



Group action and shift-compactness

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ABSTRACT

Shift-compactness has recently been found to be the foundation stone of classical, as well as topological, regular variation; most recently it has come again to prominence in new proofs of the Effros Open Mapping Principle of group action, another ingredient of topological regular variation. Using the real line under the Euclidean and density topologies as a paradigm, we develop group-action versions of shift-compactness theorems for Baire groups acting on Baire spaces under metrizable topologies and under certain refinements of these. One aim is to pursue constructive approaches rather than rely on plain Baire-category methods (so keeping more to the Banach–Mazur strategic approach). Along the way we uncover three new coarse topologies for groups of homeomorphisms. A second purpose is to establish limitations of the shift-compactness methodology.

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1. Introduction and motivation

The theorem below in real analysis, proved in the measure case by Borwein and Ditor [20] and earlier derived in slightly weaker form by Kestelman [44], has been generalized to a topological setting in [18] which permits the underlying property of ‘shift-compactness’ (defined in Section 2 after Theorem S) to be viewed as the foundation stone of classical regular variation and also the ultimate explanation for the dual measure-category framework of classical regular variation as established in [11]. Furthermore, the more general topological theorem leads to a natural development of a topological theory of regular variation embracing earlier partial attempts at generalizations of the classical theory, see e.g. [12,14]. The shift-compactness property is implied by ‘amenability at 1’ introduced recently by Solecki [69], a matter we consider elsewhere, and in turn implies Steinhaus’ Interior Point Theorem and its relatives. It and they are critical in automatic continuity (cf. [68]); for illustration, see e.g. the treatment of Jones’s theorem in [13] (and for background refer to [41]). Most recently shift-compactness has been found to imply an important result in general topology, the Effros Open Mapping Theorem [57], a result concerning group actions on a metric space, itself a further ingredient of topological regular variation (under the guise of the ‘crimping property’, for which see [15] and again [57]). Of course group action underpins the very definition of regular variation, but its implicit presence became visible only in the recent topological formulation just cited; hitherto explicit reference to group action was via the one-parameter group of affine transformations – see [11, §8.5] or [5, §18].

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The present paper resumes the study of the Kestelman–Borwein–Ditor theorem below, initiated in this journal in [12], from the newly acquired perspective of group action.

Theorem KBD (*Kestelman–Borwein–Ditor Theorem*). Let $\{z_n\} \rightarrow 0$ be a null sequence in \mathbb{R} . If T is a measurable/Baire subset of \mathbb{R} , then for generically all ($=$ almost all/quasi all) $t \in T$ there is an infinite set \mathbb{M}_t such that

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq T. \quad (1)$$

Denoting by $\tau_t(x) := t + x$, translation by t , put $Tr(S) := \{\tau_s : s \in S\}$ for $S \subseteq \mathbb{R}$, so that $Tr(\mathbb{R})$ is the group of translations on \mathbb{R} . The result above has two dual interpretations, depending on whether one views the points $t + z_m$ above as being the image sequence $\{\tau(z_m) : m \in \mathbb{M}_t\}$ for some $\tau \in Tr(T)$, or the evaluation at some $t \in T$ of the corresponding subsequence of translations τ^m with $\tau^n(x) := x + z_n$ (with pointwise limit the identity: $id(x) \equiv x$).

In the first case, the subsequence $\{z_m\}$ and its limit are embedded in the target set T by an application of the map $x \mapsto x + t$ (so that T appears as both a target set and a contributor of an action, namely translation). In the second, the sequence τ^m together with its pointwise limit id are embedded in T by an evaluation map $h \mapsto h(t)$ evaluated at some $t \in T$ (and again T appears as both a target set and a contributor of an action, namely evaluation).

The natural framework exhibiting the inherent duality of Theorem KBD is thus the action $A : Tr(\mathbb{R}) \times \mathbb{R} \rightarrow \mathbb{R}$ by the group $Tr(\mathbb{R})$, wherein A maps (τ, x) for $\tau \in Tr(\mathbb{R})$ and $x \in \mathbb{R}$, to $\tau(x) = t + x$ for $\tau = \tau_t$. More generally, denoting by $Auth(X)$ the group of self-homeomorphisms of a topological space X and equipping a subgroup G of $Auth(X)$ with some topology, we seek to establish analogues of Theorem KBD by demanding various forms of separate continuity of the action $G \times X \rightarrow X$ given by $(g, x) \rightarrow g(x)$. These allow X to include not only metric group topologies, such as the Euclidean topologies, but also such refinements as the (measure) *density topologies* (see Section 2), which embrace the measure case of Theorem KBD.

In going beyond translations in a group to homeomorphic actions in a space, we speak of ‘shift-compactness theorems’ (borrowing a term from Parthasarathy et al. [62], who used a related notion in the context of a semi-group of probability measures under convolution; see also [61,39,40]).

Given the increasing significance of shift-compactness theorems, the first aim in this paper is to identify ‘constructive’ arguments (see Section 2 for an explanation) yielding the asserted existence of either the embedding or the evaluation point of the theorem (Sections 2–4). The second aim is the other side of the same coin: understanding the limitations of these theorems (Section 5). In this we are helped by some earlier papers, which with hindsight now seem to have anticipated special forms of the current ‘action approach’: [50] (applying, albeit in the context of \mathbb{R} , actions more general than standard group actions – compare Section 5 below), [19] (interpreting the preceding paper as identifying a ‘homotopy to the identity’ – in which the function id played a leading role, just as in the opening remarks above), also [18], and van Mill’s recent paper [74]. The latter has been a very valuable help and source of inspiration.

In the interests of transparency, and because the authors of this paper come to this subject from two different but complementary points of view, we have whenever possible formulated arguments first in the Euclidean context, and then indicated the natural generalizations. One viewpoint emphasizes general “positive” results (motivated by earlier work with N.H. Bingham, much of it summarized in [17], cf. [15] on topological regular variation, and also [58]), the other what might be called counter-examples or “negative” results testing the limitations of Theorem KBD (on this cf. also Komjáth [45,46]). Each of the authors travels in his ordained direction, but the distinct approaches yield just another example, in analysis, of the interplay between concrete classical real analysis and modern geometric–topological analysis.

The structure of the paper is as follows. In Section 2 we derive generalizations of Theorem KBD in Theorem 2 and Theorem 2’ (in the context of Polish spaces) for groups of homeomorphisms that act transitively and strongly separate points from nowhere dense sets; here we conduct the argument first on the line (where the group of translations has these properties), under the Euclidean topology. The argument may be adjusted so as to apply also to the density topology. That adjustment in turn permits in Section 3 the establishment of a new general result for cometrizable refinement topologies in Theorem 3. Its format prompts a reformulation in action terms of the Category Embedding Theorem of [18], as Theorem 4. The hypotheses in the two theorems set the agenda of Section 4 – their interpretation along the lines of joint continuity uncovers three new topological structures for groups of homeomorphisms, which the two theorems require to be coarser than norm topologies (Propositions 3 and 4); Theorem 5 closes the loop, by verifying this condition for the group of translations on the line. In the setting of submetrizable topologies one verifies only *subsequence* embedding in a target set T , so in Section 5 we demonstrate circumstances involving measurability (or its absence) under which certain translated subsequences $t + z_m$ must omit the target set T .

2. Category & measure shift-compactness

A group-action framework allows the formulation of uniformity properties; a simple instance is that a non-meagre metric space supporting a transitive group action is a Baire space (see [57, Remark to Theorem B]). The more important example for us is the classical Effros Open Mapping Principle of [27] (which also implies the Banach–Schauder Open Mapping Theorem and the Banach–Steinhaus theorem on uniform boundedness in a separable context). That example has a number of more recent extensions (e.g. [73,50,56]). Another is the Lavrentieff theorem on the extension of a homeomorphism between sets to a homeomorphism between \mathcal{G}_δ sets covering the given ones (cf. Theorem 2.2.7 in [7], a text devoted to a fruitful, kindred,

programme investigating group actions from the viewpoint of descriptive set theory). Recently, one of us has shown [57] that the classical Effros theorem is a simple consequence of the following result, proved by an appeal to Baire category (see below for definitions). As intimated earlier, we seek proofs that are more constructive than this, by identifying convergent sequences (cf. especially [56]). The ultimate justification for this quest is that Choquet's (weak) α -favourability property (see [60, Ch. 6], or [42, Theorem 8.17(i)], for Oxtoby's result that weak α -favourability in the realm of separable metrizable spaces is characterized as almost completeness; compare [55]) is preferable over the plain Baire-category method, the former being a stronger form (e.g. through its preservation under products – [42, §8]). As group action is the primary focus of our investigation, we rely on completeness (e.g. in Theorem 2), leaving aside any generalization to an almost complete context (cf. Remark 2 after Theorem 3).

Theorem S (Shift-compactness Theorem). *For T a Baire non-meagre subset of a metric space X and G a separable normed group, Baire in its right-norm topology (e.g. almost complete in the norm topology), acting separately continuously and transitively on X :*

for every convergent sequence x_n with limit x_0 and any Baire non-meagre $A \subseteq G$ with $e_G \in A$ such that $Ax_0 \subseteq T$, there are $\alpha \in A$ and an integer N such that $\alpha x_0 \in T$ and

$$\{\alpha(x_n) : n > N\} \subseteq T.$$

In this general formulation we regard the application of the homeomorphism α as a topological shift, since for a group X this α may indeed be a translation. In more general circumstances (e.g. Theorem 1M), the conclusion of the theorem will assert only that $\{\alpha(x_n) : n \in \mathbb{N}\} \subseteq T$ for some infinite subset of integers. That is, a shifted subsequence converges to a limit in T , and we refer to this property as 'shift-compactness'.

Note that for X a metrizable topological group and G its group of left translations $x \mapsto gx$, the general case of the theorem for $x_n \rightarrow x_0$ and $Ax_0 \subseteq T$ reduces, via $A \subseteq Tx_0^{-1}$, to the case of null sequences $z_n \rightarrow e_X$; for these one has $az_m \in A$ infinitely often, for some $a \in A$, exactly as in Theorem KBD. (Indeed, if $z_n = x_n x_0^{-1}$, then $az_m = ax_m x_0^{-1} \in A \subseteq Tx_0^{-1}$, so that $ax_m \in T$ and likewise the limit ax_0 is in T , since we have $ax_0 \in Ax_0 \subseteq T$.)

Motivated by the recent paper of van Mill [74], we consider here group-action versions of the Shift-compactness Theorem. Our interest springs from the existence of both category and measure versions of Theorem S for the real line \mathbb{R} ; in the form given by Theorems 1E and 1M below these already improve the original KBD Theorem.

Given our aim to include measure-case variants, we necessarily take a topologically broad view of *group actions*. We will say a group G acts on a set X if, as usual, there is a map $\varphi : G \times X \rightarrow X$ such that $\varphi(gh, x) = \varphi(g, \varphi(h, x))$ and $\varphi(e, x) = x$. We refer to the *action* of an element g , i.e. the map $x \mapsto \varphi(g, x)$ as $g(x)$. The set X may have more than one topology; to identify the topology we will either refer to it directly by name, e.g. \mathcal{T}_X , or indirectly to X if context permits, or else we will write $\mathfrak{X} := (X, \mathcal{T}_X)$ for the topological space. Under these circumstances, if each action map $\varphi_g : x \mapsto \varphi(g, x)$ is continuous, then $g(\cdot)$ is a homeomorphism, and so G may be regarded as a group of (auto-) homeomorphisms of X under composition. We denote by $\text{Auth}(\mathfrak{X})$ the group of autohomeomorphisms of a topological space X under composition – but do not equip it with any topology. When also the group $G \subseteq \text{Auth}(\mathfrak{X})$ is equipped with a topology \mathcal{T}_G (not necessarily metric) one may place topological conditions on the *evaluation* map $\varphi^x : g \mapsto \varphi(g, x)$ for each x . The simplest situation is to require the group action to be *separately continuous* (so that all the pointwise evaluation functions are continuous). By a theorem of Bouziad [21], if the topology \mathcal{T}_G is metrizable and Baire, as is the case in Theorem S (though not in Theorem 3 below), a separately continuous action is necessarily jointly continuous.

In order to work with both measure and category, we must step beyond a metrizable topology \mathcal{T}_X to one which is *submetrizable*, i.e. either is or refines a metrizable topology \mathcal{T}_d on X , generated by a metric, $d = d^X$ say. For example, X may be the real line either with the Euclidean topology \mathcal{E} or the density topology \mathcal{D} (recalled in the definitions below). Given the metric d^X , an element h of $\text{Auth}(X, \mathcal{T}_d)$ will be termed *bounded* if

$$\|h\| := \sup_x d^X(h(x), x) < \infty. \quad (\text{sup})$$

The set of bounded elements of $\text{Auth}(X, \mathcal{T}_d)$ will be denoted by $\mathcal{H}(X)$ and equipped with the group-norm $\|\cdot\|$. For background, see [17], but we recall that for X an *algebraic* group $\|\cdot\| : X \rightarrow \mathbb{R}_+$ is a *group-norm* if the following properties hold:

- (i) Subadditivity (Triangle inequality): $\|xy\| \leq \|x\| + \|y\|$;
- (ii) Positivity: $\|x\| > 0$ for $x \neq e$;
- (iii) Inversion (Symmetry): $\|x^{-1}\| = \|x\|$.

The right and left *induced norm topologies* are given by the right and left invariant metrics: $d_R^X(x, y) := \|xy^{-1}\|$ and $d_L^X(x, y) := \|x^{-1}y\| = d_R^X(x^{-1}, y^{-1})$.

The *van Mill separation property* SP [74] holds in X if for any countable set D and co-meagre set T there is $h \in \text{Auth}(X)$ such that $h(D) \subseteq T$. (For our purposes this is stated as an embedding into T rather than, as originally, an omission of the meagre complement; compare the embedding property of *countable dense homogeneity* in e.g. [3, Theorem 5.2].) Likewise say that X is *shift-compact* if for any convergent sequence $x_n \rightarrow x_0$, any open subset U in X and any Baire set T co-meagre

in U , there is $h \in \mathcal{H}(X)$ such that $h(x_n) \in T \cap U$ along a subsequence. Working in metric spaces, van Mill observes that SP implies the space is Baire, by specializing D above to a single point; the same applies to a constant sequence. Thus shift-compactness is a localized version of SP (localized to ‘co-meagre on an open set’); that is:

Proposition B. (See [74, Proposition 3.1]; [57].) *If X is shift-compact, then X is Baire.*

We also find a use for *strong separation* as defined below.

Definitions. 1. Say that a subgroup $G \subseteq H(X)$ *separates (individual) points and closed nowhere dense sets* in (X, \mathcal{T}_X) if for each $p \in X$ and F closed and nowhere dense in \mathcal{T}_X there is in each neighbourhood of the identity e_G an element $g \in G$ such that $g(p) \notin F$. Here we assume that G is given either a norm topology, or some refinement of it.

2. Say that the separation of p from F , as in Definition 1 above, is *strong* if in each neighbourhood of the identity there is a non-empty open set H such that $h(p) \notin F$ for every $h \in H$.

Equivalently (when the group is right-topological), in each open neighbourhood U of e_G there are $g \in U$ and an open neighbourhood V of e_G such that $Vg \subseteq U$ and $Vg(p)$ is disjoint from F .

3. Denote by $Tr(\mathbb{R}^d)$ the group of c -translations $x \rightarrow x + c$ in \mathbb{R}^d . Under the sup-norm defined above in equation (sup) this group is isometric with \mathbb{R}^d . Thus any refinement of the Euclidean topology can be used as a topology also on $Tr(\mathbb{R}^d)$, as in the proposition below. Particularly useful refinements are provided by *density topologies*, as they permit measure properties to be handled topologically (see [18]). These were introduced in [34] and further studied [33] (see also [49], and [70]), though they can be traced back to Denjoy [24]. Recall that density-open sets are measurable sets W all of whose members are *density points*, that is $1 = \lim_{\varepsilon \rightarrow 0} |W \cap B_\varepsilon(w)|/|B_\varepsilon(w)|$ for every $w \in W$. Here $|\cdot|$ denotes Lebesgue measure and $B_\varepsilon(w)$ is the open ball of radius ε . For other density topologies in \mathbb{R}^d (e.g. using density bases other than these balls) in particular, and refinement topologies in general see [47]; for the locally compact case see [17] for topological groups, and [59] for normed groups. We recall that in the density topology a set is nowhere dense iff it is null (has measure zero).

Proposition 1 (Strong Separation). *For \mathbb{R}^d and $Tr(\mathbb{R}^d)$ both equipped with the same topology, either the Euclidean or the density topology, $Tr(\mathbb{R}^d)$ strongly separates points and closed nowhere dense sets.*

Proof. In the Euclidean case, for F closed and nowhere dense, p a given point and arbitrary $\varepsilon > 0$, there is an open interval $I = (a, b)$ with $I \subseteq B_\varepsilon(p)$ disjoint from F (by definition). For $m \in I$, put $c = m - p$; then $|c| < \varepsilon$ as $p - \varepsilon < m < p + \varepsilon$ and for the c -shift $h(x) = c + x$, one has $\|h\| = \sup_x \|h(x) - x\| = \|c\| < \varepsilon$ and $h(p) = m \notin F$. Furthermore, this holds for all the choices of $c \in I - p = (a - p, b - p)$, corresponding to $a < m < b$.

In the density case, consider M measurable and null and w.l.o.g. $p \in M$. For any $c \in B_\varepsilon(0) \setminus (M - p)$ one has $\|c\| < \varepsilon$ and $p + c \notin M$. Since $M - p$ is null, the set $B_\varepsilon(0) \setminus (M - p)$ has non-empty interior under the density topology. \square

Remarks. 1. With \mathbb{R}^d under the density topology and $Tr(\mathbb{R}^d)$ under the norm topology the separation need not be strong. On the line, for M the rationals and $p \in M$ every rational translation $p + c$ fails to avoid M (and contrarily for every irrational translation).

2. Note the following less informative argument in the density case. Suppose otherwise that for some $\varepsilon > 0$ and all c , with $\|c\| < \varepsilon$, the c -shift $p + c$ is in M . Then $B_\varepsilon(p) \subseteq M$ and so $|M| > 0$, a contradiction.

We apply Proposition 1 to show that individual separation of points from a closed nowhere dense set may be improved to a *local finitary separation* by the group of shifts, i.e. a finite collection of points in some open set may be separated by a shift from a closed nowhere dense set. (The alternative view is that any finite number of points may be shifted locally into the complement of a closed nowhere dense set; in the semigroup setting a set into which any finite set may be shifted was termed by Mitchell [52] *left thick*; for more on this see [22] and [23].)

Proposition 2 (Finitary Euclidean Strong Separation). *Let U be Euclidean open and $u_i \in U$ for $i \leq n$. Suppose that F is (Euclidean) closed and nowhere dense. Then, for each $\varepsilon > 0$, in $B_\varepsilon(0)$ there is a neighbourhood of c -shifts $x \rightarrow x + c$ such that $u_i + c \in U$ and $u_i + c \notin F$ for each $i \leq n$.*

Proof. Let $\varepsilon > 0$. By assumption $\eta := \min\{\varepsilon, \min_i\{d(u_i, X \setminus U)\}/(n + 1)\} > 0$. Let $I_0 := (-\eta, \eta)$. By induction on $i \leq n$, we select c_1, \dots, c_n and open neighbourhoods I_1, \dots, I_n of 0 such that

- (i) $c_i \in I_{i-1}$, $c_i + I_i \subseteq I_{i-1}$,
- (ii) $u_j + c_1 + \dots + c_i + c \in U \setminus F$ for $c \in I_i$ for $j < i$, and
- (iii) $u_j + c_1 + \dots + c_i + c \in U$ for $c \in I_i$ for $j \leq n$.

It will follow that $u_i + c_1 + \dots + c_n + c \in U \setminus F$ for all $i \leq n$ and each $c \in I_n$.

Choose $c_1 \in I_0 = (-\eta, \eta)$ and I_1 an open neighbourhood of 0 such that $I_1 + c_1 \subseteq (-\eta, \eta)$ so that $(u_1 + c_1) + c \in U \setminus F$ for each $c \in I_1$. For each such c and each i one has $u_i + c_1 + c \in U$, since $|c_1 + c| < 2\eta \leq \varepsilon$.

Now choose c_2 in I_1 and I_2 a neighbourhood of 0 such that $I_2 + c_2 \subseteq I_1$ so that $(u_2 + c_1) + (c_2 + c) \in U \setminus F$ for each $c \in I_2$. For any such c and each i one has $u_i + c_1 + c_2 + c \in U$ as $|c_1 + c_2 + c| < 3\eta \leq \varepsilon$ and $(u_1 + c_1) + c_2 + c \in U \setminus F$ as $c_1 + c_2 + c \in c_1 + (c_2 + I_2) \subseteq c_1 + I_1$.

Proceed similarly for any $i < n$, by selecting c_i in I_{i-1} and I_i a neighbourhood of 0 such that $I_i + c_i \subseteq I_{i-1}$ so that $(u_i + c_1 + \dots + c_{i-1}) + c_i + c \in U \setminus F$ for each $c \in I_i$.

For any such c and each $j < i$ one has $u_j + c_1 + c_2 + \dots + c_i + c \in U$ as $|c_1 + c_2 + \dots + c_i + c| < (i+1)\eta \leq n\eta < \varepsilon$ and $(u_i + c_1 + \dots + c_{i-1}) + c_i + c \in U \setminus F$ as $c_i + c \in I_{i-1}$. Likewise for each j one has $(u_j + c_0 + \dots + c_{i-1}) + c_i + c \in U$ as $c_i + c \in I_{i-1}$.

For $c'_n \in I_n$ the shift $c := c_1 + \dots + c_n + c'_n$ has $|c| < \min\{\varepsilon, d(u_i, X \setminus U)\}$, so $u_i + c \in U$ and $u_i + c \notin F$, as asserted. \square

For the following result, which is inspired by van Mill [74], we use Proposition 2 inductively, using *summable* shifts c_n (i.e. with convergent sum $\sum_n c_n$), to prove the following particularly transparent Euclidean case. The proof uses the Euclidean topology in two matching roles: it defines both the closed nowhere dense sets and the relation of convergence for the sequence $\{x_n\}$.

Theorem 1E. *For the real line under the Euclidean topology, for any convergent sequence x_n with limit x_0 , and any Baire set T which is co-meagre on an open set U , there are a c -shift $h(x) = x + c$ and an infinite set \mathbb{M} such that $h(x_0) \in T$ and*

$$h(x_m) = x_m + c \in T \quad \text{for } m \in \mathbb{M}.$$

Moreover, for S Baire and non-meagre on a non-empty open V with $(S \cap V) + x_0 \subseteq T \cap U$, the c -shift may be chosen with $c \in S$.

Furthermore, \mathbb{M} may always be taken co-finite.

Proof. For the first part, write $T := U \setminus \bigcup_n F_n$, where the sets F_n are closed and nowhere dense. We put $H_n := \bigcup_{m \leq n} F_m$. Choosing $u_0 \in U \cap T$ arbitrarily, put $h_0(x) := x + c_0$ with $c_0 = u_0 - x_0$. Then $h(x_0) = u_0 \in T$. As $u_0 \in U$ we have $u_n := h_0(x_n) \in U$ for all large enough n , so that, dropping a finite number of indices only at this point in the proof, we assume for all n that $u_n \in U$ and $u_0 \notin F_0 = H_0$.

In what follows we construct homeomorphisms η_n which shift an increasing number of the points in the sequence $\{u_n\}$ away from an increasing number of sets in the sequence $\{F_n\}$. By selecting each shift to be a small shift perturbation of the preceding one (e.g. by less than 2^{-n}), we ensure the shifts are summable, and the limiting shifted image of u_n (for each $n \geq 0$) remains outside $X \setminus U$, i.e. is in U .

Put $\eta_0(x) = h_0(x) = x + c_0$; then $\eta_0(x_0) \in U \setminus H_0$ and $\eta_0(x_n) \in U$ for all n .

We make the inductive hypothesis that there is $\eta_n(x) = x + c_0 + \dots + c_{n-1}$ with $|c_i| < 2^{-i}$ for $i < n$ such that:

$\eta_n(x_i) \in U \setminus H_n$ for $i \leq n$ and $\eta_n(x_m) \in U$ for all m .

Put $v_m := \eta_n(x_m)$. Since $v_m \rightarrow v_0 \in U$ and $v_m \in U$ for all m , one has $\min_m d(v_m, X \setminus U) > 0$.

Since $v_i \in U \setminus H_n$ for $i \leq n$, by Proposition 2 applied with U replaced by $U \setminus H_n$ and F by F_{n+1} and $\varepsilon = \min\{2^{-n}, \min_m d(v_m, X \setminus U)\}$, there is c_n such that $v'_i = v_i + c_n$ satisfies $v'_i \in U \setminus H_{n+1}$ for $i \leq n+1$. Moreover for $m > n+1$ we have $v'_m = v_m + c_n \in U$, since $c_n < \min_m d(v_m, X \setminus U)$. Put $\eta_{n+1}(x) = \eta_n(x) + c_n$. Then $\eta_{n+1}(x) = x + c_0 + \dots + c_n$ has $|c_i| < 2^{-i}$ for $i \leq n$ and is such that:

$\eta_{n+1}(x_i) = v_i + c_n \in U \setminus H_n$ for $i \leq n+1$ and $\eta_{n+1}(x_m) = v_m + c_n \in U$ for all m .

This completes the induction.

Put $c = \sum_j c_j$ and consider the c -shift $h(x) = x + c$. Fix i and j . For $n > \max\{i, j\}$, one has

$$h(x_i) = \lim_m [\eta_n(x_i) + c_{n+1} + \dots + c_m].$$

But, for each n , one has $|\sum_{j>n} c_j| < c_n \sum_{j>n} 2^{-j} = c_n/2 < 2^{-n-1}$. So $h(x_i) \notin F_j$ as $d(\eta_n(x_i), F_j) \geq 2^{-n}$. This proves the first assertion.

As for the second assertion, we sketch the proof. It is here that we use the strong separation of Proposition 1.

Let $x_n \rightarrow x_0$. We may suppose, by regularity of the Euclidean topology, that $C = V \setminus \bigcup_n G_n$, where V is a closed interval and the sets G_n are closed and nowhere dense, has the property that $C + x_0 \cap T \neq \emptyset$. So for some $c_0 \in V \cap C$ and $u_0 \in U \cap T$ one has $c_0 + x_0 = u_0$. Apply a shift through $-c_0$ to arrive at $0 \in S = V \setminus \bigcup_n G_n$ as well as $u_0 \in U \cap T$ and $x_n \rightarrow x_0 = u_0$. So taking $h_0(x) = x$ yields $h_0(x_0) = u_0$. W.l.o.g. $u_n := h_0(x_n) \in U$ for all n . By strong separation, there exists an interval $J_1 \subseteq V$ of diameter less than $1/2d(V)$ of values c_1 which is disjoint from the set G_1 above (in the expression for C) such that $u_0 + c_1 \in U \setminus F_1$ where the set F_1 comes from the expression for T (as c_1 is small enough), and $u_1 + c_1 \in U \setminus F_2$ by choice of c_2 . Apply the shift $-c_1$ to arrive at a similar situation as before. Note that $c_0 + c_1 \in V$. Continuing by induction, we obtain a limiting translation $h(x) = x + c$, where $c = \sum_j c_j$ is in C and $h(x_n) \in T$, for all n . Indeed V is closed, and one may arrange as before not only that $h(x_i) \notin F_j$ as $d(\eta_n(x_i), F_j) \geq 2^{-n}$, but also that $h(x_i) \notin G_j$ as $d(\eta_n(x_i), G_j) \geq 2^{-n}$. \square

The entire proof above is based on completeness considerations and triangle inequalities. Using a change of vocabulary, the result above generalizes as follows, by an appeal to the *Inductive Convergence Criterion* for Polish spaces (which goes back to [29] and [2, Lemma 2.1]; see [3, Lemma 5.1], cf. [72, Theorem 6.1.2], and [74]).

Theorem 2 (*Shift-compactness Theorem – Category case*). For X Polish, suppose the subgroup $G \subseteq \mathcal{H}(X)$ separates points from nowhere dense sets and is complete. Then, for any convergent sequence x_n with limit x_0 and any Baire set T which is co-meagre on an open subset $U \subseteq X$, there are $h \in G$ and an infinite set \mathbb{M} such that $h(x_0) \in T$ and

$$h(x_m) \in T \quad \text{for } m \in \mathbb{M}.$$

Furthermore, if separation by G is strong, then for any convergent sequence x_n with limit x_0 , any Baire set T which is co-meagre on an open subset $U \subseteq X$, and any Baire set H non-meagre on an open subset $V \subseteq G$ with $Hx_0 \cap T \neq \emptyset$, there exists h as above in H .

We quote now a lemma from [57], which was inspired by a close reading of [74].

Separation Lemma. If G is a separable normed group, acting transitively on a non-meagre space X , then for any given point x and closed nowhere dense set F the set $W_{x,F} := \{\alpha : \alpha(x) \notin F\}$ is dense and open. In particular, G strongly separates points from closed nowhere dense sets.

As an immediate consequence we obtain a new proof of a result given in [57].

Theorem 2' (*Shift-compactness Theorem – Category case*). For X Polish, and $G \subseteq \mathcal{H}(X)$ a separable and complete subgroup, acting transitively on X , for any convergent sequence x_n with limit x_0 and any Baire set T which is co-meagre on an open subset $U \subseteq X$, there are $h \in G$ and an infinite set \mathbb{M} such that $h(x_0) \in T$ and

$$h(x_m) \in T \quad \text{for } m \in \mathbb{M}.$$

Furthermore, for any convergent sequence x_n with limit x_0 , any Baire set T which is co-meagre on an open subset $U \subseteq X$, and any Baire set H non-meagre on an open subset $V \subseteq G$ with $Hx_0 \cap T \neq \emptyset$, there exists h as above in H .

In the case of the density topology, we need to make the connection with its Euclidean counterpart. To gain an intuition we follow Miller [50] in applying one of Littlewood's Three Principles [66]. Note that if T is bounded, measurable, and of positive measure, then by outer regularity of Lebesgue measure, for each $\varepsilon > 0$, we may choose U open and E_1 and E_2 measurable, with the sum of their measures at most $\varepsilon > 0$, to write

$$T := (U \setminus E_1) \cup E_2.$$

The sets E_i here play the role of the closed nowhere dense sets that are to be avoided. Observe that, if $p \in E = E_1 \cup E_2$ and $|E| < \varepsilon$, then there exists c with $|c| < \varepsilon/2$ such that $p + c \notin E$. Otherwise $B_{\varepsilon/2}(p) \subseteq E$, which implies that $|E| \geq \varepsilon$, a contradiction. If $p = \lim p_n$ and c is selected so that $p + c \notin E$ (possible, provided $|E_i| < \varepsilon/2$, so that $|E_1 \cup E_2| < \varepsilon$), then provided $p + c \in T$, one has $p + c \in U$, and so as U is open, $p_n + c \in U$ for large enough n .

We shall use this quantitative (as opposed to qualitative) measure-theoretic observation inductively. We employ, as before, a sequence of shifts of decreasing size. By working in a compact set T , we ensure that the limiting image points are in T . However, the situation is altered now, in that the open set U is not the same at each stage of the induction, but rather depends on ε . So here we cannot secure $p_n + c \in U$ for all n , even if we arrange that $p_n \in T$ for all n , as at the start of Theorem 1E. This explains why, in the measure case of the Shift-compactness Theorem, one must pass to a subsequence, rather than drop only an initial finite number of terms.

This approach is successful provided we make use of density points. Indeed, without the language of density topology it is difficult to state the theorem in its sharpest form (contained in the final sentence of Theorem 1M).

In the theorems of Section 3 (Theorems 3 and 4) we adapt a proof from [16] to offer two topological approaches to the Baire-category based theorems above which complete a desirable measure-category analogy with Theorems 1E and 2 (see also Theorem 1M below) and thereby unify the category and measure cases; both cases of the Shift-compactness Theorem then have “direct” (“constructive”) proof.

The next result may be viewed as an Effros open mapping theorem for the density topology. The proof relies on the completeness of the Euclidean topology of the line and the property that all small enough shifts $T + s$ of a density-open set T meet T in a density-open set. Recall that 0 is a density point of T if for each $\varepsilon > 0$ there is $\delta > 0$ so that for all symmetric intervals I about 0 of length at most δ one has $|I \cap T| > (1 - \varepsilon)|I|$. Notice that, provided $\varepsilon < 1/4$, for any such interval I and s with $|s| < \varepsilon|I|$, putting $S = (T + s) \cap I$ one has $|S \cap T \cap I| > \varepsilon|I|$. Otherwise, since $|S| \geq |T| - |I|\varepsilon$ and so

$$\begin{aligned} |I| &\geq |T \cup S| = |T| + |S| - |S \cap T| \geq |T| + |T| - \varepsilon|I| - k\varepsilon|I| \\ &\geq 2(1 - \varepsilon)|I| - 2\varepsilon|I| = |I|(2 - 4\varepsilon), \end{aligned}$$

and so $1 \geq 2 - 4\varepsilon$, i.e. $\varepsilon \geq 1/4$, a contradiction.

Theorem 1M. In \mathbb{R} , for any convergent sequence x_n with limit x_0 (in the Euclidean sense) and any non-null measurable set T , there are a c -shift $h(x) = x + c$ and an infinite set \mathbb{M} such that $h(x_0) \in T$ and

$$h(x_m) = x_m + c \in T \quad \text{for } m \in \mathbb{M}.$$

Moreover, for S and T density-open with $S + x_0 \subseteq T$ the shift may be chosen with $c \in S$.

Proof. For the first part, let T be measurable non-null. By inner regularity, we may assume that T is compact and non-null.

Suppose inductively that $\eta_n(x) = x + c_1 + \cdots + c_n$ has been selected with $|c_i| \leq 2^{-i}$ for $i \leq n$, and an increasing sequence $m(j)$ for $j < n$ such that $\eta_n(x_{m(j)})$ is a density point of T .

For each $\varepsilon = 2^{-n}$ we may choose U a finite union of open intervals and E, E' disjoint and measurable with $|E| < \varepsilon$ such that $E \subseteq U$ and $T = E' \cup (U \setminus E)$. Choose open intervals I_j with $u_j \in I_j \subseteq U$. Let $\eta := \min_j \{d(u_j, X \setminus I_j), \varepsilon\}$. Since each u_j is a density point, choose a symmetric interval V round 0 such that for $V_j := u_j + V \subseteq I_j$ one has $u_j \in V_j$ and $|V_j \cap T| \geq (1 - \eta)|V|$ for all $j < n$ and $|V| < \varepsilon$. Choose x_m with $m > m(n)$ such that $|x_0 - x_m| < \eta$. Then $u_{m(n+1)} := \eta(x_m) \in V_0$, as $\eta(x_0) = u_0 \in V_0$ and η is an isometry. Choose an open interval $V_{n+1} \subseteq I_0$.

For $j < n$ one has $|V_j \cap E| < \eta|V|$, as $V_j \setminus E \subseteq U \setminus E \subseteq T$, and so $|V_j \setminus E| > (1 - \eta)|V_j|$. Let F be a measure-zero set such that $(V_j \setminus E) \setminus F$ is a density-open subset of T (all its points are density points) for each $j < n$.

For any c note that $c + u_{m(n+1)}$ is a density point of $T \cap V_0$ iff c is a density point of $T' := T \cap V_0 - u_{m(n+1)}$, but by Lebesgue's Density Theorem off a null subset N of T' all its members are density points. In what follows we ensure that $c \notin N$.

Now choose $c_{n+1} \in V \setminus (N \cup [(E \cup F) - u_j])$ with $c_{n+1} + u_j \in V_j \setminus E_0 \subseteq T$ and $c + u_j$ a density point of T for all $j \leq n + 1$ and $|c_{n+1}| < \varepsilon$.

Setting $\eta_{n+1}(x) = \eta_n(x) + c_{n+1}$ we obtain $\eta_{n+1}(x_j) \in T$ for $j \leq n + 1$.

By compactness of T each of the limit points $\lim_n \eta_n(x_j)$ is in T for each j . Moreover, $s_n := c_1 + \cdots + c_n$ converges, to s say. Then with $\eta(x) = x + s$ we have $\eta(x_j) = \lim_n \eta_n(x_{m(j)}) \in T$ and $\lim_j \eta(x_j) = \eta(x_0) \in T$.

The second assertion follows now quite easily (cf. the comments after Theorem S). Specializing the sequence arising in the proof above to a null sequence $z_n \rightarrow z_0 = 0$ and replacing T by S we obtain $\eta(z_0) = 0 + s \in S$ and $s + z_m \in S$ for an infinite set of m , in M_s say.

Returning to a general sequence x_n with limit x_0 , put $z_n := x_n - x_0$. Then, as before, for some $s \in S$ and some infinite set M_s one has $s + z_m \in S$ for $m \in M_s$. But then $s + x_0 + z_m = s + x_n \in S + x_0 \subseteq T$ for $m \in M_s$, as asserted. \square

Remark. By outer regularity, there are a non-null \mathcal{G}_δ set H and a null set E such that $T = H \setminus E$. Let d^H be a complete metric on H . It seems plausible that one might first arrange for $\eta_n(x_j)$ to be d -Cauchy and so converge to a point v^n in H . Then a further shift would be needed to ensure that $v^n + c \in H \setminus E$.

3. Shift-compactness in cometrizably Polish spaces

In this section we develop a general group-action result, inspired by the measure-category results in Theorems 1E and 1M, which at once embraces both the Euclidean and the density topology cases on the real line.

We recall a definition which refers to the connection between two topologies \mathcal{E} and \mathcal{D} , so the notation below is subscripted according to topological context. We refer below to what we call the *canonical example*, which is the real line with \mathcal{E} the Euclidean and \mathcal{D} the density topology, hence the choice of letters in the abstract setting below.

Definition. (See [47, p. 133] and [70].) For (X, \mathcal{E}) a metrizable topology, a refinement topology $\mathcal{D} \supseteq \mathcal{E}$ is called *cometrizable* if for each $x \in U \in \mathcal{D}$ there is V with $x \in V \in \mathcal{D}$ with

$$V \subseteq \text{cl}_{\mathcal{E}} V \subseteq U.$$

Equivalently: for each non-empty \mathcal{D} -open set U , there is an \mathcal{E} -closed set K with non-empty \mathcal{D} -interior:

$$\emptyset \neq \text{int}_{\mathcal{D}}(K) \subseteq K \subseteq U.$$

Remarks. 1. This property is implied by the Luzin–Menchoff (LM) property (cf. [47, Proposition 4.1, p. 133]).

2. For the canonical example, cometrizability follows from the inner regularity of Lebesgue measure taken together with the Lebesgue Density Theorem. Other examples include the Kunen line, van Douwen's examples of S and L subspaces in $\wp(\omega)$. For more on this, see Gruenhage [37], which studies the relationship between cometrizability and “cosmicity” under the Proper Forcing Axiom (PFA).

Interest in cometrizability dates back to the study of co-topologies by de Groot [35] and Aarts, de Groot and McDowell [1], culminating in the characterization of a metrizable space as topologically complete iff it has a compact co-topology (“is co-compact”).

3. According to [47, Theorem 4.2] if, as below, \mathcal{E} is completely metrizable, and \mathcal{D} is cometrizable (e.g. if \mathcal{D} has the LM property), then any \mathcal{D}_δ subspace of X is Baire under \mathcal{D} ; in particular, \mathcal{D} itself is Baire. One may refer to this as the “Luzin–Menchoff variant of the Baire theorem”.

4. A topology \mathcal{T} has the \mathcal{H} -insertion property [47, p. 39] if for each $A \subseteq X$ there is $H \in \mathcal{H}$ with

$$\text{int}_{\mathcal{T}} A \subseteq H \subseteq \text{cl}_{\mathcal{T}} A.$$

For the proof below of Theorem 3 it is enough for \mathcal{D} to have the $\mathcal{G}_\delta(\mathcal{E})$ -insertion property. We note from [47, p. 66] that \mathcal{T} has the \mathcal{G}_δ -insertion property if there exists an ‘essential radius assignment’ for \mathcal{T} , i.e. a function r assigning to each $x \in U \in \mathcal{T}$ a number $r(x, U) > 0$ in such a way that if $d(x, y) \leq \min\{r(x, U), r(y, V)\}$, then U meets V . (Compare also [48, Corollary 4.1].)

The assumptions in the following theorem are satisfied in the case when G is the additive group of reals for the action $H(g, k) = g + k$, and \mathcal{E} and \mathcal{D} are the Euclidean and density topologies respectively. Below, the group is not required to be Baire, and the conditions placed on the associated action demand a novel ‘unbalanced’ mixture of separate continuity. Condition (i) is stronger than that $x \rightarrow g(x)$ be continuous in the submetrizable topology. On the other hand, condition (ii) is weaker than that $g \rightarrow g(x)$ be continuous. We show below in the next section that (ii) in fact demands that $g \rightarrow g(x)$ be continuous relative to a coarser topology on G , namely one modelled after the lower Vietoris topology (cf. [28, 2.7.20]).

In the canonical example of the density topology, the map $g \rightarrow g(x)$ is not continuous at $g = \text{id}$: if $x \in U$ and U is density-open, there can be arbitrarily small translations taking x out of U . Notwithstanding this, if U is Euclidean open, condition (ii) follows from openness of translations in the Euclidean sense.

Condition (ii) can be weakened further, as we shall see in the next section.

The conditions (i)–(ii) may be viewed as topological variants of the definition of a Miller homotopy due, though not under that name, to H.I. Miller (for which see [19]). The Miller conditions, which involve differentiability in nature, ensure in the real-line case that the maps $x \rightarrow H(x, y)$ and $y \rightarrow H(x, y)$ are Euclidean homeomorphisms that are bi-Lipschitz, so are also density-homeomorphisms. One of his conditions now reads $H(e_G, x) = x$, which we recall is among the defining conditions for an action. In what follows we refer to group-norms as defined in Section 2.

Theorem 3 (Shift-compactness Theorem – Action version). *Let \mathcal{D} be a cometrizable refinement of a Polish topology \mathcal{E} on a space X , and suppose that for some normed group G there exists an action $H : G \times \mathcal{D} \rightarrow \mathcal{D}$ such that:*

- (i) *each map $H_g : x \rightarrow H(g, x)$, also written $g(x)$, is in $\text{Auth}(\mathcal{D}) \cap \text{Auth}(\mathcal{E})$,*
- (ii) *if W is non-empty \mathcal{D} -open, then, for $\|g\|$ small enough, the set $W \cap g(W)$ is non-empty (and \mathcal{D} -open, by (i)).*

Then for $h_n \rightarrow e_G$ in G and W non-empty \mathcal{D} -open there is a subsequence $h_{m(n)}$ with $0 = m(0) < m(1) < \dots$ and an \mathcal{E} -convergent sequence x_n in W with limit x_0 in W such that the sequence of \mathcal{E} -homeomorphisms $\eta_n(g) := g(x_n)$ satisfies

- (a) *$\eta_n(h_{m(i)}) = h_{m(i)}(x_n) \in W$ for $i \leq n$, and*
- (b) *the limit \mathcal{E} -homeomorphism $\eta(g) := H(g, x_0) = g(x_0) = \lim_n \eta_n(g)$ satisfies $\eta(h_{m(n)}) = h_{m(n)}(x_0) \in W$ for all n .*

Proof. Let d^X be a complete metric compatible with (X, \mathcal{E}) . We write $B^X(x, r)$ for the ball $\{y \in X : d(x, y) < r\}$ and $\|g\|$ for the norm in G . Let $h_n \rightarrow e_G$ in the right-norm topology.

For W a non-empty \mathcal{D} -open set, let K be \mathcal{E} -closed with $\emptyset \neq U = \text{int}_{\mathcal{D}} K \subseteq K \subseteq W$.

Let $k_0 \in U$. Put $g_0 = h_0 = e_G$. Select $\eta_0(g) = H(g, k_0) = g(k_0)$; then $\eta_0(h_0) = e(k_0) = k_0$.

We proceed inductively. Suppose that the points $g_i := h_{m(i)}$ have been selected for $i \leq n$ in such a way that $K_m := \bigcap_{i \leq m} g_i^{-1}(K)$ includes as a non-empty intersection the \mathcal{D} -open set $\bigcap_{i \leq m} g_i^{-1}(U)$, and that points $k_i \in K_i$ have been selected so that they are \mathcal{D} -interior points of K_i with $d^X(k_i, k_{i-1}) < 2^{-i}$.

Note that $g_i(k_n) \in K$ for each $i \leq n$, since $k_n \in K_n \subseteq g_i^{-1}(K)$.

To carry through the inductive step, we note that, as K_n has k_n as a \mathcal{D} -interior point, by (ii) there is $\varepsilon_n < 2^{-n}$ such that, for $\|g\| < \varepsilon_n$ in G , the set $K_n \cap g^{-1}(K_n)$ has non-empty \mathcal{D} -interior. We refine this observation. The same is true for the subset $K'_n := K_n \cap B^X(k_n, 2^{-n-1})$, since \mathcal{D} refines \mathcal{E} . So we assume ε_n chosen so that $K'_n \cap g^{-1}(K'_n)$ has non-empty \mathcal{D} -interior for all $g \in G$ with $\|g\| < \varepsilon_n$. Choose $g_{n+1} = h_{m(n+1)}$ with $\|g_{n+1}\| < \varepsilon_n$, and also choose k_{n+1} to be a \mathcal{D} -interior point of $K'_n \cap g_{n+1}^{-1}(K'_n)$.

Putting

$$K_{n+1} = K_n \cap g_{n+1}^{-1}(K_n) = \bigcap_{i \leq n+1} g_i^{-1}(K),$$

we have $k_{n+1} \in K_{n+1}$, since $K'_n \cap g_{n+1}^{-1}(K'_n) \subseteq K_n \cap g_{n+1}^{-1}(K_n) = K_{n+1}$. Also $d^X(k_{n+1}, k_n) < 2^{-n-1}$, as $k_{n+1} \in K'_n$. Now consider

$$\eta_{n+1}(g) := g(k_{n+1}) \in X.$$

Then, by choice of k_{n+1} , one has

$$\eta_{n+1}(g_i) = g_i(k_{n+1}) \in K, \quad \text{as } k_{n+1} \in K_{n+1} \subseteq g_i^{-1}(K).$$

That is, η_{n+1} embeds g_0, g_1, \dots, g_{n+1} into K .

With the induction established, and since k_n is a d^X -Cauchy sequence, we may put $k^* := \lim_n k_n$. Taking limits pointwise we have

$$\eta(g) := \lim_n \eta_n(g) = \lim_n H(g, k_n) = H(g, k^*) = g(k^*),$$

since $x \rightarrow H(g, x)$ is continuous relative to the \mathcal{E} topology, by (i). Evidently the closed sets K_n are nested, so $k^* \in K^* := \bigcap_n K_n$. Furthermore, for $n > m > i$, one has

$$g_i(k_n) \in K_n \subseteq K_m.$$

So again, since for fixed g_i the map $x \rightarrow H(g_i, x)$ is \mathcal{E} -to- \mathcal{E} continuous, one has

$$\eta(g_i) = \lim_n H(g_i, k_n) \in K_m \quad \text{for each } m.$$

So

$$\eta(g_i) = H(g_i, k^*) \in \bigcap_m K_m = K^* \subseteq K.$$

Thus the homeomorphism η establishes our claim. Note that $\eta(g_0) = e(k^*) = k^* \in K$. \square

Remark. The following restatement of Theorem 3 above holds generically in W , by the Generic Dichotomy Theorem of [16]: W contains a point x^* such that the \mathcal{E} -homeomorphism $\eta(g, x^*) = g(x^*)$ embeds the sequence $g_{m(n)}$ as $g_{m(n)}(x^*)$ into W with \mathcal{E} -limit point x^* .

Theorem 3 prompts a reformulation in action terms of a result in [17] and [18]. We recall a definition.

Definition (*Weak category convergence*). A sequence of homeomorphisms h_n of a topological space $\mathfrak{X} = (X, \mathcal{T}_X)$ satisfies the *weak category convergence* condition (wcc) if for any non-meagre open set U there is a non-meagre open set $V \subseteq U$ such that, for each $k \in \omega$,

$$\bigcap_{n \geq k} V \setminus h_n^{-1}(V) \text{ is meagre.} \quad (\text{wcc})$$

Equivalently, for each $k \in \omega$, there is a meagre set M such that, for $t \notin M$,

$$t \in V \implies (\exists n \geq k) h_n(t) \in V.$$

The second formulation permits one to prove ([18, Theorem 2], or [58]) that if the topology \mathcal{T}_X is Baire and submetrizable, i.e. arises as the refinement of a metric topology \mathcal{T}_d (as with the density topology), then for quasi all t (under \mathcal{T}_X) one has (under \mathcal{T}_d) that $h_{m(n)}(t) \rightarrow t$, down a subsequence $m_n = m_n(t)$. (For background on submetrizability see [36].) In Section 4.2 we interpret (wcc) as a topological convergence condition.

Theorem 4 (*Bitopological Shift-compactness Theorem, aka Category Embedding Theorem*). Let \mathcal{T}_X be a submetrizable topology on X , i.e. a refinement topology of some metric topology (X, \mathcal{T}_d) .

For a subgroup $G \subseteq \mathcal{H}(X, \mathcal{T}_d) \cap \text{Auth}(\mathcal{T}_X)$ under the right-norm topology, put $H(g, x) = g(x)$ for $g \in G$ and $x \in X$.

Then the mapping $H_g : x \rightarrow g(x)$ is continuous.

Suppose further that for any $h_n \rightarrow e_G$ in norm, h_n satisfies the (wcc).

Let $T \subseteq X$ be non-meagre and Baire in \mathcal{T}_X .

Then there exists $t \in T$ such that $h_n(t) \in T$ infinitely often.

If \mathcal{T} is the density topology, the set T above may without loss of generality be a density-open set W . We shall show in Section 4.2 that under certain circumstances, which include the case of the real line under the density topology, the (wcc) condition in Theorem 4 is a continuity condition.

4. Topologies weaker than the norm topology

To bring the format of the group-action theorems into better alignment with the standard assumptions of separate continuity, we consider two *weak topologies* on a subgroup G of $\text{Auth}(X, d)$ in the two following subsections. The theorems above may then be viewed as demanding that, for $G \subseteq \mathcal{H}(X, d)$ equipped with the supremum norm, its right-norm topology refines the relevant weak topology and that the evaluation maps $g \rightarrow g(x)$ are *weakly continuous*.

4.1. Upper and lower topologies on G

The two topologies here on function spaces are inspired by the Vietoris upper and lower topology sub-bases (dating back to 1922) in the hyperspace of closed subsets of a space (for which see [28, 2.7.20]). Our approach follows the similar but later (1945) application to function spaces by Fox [30], when he defined the compact-open topology (for which see [28, §3.4]). A generalization is given by Dieudonné [25] to unify the treatment of the uniform, the compact-open and the pointwise topologies.

The more recent term *hit-and-miss* topology embraces generalizations of the two Vietoris topologies, e.g. the Fell topology: see for example [9], or [53,54], and usually (though not here) a passage between hyperspace and function-space topologies is effected by identifying a function with its graph (or epigraph, or hypograph). A version of what we call the lower topology on the homeomorphisms of X is studied in the context of Banach spaces X under the name *Mosco topology* – see particularly [8] or [4], where the miss-sets are weakly compact sets and the hit-sets are strongly open. Other function space and hyperspace topologies have been studied – see e.g. [63]. From our perspective it seems more natural to adopt a ‘capture-or-hit’ terminology.

We denote by $N_Y(y)$ the neighbourhood base at y in whatever regular space Y we consider.

We work below in $G \subseteq \text{Auth}(X, \mathcal{T})$, with \mathcal{T} a regular topology.

The upper (capture) Fox–Mosco topology \mathcal{FM}^+ on G is naturally associated with the notion of upper semicontinuity. We define first $N_G^+(e)$, the upper neighbourhood base at $e = e_G$. If $\text{cl } V \subseteq U$ for $U, V \in \mathcal{T}$ -open, then $e(V) \subseteq U$. So we regard g as close to e if $g(V) \subseteq U$. This yields sub-basic neighbourhoods of e in the form

$$[V, U]^+ := \{g \in G: g(V) \subseteq U\}, \quad \text{for } U, V \in \mathcal{T} \text{ with } \text{cl } V \subseteq U.$$

Regard h as close to g if $h(x)$ is close to $g(x)$ for all x , in some sense. Taking $y = g(x)$, we want $h(g^{-1}(y))$ to be close to y . This motivates our definition of $N_G^+(g)$ as generated by sub-basic sets of the form

$$G^+(g, V, U) := \{h \in G: hg^{-1}(V) \subseteq U\}, \quad \text{for } U, V \in \mathcal{T} \text{ with } \text{cl } V \subseteq U.$$

For later use, note that regarding h close to g when hg^{-1} is close to e is modelled after the right-norm topology.

Remark. As for upper semicontinuity, recall that g is upper semicontinuous if for every U and each x with $g(x) \in U$, there is V with $V \in N_{\mathcal{T}}(x)$ with $g(V) \subseteq U$, i.e. for each U there is $V \subseteq g^{-1}(U)$ with $V \in N(x)$ or, in the notation above, there is $V \in N(x)$ with $g \in [V, U]^+$.

Important Examples. 1. For the translations $g_n(x) = x + z_n$ with $z_n \rightarrow 0$ one has $g_n \rightarrow e$ in the upper topology. Indeed, if $B_\delta(V) \subset U$, pick N so that $|z_n| < \delta$ for $n > N$; then $g_n(V) \subset U$ for $n > N$.

2. If $\text{cl } V \subseteq U$, then for some $\varepsilon > 0$ with $B_\varepsilon(V) \subseteq U$; so if $\|h_n\| < \varepsilon$ for $n > N$, then $h_n(V) \subseteq U$ for $n > N$.

3. For (X, d) locally compact, if $g_n \rightarrow e$ in the compact-open topology and V open is precompact with $\text{cl } V \subset U$, then for some N one has $g_n(\text{cl } V) \subset U$ and so $g_n(V) \subset U$ for $n > N$.

The lower (hit) Fox–Mosco topology \mathcal{FM}^- is naturally associated with the notion of lower semicontinuity. We define $N_G^-(e)$ as the lower neighbourhood base at $e = e_G$. If $V \subseteq U$ are \mathcal{T} -open, then one has $e(V) \cap U \neq \emptyset$. So we regard g as close to e if $g(V) \cap U \neq \emptyset$. This yields sub-basic neighbourhoods of e in the form

$$[V, U]^- := \{g \in G: g(V) \cap U \neq \emptyset\}, \quad \text{for } U, V \in \mathcal{T}.$$

Our earlier considerations motivate our definition of $N_G^-(g)$ as comprising sets of the form:

$$G^-(g, V, U) := \{h \in G: hg^{-1}(V) \cap U \neq \emptyset\}, \quad \text{for } U, V \in \mathcal{T}.$$

Remarks. 1. Recall that g is lower semi-continuous if for every U and each x with $g(x) \in U$, there is V with $V \in N_{\mathcal{T}}(x) \subseteq \mathcal{T}$ with $g(V) \cap U \neq \emptyset$. So for each U there is V with $g \in [V, U]^-$.

2. Verification of condition (ii) of Theorem 3 is typically linked with the continuity of the mapping $f(x) = \mu(Bx)$ for B a Borel subset of a group carrying a translation-invariant measure μ . On this matter see [43], [38, Ch. XII, p. 266], [76], or [17, Theorem 5.5M].

We summarize our interpretation of condition (ii) as

Proposition 3. Condition (ii) of Theorem 3 is equivalent to every hit-open set being open in the sense of the right-norm topology, in symbols $\mathcal{FM}_G^- \subseteq \mathcal{T}_G$, so that \mathcal{FM}_G^- is coarser than the norm topology.

Proof. Suppose that U is non-empty and open and V is non-empty open with closure in U . Then, since $g(V)$ meeting V implies that $g(V)$ also meets U , condition (ii) applied to V yields $\varepsilon = \varepsilon(V)$ such that

$$B^G(e_G, \varepsilon) \subseteq [V, U]^-,$$

as asserted. \square

4.2. Ideal topologies

We identify a third kind of topology which applied to certain groups of homeomorphisms yields the weak category convergence condition (wcc) of Theorem CET as convergence in this topology. For this we need first to introduce an appropriate convergence structure, for which see the recent [32], or [26, p. 26]. We show below that this applies to the particular cases of the group of translations on the real line and the group of homeomorphisms under the compact-open topology.

Definition. Let (X, \mathcal{D}) be a topological space and \mathcal{I} a σ -ideal of subsets of X . (We have in mind X the line with \mathcal{D} either the Euclidean or density topology, and correspondingly $\mathcal{I} = \mathcal{M}$ the meagre sets or $\mathcal{I} = \mathcal{N}$ the null sets.) Say that h_n \mathcal{I} -converges to the identity and write $\{h_n\} \rightrightarrows_{\mathcal{I}} e_G$ if for any open U on X there is a non-empty open $W \subseteq U$ such that for every increasing sequence $\{m(n)\}$ of natural numbers

$$\bigcap_n V \setminus h_{m(n)}^{-1}(V) \in \mathcal{I}.$$

Remarks. 1. Taking in particular for $m(n) = n + k$ one retrieves the old (wcc) condition for $k = 1, 2, \dots$ as part of the new more demanding condition. For the group of translates this condition holds equally well, since $z_{m(n)}$ is a null sequence whenever z_n is a null sequence.

2a. We recall a result from [18, Theorem 2] that if there is a countable family of open \mathcal{D} -set \mathcal{B} that generates a regular coarser topology \mathcal{E} (so that \mathcal{D} is submetrizable), then for \mathcal{D} -quasi all t there is an infinite $\mathbb{N}(t)$ such that $h_n(t) \rightarrow t$ through $\mathbb{N}(t)$ under \mathcal{E} .

2b. Note that $\bigcap_n V \setminus h_{m(n)}^{-1}(V) \in \mathcal{I}$ iff for some $M \in \mathcal{I}$ (dependent on $\{m(n)\}$) one has

$$V \subseteq \bigcup_n h_{m(n)}^{-1}(V) \cup M. \quad (*)$$

3. Note that:

3a. $\{h_{m(n)}\} \rightrightarrows_{\mathcal{I}} e$ holds, by definition,

3b. $\{e_G\} \rightrightarrows_{\mathcal{I}} e_G$ holds, i.e. for $h_n := e_G$ all n $\{h_n\} \rightrightarrows_{\mathcal{I}} e_G$ holds.

Definition. Say that g_n \mathcal{I} -converges to g , and write $\{g_n\} \rightrightarrows_{\mathcal{I}} g$, iff $g_n g^{-1} \rightrightarrows_{\mathcal{I}} e_G$.

Remark. Here again the extension is modelled after convergence in the right-norm topology, where $g_n \rightarrow_R g$ iff $g_n g^{-1} \rightarrow e_G$.

Example. $g_n(x) = a_n + x$, and $g(x) = a + x$ (so that $g^{-1}(y) = y - a$), with $a_n \rightarrow a$. Then $g_n g^{-1}(x) = a_n + (x - a) = z_n + x$, where $z_n = a_n - a \rightarrow 0$. Then $g_n g^{-1} \rightrightarrows_{\mathcal{I}} e_G$.

We now have (cf. Dudley [26, p. 26], on L -convergence), that

1. $\{g\} \rightrightarrows_{\mathcal{I}} g$ (i.e. when $g_n = g$ for all n).

2. If $\{g_n\} \rightrightarrows_{\mathcal{I}} g$, then $\{g_{m(n)}\} \rightrightarrows_{\mathcal{I}} g$.

In the definition below we will need to assume a further property, which we verify in the circumstances given in Proposition 2 below.

3. If $g_n g^{-1} \rightrightarrows_{\mathcal{I}} e_G$ and $g_n h^{-1} \rightrightarrows_{\mathcal{I}} e_G$ with $g_n, g, h \in G$, then $g = h$.

Definition. (See Dudley [26, p. 27].) On the assumption that 3 holds in G , the following defines a Hausdorff topology $\mathcal{T}_{\mathcal{I}}$, to be called the *ideal topology of \mathcal{I}* :

$U \in \mathcal{T}_{\mathcal{I}}$ iff $g \in U$ whenever $\{g_n\} \rightrightarrows_{\mathcal{I}} g$ implies that $g_n \in U$ for all large n .

Thus

$$\mathcal{T}_{\mathcal{I}} := \{U \subseteq G: g \in U \Leftrightarrow (\forall \{g_n\}) \{g_n\} \rightrightarrows_{\mathcal{I}} g \implies \exists N (\forall n > N) g_n \in U\}.$$

Equivalently, $F \subseteq G$ is $\mathcal{T}_{\mathcal{I}}$ -closed iff for each $\{g_n\}$ in F if $\{g_n\} \rightrightarrows_{\mathcal{I}} g$, then $g \in F$.

Proposition 4. Let X be the line under the Euclidean or density topology and suppose G is a group of homeomorphisms of X with a topology finer than the upper Fox–Mosco, i.e. one for which $g_n \rightarrow g$ implies convergence in the upper Fox–Mosco topology.

Then, for $\mathcal{I} = \mathcal{N}$ or $\mathcal{I} = \mathcal{M}$, $g_n g^{-1} \rightrightarrows_{\mathcal{I}} e_G$ and $g_n h^{-1} \rightrightarrows_{\mathcal{I}} e_G$ with $g_n, g, h \in G$, imply $g = h$, and so $\mathcal{T}_{\mathcal{I}}$ is a well-defined Hausdorff topology.

Proof. Suppose otherwise. Then, for some x_0 one has without loss of generality $g^{-1}(x_0) < h^{-1}(x_0)$. Let $\eta := |g^{-1}(x_0) - h^{-1}(x_0)|/2$. Since g, h are \mathcal{E} -homeomorphisms, there is an \mathcal{E} -open set W of length $\delta < \eta/3$ containing x_0 such that the intervals $g^{-1}(W)$ and $h^{-1}(W)$ are at least η apart. Take k so large that $g_n(W) < B_{\delta}^X(W)$ for $n > k$. By the definition

applied twice to $U := g(W)$, there are (in X) open sets $V_g \subseteq W$ and $V_h \subseteq W$ satisfying the condition $(*)$ above. Note that $g^{-1}(V_g) \subseteq g^{-1}(W)$ and $g^{-1}(V_h) \subseteq g^{-1}(W)$, so $g^{-1}(V_h)$ is distant from $h^{-1}(V_h)$ by at least η , as the latter lies in $h^{-1}(W)$. Thus, taking $m(n) = n + k$ one has

$$g^{-1}(V_g) \subseteq \bigcup_n g_{m(n)}^{-1}(V_g) \cup g^{-1}(M_{g,k}) \subseteq \bigcup_{n>k} g_n^{-1}(W) \cup g^{-1}(M_{g,k}).$$

Put $N_{g,k} := g^{-1}(M_{g,k})$; then $N_{g,k} \in \mathcal{I}$, since g is both a Euclidean and a density-homeomorphism. Thus

$$g^{-1}(V_g) \setminus N_{g,k} \subseteq \bigcup_{n>k} g_n^{-1}(W).$$

Likewise

$$h^{-1}(V_h) \subseteq \bigcup_n g_{m(n)}^{-1}(V_h) \cup h^{-1}(M_{h,k}) \subseteq \bigcup_{n>k} g_n^{-1}(W) \cup h^{-1}(M_{h,k}),$$

and

$$h^{-1}(V_h) \setminus N_{h,k} \subseteq \bigcup_{n>k} g_n^{-1}(W).$$

Since $g_n^{-1}(W) \subseteq B_\delta(W)$, we have, modulo \mathcal{I} , both $h^{-1}(V_h) \subseteq B_\delta(W)$ and $g^{-1}(V_g) \subseteq B_\delta(W)$, i.e. modulo \mathcal{I} they both lie in an interval of length $3\delta < \eta$, so cannot be distant η apart, a contradiction. \square

Theorem 5. For the group of translations of the real line under the supremum norm defined in Eq. (sup) and with $\mathcal{I} = \mathcal{N}$ or $\mathcal{I} = \mathcal{M}$, the corresponding topology $\mathcal{T}_{\mathcal{I}}$ on G is coarser than the right-norm topology.

Proof. By the First and Second Verification Theorem [17] the (wcc) holds for the group of translations on the line with the right-norm topology. That is, if $g_n \rightarrow g$ in norm then also $g_n \rightrightarrows g$. By Proposition 4 the latter convergence is equivalent to convergence under the topology $\mathcal{T}_{\mathcal{I}}$. \square

Remark. The inclusion $(*)$ may be interpreted as an almost inclusion, for which sets in \mathcal{I} are neglected; see [55] for an investigation of ideal-neglecting topologies (also compare this with the ‘ \mathcal{I} -essential topology’ studied in [16], and the ideal topologies of [47]). While we do not pursue this here, we note that other modes of convergence could be studied in a similar fashion (cf. Wilczyński [75]). However, not all convergence structures are topological; for example, almost sure convergence on $[0, 1]$ with Lebesgue measure is not topological – see [26, §9.2, Pb. 2]. (Compare the observations on this point in [4].)

5. Subsequence omissions

D. Borwein and S.Z. Ditor [20] have proved the following theorem.

Theorem B and D. 1) If A is a measurable set of the real numbers with $m(A) > 0$ and (d_n) is a sequence converging to zero, then for almost all $x \in A$, $x + d_n \in A$ for infinitely many n .

2) There is a measurable set A , with $m(A) > 0$ and a monotonic sequence (d_n) of positive reals converging to zero such that, for each x , $x + d_n \notin A$ for infinitely many n .

In the previous paragraphs we have proved a series of results related to 1). In this paragraph we prove several results related to 2). In [20], by a clever construction, the authors have obtained a pair A and (d_n) satisfying 2). The set A , in their paper, is in fact a closed nowhere dense subset of $[0, 1]$. Our first and main result in this paragraph shows that for any A , $A \subseteq [0, 1]$, A closed and nowhere dense, there exists a decreasing null sequence (d_n) such that A and (d_n) satisfy 2). Namely we have the following.

Theorem 6. Given $A \subseteq [1, 0]$ closed and nowhere dense, then there exists a monotonic sequence (d_n) , converging to zero such that for every $x \in \mathbb{R}$, $x + d_n \notin A$ for infinitely many n .

Proof. For each n , divide $[0, 1]$ into 2^n abutting subintervals, each of length $1/2^n$. There is an ε_n , $0 < \varepsilon_n < \frac{1}{2^n}$ such that each interval $[\frac{1}{2^n}, \frac{2}{2^n}]$, $[\frac{2}{2^n}, \frac{3}{2^n}]$, \dots , $[\frac{2^n-1}{2^n}, 1]$ contains, respectively, an open subinterval J_k , $k = 1, 2, \dots, 2^n - 1$, of length greater than ε_n such that $J_k \cap A = \emptyset$. Now, consider the set $\{\varepsilon_n, 2\varepsilon_n, \dots, m(n)\varepsilon_n\} := F_n$ where $m(n)$ is the smallest integer such that $m(n)\varepsilon_n > \frac{2}{2^n}$. Notice that this implies that for each $x \in [0, \frac{2^n-1}{2^n}]$, there is an element $y(x) \in F_n$ such that $x + y(x) \notin A$. Write $\bigcup_{n=1}^{\infty} F_n$ as a nonincreasing sequence (d_n) . Notice that $m(n)\varepsilon_n < \frac{3}{2^n}$ and hence $\lim_{n \rightarrow \infty} d_n = 0$. Clearly, if $x \in [0, 1)$, $x + d \notin A$ infinitely often. Also

$1 + d_n \notin A$ for all n and
 $y + d_n \notin A$ infinitely often if $y \notin [0, 1]$.

Hence, for each $x \in \mathbb{R}$, $x + d_n \notin A$ infinitely often. \square

Remark. If $A \subseteq [0, 1]$ is closed and not nowhere dense, then it is clear that there is no null sequence (d_n) such that A and (d_n) satisfy 2).

For non-measurable sets we have the following.

Theorem 7. *There exists a non-measurable set A such that*

$$x + \frac{1}{n} \notin A, \quad \text{for each } x \in A \text{ and for each } n \in \mathbb{N}.$$

Proof. First observe that if T is measurable, then either T or $\mathbb{R} \setminus T$ contains a closed set of positive measure. Write the collection of all closed sets of positive measure as $\{F_\alpha\}_{\alpha < \omega}$, where ω is the first uncountable ordinal (if we assume the CH). We construct, using transfinite induction, two transfinite sequences of reals

$\{x_\alpha\}_{\alpha < \omega}$, $\{y_\alpha\}_{\alpha < \omega}$ satisfying:

- (a) $x_\alpha, y_\alpha \in F_\alpha$ for each $\alpha < \omega$.
- (b) The two sequences contain no common element.
- (c) $x_\alpha + \frac{1}{n} \neq x_\beta$ for every $\alpha, \beta < \omega$ and for every $n \in \mathbb{N}$.

If we can do this then take $A = \{x_\alpha : \alpha < \omega\}$ and we are done.

To construct $\{x_\alpha\}_{\alpha < \omega}$ and $\{y_\alpha\}_{\alpha < \omega}$, first let $x_1, y_1 \in F_1$, $x_1 \neq y_1$ be arbitrary. Suppose $\beta < \omega$ and we have $\{x_\alpha : \alpha < \beta\}$ and $\{y_\alpha : \alpha < \beta\}$ disjoint from each other and such that $x_\alpha, y_\alpha \in F_\alpha$, for every $\alpha < \beta$, and $x_\alpha + \frac{1}{n} \neq x_{\alpha'}$, for every $\alpha, \alpha' < \beta$ and for every $n \in \mathbb{N}$.

Next we pick $x_\beta, y_\beta \in F_\beta$ such that $\{x_\alpha : \alpha \leq \beta\}$ and $\{y_\alpha : \alpha \leq \beta\}$ are disjoint and $x_\alpha + \frac{1}{n} \neq x_{\alpha'}$ for every $\alpha, \alpha' \leq \beta$ and for every $n \in \mathbb{N}$. This is certainly possible since F_β has cardinality of the continuum and the cardinal of β is at most countable. Therefore by transfinite induction we have two “full” transfinite sequences satisfying (a), (b) and (c). \square

Remark. The above argument goes through without CH, using ω as the smallest ordinal having the same cardinal as the continuum.

In [50] a generalization of the Borwein–Ditor result is presented using a general, “nice”, 2-place function $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ instead of the binary operation $(+)$, used in the Borwein–Ditor paper). Also see [51] for n -dimensional analogues. Our first two results, in this paragraph, again considered the binary operation $f(x, y) = x + y$. We now present a type 2), i.e. a negative result, for a “wild” 2-place function f . The function f is built using a 1-place function $T : (0, 1] \rightarrow (0, 1]$ defined by

$$T(x) = \begin{cases} 2x, & 0 < x \leq \frac{1}{2}, \\ 2x - 1, & \frac{1}{2} < x \leq 1. \end{cases}$$

Theorem 8. *Let T^n denote the n th iterate of T . Suppose that (d_n) is a strictly monotonic converging to zero. Suppose further that*

$$f(x, y) = \begin{cases} T^n(x), & \text{if } y = d_n, n = 1, 2, \dots, \\ g(x, y), & \text{if } y \neq d_n, n = 1, 2, \dots \end{cases}$$

where $g : (0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is an arbitrary function. Then for almost all $x \in (0, 1]$

$f(x, d_n) \notin A$ for infinitely many $n \in \mathbb{N}$ and
 $f(x, d_n) \in A$ for infinitely many $n \in \mathbb{N}$

if A is any measurable subset of $(0, 1]$ satisfying $0 < m(A) < 1$.

Proof. T preserves Lebesgue measure on $(0, 1]$ and is ergodic. Then the ergodic theorem [10] implies $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n I_A(T^{k-1}(x)) = m(A)$ for almost all $x \in (0, 1]$, where I_A is the indicator function of A . But, since $0 < m(A) < 1$, $f(x, d_n) \notin A$ infinitely often and $f(x, d_n) \in A$ infinitely often and this is true for almost all $x \in (0, 1]$. \square

If A is measurable and $m(A) > 0$, then by a well-known result of Steinhaus ([11, Theorem 1.1.1], or [6]): $A - A = \{a - a' \mid a, a' \in A\}$ contains an interval. Also, if F is any finite set of reals then A contains a set \tilde{F} that is similar to F (i.e. there is a linear transformation $(ax + b)$ that maps \tilde{F} onto F). These two properties say that A , besides being big in measure (greater than 0), are also big in two other senses. We now return to the binary operation plus and construct (by transfinite induction) a set A that is “very small” in the sense of measure, yet large in the two senses mentioned above and A together with the sequence $(\frac{1}{n})$ more or less satisfies 2) in the Borwein–Ditor theorem.

In our construction we use a method presented in [64, pp. 73–76]. We need a few preliminaries before we proceed. Let

$$\mathcal{B} := \{A \mid A - A = \mathbb{R}\},$$

$$\mathcal{C} := \{A \mid \text{for each finite set } F, \text{ there exists } \tilde{F} \subseteq A \text{ such that } \tilde{F} \text{ is similar to } F\},$$

$$\mathcal{R} := \{R \mid R \text{ is open and } R \text{ contains the rationals}\}.$$

\mathcal{R} has cardinality of the continuum and hence \mathcal{R} can be written as $\mathcal{R} = \{R_\alpha\}_{\alpha < \omega}$ where ω is the least ordinal having the cardinal of \mathbb{R} .

In the next theorem we assume the following holds.

Assumption. The union $\bigcup_{\alpha < \beta} F_\alpha$ is of the first Baire category (and measure zero) if $\beta < \omega$ and F_α is of the first category (and measure zero) for each $\alpha < \beta$.

Note: CH implies Martin’s axiom which in turn implies our assumption [67,31].

Theorem 9. Under our assumption, there exists a set A satisfying:

- a) $A \in \mathcal{B}$,
- b) $A \in \mathcal{C}$,
- c) for each $x \in A$, $x + \frac{1}{n} \notin A$ for all n with at most one exception,
- d) $A \setminus R$ is countable for each $R \in \mathcal{R}$ if CH holds.

Proof. First select $(y_n), (x_n)$ such that all of the terms of these two sequences are larger than 2 and such that

$$y_n - x_n = \frac{1}{n},$$

$$x_{n_1} - x_{n_2} \neq \pm \frac{1}{n},$$

$$y_{n_1} - y_{n_2} \neq \pm \frac{1}{n},$$

$$y_{n_1} - x_{n_2} \neq \pm \frac{1}{n}$$

for all n in \mathbb{N} if $n_1 \neq n_2$.

Set $A_0 = \{x_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\}$. Write the reals less the set $\{+\frac{1}{n} : n \in \mathbb{N}\} \cup \{-\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ as $\{r_\alpha\}_{\alpha < \omega}$.

We now proceed, by transfinite induction, to construct a set A , satisfying a), b), c) and d). We denote all finite subsets by $\{\tilde{F}_\alpha\}_{\alpha < \omega}$.

Step 1. Consider r_1 and $F_1 = \{f_{11}, f_{12}, \dots, f_{1n(1)}\}$. We can assume $0 < f_{11} < f_{12} < \dots < f_{1n(1)}$. An easy argument shows that there is a $\delta_1 > 0$ such that the set $\{a \mid a > 0 \text{ such that } af_{1i} \in R_1 \text{ for all } i = 1, 2, \dots, n(1)\}$ contains $(0, \delta_1)$ less a meager (first category) set call it T_{δ_1} , and $a_0 + \frac{1}{n} \notin \{af_{11}, af_{12}, \dots, af_{1n(1)}\}$ for each $a \in (0, \delta_1)$ and for each $a_0 \in A_0$ and $af_{1i} + \frac{1}{n} \notin \{af_{11}, af_{12}, \dots, af_{1n(1)}\}$ for every $n \in \mathbb{N}$ and for every $a \in T_{\delta_1}$.

Let $\{\tilde{f}_{11}, \tilde{f}_{12}, \dots, \tilde{f}_{1n(1)}\} = \tilde{F}_1$ denote the set $\{af_{11}, af_{12}, \dots, af_{1n(1)}\}$ for a fixed “ a ” in T_{δ_1} . We now select $u_1, v_1 \in R_1$ such that $v_1 - u_1 = r_1$ and such that

$$A_1 = A_0 \cup \{u_1, v_1\} \cup \tilde{F}_1$$

satisfies $w + \frac{1}{n} \notin A_1$ for all $w \in A_1$ and all $n \in \mathbb{N}$, except for when $w = x_n$ in which case $x_n + \frac{1}{n} \in A_1$ and $x_n + \frac{1}{m} \notin A_1$ if $m \neq n$.

It is possible to find a pair (u_1, v_1) with the required conditions since $\{\sigma \mid \sigma, \sigma + r_1 \in R_1\}$ is all of the reals, less a meager set.

Step 2. The inductive step. For each $\sigma, \sigma < \omega$, let $C_\sigma = \bigcap_{\gamma \leq \sigma} R_\gamma$. Notice by our assumption that C_σ is \mathbb{R} less a meager set. Suppose now that $\beta < \omega$ and that for every $\alpha < \beta$ we have $u_\alpha, v_\alpha \in C_\alpha$ and $\bar{f}_{\alpha 1}, \dots, \bar{f}_{\alpha n(\alpha)} \in C_\alpha$ and such that $v_\alpha - u_\alpha = r_\alpha$ and $\{\bar{f}_{\alpha 1}, \dots, \bar{f}_{\alpha n(\alpha)}\} = \bar{F}_\alpha$ is similar to F_α and that the following holds

$$w + \frac{1}{n} \notin A_\alpha \quad \text{for all } w \in A_\alpha \text{ and all } n \in \mathbb{N},$$

except for when $w = x_n$ in which case

$$\begin{aligned} x_n + \frac{1}{n} &\in A_\alpha \quad \text{and} \\ x_n + \frac{1}{m} &\notin A_\alpha \quad \text{if } m \neq n \end{aligned}$$

where

$$A_\alpha = A_0 \cup \bigcup_{\gamma \leq \alpha} \{u_\gamma, v_\gamma\} \cup \bigcup_{\gamma \leq \alpha} \{\bar{f}_{\gamma 1}, \dots, \bar{f}_{\gamma n(\gamma)}\}.$$

Consider $F_\beta = \{f_{\beta 1}, \dots, f_{\beta n(\beta)}\}$. We may assume the numbers in F_β are positive and increasing. Now, there is a $\delta_\beta > 0$ such that $\{a \mid a > 0 \text{ such that } af_{\beta i} \in C_\beta \text{ for all } i = 1, 2, \dots, n(\beta)\}$ containing $(0, \delta_\beta)$ less a meager set, call it T_{δ_β} , and such that

$$w + \frac{1}{n} \notin B_\beta(a) \quad \text{for all } w \in B_\beta(a) \text{ and all } n \in \mathbb{N}$$

except for when $w = x_n$ in which case

$$\begin{aligned} x_n + \frac{1}{n} &\in B_\beta(a) \quad \text{and} \\ x_n + \frac{1}{m} &\notin B_\beta(a) \quad \text{if } m \neq n, \end{aligned}$$

for all $a \in T_{\delta_\beta}$ where

$$B_\beta(a) = A_0 \cup \bigcup_{\alpha < \beta} (u_\alpha v_\alpha) \cup \bigcup_{\alpha < \beta} \bar{F}_\alpha \cup (af_{\beta 1}, af_{\beta 2}, \dots, af_{\beta n(\beta)}).$$

This is true since, by our assumption $A_0 \cup \bigcup_{\alpha < \beta} (u_\alpha v_\alpha) \cup \bigcup_{\alpha < \beta} \bar{F}_\alpha$ is of the first category and our inductive hypothesis. Let $\{f_{\beta 1}, f_{\beta 2}, \dots, f_{\beta n(\beta)}\} = \bar{F}_\beta$ denote the set $\{af_{\beta 1}, af_{\beta 2}, \dots, af_{\beta n(\beta)}\}$ for a fixed “ a ” $\in T_{\delta_\beta}$. We now select $u_\beta, v_\beta \in \bigcap_{\alpha \leq \beta} R_\alpha$ such that $v_\beta - u_\beta = r_\beta$ and such that

$$A_\beta = A_0 \cup \bigcup_{\alpha \leq \beta} (u_\alpha v_\alpha) \cup \bigcup_{\alpha \leq \beta} \bar{F}_\alpha$$

satisfies $w + \frac{1}{n} \notin A_\beta$ for all $w \in A_\beta$ and all $n \in \mathbb{N}$, except for when $w = x_n$ in which case $x_n + \frac{1}{n} \in A_\beta$ and $x_n + \frac{1}{m} \notin A_\beta$ if $m \neq n$.

Again, we can find such a pair u_β, v_β with the properties above, since $\{\delta \mid \delta, \delta + r_\beta \in C_\beta\}$ is \mathbb{R} less a meager set and by our assumption, $A_0 \cup \bigcup_{\alpha < \beta} (u_\alpha v_\alpha) \cup \bigcup_{\alpha < \beta} \bar{F}_\alpha$ is of the first category.

Finally, the set

$$A = A_0 \cup \bigcup_{\beta < \omega} (u_\beta, v_\beta) \cup \bigcup_{\beta < \omega} \bar{F}_\beta$$

satisfies a), b), c) and

$$(A \setminus A_0) \setminus R_\beta \subseteq \bigcup_{\alpha < \beta} (u_\alpha v_\alpha) \cup \bigcup_{\alpha < \beta} \bar{F}_\alpha.$$

A_0 is countable, so if we assume CH we have that $A \setminus R_\beta$ is countable for each $\beta < \omega$, and d) holds. \square

Remark. Sets A satisfying d) are said to be concentrated on the rationals. This implies (see Theorem 39, p. 77 of [64]), that the “measure” of A is zero for many different measures and in some sense we might call such set a “universal null (in measure) set”.

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