dx + \frac{\kappa_1^{-h_1}\alpha}{C(\varepsilon_1)}\left(\int_\Omega u^{m+1}\,dx\right)^{\frac{h_1}{m+1}} \leqslant 0. \tag{2.6}$$

Since $\lambda<\lambda_1$, we can choose ε_1 small enough such that $1-\frac{\lambda}{\lambda_1}-\frac{\varepsilon_1\beta}{C(\varepsilon_1)}\geqslant 0$. Once ε_1 is fixed, let $K_1=\frac{\kappa_1^{-h_1}\alpha}{C(\varepsilon_1)}>0$. Thus, from (2.6) we obtain

$$\frac{1}{m+1}\frac{d}{dt}\int_\Omega u^{m+1}\,dx + K_1\left(\int_\Omega u^{m+1}\,dx\right)^{\frac{h_1}{m+1}} \leqslant 0. \tag{2.7}$$

Integrating this inequality, we derive

$$\left(\int_\Omega u^{m+1}\,dx\right)^{\frac{m+1-h_1}{m+1}} \leqslant \left(\int_\Omega u_0^{m+1}\,dx\right)^{\frac{m+1-h_1}{m+1}} - K_1(m+1-h_1)t. \tag{2.8}$$

Since $h_1 < m+1$, there exists a positive constant T, such that

$$\int_\Omega u^{m+1}\,dx \equiv 0, \quad t \in [T, +\infty),$$

which implies $u(x,t)$ vanishes in finite time.

Next, we study the case $1 < p < \frac{Nm+N}{Nm+m+1}$ and $\lambda < lm^{p-1}(\frac{p}{l+m(p-1)})^p\lambda_1$. Multiplying both sides of (1.1) by u^l, where $l > \frac{N(m+1)-(Nm+1)p}{p} > m$, and integrating over Ω, we conclude

$$\frac{1}{l+1}\frac{d}{dt}\int_\Omega u^{l+1}\,dx + lm^{p-1}\left(\frac{p}{l+m(p-1)}\right)^p\int_\Omega \left|\nabla u^{\frac{l+m(p-1)}{p}}\right|^p dx + \alpha\int_\Omega u^{l+q}\,dx = \lambda\int_\Omega u^{l+r}\,dx. \tag{2.9}$$

Since $r = m(p-1)$ and $\lambda < lm^{p-1}(\frac{p}{l+m(p-1)})^p\lambda_1$, we obtain

$$\frac{1}{l+1}\frac{d}{dt}\int_\Omega u^{l+1}\,dx + \left(lm^{p-1}\left(\frac{p}{l+m(p-1)}\right)^p - \frac{\lambda}{\lambda_1}\right)\int_\Omega \left|\nabla u^{\frac{l+m(p-1)}{p}}\right|^p dx + \alpha\int_\Omega u^{l+q}\,dx \leqslant 0. \tag{2.10}$$

According to Lemma 1 of [18] and $m \geqslant 1$, we have

$$\|u\|_{l+1} \leqslant \kappa_2 \left\|\nabla u^{\frac{l+m(p-1)}{p}}\right\|_p^{\frac{\nu_2 p}{l+m(p-1)}} \|u\|_{l+q}^{1-\nu_2}, \tag{2.11}$$

where

$$\begin{aligned}\nu_2 &= \frac{l+m(p-1)}{p}\left(\frac{1}{l+q} - \frac{1}{l+1}\right)\left(\frac{1}{N} - \frac{1}{p} + \frac{l+m(p-1)}{p(l+q)}\right)^{-1}\\ &= \frac{N(1-q)(l+m(p-1))}{(l+1)[p(l+q) - N(q-m(p-1))]}\end{aligned}$$

and κ_2 is a positive constant independent of q and ν_2.

Since $l > \frac{N(m+1)-(Nm+1)p}{p}$ and $q < 1$, it is not difficult to show that $\nu_2 \in (0,1)$. Then using Young's inequality with ε, we derive from (2.11)

$$\begin{aligned}\|u\|_{l+1}^{h_2} &\leqslant \kappa_2^{h_2} \left\|\nabla u^{\frac{l+m(p-1)}{p}}\right\|_p^{\frac{ph_2\nu_2}{l+m(p-1)}} \|u\|_{l+q}^{h_2(1-\nu_2)}\\ &\leqslant \kappa_2^{h_2}\left(\varepsilon_2\left\|\nabla u^{\frac{l+m(p-1)}{p}}\right\|_p^p + C(\varepsilon_2)\|u\|_{l+q}^{\frac{h_2(1-\nu_2)(l+m(p-1))}{l+m(p-1)-h_2\nu_2}}\right),\end{aligned} \tag{2.12}$$

where $\varepsilon_2 > 0$ and $h_2 > 0$ are to be determined later. We choose

$$h_2 = \frac{(l+q)(l+m(p-1))}{(l+m(p-1))(1-\nu_2) + (l+q)\nu_2} = \frac{(l+1)[p(l+q) - N(q-m(p-1))]}{(l+1)p - N(q-m(p-1))}.$$

After a simple computation, we get $h_2 \in (l, l+1)$ and $\frac{h_2(1-\nu_2)(l+m(p-1))}{l+m(p-1)-h_2\nu_2} = q+l$. Thus, from (2.12) we have

$$\frac{\kappa_2^{-h_2}}{C(\varepsilon_2)}\|u\|_{l+1}^{h_2} \leqslant \frac{\varepsilon_2}{C(\varepsilon_2)}\left\|\nabla u^{\frac{l+m(p-1)}{p}}\right\|_p^p + \|u\|_{l+q}^{l+q}. \tag{2.13}$$

Combining (2.10) and (2.13), we derive

$$\frac{1}{l+1}\frac{d}{dt}\int_\Omega u^{l+1}\,dx + \left(lm^{p-1}\left(\frac{p}{l+m(p-1)}\right)^p - \frac{\lambda}{\lambda_1} - \frac{\varepsilon_2\alpha}{C(\varepsilon_2)}\right)\int_\Omega \left|\nabla u^{\frac{l+m(p-1)}{p}}\right|^p dx + \frac{\kappa_2^{-h_2}\alpha}{C(\varepsilon_2)}\left(\int_\Omega u^{l+1}\,dx\right)^{\frac{h_2}{l+1}} \leqslant 0. \tag{2.14}$$

Since $\lambda < lm^{p-1}(\frac{p}{l+m(p-1)})^p\lambda_1$, we can choose ε_2 small enough such that $lm^{p-1}(\frac{p}{l+m(p-1)})^p - \frac{\lambda}{\lambda_1} - \frac{\varepsilon_2\alpha}{C(\varepsilon_2)} \geqslant 0$. Once ε_2 is fixed, let $K_2 = \frac{\kappa_2^{-h_2}\alpha}{C(\varepsilon_2)} > 0$. Then from (2.14) we get

$$\frac{1}{l+1}\frac{d}{dt}\int_\Omega u^{l+1}\,dx + K_2\left(\int_\Omega u^{l+1}\,dx\right)^{\frac{h_2}{l+1}} \leqslant 0. \tag{2.15}$$

Integrating this inequality, we obtain

$$\left(\int_\Omega u^{l+1}\,dx\right)^{\frac{l+1-h_2}{l+1}} \leqslant \left(\int_\Omega u_0^{l+1}\,dx\right)^{\frac{l+1-h_2}{l+1}} - K_2(l+1-h_2)t.$$

Since $h_2 < l+1$, we obtain the desired result. The proof of Theorem 1.5 is complete. □

Proof of Theorem 1.6. We will divide the proof into two cases: $r \leqslant 1$ and $r > 1$.

In the first case $r \leqslant 1$. For $\frac{Nm+N}{Nm+m+1} \leqslant p < 1 + \frac{1}{m}$. Combining (2.5) and applying the same computation as for (2.1) and by Hölder's inequality, we conclude

$$\frac{1}{m+1}\frac{d}{dt}\int_\Omega u^{m+1}\,dx + \left(1 - \frac{\varepsilon_1\alpha}{C(\varepsilon_1)}\right)\int_\Omega \left|\nabla u^m\right|^p dx + \frac{\kappa_1^{-h_1}\alpha}{C(\varepsilon_1)}\left(\int_\Omega u^{m+1}\,dx\right)^{\frac{h_1}{m+1}}$$
$$\leqslant \lambda|\Omega|^{\frac{1-r}{m+1}}\left(\int_\Omega u^{m+1}\,dx\right)^{\frac{m+r}{m+1}}. \tag{2.16}$$

Choosing ε_1 small enough such that $1 - \frac{\varepsilon_1\alpha}{C(\varepsilon_1)} \geqslant 0$, and by a direct calculation, we obtain

$$\frac{1}{m+1-h_1}\frac{d}{dt}\left(\int_\Omega u^{m+1}\,dx\right)^{\frac{m+1-h_1}{m+1}} \leqslant \lambda|\Omega|^{\frac{1-r}{m+1}}\left(\int_\Omega u^{m+1}\,dx\right)^{\frac{m+r-h_1}{m+1}} - \frac{\kappa_1^{-h_1}\alpha}{C(\varepsilon_1)}. \tag{2.17}$$

To simplify, we denote

$$f\big(u(t)\big) = \lambda|\Omega|^{\frac{1-r}{m+1}}\left(\int_\Omega u^{m+1}\,dx\right)^{\frac{m+r-h_1}{m+1}} - \frac{\kappa_1^{-h_1}\alpha}{C(\varepsilon_1)}. \tag{2.18}$$

Assume that $f(u_0) < 0$, so

$$\left(\int_\Omega u_0^{m+1}\,dx\right)^{\frac{m+r-h_1}{m+1}} < \frac{\kappa_1^{-h_1}\alpha}{\lambda C(\varepsilon_1)}|\Omega|^{\frac{r-1}{m+1}}.$$

We claim that $f(u(t))$ is decreasing with t. The proof is almost identical to the corresponding proof in [25], for the convenience of the readers, we give the proof. If the claim is not correct, we say that there exists some time $T > 0$ such that $\frac{d}{dt}f(u(t))|_{t=T} \geqslant 0$. Since $f(u_0) < 0$, so from (2.17) we can obtain $\frac{d}{dt}(\int_\Omega u^{m+1}\,dx)|_{t=0} < 0$. Then, using

$$r > \frac{pq(m+1) + N(mp-m-q)}{p(m+1) + N(mp-m-q)} = h_1 - m$$

and (2.18), it is easy to show that $\frac{d}{dt}f(u(t))|_{t=0} < 0$. So there exist time $T^* \in (0, T]$ such that $f(u(T^*)) < f(u_0) < 0$ and $\frac{d}{dt}f(u(t))|_{t=T^*} = 0$. If $u(x, T^*) = 0$, we complete our proof. If $u(x, T^*) \neq 0$, we have $\frac{d}{dt}(\int_\Omega u^{m+1}\,dx)|_{t=T^*} = 0$. Then from (2.17), we derive

$$0 = \frac{1}{m+1-h_1}\frac{d}{dt}\left(\int_\Omega u^{m+1}\,dx\right)^{\frac{m+1-h_1}{m+1}}\Bigg|_{t=T^*} \leqslant f\big(u\big(T^*\big)\big) < 0.$$

So we get a contradiction.

Therefore, we arrive at the inequality

$$\frac{1}{m+1-h_1}\frac{d}{dt}\left(\int_\Omega u^{m+1}\,dx\right)^{\frac{m+1-h_1}{m+1}} \leqslant \lambda|\Omega|^{\frac{1-r}{m+1}}\left(\int_\Omega u_0^{m+1}\,dx\right)^{\frac{m+r-h_1}{m+1}} - \frac{\kappa_1^{-h_1}\alpha}{C(\varepsilon_1)}. \tag{2.19}$$

Integrating (2.19), we derive

$$\left(\int_\Omega u^{m+1}\,dx\right)^{\frac{m+1-h_1}{m+1}} \leqslant \left(\int_\Omega u_0^{m+1}\,dx\right)^{\frac{m+1-h_1}{m+1}} + f(u_0)(m+1-h_1)t.$$

Since $h_1 \in (m, m+1)$ and $f(u_0) < 0$, we obtain our result.

For $1 < p < \frac{Nm+N}{Nm+m+1}$. Combining (2.13) and applying the same computation as for (2.9) and by Hölder's inequality, we arrive at the inequality

$$\frac{1}{l+1}\frac{d}{dt}\int_\Omega u^{l+1}\,dx + \left(lm^{p-1}\left(\frac{p}{l+m(p-1)}\right)^p - \frac{\varepsilon_2\alpha}{C(\varepsilon_2)}\right)\int_\Omega \left|\nabla u^{\frac{l+m(p-1)}{p}}\right|^p dx + \frac{\kappa_2^{-h_2}\alpha}{C(\varepsilon_2)}\left(\int_\Omega u^{l+1}\,dx\right)^{\frac{h_2}{l+1}}$$
$$\leqslant \lambda|\Omega|^{\frac{1-r}{l+1}}\left(\int_\Omega u^{l+1}\,dx\right)^{\frac{l+r}{l+1}}. \tag{2.20}$$

Choosing ε_2 small enough such that $lm^{p-1}(\frac{p}{l+m(p-1)})^p - \frac{\varepsilon_2\alpha}{C(\varepsilon_2)} \geqslant 0$, we get

$$\frac{1}{l+1-h_2}\frac{d}{dt}\left(\int_\Omega u^{l+1}\,dx\right)^{\frac{l+1-h_2}{l+1}} \leqslant \lambda|\Omega|^{\frac{1-r}{l+1}}\left(\int_\Omega u^{l+1}\,dx\right)^{\frac{l+r-h_2}{l+1}} - \frac{\kappa_2^{-h_2}\alpha}{C(\varepsilon_2)}. \tag{2.21}$$

We denote

$$\psi\big(u(t)\big) = \lambda|\Omega|^{\frac{1-r}{l+1}}\left(\int_\Omega u^{l+1}\,dx\right)^{\frac{l+r-h_2}{l+1}} - \frac{\kappa_2^{-h_2}\alpha}{C(\varepsilon_2)}.$$

Assume that $\psi(u_0) < 0$ and using $r > \frac{pq(l+1)+N(mp-m-q)}{p(l+1)+N(mp-m-q)} = h_2 - l$, we claim that $\psi(u(t))$ is decreasing with t. The proof is similar to above, so we omit it here. Therefore, we have

$$\frac{1}{l+1-h_2}\frac{d}{dt}\left(\int_\Omega u^{l+1}\,dx\right)^{\frac{l+1-h_2}{l+1}} \leqslant \lambda|\Omega|^{\frac{1-r}{l+1}}\left(\int_\Omega u_0^{l+1}\,dx\right)^{\frac{l+r-h_2}{l+1}} - \frac{\kappa_2^{-h_2}\alpha}{C(\varepsilon_2)}. \tag{2.22}$$

Integrating (2.22), and applying $h_2 \in (l, l+1)$, it is easy to show that there exists a positive constant T, such that

$$\int_\Omega u^{l+1}\,dx \equiv 0 \quad \text{for any } t \geqslant T,$$

which implies $u(x,t)$ vanishes in finite time.

In the second case $r > 1$, by the comparison principle (see [9,20,24,25]), for sufficiently small $d > 0$, if $u_0(x) \leqslant d\phi(x)$ in Ω, where $\phi(x)$ is the first eigenfunction of the p-Laplacian equation with homogeneous Dirichlet boundary-value condition, it can be easily verified that $d\phi(x)$ is a supersolution of problem (1.1), and thus that $u(x,t) < d$ in $\Omega \times (0, +\infty)$. Therefore, (2.16) and (2.20) can be rewritten as (see for example (2.16))

$$\frac{1}{m+1}\frac{d}{dt}\int_\Omega u^{m+1}\,dx + \left(1 - \frac{\varepsilon_1\alpha}{C(\varepsilon_1)}\right)\int_\Omega \left|\nabla u^m\right|^p dx + \frac{\kappa_1^{-h_1}\alpha}{C(\varepsilon_1)}\left(\int_\Omega u^{m+1}\,dx\right)^{\frac{h_1}{m+1}}$$
$$\leqslant \lambda d^{r-1}\int_\Omega u^{m+1}\,dx. \tag{2.23}$$

So the above argument can also be applied. The proof of Theorem 1.6 is complete. □

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Appendix A. Proofs of Theorems 1.1–1.4

In this appendix, for the case $q = 1$, we prove Theorems 1.1–1.4. Note that the proof of Theorem 1.3(i) is almost identical to the corresponding proof in [9] (see also [25]), so we omit the proof.

Proof of Theorem 1.1. We will prove the theorem by two cases: $r \leqslant 1$ and $r > 1$.

In the first case $r \leqslant 1$. For $\frac{Nm+N}{Nm+m+1} \leqslant p < 1 + \frac{1}{m}$, multiplying both sides of (1.1) by u^m and integrating over Ω, we arrive at the equality

$$\frac{1}{m+1}\frac{d}{dt}\int_\Omega u^{m+1}\,dx+\int_\Omega\left|\nabla u^m\right|^p dx+\alpha\int_\Omega u^{m+1}\,dx=\lambda\int_\Omega u^{m+r}\,dx. \tag{A.1}$$

By Hölder's inequality and the Sobolev embedding theorem ($N>p$), we obtain

$$\begin{aligned}\int_\Omega u^{m+1}\,dx &\leqslant |\Omega|^{1-\frac{(m+1)(N-p)}{mNp}}\left(\int_\Omega u^{\frac{mNp}{N-p}}\,dx\right)^{\frac{(m+1)(N-p)}{mNp}}\\ &\leqslant C|\Omega|^{1-\frac{(m+1)(N-p)}{mNp}}\left(\int_\Omega\left|\nabla u^m\right|^p dx\right)^{\frac{m+1}{mp}},\end{aligned} \tag{A.2}$$

where C is the embedding constant.

Combining (A.1) and (A.2) and applying Hölder's inequality, we get

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^{m+1}\,dx+(m+1)C^{-\frac{mp}{m+1}}|\Omega|^{\frac{N-p}{N}-\frac{mp}{m+1}}\left(\int_\Omega u^{m+1}\,dx\right)^{\frac{mp}{m+1}}+\alpha(m+1)\int_\Omega u^{m+1}\,dx\\ &\quad\leqslant\lambda(m+1)|\Omega|^{\frac{1-r}{m+1}}\left(\int_\Omega u^{m+1}\,dx\right)^{\frac{m+r}{m+1}}.\end{aligned} \tag{A.3}$$

From (A.3) and Lemma 4 in [11], we know there exists a $\sigma>(m+1)\alpha$ such that

$$0\leqslant\int_\Omega u^{m+1}\,dx\leqslant\left(\int_\Omega u_0^{m+1}\,dx\right)e^{-\sigma t},\quad t\geqslant 0,$$

provided that

$$\left(\int_\Omega u_0^{m+1}\,dx\right)^{\frac{m+r-mp}{m+1}}<\lambda^{-1}C^{-\frac{mp}{m+1}}|\Omega|^{\frac{N-p}{N}-\frac{mp+1-r}{m+1}}.$$

Since $r>m(p-1)$, there exists a T^*, when $t\in[T^*,+\infty)$, we have

$$\begin{aligned}&C^{-\frac{mp}{m+1}}|\Omega|^{\frac{N-p}{N}-\frac{mp}{m+1}}-\lambda|\Omega|^{\frac{1-r}{m+1}}\left(\int_\Omega u^{m+1}\,dx\right)^{\frac{m+r-mp}{m+1}}\\ &\quad\geqslant C^{-\frac{mp}{m+1}}|\Omega|^{\frac{N-p}{N}-\frac{mp}{m+1}}-\lambda|\Omega|^{\frac{1-r}{m+1}}\left[\left(\int_\Omega u_0^{m+1}\,dx\right)e^{-\sigma t}\right]^{\frac{m+r-mp}{m+1}}=C_1>0.\end{aligned} \tag{A.4}$$

Substituting (A.4) into (A.3), we obtain

$$\frac{d}{dt}\int_\Omega u^{m+1}\,dx+(m+1)C_1\left(\int_\Omega u^{m+1}\,dx\right)^{\frac{mp}{m+1}}+\alpha(m+1)\int_\Omega u^{m+1}\,dx\leqslant 0. \tag{A.5}$$

Combining (A.5) and Lemma 3 in [11], we derive

$$\begin{cases}\displaystyle\int_\Omega u^{m+1}\,dx\leqslant\left(\left(\left(\int_\Omega u_0^{m+1}\,dx\right)^{\frac{m+1-mp}{m+1}}+\frac{C_1}{\alpha}\right)e^{(mp-m-1)\alpha t}-\frac{C_1}{\alpha}\right)^{\frac{m+1}{m+1-mp}}, & t\in[0,T_1),\\ \displaystyle\int_\Omega u^{m+1}\,dx\equiv 0,\quad t\in[T_1,+\infty),\end{cases}$$

where

$$T_1=\frac{1}{\alpha(m+1-mp)}\ln\left(1+\frac{\alpha}{C_1}\left(\int_\Omega u_0^{m+1}\,dx\right)^{\frac{m+1-mp}{m+1}}\right).$$

For $1 < p < \frac{Nm+N}{Nm+m+1}$. Multiplying both sides of (1.1) by u^s, where $s = \frac{N(m+1)-(Nm+1)p}{p} > m$, and integrating over Ω, we arrive at the equality

$$\frac{1}{s+1}\frac{d}{dt}\int_\Omega u^{s+1}\,dx + sm^{p-1}\left(\frac{p}{s+m(p-1)}\right)^p \int_\Omega \left|\nabla u^{\frac{s+m(p-1)}{p}}\right|^p dx + \alpha\int_\Omega u^{s+1}\,dx$$
$$= \lambda\int_\Omega u^{s+r}\,dx. \tag{A.6}$$

Applying the Sobolev embedding theorem ($N > p$) and inserting our choice of s, we derive

$$\left(\int_\Omega u^{s+1}\,dx\right)^{\frac{s+m(p-1)}{p(s+1)}} = \left(\int_\Omega u^{\frac{Ns+Nm(p-1)}{N-p}}\,dx\right)^{\frac{N-p}{Np}} \leqslant C\left(\int_\Omega \left|\nabla u^{\frac{s+m(p-1)}{p}}\right|^p dx\right)^{\frac{1}{p}}. \tag{A.7}$$

Combining (A.6) and (A.7) and applying Hölder's inequality, we conclude

$$\frac{d}{dt}\int_\Omega u^{s+1}\,dx + \frac{C^{-p}(s+1)sm^{p-1}p^p}{(s+m(p-1))^p}\left(\int_\Omega u^{s+1}\,dx\right)^{\frac{s+m(p-1)}{s+1}} + \alpha(s+1)\int_\Omega u^{s+1}\,dx$$
$$\leqslant \lambda(s+1)|\Omega|^{\frac{1-r}{1+s}}\left(\int_\Omega u^{s+1}\,dx\right)^{\frac{s+r}{s+1}}. \tag{A.8}$$

Applying (A.8) and Lemma 4 in [11], we know there exists a $\varsigma > (s+1)\alpha$ such that

$$0 \leqslant \int_\Omega u^{s+1}\,dx \leqslant \left(\int_\Omega u_0^{s+1}\,dx\right)e^{-\varsigma t}, \quad t \geqslant 0,$$

provided that

$$\left(\int_\Omega u_0^{s+1}\,dx\right)^{\frac{m+r-mp}{s+1}} < \lambda^{-1}C^{-p}|\Omega|^{\frac{r-1}{s+1}}sm^{p-1}\left(\frac{p}{s+m(p-1)}\right)^p.$$

Since $r > m(p-1)$, there exists a T', when $t \in [T', +\infty)$, we have

$$C^{-p}sm^{p-1}\left(\frac{p}{s+m(p-1)}\right)^p - \lambda|\Omega|^{\frac{1-r}{s+1}}\left(\int_\Omega u^{s+1}\,dx\right)^{\frac{m+r-mp}{s+1}}$$
$$\geqslant C^{-p}sm^{p-1}\left(\frac{p}{s+m(p-1)}\right)^p - \lambda|\Omega|^{\frac{1-r}{s+1}}\left[\left(\int_\Omega u_0^{s+1}\,dx\right)e^{-\varsigma t}\right]^{\frac{m+r-mp}{s+1}} = C_2 > 0. \tag{A.9}$$

Combining (A.8) and (A.9), we have

$$\frac{d}{dt}\int_\Omega u^{s+1}\,dx + (s+1)C_2\left(\int_\Omega u^{s+1}\,dx\right)^{\frac{s+m(p-1)}{s+1}} + \alpha(s+1)\int_\Omega u^{s+1}\,dx \leqslant 0. \tag{A.10}$$

Applying (A.10) and Lemma 3 in [11], we obtain

$$\begin{cases} \displaystyle\int_\Omega u^{s+1}\,dx \leqslant \left(\left(\left(\int_\Omega u_0^{s+1}\,dx\right)^{\frac{m+1-mp}{s+1}} + \frac{C_2}{\alpha}\right)e^{(mp-m-1)\alpha t} - \frac{C_2}{\alpha}\right)^{\frac{s+1}{m+1-mp}}, & t \in [0, T_2), \\ \displaystyle\int_\Omega u^{s+1}\,dx \equiv 0, & t \in [T_2, +\infty), \end{cases}$$

where

$$T_2 = \frac{1}{\alpha(m+1-mp)}\ln\left(1 + \frac{\alpha}{C_2}\left(\int_\Omega u_0^{m+1}\,dx\right)^{\frac{m+1-mp}{s+1}}\right).$$

In the second case $r > 1$, since $d\phi(x)$ is a supersolution of problem (1.1). Therefore, (A.1) and (A.6) can be rewritten as (see for example (A.1))

$$\frac{1}{m+1}\frac{d}{dt}\int_\Omega u^{m+1}\,dx + \int_\Omega \left|\nabla u^m\right|^p dx + \alpha\int_\Omega u^{m+1}\,dx \leqslant \lambda d^{r-1}\int_\Omega u^{m+1}\,dx,$$

to which the above argument can be applied. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. For $q = 1$, let $Y(x,t) = C_0 k(t)\phi(x)$, where $C_0 > 0$ is to be determined later and $k(t)$ satisfies

$$\begin{cases} k'(t) = -\lambda_1 k^{m(p-1)}(t) - \alpha k(t) + \lambda k^r(t), & t \geqslant 0, \\ k(t) > 0, & t > 0, \\ k(0) = 0. \end{cases} \tag{A.11}$$

Then we have

$$\begin{aligned} &\int_0^t\int_\Omega \left(Y_\tau\varphi + \left|\nabla Y^m\right|^{p-2}\nabla Y^m\nabla\varphi + \alpha Y\varphi - \lambda Y^r\varphi\right) dx\,d\tau \\ &\quad= \int_0^t\int_\Omega \left[C_0\varphi\phi\left(-\lambda_1 k^{m(p-1)} + \lambda k^r\right) + \lambda_1 C_0^{m(p-1)} k^{m(p-1)}\phi^{m(p-1)}\varphi - \lambda C_0^r k^r\phi^r\varphi\right] dx\,d\tau \\ &\quad= \int_0^t\int_\Omega \left[\lambda_1 k^{m(p-1)}\left(C_0^{m(p-1)}\phi^{m(p-1)} - C_0\phi\right) - \lambda k^r\left(C_0^r\phi^r - C_0\phi\right)\right]\varphi\,dx\,d\tau. \end{aligned}$$

Since $r < m(p-1)$ and $0 < \phi(x) \leqslant 1$, we can choose C_0 sufficiently small such that

$$\lambda_1 k^{m(p-1)}\left(C_0^{m(p-1)}\phi^{m(p-1)} - C_0\phi\right) \leqslant \lambda k^r\left(C_0^r\phi^r - C_0\phi\right). \tag{A.12}$$

Let $F(x) = \frac{x^r - x}{x^{m(p-1)} - x}$, it is easy to check that $F(x)$ is decreasing in (0,1), and $\lim_{x\to 0^+} F(x) = +\infty$. In addition, it is not difficult to show that $k(t)$ is a bounded function from (A.11) (see [11]). Thus we can choose a sufficiently small $C_0 > 0$ such that (A.12) holds.

Moreover, $Y(x,0) = C_0 k(0)\phi(x) = 0 \leqslant u_0(x)$ in Ω, and $Y(x,t) = 0$ in $\partial\Omega\times(0,+\infty)$. Therefore, by the comparison principle (see [9,20,24,25]), we have $u(x,t) \geqslant Y(x,t) > 0$ in $\Omega\times(0,+\infty)$. The proof of Theorem 1.2 is complete. □

Proof of Theorem 1.3. (ii) For $q = 1$ and $r = m(p-1)$. We consider the first case $\frac{Nm+N}{Nm+m+1} \leqslant p < 1 + \frac{1}{m}$ and $\lambda < \lambda_1$. Applying (A.1) and $\lambda_1\int_\Omega u^{mp}\,dx \leqslant \int_\Omega |\nabla u^m|^p\,dx$, we get

$$\frac{1}{m+1}\frac{d}{dt}\int_\Omega u^{m+1}\,dx + \left(1 - \frac{\lambda}{\lambda_1}\right)\int_\Omega \left|\nabla u^m\right|^p dx + \alpha\int_\Omega u^{m+1}\,dx \leqslant 0. \tag{A.13}$$

It follows from (A.2) and (A.13) that

$$\frac{d}{dt}\int_\Omega u^{m+1}\,dx + (m+1)C_3\left(\int_\Omega u^{m+1}\,dx\right)^{\frac{mp}{m+1}} + \alpha(m+1)\int_\Omega u^{m+1}\,dx \leqslant 0, \tag{A.14}$$

where

$$C_3 = C^{-\frac{mp}{m+1}}\left(1 - \frac{\lambda}{\lambda_1}\right)|\Omega|^{\frac{N-p}{N} - \frac{mp}{m+1}} > 0.$$

From (A.14) and Lemma 3 in [11], we obtain

$$\begin{cases} \displaystyle\int_\Omega u^{m+1}\,dx \leqslant \left(\left(\left(\int_\Omega u_0^{m+1}\,dx\right)^{\frac{1-r}{m+1}} + \frac{C_3}{\alpha}\right)e^{(r-1)\alpha t} - \frac{C_3}{\alpha}\right)^{\frac{m+1}{1-r}}, & t \in [0, T_3), \\ \displaystyle\int_\Omega u^{m+1}\,dx \equiv 0, & t \in [T_3, +\infty), \end{cases}$$

where

$$T_3 = \frac{1}{\alpha(1-r)} \ln\left(1 + \frac{\alpha}{C_3}\left(\int_\Omega u_0^{m+1}\,dx\right)^{\frac{1-r}{m+1}}\right).$$

Next, we consider the case $1 < p < \frac{Nm+N}{Nm+m+1}$ and $\lambda < \frac{(Nm+N-Nmp-p)m^{p-1}p^{2p-1}}{[(N-p)(m+1-mp)]^p}\lambda_1 = sm^{p-1}(\frac{p}{s+m(p-1)})^p\lambda_1$. Applying (A.6) and inserting $r = m(p-1)$, we get

$$\frac{1}{s+1}\frac{d}{dt}\int_\Omega u^{s+1}\,dx + \left(\frac{sm^{p-1}p^p}{(s+m(p-1))^p} - \frac{\lambda}{\lambda_1}\right)\int_\Omega \left|\nabla u^{\frac{s+m(p-1)}{p}}\right|^p dx + \alpha\int_\Omega u^{s+1}\,dx \leqslant 0. \tag{A.15}$$

Combining (A.7) and (A.15), we conclude

$$\frac{d}{dt}\int_\Omega u^{s+1}\,dx + C_4(s+1)\left(\int_\Omega u^{s+1}\,dx\right)^{\frac{s+m(p-1)}{s+1}} + \alpha(s+1)\int_\Omega u^{s+1}\,dx \leqslant 0, \tag{A.16}$$

where

$$C_4 = C^{-p}\left[sm^{p-1}\left(\frac{p}{s+m(p-1)}\right)^p - \frac{\lambda}{\lambda_1}\right] > 0.$$

From (A.16) and Lemma 3 in [11], we obtain

$$\begin{cases} \displaystyle\int_\Omega u^{s+1}\,dx \leqslant \left(\left(\left(\int_\Omega u_0^{s+1}\,dx\right)^{\frac{1-r}{s+1}} + \frac{C_4}{\alpha}\right)e^{(r-1)\alpha t} - \frac{C_4}{\alpha}\right)^{\frac{s+1}{1-r}}, & t \in [0, T_4), \\ \displaystyle\int_\Omega u^{s+1}\,dx \equiv 0, \quad t \in [T_4, +\infty), \end{cases}$$

where

$$T_4 = \frac{1}{\alpha(1-r)} \ln\left(1 + \frac{\alpha}{C_4}\left(\int_\Omega u_0^{m+1}\,dx\right)^{\frac{1-r}{s+1}}\right).$$

The proof of Theorem 1.3 is complete. □

Proof of Theorem 1.4. (i) When $r = m(p-1)$, $\lambda > \lambda_1$ and $q = 1$. Let $w(x,t) = g(t)\phi(x)$, and $g(t)$ satisfies

$$\begin{cases} g'(t) = (\lambda - \lambda_1)g^r(t) - \alpha g(t), & t \geqslant 0, \\ g(t) > 0, & t > 0, \\ g(0) = 0. \end{cases}$$

Since $0 < \phi(x) \leqslant 1$, we obtain

$$\begin{aligned} &\int_0^t\int_\Omega \left(w_\tau\varphi + |\nabla w^m|^{p-2}\nabla w^m\nabla\varphi + \alpha w\varphi - \lambda w^r\varphi\right)dx\,d\tau \\ &\quad = \int_0^t\int_\Omega \left[\left((\lambda-\lambda_1)g^r - \alpha g\right)\phi\varphi + \lambda_1 g^{m(p-1)}\phi^{m(p-1)}\varphi + \alpha g\phi\varphi - \lambda g^r\phi^r\varphi\right]dx\,d\tau \\ &\quad = \int_0^t\int_\Omega (\lambda-\lambda_1)\left(\phi - \phi^r\right)g^r\varphi\,dx\,d\tau \leqslant 0. \end{aligned}$$

Moreover, $w(x,0) = g(0)\phi(x) = 0 \leqslant u_0(x)$ in Ω, and $w(x,t) = 0$ in $\partial\Omega \times (0,+\infty)$. Therefore, by the comparison principle (see [9,20,24,25]), we have $u(x,t) \geqslant w(x,t) > 0$ in $\Omega \times (0,+\infty)$.

(ii) For the case $\lambda = \lambda_1$, it is easily proved that $\underline{u}(x,t) = h(t)\phi(x)$ is a solution of problem (1.1), where $h(t) = h_0e^{-\alpha t}$, $h_0 > 0$. Then for any positive initial data $u_0(x)$, we can choose h_0 sufficiently small such that $u_0(x) \geqslant h_0\phi(x)$ in Ω and by the comparison principle (see [9,20,24,25]), $\underline{u}(x,t)$ is a non-extinction subsolution of problem (1.1). The proof of Theorem 1.4 is complete. □

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