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On decay and blow-up of solutions for a system of nonlinear wave equations

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ABSTRACT

The initial boundary value problem for a system of viscoelastic wave equations of Kirchhoff type with the nonlinear damping and the source terms in a bounded domain is considered. We prove that, under suitable conditions on the nonlinearity of the damping and the source terms and certain initial data in the stable set and for a wider class of relaxation functions, the decay estimates of the energy function is exponential or polynomial depending on the exponents of the damping terms in both equations by using Nakao's method. Conversely, for certain initial data in the unstable set, we obtain the blow-up of solutions in finite time when the initial energy is nonnegative. This improves earlier results in the literature.

1. Introduction

We consider the initial boundary value problem for the following nonlinear wave equations of Kirchhoff type:

$$u_{tt} - M\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)\Delta u + \int_0^t g(t-s)\Delta u(s)ds + |u_t|^{p-1}u_t = f_1(u,v) \quad \text{in } \Omega \times [0,\infty), \tag{1.1}$$

$$v_{tt} - M\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)\Delta v + \int_0^t h(t-s)\Delta u(s)ds + |v_t|^{q-1}v_t = f_2(u,v) \quad \text{in } \Omega \times [0,\infty), \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.3}$$

$$v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \quad x \in \Omega, \tag{1.4}$$

$$u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \tag{1.5}$$

where Ω is a bounded domain in R^n $(n = 1, 2, 3)$ with a smooth boundary $\partial\Omega$, $M(r)$ is a nonnegative C^1 function like $M(s) = m_0 + \alpha s^\gamma$, with $m_0 \geq 0, \alpha \geq 0,\ m_0 + \alpha > 0,\ \gamma > 0$, and $g, h : R^+ \to R^+,\ f_i(\cdot,\cdot) : R^2 \to R,\ i = 1, 2$, are given functions which will be specified later.

To motivate our work, let us recall some results regarding viscoelastic wave equations of Kirchhoff type. The single wave equation of Kirchhoff type of the form:

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + \int_0^t g(t-s)\Delta u(s)ds + h(u_t) = f(u) \quad \text{in } \Omega \times [0,\infty), \tag{1.6}$$

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is the model to describe the motion of deformable solids as hereditary effect is incorporated. Eq. (1.6) was first studied by Torrejon and Yong [1] who proved the existence of weakly asymptotic stable solution for large analytical datum. Later, Rivera [2] showed the existence of global solutions for small datum and the total energy decays to zero exponentially under some restrictions. Then, Wu and Tsai [3] studied (1.6) for $h(u_t) = -\Delta u_t$ and f is a power like function. The authors established the global existence and energy decay under the assumption $g'(t) \leq -rg(t),\ \forall t \geq 0$ for some $r > 0$. Recently, this decay estimate of the energy function was improved by Wu in [4] under a weaker condition on g i.e. $g'(t) \leq 0,\ \forall t \geq 0$.

When $g \equiv 0$, problem (1.6) reduces to the following form:

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + h(u_t) = f(u) \quad \text{in } \Omega \times [0, \infty). \tag{1.7}$$

For the case that $M \equiv 1$, it is a nonlinear wave equation which has been extensively studied and several results concerning existence and nonexistence have been established [5–9]. When M is not a constant function, Eq. (1.7) without damping and the source terms is often called the Kirchhoff type equation; it was first introduced by Kirchhoff [10] in order to describe the nonlinear vibrations of an elastic string. In this regard, there are numerous results related to global existence, decay result and blowup properties, we refer the reader to references [11–15].

Concerning the system of wave equations, many authors considered the initial boundary value problem as follows:

$$\begin{aligned} &u_{tt} - \Delta u + \int_0^t g_1(t-s)\Delta u(s)ds + h_1(u_t) = f_1(u, v) \quad \text{in } \Omega \times [0, \infty), \\ &v_{tt} - \Delta v + \int_0^t g_2(t-s)\Delta u(s)ds + h_2(v_t) = f_2(u, v) \quad \text{in } \Omega \times [0, \infty), \end{aligned} \tag{1.8}$$

where Ω is a bounded domain in R^n $(n = 1, 2, 3)$ with a smooth boundary $\partial\Omega$. When the viscoelastic terms g_i $(i = 1, 2)$ are absent in (1.8), Agre and Rammaha [16] obtained several results related to local existence and global existence of a weak solution. By using the same technique as in [5], they showed that any weak solution blow up in finite with negative initial energy. Later, Said-Houari [17] extended this blow up result to positive initial energy. Conversely, in the presence of the memory term ($g_i \neq 0,\ i = 1, 2$), there are numerous results related to the asymptotic behavior and blow up of solutions of viscoelastic systems. For example, Liang and Gao [18] studied problem (1.8) with $h_1(u_t) = -\Delta u_t$ and $h_2(v_t) = -\Delta v_t$. They obtained that, under suitable conditions on the functions $g_i,\ f_i,\ i = 1, 2$, and certain initial data in the stable set, the decay rate of the energy function is exponential. On the contrary, for certain initial data in the unstable set, there are solutions with positive initial energy that blow up in finite time. For $h_1(u_t) = |u_t|^{m-1} u_t$ and $h_2(v_t) = |v_t|^{r-1} v_t$, Han and Wang [19] established several results related to local existence, global existence and finite time blow-up (the initial energy $E(0) < 0$). This latter blow-up result has been improved by Messaoudi and Said-Houari [20] by considering a larger class of initial data for which the initial energy can take positive values. On the other hand, Messaoudi and Tatar [21] considered the following problem:

$$\begin{aligned} &u_{tt} - \Delta u + \int_0^t g_1(t-s)\Delta u(s)ds + f_1(u, v) = 0 \quad \text{in } \Omega \times [0, \infty), \\ &v_{tt} - \Delta v + \int_0^t g_2(t-s)\Delta u(s)ds + f_2(u, v) = 0 \quad \text{in } \Omega \times [0, \infty), \end{aligned} \tag{1.9}$$

where the functions f_1 and f_2 satisfy the following assumptions

$$|f_1(u, v)| \leq d\left(|u|^{\beta_1} + |v|^{\beta_2}\right),$$
$$|f_2(u, v)| \leq d\left(|u|^{\beta_3} + |v|^{\beta_4}\right),$$

for some constant $d > 0$ and $\beta_i \geq 1,\ \beta_i \leq \frac{n}{n-2},\ i = 1, 2, 3, 4$. They obtained that the solution goes to zero with an exponential or polynomial rate, depending on the decay rate of the relaxation functions $g_i,\ i = 1, 2$. Their result improves the one in [22] to weaker conditions on the relaxation functions g_1 and g_2 and more general coupling functions f_1 and f_2.

Motivated by previous works, it is interesting to investigate the global existence, decay and blow-up of solutions to problem (1.1)–(1.5). Firstly, we show that, under suitable conditions on the function g, h and $f_i, i = 1, 2$, and certain initial data in the stable set, the solutions are global in time. After that, we establish the rate of decay of solutions by a difference inequality given by Nakao [23]. Precisely, we show that the decay rate of energy function is exponential or polynomial depending on the parameters p and q. Secondly, we intend to study the blow up phenomena of problem (1.1)–(1.5). By adopting and modifying the methods used in [20], we prove the blow-up of solutions when the energy is negative or nonnegative at less than the critical value E_1 (given in (4.6)). In this way, our results allow a bigger region for the blow up results and improve the result of Messaoudi and Said-Houari [20], who considered problem (1.1)–(1.5) for the case of $M \equiv 1$. Additionally, to the best of our knowledge, the global existence and asymptotic behavior for systems of viscoelastic wave equations of Kirchhoff type have not been well studied. In [24], Said-Houari considered (1.1)–(1.5) with $M \equiv 1$ and without imposing the memory terms ($g = h = 0$). They proved that the rate of decay of the energy is exponential or polynomial depending on exponents of the damping terms in both equations. In this regard, our decay result extends the one in [24] to our problem, where we consider M is not a cons tan t function and we have more dissipations. We also improve the decay result of [13] to weaker conditions on the relaxation functions.

The paper is organized as follows. In Section 2, we present the preliminaries and some lemmas. In Section 3, the global existence and decay property are derived. Finally, the blow-up results of (1.1)–(1.5) are obtained in the case of the initial energy being nonnegative.

2. Preliminaries

In this section, we shall give some lemmas and assumptions which will be used throughout this work. We use the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H_0^1(\Omega)$ with their usual products and norms. We will use the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ for $2 \le p \le \frac{2n}{n-2}$, if $n \ge 3$ or $2 \le p$, if $n = 1, 2$. In this case, the embedding constant is denoted by c_*, i.e.

$$\|u\|_p \le c_* \|\nabla u\|_2 . \tag{2.1}$$

Next, we give the assumptions for problem (1.1)–(1.5).

(**A1**) $M(s)$ is a nonnegative C^1 function for $s \ge 0$ satisfying

$$M(s) = m_0 + \alpha s^\gamma, \quad m_0 > 0, \alpha \ge 0 \text{ and } \gamma > 0.$$

(**A2**) The relaxation functions g and h are of class C^1 and satisfy, for $s \ge 0$,

$$g(s) \ge 0, \quad m_0 - \int_0^\infty g(s)ds = l > 0,$$

$$h(s) \ge 0, \quad m_0 - \int_0^\infty h(s)ds = k > 0,$$

and

$$g'(s) \le 0, \qquad h'(s) \le 0.$$

Concerning the functions $f_1(u, v)$ and $f_2(u, v)$, we take (see [20])

$$f_1(u, v) = (m + 1) \left(a |u + v|^{m-1} (u + v) + b |u|^{\frac{m-3}{2}} |v|^{\frac{m+1}{2}} u \right), \tag{2.2}$$

$$f_2(u, v) = (m + 1) \left(a |u + v|^{m-1} (u + v) + b |v|^{\frac{m-3}{2}} |u|^{\frac{m+1}{2}} v \right), \tag{2.3}$$

with $a, b > 0$. One can easily verify that

$$uf_1(u, v) + vf_2(u, v) = (m + 1)F(u, v), \quad \forall (u, v) \in R^2,$$

where

$$F(u, v) = a |u + v|^{m+1} + 2b |uv|^{\frac{m+1}{2}} .$$

(**A3**) For the nonlinearity, we suppose that

$$m > 1, \quad \text{if } n = 1, 2 \text{ or } 1 < m \le 3, \ \text{if } n = 3. \tag{2.4}$$

and

$$p, q \ge 1, \quad \text{if } n = 1, 2 \text{ or } 1 \le p, q \le 5, \ \text{if } n = 3. \tag{2.5}$$

As in [20], we still have the following results.

Lemma 2.1. *There exist two positive constants c_0 and c_1 such that*

$$c_0 \left(|u|^{m+1} + |v|^{m+1} \right) \le F(u, v) \le c_1 \left(|u|^{m+1} + |v|^{m+1} \right), \quad \forall (u, v) \in R^2.$$

Lemma 2.2. *Suppose that (2.4) holds. Then there exists $\eta > 0$ such that for any $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$, we have*

$$\|u + v\|_{m+1}^{m+1} + 2 \|uv\|_{\frac{m+1}{2}}^{\frac{m+1}{2}} \le \eta \left(l \|\nabla u\|_2^2 + k \|\nabla v\|_2^2 \right)^{\frac{m+1}{2}} .$$

We also need the following technical lemma in the course of the investigation.

Lemma 2.3 ([20]). *For any $g \in C^1$ and $\phi \in H^1(0, T)$, we have*

$$-2 \int_0^t \int_\Omega g(t - s)\phi\phi_t dxds = \frac{d}{dt} \left((g \diamond \phi)(t) - \int_0^t g(s)ds \|\phi\|_2^2 \right) + g(t) \|\phi\|_2^2 - \left(g' \diamond \phi \right)(t),$$

where

$$(g \diamond \phi)(t) = \int_0^t g(t-s) \int_\Omega |\phi(s) - \phi(t)|^2 \, dxds.$$

Now, we are in a position to state the local existence result to problem (1.1)–(1.5), which can be established by combining arguments of [13,15,16,19].

Theorem 2.4. *Let $u_0, v_0 \in H_0^1 \cap H^2(\Omega)$ and $u_1, v_1 \in H_0^1(\Omega)$ be given. Assume that* (A1)–(A3) *are satisfied, then there exists a couple solution (u, v) of problem* (1.1)–(1.5) *such that*

$$u, v \in C([0,T], H^2(\Omega) \cap H_0^1(\Omega)),$$
$$u_t \in C\left([0,T], H_0^1(\Omega)\right) \cap L^{p+1}(\Omega), \qquad v_t \in C\left([0,T], H_0^1(\Omega)\right) \cap L^{q+1}(\Omega),$$

for some $T > 0$.

We conclude this section by stating Nakao's Lemma, which will be used in establishing the decay rate of solutions to problem (1.1)–(1.5).

Lemma 2.5 ([23]). *Let $\phi(t)$ be a nonincreasing and nonnegative function on $[0, T]$, $T > 1$, such that*

$$\phi(t)^{1+r} \le \omega_0 \left(\phi(t) - \phi(t+1)\right) \quad \text{on } [0, T],$$

where $\omega_0 > 1$ and $r \ge 0$. Then we have, for all $t \in [0, T]$

(i) *if $r = 0$, then*

$$\phi(t) \le \phi(0) e^{-\omega_1 [t-1]^+};$$

(ii) *if $r > 0$, then*

$$\phi(t) \le \left(\phi(0)^{-r} + \omega_0^{-1} r [t-1]^+\right)^{-\frac{1}{r}},$$

where $\omega_1 = \ln(\frac{\omega_0}{\omega_0 - 1})$ and $[t-1]^+ = \max(t-1, 0)$.

Remark 2.6. For the sake of simplicity, we take $a = b = 1$ in (2.2) and (2.3) throughout this paper.

3. Global existence and energy decay

In this section, we focus our attention to the global existence and decay rate of the solutions to problem (1.1)–(1.5). In order to do so, we first define

$$\begin{aligned} I_1(t) &\equiv I_1(u(t), v(t)) \\ &= \left(m_0 - \int_0^t g(s)ds\right) \|\nabla u\|_2^2 + \left(m_0 - \int_0^t h(s)ds\right) \|\nabla v\|_2^2 \\ &\quad + (g \diamond \nabla u)(t) + (h \diamond \nabla v)(t) - (m+1) \int_\Omega F(u, v) dx, \end{aligned} \tag{3.1}$$

$$\begin{aligned} I_2(t) &\equiv I_2(u(t), v(t)) \\ &= \left(m_0 - \int_0^t g(s)ds\right) \|\nabla u\|_2^2 + \left(m_0 - \int_0^t h(s)ds\right) \|\nabla v\|_2^2 + \alpha \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{\gamma+1} + (g \diamond \nabla u)(t) \\ &\quad + (h \diamond \nabla v)(t) - (m+1) \int_\Omega F(u, v) dx, \end{aligned} \tag{3.2}$$

$$\begin{aligned} J(t) &\equiv J(u(t), v(t)) \\ &= \frac{1}{2} \left(m_0 - \int_0^t g(s)ds\right) \|\nabla u\|_2^2 + \frac{1}{2} \left(m_0 - \int_0^t h(s)ds\right) \|\nabla v\|_2^2 \\ &\quad + \frac{\alpha}{2(\gamma+1)} \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{\gamma+1} + \frac{1}{2} (g \diamond \nabla u)(t) + \frac{1}{2} (h \diamond \nabla v)(t) - \int_\Omega F(u, v) dx, \end{aligned} \tag{3.3}$$

and define the energy function as

$$E(t) \equiv E(u(t), v(t)) = \frac{1}{2} \left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + J(t). \tag{3.4}$$

Lemma 3.1. *Suppose that* (A1), (A2) *and* (2.4) *hold. Let* (u, v) *be the solution of problem* (1.1)–(1.5), *then* $E(t)$ *is a nonincreasing function, that is,*

$$\frac{d}{dt}E(t) = -\|u_t\|_{p+1}^{p+1} - \|v_t\|_{q+1}^{q+1} + \frac{1}{2}\left(g' \diamond \nabla u\right)(t) + \frac{1}{2}\left(h' \diamond \nabla v\right)(t) - \frac{1}{2}g(t)\|\nabla u\|_2^2 - \frac{1}{2}h(t)\|\nabla v\|_2^2 \le 0, \quad \forall t \ge 0. \tag{3.5}$$

Proof. Multiplying Eq. (1.1) by u_t and Eq. (1.2) by v_t, integrating over Ω, summing up and then using integration by parts, we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2 + \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + \frac{\alpha}{\gamma+1}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{\gamma+1}\right) - \int_\Omega F(u,v)dx\right]$$
$$= -\|u_t\|_{p+1}^{p+1} - \|v_t\|_{q+1}^{q+1} + \int_0^t \int_\Omega g(t-s)\nabla u(s)\cdot\nabla u_t dxds + \int_0^t\int_\Omega h(t-s)\nabla v(s)\cdot\nabla v_t dxds.$$

Exploiting Lemma 2.3 on the third term and fourth term on the right hand side of the above equality, we have the result. □

Lemma 3.2. *Suppose that* (A1), (A2) *and* (2.4) *hold. Let* (u, v) *be the solution of problem* (1.1)–(1.5). *Assume further that* $I_1(0) > 0$ *and*

$$\alpha_1 = (m+1)\eta\left(\frac{2(m+1)}{m-1}E(0)\right)^{\frac{m-1}{2}} < 1. \tag{3.6}$$

Then

$$I_1(t) > 0 \quad \text{for all } t \ge 0. \tag{3.7}$$

Proof. Since $I_1(0) > 0$, then by continuity, there exists a maximal time $t_{\max} > 0$ (possibly $t_{\max} = T$) such that

$$I_1(t) \ge 0, \quad \text{for } t \in [0, t_{\max}],$$

which implies that, for $t \in [0, t_{\max}]$,

$$\begin{aligned} J(t) &\ge \frac{m-1}{2(m+1)}\left[\left(m_0 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \left(m_0 - \int_0^t h(s)ds\right)\|\nabla v\|_2^2\right] \\ &\quad + \frac{m-1}{2(m+1)}\left((g\diamond\nabla u)(t) + (h\diamond\nabla v)(t)\right) + \frac{1}{m+1}I_1(t) \\ &\ge \frac{m-1}{2(m+1)}\left[\left(m_0 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \left(m_0 - \int_0^t h(s)ds\right)\|\nabla v\|_2^2\right] \\ &\quad + \frac{m-1}{2(m+1)}\left((g\diamond\nabla u)(t) + (h\diamond\nabla v)(t)\right) \\ &\ge \frac{m-1}{2(m+1)}\left(l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2\right). \end{aligned} \tag{3.8}$$

Using (3.4), (3.8) and $E(t)$ is nonincreasing by (3.5), we get, for $t \in [0, t_{\max}]$,

$$l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2 \le \frac{2(m+1)}{m-1}J(t) \le \frac{2(m+1)}{m-1}E(t) \le \frac{2(m+1)}{m-1}E(0). \tag{3.9}$$

Employing Lemma 2.2, (3.9), (3.6) and (A2), we obtain, for $t \in [0, t_{\max}]$,

$$\begin{aligned} (m+1)\int_\Omega F(u,v)dx &\le (m+1)\eta\left(l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2\right)^{\frac{m+1}{2}} \\ &\le (m+1)\eta\left(\frac{2(m+1)}{m-1}E(0)\right)^{\frac{m-1}{2}}\left(l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2\right) \\ &= \alpha_1\left(l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2\right) \\ &< \left(m_0 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \left(m_0 - \int_0^t h(s)ds\right)\|\nabla v\|_2^2. \end{aligned} \tag{3.10}$$

Thus

$$I_1(t) = \left(m_0 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \left(m_0 - \int_0^t h(s)ds\right)\|\nabla v\|_2^2 + (g \diamond \nabla u)(t)$$
$$+ (h \diamond \nabla v)(t) - (m+1)\int_\Omega F(u,v)dx > 0 \quad \text{on } [0, t_{\max}].$$

By repeating these steps and using the fact that

$$\lim_{t\to t_{\max}} (m+1)\eta\left(\frac{2(m+1)}{m-1}E(t)\right)^{\frac{m-1}{2}} \le \alpha_1 < 1.$$

This implies that we can take $t_{\max} = T$. $\square$

Lemma 3.3. *Let the assumptions of Lemma 3.2 hold. Then there exists* $0 < \eta_1 < 1$ *such that*

$$(m+1)\int_\Omega F(u,v)dx \le (1-\eta_1)\left[\left(m_0 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \left(m_0 - \int_0^t h(s)ds\right)\|\nabla v\|_2^2\right] \quad \text{on } [0,T], \tag{3.11}$$

where $\eta_1 = 1 - \alpha_1$.

Proof. From (3.10), we have

$$(m+1)\int_\Omega F(u,v)dx \le \alpha_1\left(l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2\right).$$

Letting $\eta_1 = 1-\alpha_1$ and using (A2), we obtain (3.11). $\square$

Theorem 3.4. *Suppose that* (A1), (A2) *and* (A3) *hold. Let* $u_0, v_0 \in H_0^1(\Omega)\cap H^2(\Omega)$ *and* $u_1, v_1 \in H_0^1(\Omega)$ *be given which satisfy* $I_1(0) > 0$ *and* (3.6). *Then the solution of problem* (1.1)–(1.5) *is global and bounded. Furthermore, if*

$$m_0 > \frac{5+2\eta_1}{2\eta_1}\max\left(\int_0^\infty g(s)ds, \int_0^\infty h(s)ds\right), \tag{3.12}$$

then we have the following decay estimates:

(i) *if* $p = q = 1$, *then, for all* $t \ge 0$,

$$E(t) \le E(0)e^{-\tau_1 t}.$$

(ii) *If max* $(p,q) > 1$, *then, for all* $t \ge 0$,

$$E(t) \le \left(E(0)^{-\max\left(\frac{p-1}{2},\frac{q-1}{2}\right)} + \tau_2 \max\left(\frac{p-1}{2}, \frac{q-1}{2}\right)[t-1]^+\right)^{-\frac{2}{\max(p,q)-1}},$$

where $\tau_1 = \tau_1(m_0, \alpha, \gamma)$ *and* $\tau_2 = \tau_2(m_0, \alpha, \gamma, E(0))$ *are positive constants given in the proof.*

Proof. First, we prove $T = \infty$, it is sufficient to show that $\|u_t\|_2^2 + \|v_t\|_2^2 + l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2$ is bounded independently of t. Thanks to (3.4) and (3.8), we have

$$E(0) \ge E(t) = \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + J(t)$$
$$\ge \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + \frac{m-1}{2(m+1)}\left(l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2\right).$$

Therefore

$$\|u_t\|_2^2 + \|v_t\|_2^2 + l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2 \le \alpha_2 E(0),$$

where α_2 is a positive constant which depends only on m. Thus, we obtain the global existence result.

Following, we will derive the decay rate of the energy function for problem (1.1)–(1.5) by Nakao's method [23]. For this purpose, we have to show that the energy function defined by (3.4) satisfies the hypothesis of Lemma 2.5. By integrating (3.5) over $[t, t+1]$, we have

$$E(t) - E(t+1) = D_1(t)^{p+1} + D_2(t)^{q+1}, \tag{3.13}$$

where

$$D_1(t)^{p+1} = \int_t^{t+1} \|u_t\|_{p+1}^{p+1} ds - \frac{1}{2}\int_t^{t+1} \left(g' \diamond \nabla u\right)(s) ds + \frac{1}{2}\int_t^{t+1} g(s)\,\|\nabla u\|_2^2\, ds, \tag{3.14}$$

$$D_2(t)^{q+1} = \int_t^{t+1} \|v_t\|_{q+1}^{q+1} ds - \int_t^{t+1} \frac{1}{2}\left(h' \diamond \nabla v\right)(s) ds + \frac{1}{2}\int_t^{t+1} h(s)\,\|\nabla v\|_2^2\, ds. \tag{3.15}$$

By virtue of (3.14), (3.15) and Hölder inequality, we observe that

$$\int_t^{t+1}\int_\Omega |u_t|^2\, dxdt + \int_t^{t+1}\int_\Omega |v_t|^2\, dxdt \le c_1(\Omega)D_1(t)^2 + c_2(\Omega)D_2(t)^2, \tag{3.16}$$

where $c_1(\Omega) = \mathrm{vol}(\Omega)^{\frac{p-1}{p+1}}$ and $c_2(\Omega) = \mathrm{vol}(\Omega)^{\frac{q-1}{q+1}}$. Applying the mean value theorem, there exist $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|u_t(t_i)\|_2^2 + \|u_t(t_i)\|_2^2 \le 4c_1(\Omega)D_1(t)^2 + 4c_2(\Omega)D_2(t)^2, \quad i = 1, 2. \tag{3.17}$$

Next, multiplying Eq. (1.1) by u, Eq. (1.2) by v, integrating over $\Omega \times [t_1, t_2]$, using integration by parts, Hölder inequality and adding them together, we obtain

$$\begin{aligned}
\int_{t_1}^{t_2} I_2(t)dt \le{} & \sum_{i=1}^{2} \|u_t(t_i)\|_2\, \|u(t_i)\|_2 + \sum_{i=1}^{2} \|v_t(t_i)\|_2\, \|v(t_i)\|_2 + \int_{t_1}^{t_2} \left(\|u_t\|_2^2 + \|v_t\|_2^2\right) dt \\
& - \int_{t_1}^{t_2}\int_\Omega \left(|u_t|^{p-1} u_t u + |v_t|^{q-1} v_t v\right) dxdt + \int_{t_1}^{t_2} ((g \diamond \nabla u)(t) + (h \diamond \nabla u)(t))\, dt \\
& + \int_{t_1}^{t_2}\int_\Omega \int_0^t g(t-s)\nabla u(t)\cdot[\nabla u(s) - \nabla u(t)]\, dsdxdt \\
& + \int_{t_1}^{t_2}\int_\Omega \int_0^t h(t-s)\nabla v(t)\cdot[\nabla v(s) - \nabla v(t)]\, dsdxdt.
\end{aligned} \tag{3.18}$$

Since

$$\begin{aligned}
&\int_\Omega \int_0^t g(t-s)\nabla u(t)\cdot[\nabla u(s) - \nabla u(t)]\, dsdx \\
&= \frac{1}{2}\left[\int_0^t g(t-s)\left(\|\nabla u(t)\|_2^2 + \|\nabla u(s)\|_2^2\right) ds - \int_0^t g(t-s)\,\|\nabla u(t) - \nabla u(s)\|_2^2\, ds\right] \\
&\quad - \int_\Omega \int_0^t g(s)\,|\nabla u(t)|^2\, dsdx \\
&= -\frac{1}{2}\int_\Omega \int_0^t g(s)\,|\nabla u(t)|^2\, dsdx + \frac{1}{2}\int_0^t g(t-s)\,\|\nabla u(s)\|_2^2\, ds - \frac{1}{2}\left(g \diamond \nabla u\right)(t),
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega \int_0^t h(t-s)\nabla v(t)\cdot[\nabla v(s) - \nabla v(t)]\, dsdx \\
&= -\frac{1}{2}\int_\Omega \int_0^t h(s)\,|\nabla v(t)|^2\, dsdx + \frac{1}{2}\int_0^t h(t-s)\,\|\nabla v(s)\|_2^2\, ds - \frac{1}{2}\left(h \diamond \nabla v\right)(t),
\end{aligned}$$

hence (3.18) takes the form

$$\begin{aligned}
\int_{t_1}^{t_2} I_2(t)dt \le{} & \sum_{i=1}^{2} \|u_t(t_i)\|_2\, \|u(t_i)\|_2 + \sum_{i=1}^{2} \|v_t(t_i)\|_2\, \|v(t_i)\|_2 + \int_{t_1}^{t_2} \left(\|u_t\|_2^2 + \|v_t\|_2^2\right) dt \\
& - \int_{t_1}^{t_2}\int_\Omega \left(|u_t|^{p-1} u_t u + |v_t|^{q-1} v_t v\right) dxdt + \frac{1}{2}\int_{t_1}^{t_2} ((g \diamond \nabla u)(t) + (h \diamond \nabla u)(t))\, dt \\
& + \frac{1}{2}\int_{t_1}^{t_2}\int_0^t g(t-s)\,\|\nabla u(s)\|_2^2\, dsdt + \frac{1}{2}\int_{t_1}^{t_2}\int_0^t h(t-s)\,\|\nabla v(s)\|_2^2\, dsdt.
\end{aligned} \tag{3.19}$$

Now, we will estimate the right-hand side of (3.19). First, by (3.17), (2.1) and (3.9), we have

$$\begin{aligned}\|u_t(t_i)\|_2\,\|u(t_i)\|_2 &\le c_*\sqrt{4c_1(\Omega)D_1(t)^2+4c_2(\Omega)D_2(t)^2}\sup_{t_1\le s\le t_2}\|\nabla u(s)\|_2\\ &\le c_*\left(\frac{2(m+1)}{l(m-1)}\right)^{\frac{1}{2}}\sqrt{4c_1(\Omega)D_1(t)^2+4c_2(\Omega)D_2(t)^2}\sup_{t_1\le s\le t_2}E(s)^{\frac{1}{2}}\\ &\le c_*\left(\frac{2(m+1)}{\beta(m-1)}\right)^{\frac{1}{2}}\sqrt{4c_1(\Omega)D_1(t)^2+4c_2(\Omega)D_2(t)^2}E(t)^{\frac{1}{2}},\end{aligned} \tag{3.20}$$

and

$$\|v_t(t_i)\|_2\,\|v(t_i)\|_2 \le c_*\left(\frac{2(m+1)}{\beta(m-1)}\right)^{\frac{1}{2}}\sqrt{4c_1(\Omega)D_1(t)^2+4c_2(\Omega)D_2(t)^2}E(t)^{\frac{1}{2}}, \tag{3.21}$$

where

$$\beta=\min(l,k).$$

Using Hölder inequality, (2.1), (2.5) and (3.9), we have

$$\begin{aligned}\left|\int_{t_1}^{t_2}\int_\Omega |u_t|^{p-1}u_t u\,dxdt\right| &\le \int_{t_1}^{t_2}\|u_t(t)\|_{p+1}^p\,\|u\|_{p+1}\,dt\\ &\le c_*\int_{t_1}^{t_2}\|u_t(t)\|_{p+1}^p\,\|\nabla u\|_2\,dt\\ &\le c_*\left(\frac{2(m+1)}{l(m-1)}\right)^{\frac{1}{2}}\sup_{t_1\le s\le t_2}E(s)^{\frac{1}{2}}\int_{t_1}^{t_2}\|u_t(t)\|_{p+1}^p\,dt\\ &\le c_*\left(\frac{2(m+1)}{\beta(m-1)}\right)^{\frac{1}{2}}E(t)^{\frac{1}{2}}D_1(t)^p,\end{aligned} \tag{3.22}$$

and similarly

$$\left|\int_{t_1}^{t_2}\int_\Omega |v_t|^{q-1}v_t v\,dxdt\right| \le c_*\left(\frac{2(m+1)}{\beta(m-1)}\right)^{\frac{1}{2}}E(t)^{\frac{1}{2}}D_2(t)^q. \tag{3.23}$$

Employing Young's inequality for convolution $\|\phi*\psi\|_q\le\|\phi\|_r\,\|\psi\|_s$, with $\frac{1}{q}=\frac{1}{r}+\frac{1}{s}-1,\ 1\le q,\ r,\ s\le\infty$, noting that if $q=1$, then $r=1$ and $s=1$, we get

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^t g(t-s)\|\nabla u(s)\|_2^2\,dsdt &\le \int_{t_1}^{t_2}g(t)dt\int_{t_1}^{t_2}\|\nabla u(t)\|_2^2\,dt\\ &\le (m_0-l)\int_{t_1}^{t_2}\|\nabla u(t)\|_2^2\,dt\\ &\le (m_0-\beta)\int_{t_1}^{t_2}\|\nabla u(t)\|_2^2\,dt,\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^t h(t-s)\|\nabla v(s)\|_2^2\,dsdt &\le \int_{t_1}^{t_2}h(t)dt\int_{t_1}^{t_2}\|\nabla v(t)\|_2^2\,dt\\ &\le (m_0-\beta)\int_{t_1}^{t_2}\|\nabla v(t)\|_2^2\,dt.\end{aligned} \tag{3.25}$$

Adding (3.24) and (3.25) together and noting that

$$l\|\nabla u\|_2^2+k\|\nabla v\|_2^2\le\frac{1}{\eta_1}I_2(t) \tag{3.26}$$

from (3.11) and the definition of $I_2(t)$ by (3.2), we have

$$\begin{aligned}
&\frac{1}{2}\left(\int_{t_1}^{t_2}\int_0^t g(t-s)\,\|\nabla u(s)\|_2^2\,dsdt+\int_{t_1}^{t_2}\int_0^t h(t-s)\,\|\nabla v(s)\|_2^2\,dsdt\right)\\
&\quad\le \frac{m_0-\beta}{2\beta}\int_{t_1}^{t_2}\left(l\,\|\nabla u(t)\|_2^2+k\,\|\nabla v(t)\|_2^2\right)dt\\
&\quad\le \frac{m_0-\beta}{2\beta\eta_1}\int_{t_1}^{t_2} I_2(t)dt.
\end{aligned}\tag{3.27}$$

To estimate the last two terms on the right-hand side of (3.19), we exploit (3.24)–(3.26) to obtain

$$\begin{aligned}
\frac{1}{2}\int_{t_1}^{t_2}((g\diamond\nabla u)(t)+(h\diamond\nabla v)(t))\,dt &= \frac{1}{2}\int_{t_1}^{t_2}\int_0^t g(t-s)\left(\|\nabla u(s)-\nabla u(t)\|_2^2\right)dsdt\\
&\quad+\frac{1}{2}\int_{t_1}^{t_2}\int_0^t h(t-s)\left(\|\nabla v(s)-\nabla v(t)\|_2^2\right)dsdt\\
&\le \int_{t_1}^{t_2}\int_0^t g(t-s)\left(\|\nabla u(s)\|_2^2+\|\nabla u(t)\|_2^2\right)dsdt\\
&\quad+\int_{t_1}^{t_2}\int_0^t h(t-s)\left(\|\nabla v(s)\|_2^2+\|\nabla v(t)\|_2^2\right)dsdt\\
&\le \frac{2(m_0-\beta)}{\beta}\int_{t_1}^{t_2}\left(l\,\|\nabla u(t)\|_2^2+k\,\|\nabla v(t)\|_2^2\right)dt\\
&\le \frac{2(m_0-\beta)}{\beta\eta_1}\int_{t_1}^{t_2} I_2(t)dt.
\end{aligned}\tag{3.28}$$

Therefore, from (3.16), (3.20)–(3.23), (3.27) and (3.28), (3.19) becomes

$$\begin{aligned}
\int_{t_1}^{t_2} I_2(t)dt &\le c_1(\Omega)D_1(t)^2+c_2(\Omega)D_2(t)^2\\
&\quad+4c_3\sqrt{4c_1(\Omega)D_1(t)^2+4c_2(\Omega)D_2(t)^2}E(t)^{\frac{1}{2}}+c_3E(t)^{\frac{1}{2}}\left(D_1(t)^p+D_2(t)^q\right)+c_4\int_{t_1}^{t_2} I_2(t)dt,
\end{aligned}\tag{3.29}$$

where $c_3=c_*\left(\frac{2(m+1)}{\beta(m-1)}\right)^{\frac{1}{2}}$ and $c_4=\frac{5(m_0-\beta)}{2\beta\eta_1}$. Then, rewriting (3.29), we have

$$\begin{aligned}
\beta_2\int_{t_1}^{t_2} I_2(t)dt &\le c_1(\Omega)D_1(t)^2+c_2(\Omega)D_2(t)^2\\
&\quad+4c_3\sqrt{4c_1(\Omega)D_1(t)^2+4c_2(\Omega)D_2(t)^2}E(t)^{\frac{1}{2}}+c_3E(t)^{\frac{1}{2}}\left(D_1(t)^p+D_2(t)^q\right),
\end{aligned}$$

with $\beta_2=1-\frac{5(m_0-\beta)}{2\beta\eta_1}$. Observing that the assumption $m_0>\frac{5+2\eta_1}{2\eta_1}\max\left(\int_0^\infty g(s)ds,\int_0^\infty h(s)ds\right)$ given in (3.12) is equivalent to $\beta_2>0$. Thus,

$$\int_{t_1}^{t_2} I_2(t)dt\le c_5\left[\sqrt{4c_1(\Omega)D_1(t)^2+4c_2(\Omega)D_2(t)^2}E(t)^{\frac{1}{2}}+D_1(t)^2+D_2(t)^2+E(t)^{\frac{1}{2}}\left(D_1(t)^p+D_2(t)^q\right)\right],\tag{3.30}$$

where $c_5=\frac{\max(c_1(\Omega),c_2(\Omega),4c_3)}{\beta_2}$.

On the other hand, from the definition of $E(t)$ by (3.4) and $I_2(t)=I_1(t)+\alpha\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)^{\gamma+1}$ by (3.1) and (3.2), we have

$$\begin{aligned}
E(t) &= \frac{1}{2}\left(\|u_t\|_2^2+\|v_t\|_2^2\right)+\frac{m-1}{2(m+1)}\left[\left(m_0-\int_0^t g(s)ds\right)\|\nabla u\|_2^2\right.\\
&\quad\left.+\left(m_0-\int_0^t h(s)ds\right)\|\nabla v\|_2^2\right]+\frac{m-1}{2(m+1)}((g\diamond\nabla u)(t)+(h\diamond\nabla v)(t))\\
&\quad+\frac{\alpha}{2(\gamma+1)}\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)^{\gamma+1}+\frac{1}{m+1}I_1(t)
\end{aligned}$$

$$\le \frac{1}{2}\left(\|u_t\|_2^2+\|v_t\|_2^2\right)+\frac{m-1}{2(m+1)}\left[\left(m_0-\int_0^t g(s)ds\right)\|\nabla u\|_2^2\right.$$
$$\left.+\left(m_0-\int_0^t h(s)ds\right)\|\nabla v\|_2^2\right]+\frac{m-1}{2(m+1)}\left((g\diamond\nabla u)(t)+(h\diamond\nabla v)(t)\right)$$
$$+\left(\frac{1}{m+1}+\frac{1}{2(\gamma+1)}\right)I_2(t). \tag{3.31}$$

Hence, integrating (3.31) over (t_1,t_2) and then utilizing (3.16), (3.26), (3.28) and (3.30), we deduce that

$$\int_{t_1}^{t_2}E(t)dt \le \frac{1}{2}\int_{t_1}^{t_2}\left(\|u_t\|_2^2+\|v_t\|_2^2\right)dt+\frac{m-1}{2(m+1)}\int_{t_1}^{t_2}\left[\left(m_0-\int_0^t g(s)ds\right)\|\nabla u\|_2^2\right.$$
$$\left.+\left(m_0-\int_0^t h(s)ds\right)\|\nabla v\|_2^2\right]dt+\frac{m-1}{2(m+1)}\int_{t_1}^{t_2}((g\diamond\nabla u)(t)$$
$$+(h\diamond\nabla v)(t))\,dt+\left(\frac{1}{m+1}+\frac{1}{2(\gamma+1)}\right)\int_{t_1}^{t_2}I_2(t)dt$$
$$\le c_1(\Omega)D_1(t)^2+c_2(\Omega)D_2(t)^2+c_6\int_{t_1}^{t_2}I_2(t)dt$$
$$\le c_7\left[\sqrt{4c_1(\Omega)D_1(t)^2+4c_2(\Omega)D_2(t)^2}E(t)^{\frac{1}{2}}+D_1(t)^2+D_2(t)^2\right.$$
$$\left.+E(t)^{\frac{1}{2}}\left(D_1(t)^p+D_2(t)^q\right)\right], \tag{3.32}$$

where $c_6=\frac{1}{m+1}+\frac{1}{2(\gamma+1)}+\frac{m-1}{2(m+1)\eta_1}+\frac{2(m-1)(m_0-\beta)}{(m+1)\beta\eta_1}$ and $c_7=\max\left(c_1(\Omega),c_2(\Omega),c_6c_5\right)$. Moreover, integrating (3.5) over (t,t_2) and using (3.14), (3.15) and the fact that

$$E(t_2)\le 2\int_{t_1}^{t_2}E(t)dt$$

due to $t_2-t_1\ge\frac{1}{2}$, we obtain

$$E(t) = E(t_2)+\int_t^{t_2}\|u_t\|_{p+1}^{p+1}ds-\frac{1}{2}\int_t^{t_2}\left(g'\diamond\nabla u\right)(s)ds+\frac{1}{2}\int_t^{t_2}g(s)\,\|\nabla u\|_2^2\,ds+\int_t^{t_2}\|v_t\|_{q+1}^{q+1}ds$$
$$-\int_t^{t_2}\frac{1}{2}\left(h'\diamond\nabla v\right)(s)ds+\frac{1}{2}\int_t^{t_2}h(s)\,\|\nabla v\|_2^2\,ds$$
$$\le 2\int_{t_1}^{t_2}E(t)dt+D_1(t)^{p+1}+D_2(t)^{q+1}. \tag{3.33}$$

Consequently, combining (3.33) with (3.32), we obtain

$$E(t) \le c_8\left[\sqrt{4c_1(\Omega)D_1(t)^2+4c_2(\Omega)D_2(t)^2}E(t)^{\frac{1}{2}}+D_1(t)^2+D_2(t)^2\right.$$
$$\left.+E(t)^{\frac{1}{2}}D_1(t)^p+E(t)^{\frac{1}{2}}D_2(t)^q+D_1(t)^{p+1}+D_2(t)^{q+1}\right].$$

Then, a simple application of Young's inequality gives, for all $t\ge 0$,

$$E(t)\le c_9\left[D_1(t)^2+D_2(t)^2+D_1(t)^{2p}+D_2(t)^{2q}+D_1(t)^{p+1}+D_2(t)^{q+1}\right], \tag{3.34}$$

where c_8 and c_9 are some positive constants.

Therefore, we have the following decay estimate:

(i) $p=q=1$

From (3.34) and (3.13), we get

$$E(t)\le c_{10}\left[E(t)-E(t+1)\right],$$

here we choose $c_{10}>1$. Thus, by Lemma 2.5, we obtain

$$E(t)\le E(0)e^{-\tau_1 t}\quad\text{for } t\ge 0,$$

with $\tau_1=\ln\frac{c_{10}}{c_{10}-1}$.

(ii) If $\max(p, q) > 1$, it follows from (3.34) that, for all $t \geq 0$,

$$E(t) \leq c_9 \left[\left(1 + D_1(t)^{p-1} + D_1(t)^{2p-2}\right) D_1(t)^2 + \left(1 + D_2(t)^{q-1} + D_2(t)^{2q-2}\right) D_2(t)^2 \right].$$

As $D_1(t) \leq E(t)^{\frac{1}{p+1}} \leq E(0)^{\frac{1}{p+1}}$ and $D_2(t) \leq E(t)^{\frac{1}{q+1}} \leq E(0)^{\frac{1}{q+1}}$ by (3.13) and (3.5), we have, for all $t \geq 0$,

$$\begin{aligned} E(t) &\leq c_9 \left[\left(1 + E(0)^{\frac{p-1}{p+1}} + E(0)^{\frac{2p-2}{p+1}}\right) D_1(t)^2 + \left(1 + E(0)^{\frac{q-1}{q+1}} + E(0)^{\frac{2q-2}{q+1}}\right) D_2(t)^2 \right] \\ &\leq c_9 \left(1 + E(0)^{\frac{p-1}{p+1}} + E(0)^{\frac{2p-2}{p+1}} + E(0)^{\frac{q-1}{q+1}} + E(0)^{\frac{2q-2}{q+1}}\right) \left(D_1(t)^2 + D_2(t)^2\right) \\ &= c_{10}(E(0)) \left(D_1(t)^2 + D_2(t)^2\right), \end{aligned}$$

where $\lim_{E(0)\to 0} c_{10}(E(0)) = c_9$. Setting $\rho = \max\left(\frac{p-1}{2}, \frac{q-1}{2}\right)$, then, we obtain

$$\begin{aligned} E(t)^{1+\rho} &\leq \left[c_{10}(E(0)) \left(D_1(t)^2 + D_2(t)^2\right)\right]^{1+\rho} \\ &\leq c_{11}(E(0)) \left(D_1(t)^{2\rho+2} + D_2(t)^{2\rho+2}\right), \\ &= c_{11}(E(0)) \left(D_1(t)^{2\rho-p+1} D_1(t)^{p+1} + D_2(t)^{2\rho-q+1} D_2(t)^{q+1}\right) \\ &\leq c_{11}(E(0)) \left(E(0)^{\frac{2\rho-p+1}{p+1}} D_1(t)^{p+1} + E(0)^{\frac{2\rho-q+1}{q+1}} D_2(t)^{q+1}\right) \\ &\leq c_{12}(E(0)) \left(D_1(t)^{p+1} + D_2(t)^{q+1}\right) \\ &= c_{12}(E(0)) \left(E(t) - E(t+1)\right), \end{aligned} \tag{3.35}$$

where $c_{11}(E(0)) = 2^\rho \cdot (c_{10}(E(0)))^{1+\rho}$ and $c_{12}(E(0)) = c_{11}(E(0)) \max\left(E(0)^{\frac{2\rho-p+1}{p+1}}, E(0)^{\frac{2\rho-q+1}{q+1}}\right)$. The application of Lemma 2.5 to (3.35) yields

$$E(t) \leq \left(E(0)^{-\rho} + \tau_2 \rho [t-1]^+\right)^{-\frac{1}{\rho}} \quad \text{for } t \geq 0,$$

with $\tau_2 = c_{12}^{-1}(E(0))$. The proof of Theorem 3.4 is completed. □

Remark 3.5. If the condition $I_1(0) > 0$ in Lemma 3.2 and Theorem 3.4 is replaced by $I_2(0) > 0$, we need the assumption $m > 2\gamma + 1$ to prove Theorem 3.4. Hence, the condition $I_1(0) > 0$ provides the decay result to problem (1.1)–(1.5) without imposing $m > 2\gamma + 1$, but at the expense of restricting the initial data by a strong condition.

4. Blow-up of solutions

In this section, we investigate the blow up properties of solutions for a kind of problem:

$$u_{tt} - M\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \Delta u + \int_0^t g(t-s) \Delta u(s) ds + |u_t|^{p-1} u_t = f_1(u, v), \quad \text{in } \Omega \times [0, \infty), \tag{4.1}$$

$$v_{tt} - M\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \Delta v + \int_0^t h(t-s) \Delta u(s) ds + |v_t|^{q-1} v_t = f_2(u, v), \quad \text{in } \Omega \times [0, \infty), \tag{4.2}$$

where $M(s) = 1 + \alpha s^\gamma$, with $\alpha > 0$, $\gamma > 0$, $s \geq 0$. In order to state our result, we make an extra assumption on g and h:

$$\max\left(\int_0^\infty g(s) ds, \int_0^\infty h(s) ds\right) < \min\left(\frac{2(m-1)}{2m-1}, \frac{2(m+1)(E_1 - E(0))}{2(m-1)\lambda_1^2}\right), \tag{4.3}$$

where λ_1 and E_1 are given in (4.5) and (4.6), respectively.

Next, we define a functional G which helps in establishing desired results. Setting

$$G(x) = \frac{1}{2} x^2 - \eta x^{m+1}, \quad x > 0, \tag{4.4}$$

where η is the constant appeared in Lemma 2.2.

Remark 4.1. (i) We can verify that the functional G is increasing in $(0, \lambda_1)$, decreasing in (λ_1, ∞), $G(\lambda) \to -\infty$ as $\lambda \to \infty$ and G has a maximum value at

$$\lambda_1 = \left(\frac{1}{\eta(m+1)}\right)^{\frac{1}{m-1}} \tag{4.5}$$

with the maximum value

$$E_1 = G(\lambda_1) = \frac{m-1}{2(m+1)}\lambda_1^2. \tag{4.6}$$

(ii) We observe from (3.4), Lemma 2.2 and (4.4) that

$$\begin{aligned} E(t) &\geq J(t) = \frac{1}{2}w(t)^2 - \int_\Omega F(u,v)dx \\ &\geq \frac{1}{2}w(t)^2 - \eta\left(l\left\|\nabla u\right\|_2^2 + k\left\|\nabla v\right\|_2^2\right)^{\frac{m+1}{2}} \\ &\geq \frac{1}{2}w(t)^2 - \eta w(t)^{m+1} = G\left(w(t)\right), \quad t \geq 0, \end{aligned} \tag{4.7}$$

where

$$w(t) = \left[l\left\|\nabla u(t)\right\|_2^2 + k\left\|\nabla v\right\|_2^2 + \frac{\alpha}{\gamma+1}\left(\left\|\nabla u\right\|_2^2 + \left\|\nabla v\right\|_2^2\right)^{\gamma+1} + (g\diamond\nabla u)(t) + (h\diamond\nabla v)(t)\right]^{\frac{1}{2}}. \tag{4.8}$$

Before we state and prove our main result, we need the following lemma, and it is similar to a lemma used firstly by Vitillaro [9] to study some classes of a single equation.

Lemma 4.2. *Assume that* (A2) *and* (2.4) *hold,* $u_0, v_0 \in H_0^1(\Omega)\cap H^2(\Omega)$ *and* $u_1, v_1 \in H_0^1(\Omega)$. *Let* (u, v) *be a solution of* (4.1)–(4.2), (1.3)–(1.5) *with initial data satisfying* $E(0) < E_1$ *and* $w(0) > \lambda_1$, *i.e.*

$$\left(l\left\|\nabla u_0\right\|_2^2 + k\left\|\nabla v_0\right\|_2^2 + \frac{\alpha}{\gamma+1}\left(\left\|\nabla u_0\right\|_2^2 + \left\|\nabla v_0\right\|_2^2\right)^{(\gamma+1)}\right)^{\frac{1}{2}} > \lambda_1. \tag{4.9}$$

Then there exists $\lambda_2 > \lambda_1$ *such that, for all* $t \geq 0$

$$w(t) \geq \lambda_2. \tag{4.10}$$

Proof. From Remark 4.1 (i), we see that G is increasing for $0 < \lambda < \lambda_1$, decreasing for $\lambda > \lambda_1$ and $G(\lambda) \to -\infty$ as $\lambda \to \infty$. Thus, as $E(0) < E_1$, there exist $\lambda_2' < \lambda_1 < \lambda_2$ such that $G\left(\lambda_2'\right) = G\left(\lambda_2\right) = E(0)$, which together with $w(0) > \lambda_1$ and (4.7) infer that

$$G\left(w(0)\right) \leq E(0) = G\left(\lambda_2\right).$$

This implies that $w(0) \geq \lambda_2$.

To establish (4.10), we argue by contradiction. Suppose that (4.10) does not hold, then there exists $t^* > 0$ such that

$$w(t^*) < \lambda_2.$$

Case 1: If $\lambda_1 < w(t^*) < \lambda_2$, then

$$G\left(w(t^*)\right) > G(\lambda_2) = E(0) \geq E(t^*).$$

This contradicts (4.7).

Case 2: If $w(t^*) \leq \lambda_1$, then by continuity of the function $w(t)$, there exists $0 < t_1 < t^*$ such that

$$\lambda_1 < w(t_1) < \lambda_2,$$

then

$$G\left(w(t_1)\right) > G(\lambda_2) = E(0) \geq E(t_1).$$

This is also a contradiction of (4.7). Thus, we have proved (4.10). □

Theorem 4.3. *Suppose that* (A2), (2.4) *and* (4.3) *hold. Assume further that* $m > \max(p,q)$ *and* $\gamma < \max\left(\frac{m-1}{2}, \kappa\right)$ *with*

$$\kappa = \begin{cases} \dfrac{(2m-1)(1-k)}{4}, & \text{if } l > k, \\ \dfrac{(2m-1)(1-l)}{4}, & \text{if } l < k. \end{cases} \tag{4.11}$$

If one of the following is satisfied

(i) $E(0) < 0$,
(ii) $0 \le E(0) < E_1$ *and* $w(0) > \lambda_1$.

Then any solution of problem (4.1)–(4.2), (1.3)–(1.5) blows up at a finite time T. The lifespan T is estimated by

$$0 < T \le \frac{1-\sigma}{c_{23}\sigma A(0)^{\frac{\sigma}{1-\sigma}}},$$

where $A(t)$ and c_{23} are given in (4.15) and (4.30) respectively. σ is a constant given in (4.22).

Proof. (I) For case $0 \le E(0) < E_1$

We suppose that the solution exists for all time and we reach to a contradiction. For this purpose, we set

$$H(t) = E_2 - E(t), \quad t \ge 0, \tag{4.12}$$

where $E_2 = \frac{E_1+E(0)}{2}$. By (3.5), we see that $H'(t) \ge 0$. Thus, we obtain

$$H(t) \ge H(0) = E_2 - E(0) > 0, \quad t \ge 0. \tag{4.13}$$

Moreover, from (4.7), (4.10) and (4.6), we see that

$$\begin{aligned} H(t) &= E_2 - E(t) \\ &\le E_1 - \frac{1}{2}w(t)^2 + \int_\Omega F(u, v)dx \\ &\le E_1 - \frac{1}{2}\lambda_1^2 + \|u+v\|_{m+1}^{m+1} + 2\,\|uv\|_{\frac{m+1}{2}}^{\frac{m+1}{2}} \\ &= -\frac{\lambda_1^2}{m+1} + \|u+v\|_{m+1}^{m+1} + 2\,\|uv\|_{\frac{m+1}{2}}^{\frac{m+1}{2}}. \end{aligned}$$

Then, by (4.13) and Lemma 2.1, we have

$$\begin{aligned} 0 &< H(0) \le H(t) \le \|u+v\|_{m+1}^{m+1} + 2\,\|uv\|_{\frac{m+1}{2}}^{\frac{m+1}{2}} \\ &\le c_1\left(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\right), \quad \text{for all } t \ge 0. \end{aligned} \tag{4.14}$$

Let

$$A(t) = H^{1-\sigma}(t) + \varepsilon\int_\Omega (uu_t + vv_t)\,dx, \tag{4.15}$$

where ε and σ are positive constants to be specified later. By taking a derivative of (4.15) and using Eqs. (4.1) and (4.2), we get

$$\begin{aligned} A'(t) =&\ (1-\sigma)H(t)^{-\sigma}H'(t) + \varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) - \varepsilon\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\ &- \varepsilon\alpha\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{\gamma+1} + \varepsilon\int_\Omega\int_0^t g(t-s)\nabla u(s)\cdot\nabla u(t)dsdx \\ &+ \varepsilon\int_\Omega\int_0^t h(t-s)\nabla v(s)\cdot\nabla v(t)dsdx - \varepsilon\int_\Omega (u\,|u_t|^{p-1}\,u_t + v\,|v_t|^{q-1}\,v_t)dx \\ &+ \varepsilon(m+1)\int_\Omega F(u, v)dx. \end{aligned} \tag{4.16}$$

Exploiting Hölder inequality and Young's inequality, we observe that

$$\begin{aligned} &\int_\Omega\int_0^t g(t-s)\nabla u(s)\cdot\nabla u(t)dsdx \\ &= \int_\Omega\int_0^t g(t-s)\nabla u(t)\cdot(\nabla u(s) - \nabla u(t))\,dsdx + \int_0^t g(t-s)ds\,\|\nabla u(t)\|_2^2 \\ &\ge -(g\diamond\nabla u)(t) + \frac{3}{4}\int_0^t g(s)ds\,\|\nabla u(t)\|_2^2, \end{aligned} \tag{4.17}$$

and

$$\int_{\Omega}\int_0^t h(t-s)\nabla v(s)\cdot\nabla v(t)dsdx \geq -(h\diamond\nabla v)(t)+\frac{3}{4}\int_0^t h(s)ds\,\|\triangledown v(t)\|_2^2. \tag{4.18}$$

Taking (4.17)–(4.18) into account, using (4.12) and the definition of $E(t)$ by (3.4) to substitute for $\int_{\Omega}F(u,v)dx$, (4.16) becomes

$$\begin{aligned}A'(t) \geq\ & (1-\sigma)H(t)^{-\sigma}H'(t)+\varepsilon a_1\left(\|u_t\|_2^2+\|v_t\|_2^2\right)+\varepsilon a_2\left((g\diamond\nabla u)(t)+(h\diamond\nabla u)(t)\right)\\ &+\varepsilon a_3\frac{\alpha}{(\gamma+1)}\left(\|\triangledown u\|_2^2+\|\triangledown v\|_2^2\right)^{\gamma+1}+\varepsilon a_4\left(\|\triangledown u\|_2^2+\|\triangledown v\|_2^2\right)\\ &-\varepsilon\int_{\Omega}(u\,|u_t|^{p-1}u_t+v\,|v_t|^{q-1}v_t)dx+(m+1)\varepsilon H(t)-(m+1)\varepsilon E_2,\end{aligned}$$

where $a_1=\frac{m+3}{2}$, $a_2=\frac{m-1}{2}$, $a_3=\frac{m-1-2\gamma}{2}$ and $a_4=\frac{m-1}{2}-\frac{2m-1}{4}\max\left(\int_0^\infty g(s)ds,\int_0^\infty h(s)ds\right)$. By (4.3), we observe that $a_4>0$ and then by the restriction on γ and the definition of $w(t)$ by (4.8), we have

$$\begin{aligned}A'(t) \geq\ & (1-\sigma)H(t)^{-\sigma}H'(t)+\varepsilon a_1\left(\|u_t\|_2^2+\|v_t\|_2^2\right)+\varepsilon a_4\left((g\diamond\nabla u)(t)+(h\diamond\nabla u)(t)\right)\\ &+\varepsilon a_4\frac{\alpha}{(\gamma+1)}\left(\|\triangledown u\|_2^2+\|\triangledown v\|_2^2\right)^{\gamma+1}+\varepsilon a_4\left(l\,\|\triangledown u\|_2^2+k\,\|\triangledown v\|_2^2\right)\\ &-\varepsilon\int_{\Omega}(u\,|u_t|^{p-1}u_t+v\,|v_t|^{q-1}v_t)dx+(m+1)\varepsilon H(t)-(m+1)\varepsilon E_2\\ =\ & (1-\sigma)H(t)^{-\sigma}H'(t)+\varepsilon a_1\left(\|u_t\|_2^2+\|v_t\|_2^2\right)+\varepsilon a_4 w(t)^2+(m+1)\varepsilon H(t)\\ &-\varepsilon\int_{\Omega}(u\,|u_t|^{p-1}u_t+v\,|v_t|^{q-1}v_t)dx-(m+1)\varepsilon E_2.\end{aligned}$$

As $w(t)\geq\lambda_2$ by (4.10) and $\lambda_2>\lambda_1$ by Lemma 4.2, we note that

$$\begin{aligned}a_4w(t)^2-(m+1)E_2 &= a_4\frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2}w(t)^2+a_4\lambda_1^2\frac{w(t)^2}{\lambda_2^2}-(m+1)E_2\\ &\geq c_2w(t)^2+c_3,\end{aligned}$$

where $c_2=a_4\frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2}>0$ and $c_3=a_4\lambda_1^2-(m+1)E_2$. Further, employing the definition of E_1 by (4.6), $E_2=\frac{E_1+E(0)}{2}$ and the assumption (4.3), we see that

$$\begin{aligned}c_3 &= a_4\lambda_1^2-(m+1)E_2\\ &= \left(\frac{m-1}{2}-\frac{2m-1}{4}\max\left(\int_0^\infty g(s)ds,\int_0^\infty h(s)ds\right)\right)\lambda_1^2-(m+1)E_2\\ &= \frac{(m+1)\left(E_1-E(0)\right)}{2}-\frac{(2m-1)\,\lambda_1^2}{4}\max\left(\int_0^\infty g(s)ds,\int_0^\infty h(s)ds\right)>0.\end{aligned}$$

Therefore, based on above arguments, we conclude that

$$\begin{aligned}A'(t) \geq\ & (1-\sigma)H(t)^{-\sigma}H'(t)+\varepsilon a_1\left(\|u_t\|_2^2+\|v_t\|_2^2\right)+\varepsilon c_2w(t)^2\\ &-\varepsilon\int_{\Omega}(u\,|u_t|^{p-1}u_t+v\,|v_t|^{q-1}v_t)dx+(m+1)\varepsilon H(t).\end{aligned} \tag{4.19}$$

To proceed further, by Hölder inequality and Young's inequality, we have

$$\left|\int_{\Omega}|u_t|^{p-1}u_tudx\right|\leq\frac{\delta_1^{p+1}}{p+1}\|u\|_{p+1}^{p+1}+\frac{p\delta_1^{-\frac{p+1}{p}}}{p+1}\|u_t\|_{p+1}^{p+1},$$

and

$$\left|\int_{\Omega}|v_t|^{q-1}v_tvdx\right|\leq\frac{\delta_2^{q+1}}{q+1}\|v\|_{q+1}^{q+1}+\frac{q\delta_2^{-\frac{q+1}{q}}}{q+1}\|v_t\|_{q+1}^{q+1},$$

where δ_1 and δ_2 are positive constants depending on t and will be specified later. Then, inserting the last two inequalities into (4.19), we obtain

$$A'(t) \geq (1-\sigma)H(t)^{-\sigma}H'(t) + \varepsilon a_1 \left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + \varepsilon c_2 w(t)^2$$
$$- \varepsilon \left(\frac{\delta_1^{p+1}}{p+1} \|u\|_{p+1}^{p+1} + \frac{p\delta_1^{-\frac{p+1}{p}}}{p+1} \|u_t\|_{p+1}^{p+1} + \frac{\delta_2^{q+1}}{q+1} \|v\|_{q+1}^{q+1} + \frac{q\delta_2^{-\frac{q+1}{q}}}{q+1} \|v_t\|_{q+1}^{q+1} \right) + (m+1)\varepsilon H(t).$$

At this point, choosing δ_1 and δ_2 such that

$$\delta_1^{-\frac{p+1}{p}} = M_1 H(t)^{-\sigma} \quad \text{and} \quad \delta_2^{-\frac{q+1}{q}} = M_2 H(t)^{-\sigma},$$

and using $H'(t) = -E'(t)$ by (4.12) and $E'(t) \leq -\left(\|u_t\|_{p+1}^{p+1} + \|v_t\|_{q+1}^{q+1}\right)$ by (3.5), we get that

$$A'(t) \geq (1-\sigma-M\varepsilon)H(t)^{-\sigma}H'(t) + \varepsilon a_1 \left(\|u_t\|_2^2 + \|v_t\|_2^2\right)$$
$$+ \varepsilon c_2 w(t)^2 - \varepsilon M_1^{-p} H(t)^{\sigma p} \|u\|_{p+1}^{p+1} - \varepsilon M_2^{-q} H(t)^{\sigma q} \|v\|_{q+1}^{q+1} + (m+1)\varepsilon H(t), \tag{4.20}$$

where $M_1,\ M_2$ are positive constants and $M = \frac{pM_1}{p+1} + \frac{qM_2}{q+1}$. It follows from (4.14) that

$$M_1^{-p} H(t)^{\sigma p} \leq M_1^{-p} c_1^{\sigma p} \left(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\right)^{\sigma p},$$
$$M_2^{-q} H(t)^{\sigma q} \leq M_2^{-q} c_1^{\sigma q} \left(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\right)^{\sigma q}.$$

A substitution of these two inequalities into (4.20) yields

$$A'(t) \geq (1-\sigma-M\varepsilon)H(t)^{-\sigma}H'(t) + \varepsilon a_1 \left(\|u_t\|_2^2 + \|v_t\|_2^2\right)$$
$$+ \varepsilon c_2 w(t)^2 - \varepsilon M_1^{-p} c_1^{\sigma p} \left(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\right)^{\sigma p} \|u\|_{p+1}^{p+1}$$
$$- \varepsilon M_2^{-q} c_1^{\sigma q} \left(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\right)^{\sigma q} \|v\|_{q+1}^{q+1} + (m+1)\varepsilon H(t).$$

Since $p < m$ and $q < m$, we note that

$$\|u\|_{p+1}^{p+1} \leq c_4 \|u\|_{m+1}^{p+1} \leq c_4 \left(\|u\|_{m+1} + \|v\|_{m+1}\right)^{p+1},$$
$$\|v\|_{p+1}^{p+1} \leq c_5 \|v\|_{m+1}^{q+1} \leq c_5 \left(\|u\|_{m+1} + \|v\|_{m+1}\right)^{q+1},$$

where $c_4 = vol\,(\Omega)^{\frac{m-p}{m+1}}$ and $c_5 = vol\,(\Omega)^{\frac{m-q}{m+1}}$. Thus,

$$A'(t) \geq (1-\sigma-M\varepsilon)H(t)^{-\sigma}H'(t) + \varepsilon a_1 \left(\|u_t\|_2^2 + \|v_t\|_2^2\right)$$
$$+ \varepsilon c_2 w(t)^2 - \varepsilon M_1^{-p} c_1^{\sigma p} c_4 \left(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\right)^{\sigma p} \left(\|u\|_{m+1} + \|v\|_{m+1}\right)^{p+1}$$
$$- \varepsilon M_2^{-q} c_1^{\sigma q} c_5 \left(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\right)^{\sigma q} \left(\|u\|_{m+1} + \|v\|_{m+1}\right)^{q+1} + (m+1)\varepsilon H(t)$$
$$\geq (1-\sigma-M\varepsilon)H(t)^{-\sigma}H'(t) + \varepsilon a_1 \left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + \varepsilon c_2 w(t)^2$$
$$- \varepsilon M_1^{-p} c_1^{\sigma p} c_7 \left(\|u\|_{m+1} + \|v\|_{m+1}\right)^{\sigma p(m+1)+p+1} + (m+1)\varepsilon H(t)$$
$$- \varepsilon M_2^{-q} c_1^{\sigma q} c_8 \left(\|u\|_{m+1} + \|v\|_{m+1}\right)^{\sigma q(m+1)+q+1}, \tag{4.21}$$

where the last inequality is derived by

$$\left(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\right)^{\sigma p} \leq 2c_6 \left(\|u\|_{m+1} + \|v\|_{m+1}\right)^{\sigma p(m+1)},$$
$$\left(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\right)^{\sigma q} \leq 2c_6 \left(\|u\|_{m+1} + \|v\|_{m+1}\right)^{\sigma q(m+1)},$$

because of

$$(x+y)^\lambda \leq c_6 \left(x^\lambda + y^\lambda\right), \quad x, y \geq 0,\ \lambda > 0,\ c_6 > 0,$$

and the constants $c_7 = 2c_6c_4$ and $c_8 = 2c_6c_5$.

Now, letting

$$0 < \sigma < \min\left\{ \frac{m-p}{p(m+1)}, \frac{m-q}{q(m+1)}, \frac{m-1}{2(m+1)} \right\}, \tag{4.22}$$

then

$$2 \le \sigma p(m+1) + p + 1 \le m+1 \quad \text{and} \quad 2 \le \sigma q(m+1) + q + 1 \le m+1,$$

hence, by the following inequality,

$$\|v\|_{m+1}^{s} \le c(vol(\Omega), m) \left(\|\nabla v\|_2^2 + \|v\|_{m+1}^{m+1}\right), \quad \forall v \in H_0^1(\Omega),\ 2 \le s \le m+1, \tag{4.23}$$

we have

$$\|u\|_{m+1}^{\sigma p(m+1)+p+1} \le c_9 \left(\|\nabla u\|_2^2 + \|u\|_{m+1}^{m+1}\right), \tag{4.24}$$

$$\|v\|_{m+1}^{\sigma q(m+1)+p+1} \le c_{10} \left(\|\nabla v\|_2^2 + \|v\|_{m+1}^{m+1}\right), \tag{4.25}$$

where c_9 and c_{10} are some positive constants. Taking (4.24)–(4.25) into consideration and using the definition of $w(t)$ by (4.8), (4.21) takes the form

$$\begin{aligned} A'(t) \ge\ & (1-\sigma-M\varepsilon)H(t)^{-\sigma}H'(t) + \varepsilon a_1 \left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + \varepsilon \left(c_2 l - M_1^{-p} c_1^{\sigma p} c_{11}\right) \|\triangledown u\|_2^2 \\ & + \varepsilon \left(c_2 k - M_2^{-q} c_1^{\sigma q} c_{12}\right) \|\triangledown v\|_2^2 + \varepsilon c_2 \left((g \diamond \nabla u)(t) + (h \diamond \nabla u)(t)\right) \\ & + \varepsilon c_2 \frac{\alpha}{(\gamma+1)} \left(\|\triangledown u\|_2^2 + \|\triangledown v\|_2^2\right)^{\gamma+1} + (m+1)\varepsilon H(t) \\ & - \varepsilon \left(M_1^{-p} c_1^{\sigma p} c_{11} \|u\|_{m+1}^{m+1} + M_2^{-q} c_1^{\sigma q} c_{12} \|v\|_{m+1}^{m+1}\right), \end{aligned} \tag{4.26}$$

where $c_{11} = c_7 \cdot c_9$ and $c_{12} = c_8 \cdot c_{10}$. At this moment, setting $a_5 = \min\{c_2 l, c_2 k, \frac{m+1}{2}\}$, decomposing $\varepsilon(m+1)H(t)$ in (4.26) by $\varepsilon(m+1)H(t) = 2a_5\varepsilon H(t) + (m+1-2a_5)\varepsilon H(t)$, using $H(t) = E_2 - E(t)$ by (4.12) and $F(u,v) \ge c_0 \left(|u|^{m+1} + |v|^{m+1}\right)$ by Lemma 2.1, we obtain

$$\begin{aligned} A'(t) \ge\ & (1-\sigma-M\varepsilon)H(t)^{-\sigma}H'(t) + \varepsilon (a_1 - a_5) \left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ & + \varepsilon (c_2 - a_5) \left((g \diamond \nabla u)(t) + (h \diamond \nabla u)(t)\right) \\ & + \varepsilon \left(c_2 l - M_1^{-p} c_1^{\sigma p} c_{11} - a_5\right) \|\triangledown u\|_2^2 + \varepsilon \left(c_2 k - M_2^{-q} c_1^{\sigma q} c_{12} - a_5\right) \|\triangledown v\|_2^2 \\ & + \varepsilon (c_2 - a_5) \frac{\alpha}{(\gamma+1)} \left(\|\triangledown u\|_2^2 + \|\triangledown v\|_2^2\right)^{\gamma+1} + (m+1-2a_5)\varepsilon H(t) \\ & + \varepsilon \left(2a_5 c_0 - \left(M_1^{-p} c_1^{\sigma p} c_{11} + M_2^{-q} c_1^{\sigma q} c_{12}\right)\right) \left(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\right). \end{aligned}$$

Choosing M_1 and M_2 large enough such that

$$\begin{aligned} & c_2 l - M_1^{-p} c_1^{\sigma p} c_{11} - a_5 > \frac{c_2 l - a_5}{2}, \\ & c_2 k - M_2^{-q} c_1^{\sigma q} c_{12} - a_5 > \frac{c_2 k - a_5}{2}, \\ & 2a_5 c_0 - \left(M_1^{-p} c_1^{\sigma p} c_{11} + M_2^{-q} c_1^{\sigma q} c_{12}\right) > a_5 c_0. \end{aligned}$$

Hence,

$$\begin{aligned} A'(t) \ge\ & (1-\sigma-M\varepsilon)H(t)^{-\sigma}H'(t) + \varepsilon c_{13} \left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ & + \varepsilon c_{14} \left((g \diamond \nabla u)(t) + (h \diamond \nabla u)(t)\right) + \varepsilon c_{15} \left(\|\triangledown u\|_2^2 + \|\triangledown v\|_2^2\right) \\ & + \varepsilon c_{16} \left(\|\triangledown u\|_2^2 + \|\triangledown v\|_2^2\right)^{\gamma+1} + \varepsilon c_{17} H(t) + \varepsilon c_{18} \left(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\right), \end{aligned}$$

for some positive constants c_i, $i = 13, 14, \ldots, 18$. Once M_1 and M_2 are fixed, we pick $\varepsilon > 0$ small enough such that

$$1 - \sigma - M\varepsilon \ge 0$$

and

$$A(0) = H^{1-\sigma}(0) + \varepsilon \int_\Omega (u_0 u_1 + v_0 v_1)\, dx > 0. \tag{4.27}$$

Thus, there exists $K > 0$ such that

$$A'(t) \ge \varepsilon K \left(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1} + \|u_t\|_2^2 + \|v_t\|_2^2 + H(t) + \|\triangledown u\|_2^2 + \|\triangledown v\|_2^2\right), \tag{4.28}$$

which together with (4.27) implies that

$$A(t) \ge A(0) > 0, \quad \text{for } t \ge 0.$$

On the other hand, we have by Hölder inequality, Young's inequality, (4.22) and (4.23) that

$$\begin{aligned}\left(\int_\Omega (u_t u + v_t v)\,dx\right)^{\frac{1}{1-\sigma}} &\le 2^{\frac{\sigma}{1-\sigma}}\left(\|u_t\|_2^{\frac{1}{1-\sigma}}\,\|u\|_2^{\frac{1}{1-\sigma}} + \|v_t\|_2^{\frac{1}{1-\sigma}}\,\|v\|_2^{\frac{1}{1-\sigma}}\right)\\ &\le c_{19}\left(\|u_t\|_2^{\frac{1}{1-\sigma}}\,\|u\|_{m+1}^{\frac{1}{1-\sigma}} + \|v_t\|_2^{\frac{1}{1-\sigma}}\,\|v\|_{m+1}^{\frac{1}{1-\sigma}}\right)\\ &\le c_{20}\left(\|u\|_{m+1}^{\frac{2}{1-2\sigma}} + \|v\|_{m+1}^{\frac{2}{1-2\sigma}} + \|u_t\|_2^2 + \|v_t\|_2^2\right)\\ &\le c_{21}\left(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1} + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|u_t\|_2^2 + \|v_t\|_2^2\right),\end{aligned}$$

which implies that

$$\begin{aligned}A(t)^{\frac{1}{1-\sigma}} &= \left(H^{1-\sigma}(t) + \varepsilon\int_\Omega (uu_t + vv_t)\,dx\right)^{\frac{1}{1-\sigma}}\\ &\le 2^{\frac{\sigma}{1-\sigma}}\left(H(t) + \left(\int_\Omega (uu_t + vv_t)\,dx\right)^{\frac{1}{1-\sigma}}\right)\\ &\le c_{22}\left(H(t) + \|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1} + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|u_t\|_2^2 + \|v_t\|_2^2\right), \quad t \ge 0,\end{aligned} \tag{4.29}$$

where $c_i,\ i = 19, 20, 21, 22$, are some positive constants. Combining (4.28) with (4.29), we get

$$A'(t) \ge c_{23}A(t)^{\frac{1}{1-\sigma}}, \quad t \ge 0, \tag{4.30}$$

here $c_{23} = \frac{\varepsilon K}{c_{22}}$. An integration of (4.30) over $(0, t)$ then yields

$$A(t) \ge \left(A(0)^{\frac{-\sigma}{1-\sigma}} - \frac{\sigma c_{23}}{1-\sigma}t\right)^{-\frac{1-\sigma}{\sigma}}.$$

Since $A(0) > 0$, (4.30) shows that A becomes infinite in a finite time T with $0 < T \le \frac{1-\sigma}{c_{23}\sigma A(0)^{\frac{\sigma}{1-\sigma}}}$.

(II) For $E(0) < 0$, we set $H(t) = -E(t)$, instead of (4.12). Then, applying the same arguments as in part (I), we have our result. □

Remark 4.4. When $M \equiv 1$, problem (1.1)–(1.5) reduces to the same problem with Messaoudi and Said-Houari [20], in which they obtained a blow-up result for solutions with initial energy $E(0) < E_2 = \left(\frac{1}{k} - \frac{1}{m+1}\right)\lambda_1^2,\ 2 < k < m+1$. However, based on Theorem 4.3, we established a blow-up result with initial energy $E(0) < \left(\frac{1}{2} - \frac{1}{m+1}\right)\lambda_1^2 = E_1$, which is bigger than E_2. Hence, our results extend the one in [20] to our problem, where we consider more general form.

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