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Journal of Mathematical Analysis and Applications

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# Canonical variation of a Lorentzian metric <sup>☆</sup>

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## ARTICLE INFO

### Article history:

Received 30 January 2014

Available online xxxx

Submitted by W. Sarlet

### Keywords:

Canonical variation

Lorentzian metric

Killing vector field

Closed vector field

Lightlike hypersurfaces

## ABSTRACT

Given a Lorentzian manifold  $(M, g_L)$  and a timelike unitary vector field  $E$ , we can construct the Riemannian metric  $g_R = g_L + 2\omega \otimes \omega$ ,  $\omega$  being the metrically equivalent one form to  $E$ . We relate the curvature of both metrics, especially in the case of  $E$  being Killing or closed, and we use the relations obtained to give some results about  $(M, g_L)$ .

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## 1. Introduction

Given a Lorentzian manifold  $(M, g_L)$  and a timelike unitary vector field  $E \in \mathfrak{X}(M)$  we can construct the Riemannian metric

$$g_R = g_L + 2\omega \otimes \omega, \quad (1)$$

$\omega$  being the  $g_L$ -metrically equivalent one form to  $E$ . This construction is frequently used to exploit the positive definiteness of  $g_R$ , which provides some conclusions about  $g_L$ . For example, in [9] it is used to prove, under suitable conditions, the existence of periodic timelike geodesics in a compact Lorentzian manifold and in [3] to give a Bernstein theorem for lightlike hypersurfaces in  $\mathbb{R}_1^n$ . A similar construction has been used to induce a Riemannian metric on a lightlike hypersurface of a Lorentzian manifold, which allows to define its extrinsic scalar curvature, [1]. Some aspects of Lorentzian metrics constructed in this way have been studied in [19] and [18].

The dual construction of (1), i.e., given a Riemannian manifold  $(M, g_R)$  defines the Lorentzian metric

$$g_L = g_R - 2\omega \otimes \omega, \quad (2)$$

is also interesting because it provides some important examples of Lorentzian manifold, [15,13,25].

<sup>☆</sup> This paper was supported in part by Junta de Andalucía research group FQM-324.

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In this paper, we consider a Riemannian or a Lorentzian manifold  $(M, g)$  and a (timelike in the Lorentzian case) unitary vector field  $E \in \mathfrak{X}(M)$ . We call  $\varepsilon = g(E, E)$  and define, for  $t \neq -\varepsilon$ ,

$$g_t = g + t\omega \otimes \omega.$$

This metric is Lorentzian if  $t < -\varepsilon$  and Riemannian if  $-\varepsilon < t$ . We call it *the canonical variation of  $g$  along  $E$*  due to its analogy with the canonical variation of a Riemannian submersion, where the metric of the fibres is multiplied by a parameter  $t$ , [6]. In [23], the metric  $g_t$  is also called variation of  $g$  and it is used to construct a Riemann metric with strictly positive sectional curvature from another Riemann metric with nonnegative sectional curvature.

Obviously, the construction (1) corresponds to  $t = 2$  and  $\varepsilon = -1$  and (2) to  $t = -2$  and  $\varepsilon = 1$  and we call them *standard canonical variation*.

Observe that metrics  $g_R$  and  $g_L$  in (1) and (2) are related in the same way and therefore it would be sufficient just to study, for example, the metric  $g_R$  constructed in (1) to obtain analogous results for  $g_L$  constructed in (2). Nevertheless, the introduction of the parameters  $t$  and  $\varepsilon$  allows us to handle jointly (1) and (2), obtaining formulas easily adaptable to each case.

We use  $X, Y, Z$  letters for orthogonal vector fields to  $E$  and  $U, V, W$  for arbitrary vector fields. We write the geometric objects derived from  $g_t$  with a  $t$  subscript, except for the connection which will be denoted by  $\nabla^t$ . For example  $K_t$ ,  $Ric_t$  and  $S_t$  are the sectional, Ricci and scalar curvatures of  $g_t$  respectively. When we deal with  $g_L$ ,  $g_R$  or  $g$  we use an  $L$ ,  $R$  or no subscript respectively.

In the second section, we show a formula relating the difference tensor of the connections  $\nabla^t$  and  $\nabla$ , the exterior differential of  $\omega$  and the Lie derivative  $L_E g$ . We also introduce the notion of vector field with *normal associated endomorphism*, which extends the notion of closed or conformal vector field. This concept will be useful to simplify the computation of the curvature of the canonical variation that are made in the third section. In Section 4 we consider the standard canonical variation along a Killing unitary vector field and we obtain some inequalities about the curvature. We use the Berger theorem to show that if there exists a timelike unitary Killing vector field in a compact Lorentzian manifold with negative sectional curvature on timelike planes, then it has odd dimension. We also use Bochner techniques to give an integral inequality in a compact Lorentzian manifold furnished with a timelike Killing vector field. In the fifth section we give a result about the geodesic completeness of the canonical variation along a closed vector field and in the last section we consider how a lightlike hypersurface is transformed under standard canonical variation. We also give some sufficient conditions for a compact lightlike hypersurface to be totally geodesic.

## 2. Preliminaries

From now on, let  $(M, g)$  be a Riemannian or a Lorentzian manifold,  $E \in \mathfrak{X}(M)$  a (timelike in the Lorentzian case) unitary vector field,  $\omega$  its metrically equivalent one form and  $\varepsilon = g(E, E)$ .

**Definition 2.1.** Fixed  $t \in \mathbb{R} - \{-\varepsilon\}$ , the canonical variation of  $g$  along  $E$  is defined as

$$g_t = g + t\omega \otimes \omega. \quad (3)$$

If  $t = -2\varepsilon$ , then it is called the standard canonical variation of  $g$ .

**Example 2.2.** We give some examples of the standard canonical variation of a Riemannian manifold.

1. The standard canonical variation along any parallel vector field in the Euclidean space gives us the Minkowski space.

2. Consider the hyperbolic space  $\mathbb{H}^n = (\mathbb{R}^n, dx_1^2 + e^{2x_1} \sum_{i=2}^n dx_i^2)$ . The standard canonical variation along  $E = \partial_{x_1}$  gives us the metric  $-dx_1^2 + e^{2x_1} \sum_{i=2}^n dx_i^2$ , which is a piece of  $\mathbb{S}_1^n$ . On the other hand, if we take  $E = e^{-x_1} \partial_{x_n}$ , the standard canonical variation is  $dx_1^2 + e^{2x_1} (\sum_{i=2}^{n-1} dx_i^2 - dx_n^2)$ , which is a piece of  $\mathbb{H}_1^n$ .
3. Fix  $v \in \mathbb{R}^n$  and let  $f : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$  be given by  $f(p) = p \cdot v$ . Take  $\mathbb{S}_+^{n-1} = \{p \in \mathbb{S}^{n-1} : f(p) > 0\}$  with its induced metric  $g_0$ . Since  $\text{Hess}_f = -f g_0$ , we have that  $(\mathbb{S}_+^{n-1} \times \mathbb{R}, g_0 + f^2 dt^2)$  is an open set of  $\mathbb{S}^n$ . The standard canonical variation along  $E = \frac{1}{f} \partial_t$  is the piece of  $\mathbb{S}_1^n$  given by  $(\mathbb{S}_+^{n-1} \times \mathbb{R}, g_0 - f^2 dt^2)$ .
4. The Lorentzian Berger spheres are obtained as the standard canonical variation along the Hopf vector field of the Euclidean spheres  $\mathbb{S}^{2n+1}$ .
5. The Lie group  $SL_2(\mathbb{R})$  can be furnished with the bi-invariant Lorentzian metric given by  $\langle X, Y \rangle_L = \frac{1}{2} \text{tr}(XY)$  or with the Riemannian left-invariant metric given by  $\langle X, Y \rangle_R = \frac{1}{2} \text{tr}(XY^t)$ , where  $X, Y \in \mathfrak{sl}_2(\mathbb{R})$ . If we call

$$E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R}),$$

then it is easy to show that  $\langle \rangle_R$  is the standard canonical variation of  $\langle \rangle_L$  along  $E$ . Recall that  $(SL_2(\mathbb{R}), \langle \rangle_L)$  is isometric to  $\mathbb{H}_1^3$ .

We call  $D^t = \nabla^t - \nabla$  the difference tensor. We can relate this tensor,  $L_E g$  and  $\omega$  as follows.

**Proposition 2.3.** *Given  $U, V, W \in \mathfrak{X}(M)$ , it holds*

$$g_t(D^t(U, V), W) = \frac{t}{2} (\omega(W)(L_E g)(U, V) + \omega(U)d\omega(V, W) + \omega(V)d\omega(U, W)). \quad (4)$$

**Proof.** Using the Koszul formula and Eq. (3) we have

$$g_t(D^t(U, V), W) = \frac{t}{2} (\nabla_U(\omega \otimes \omega)(V, W) + \nabla_V(\omega \otimes \omega)(U, W) - \nabla_W(\omega \otimes \omega)(U, V)).$$

Now, we get the result using that  $d\omega(U, V) = (\nabla_U \omega)(V) - (\nabla_V \omega)(U)$  and  $(L_E g)(U, V) = (\nabla_U \omega)(V) + (\nabla_V \omega)(U)$ .  $\square$

We have the following consequences.

**Corollary 2.4.** *Take  $V, W, X, Y \in \mathfrak{X}(M)$  with  $X, Y \in E^\perp$ .*

1.  $g(D^t(X, V), X) = 0$ .
2.  $g(D^t(V, E), E) = 0$ .
3.  $g_t(D^t(V, E), W) + g_t(D^t(W, E), V) = t(\omega(V)g(W, \nabla_E E) + \omega(W)g(V, \nabla_E E))$ .
4.  $\nabla_E E = (1 + \varepsilon t) \nabla_E E$ .
5.  $D^t(X, Y) = \frac{t}{2(1+\varepsilon t)} (L_E g)(X, Y) E$ .
6.  $(L_E g_t)(X, Y) = (L_E g)(X, Y)$ . In particular, if  $E$  is orthogonally conformal for  $g$ , then it is also orthogonally conformal for  $g_t$ .
7.  $\text{div}_t V = \text{div } V$ .

We need to introduce the following concepts for the next sections.

**Definition 2.5.** The associated endomorphism to a vector field  $U \in \mathfrak{X}(M)$  is  $A_U : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$  given by  $A_U(V) = \nabla_V U$ .

Let  $A_U^*$  be the adjoint endomorphism of  $A_U$ . If  $g(A_U(V), A_U(V)) = g(A_U^*(V), A_U^*(V))$  for all  $V \in \mathfrak{X}(M)$ , or equivalently,  $A_U$  and  $A_U^*$  commute, then  $A_U$  is called normal.

**Definition 2.6.** Let  $E \in \mathfrak{X}(M)$  be a unitary vector field.  $A_E$  is orthogonally normal if  $g(A_E(X), A_E(X)) = g(A_E^{\perp}(X), A_E^{\perp}(X))$  for all  $X \in E^{\perp}$ ,  $A_E^{\perp}(X)$  being the orthogonal component to  $E$ .

If  $A_E$  is orthogonally normal, then it is easy to show that  $A_E$  is normal if and only if  $E$  is geodesic. Moreover, if  $U \in \mathfrak{X}(M)$  with  $A_U$  normal and  $E$  is its unitary, then  $A_E$  is orthogonally normal.

**Example 2.7.** We give some examples of vector field with normal associated endomorphism.

1. The associated endomorphism of a closed vector field is normal.
2. If  $U$  is a conformal vector field, then  $A_U^* = 2\rho \cdot id - A_U$  for certain  $\rho \in C^\infty(M)$  and thus it is straightforward to check that  $A_U$  is normal.
3.  $A_U$  is normal if and only if its associated matrix with respect to a frame field is normal (it commutes with its transpose). Using this, it is easy to check that  $U = (x+y)\partial_x + (y+z)\partial_y + (x+z)\partial_z$  has normal associated endomorphism in  $\mathbb{R}^3$  and so its unitary  $E$  has orthogonally normal associated endomorphism in  $\mathbb{R}^3 - \{0\}$ . Observe that  $U$  is not closed neither conformal.
4. Let  $G$  be a Lie group with a bi-invariant metric  $g$ . If  $U$  is any left-invariant vector field, then  $A_U^*(X) = -A_U(X)$  and hence it is normal.

**Proposition 2.8.** Let  $(M, g)$  be a semi-Riemannian manifold,  $U \in \mathfrak{X}(M)$  a vector field with normal associated endomorphism and  $\overline{M}$  a nondegenerate hypersurface of  $M$  with unitary normal  $N$ . The projection of  $U$  onto  $\overline{M}$  has normal associated endomorphism if and only if

$$g(A_U(X), N)^2 - g(X, A_U(N))^2 = 2g(U, N)(g(A_U(X), S(X)) - g(A_U(S(X)), X)) \quad (5)$$

for all  $X \in \mathfrak{X}(\overline{M})$ ,  $S$  being the shape operator of  $\overline{M}$ .

**Proof.** Take  $\delta = g(N, N) = \pm 1$  and write  $U = V + \delta g(U, N)N$  with  $V \in \mathfrak{X}(\overline{M})$ . Call  $A_V(X) = \tan(\nabla_X V)$ , where  $\tan(\cdot)$  denotes the projection onto  $\overline{M}$ . Given  $X \in \mathfrak{X}(\overline{M})$ ,

$$\begin{aligned} A_V(X) &= \tan(A_U(X)) - \delta g(U, N)\nabla_X N, \\ A_V^*(X) &= \tan(A_U^*(X)) - \delta g(U, N)\nabla_X N. \end{aligned}$$

Therefore,

$$\begin{aligned} g(A_V(X), A_V(X)) &= g(A_U(X), A_U(X)) - \delta g(A_U(X), N)^2 \\ &\quad - 2\delta g(U, N)g(\nabla_X N, A_U(X)) + g(U, N)^2 g(\nabla_X N, \nabla_X N) \end{aligned}$$

and

$$\begin{aligned} g(A_V^*(X), A_V^*(X)) &= g(A_U^*(X), A_U^*(X)) - \delta g(A_U^*(X), N)^2 \\ &\quad - 2\delta g(U, N)g(\nabla_X N, A_U^*(X)) + g(U, N)^2 g(\nabla_X N, \nabla_X N). \end{aligned}$$

Hence,  $A_V$  is normal if and only if Eq. (5) holds.  $\square$

**Example 2.9.** The above proposition provides more examples of vector fields with normal associated endomorphism.

1. If  $U$  is tangent to  $\overline{M}$  and it has normal associated endomorphism, then the restriction of  $U$  to  $\overline{M}$  also has normal associated endomorphism since Eq. (5) holds trivially.
2. The projection of a Killing vector field onto an umbilic hypersurface has normal associated endomorphism, since Eq. (5) holds in this case. Observe that the projection of a Killing vector field onto a hypersurface is also a Killing vector field if and only if the hypersurface is totally geodesic.

### 3. Curvature of the canonical variation

To relate the curvature tensor  $R$  of a metric  $g$  and the curvature tensor  $R^t$  of its canonical variation  $g_t$  we need the following general lemma.

**Lemma 3.1.** *Let  $\nabla$  and  $\nabla^t$  be two arbitrary connections on a manifold  $M$  with curvature tensors  $R$  and  $R^t$  respectively. Given  $U, V, W \in \mathfrak{X}(M)$  we have*

$$\begin{aligned} R_{UV}^t W &= R_{UV} W + (\nabla_U D^t)(V, W) - (\nabla_V D^t)(U, W) \\ &\quad + D^t(U, D^t(V, W)) - D^t(V, D^t(U, W)). \end{aligned} \quad (6)$$

**Theorem 3.2.** *Let  $(M, g)$  be a Riemannian or a Lorentzian manifold and  $g_t$  the canonical variation along a (timelike) unitary vector field  $E$  with  $A_E$  orthogonally normal. Given  $X \in E^\perp$*

$$\begin{aligned} g_t(R_{XE}^t E, X) &= g(R_{XE} E, X) + t(\varepsilon g(\nabla_X A_E(E), X) - g(A_E(E), X)^2) \\ &\quad + \frac{t(2\varepsilon + t)}{2}(g(A_E(X), A_E(X)) - g(A_E^2(X), X)). \end{aligned}$$

**Proof.** Using Eq. (6), we have

$$\begin{aligned} g(R_{XE}^t E - R_{XE} E, X) &= g((\nabla_X D^t)(E, E), X) - g((\nabla_E D^t)(X, E), X) \\ &\quad + g(D^t(X, D^t(E, E)), X) - g(D^t(E, D^t(X, E)), X). \end{aligned}$$

We compute each term applying repeatedly Proposition 2.3. The first one is  $g((\nabla_X D^t)(E, E), X) = \varepsilon t g(\nabla_X \nabla_E E, X) - \varepsilon t d\omega(\nabla_X E, X)$  and, using also Corollary 2.4, the second one is  $g((\nabla_E D^t)(X, E), X) = t g(\nabla_E E, X)^2$ . The third one is zero since  $D^t(E, E) \perp E$ . Finally, the last one is

$$g(D^t(E, D^t(X, E)), X) = \frac{t^2}{4}(g(A_E(X), A_E^{*\perp}(X)) - g(A_E^{*\perp}(X), A_E^{*\perp}(X)) - d\omega(X, \nabla_X E)).$$

Since  $A_E$  is orthogonally normal, we obtain the desired result.  $\square$

**Corollary 3.3.** *Let  $(M, g)$  be a Riemannian or a Lorentzian surface and  $g_t$  the canonical variation of  $g$  along a (timelike) unitary and geodesic vector field  $E$ . If  $K$  and  $K^t$  denote the Gauss curvature of  $g$  and  $g_t$  respectively, then  $K^t = \frac{1}{1+\varepsilon t} K$ .*

**Corollary 3.4.** *Let  $(M, g_R)$  be a Riemannian manifold and  $g_t$  the canonical variation along a unitary vector field  $E$  with  $A_E$  normal. If  $\Pi$  is a plane containing  $E$ , then*

$$\begin{aligned} K_t(\Pi) &\leq \frac{1}{1+t} K_R(\Pi) \quad \text{for } t \in (-\infty, -2) \cup (-1, 0), \\ K_t(\Pi) &\geq \frac{1}{1+t} K_R(\Pi) \quad \text{for } t \in (-2, -1) \cup (0, \infty). \end{aligned}$$

In particular, if  $g_L$  is the standard canonical variation, then  $K_L(\Pi) = -K_R(\Pi)$ .

**Proof.** Use that  $g(A_E^2(X), X) \leq g(A_E(X), A_E(X))$  in [Theorem 3.2](#).  $\square$

**Corollary 3.5.** Let  $(M, g)$  be a Lorentzian or a Riemannian manifold and  $g_t$  the canonical variation along a (timelike) unitary vector field  $E$  with  $A_E$  orthogonally normal. Then

$$\text{Ric}_t(E, E) = \text{Ric}(E, E) + \varepsilon t \operatorname{div} \nabla_E E + \frac{t(2\varepsilon + t)}{2} (\|A'_E\|^2 - \operatorname{tr}(A_E'^2)),$$

where  $A'_E$  is the restriction to  $E^\perp$  of  $A_E$ . In particular, if  $(M, g_R)$  is a Riemannian compact manifold,

$$\begin{aligned} \int_M \text{Ric}_t(E, E) dg_t &\leq \sqrt{|1+t|} \int_M \text{Ric}(E, E) dg_R \quad \text{for } t \in (-\infty, -2) \cup (0, \infty), \\ \int_M \text{Ric}_t(E, E) dg_t &\geq \sqrt{|1+t|} \int_M \text{Ric}(E, E) dg_R \quad \text{for } t \in (-2, -1) \end{aligned}$$

and if  $g_L$  is the standard canonical variation, then  $\int_M \text{Ric}_L(E, E) dg_L = \int_M \text{Ric}(E, E) dg_R$ .

**Proof.** For the second part, take into account that the canonical volume forms of  $g_t$  and  $g$  are related by  $dg_t = \sqrt{|1+t|} dg_R$  and that  $\|A'_E\|^2 - \operatorname{tr}(A_E'^2) \geq 0$  because  $A_E$  is orthogonally normal.  $\square$

**Theorem 3.6.** Let  $(M, g)$  be a Riemannian or a Lorentzian manifold and  $g_t$  the canonical variation along a (timelike) unitary vector field  $E$ . If  $X, Y \in E^\perp$ , then

$$\begin{aligned} g_t(R_{XY}^t Y, X) &= g(R_{XY} Y, X) + \frac{t}{1+\varepsilon t} \left( g(A_E(X), X) g(A_E(Y), Y) \right. \\ &\quad \left. - g(A_E(X), Y) g(A_E(Y), X) - \frac{4+3\varepsilon t}{4} d\omega(X, Y)^2 \right). \end{aligned}$$

**Proof.** From [Eq. \(6\)](#),

$$\begin{aligned} g(R_{XY}^t Y - R_{XY} Y, X) &= g((\nabla_X D^t)(Y, Y), X) - g((\nabla_Y D^t)(X, Y), X) \\ &\quad + g(D^t(X, D^t(Y, Y)), X) - g(D^t(Y, D^t(X, Y)), X). \end{aligned}$$

As before, we compute each term using repeatedly [Proposition 2.3](#). The first one is

$$\begin{aligned} g((\nabla_X D^t)(Y, Y), X) &= \frac{t}{1+\varepsilon t} g(\nabla_Y E, Y) g(X, \nabla_X E) \\ &\quad + t g(Y, \nabla_X E) g(\nabla_Y E, X) - t g(\nabla_X E, Y)^2. \end{aligned}$$

The second term is

$$\begin{aligned} g((\nabla_Y D^t)(X, Y), X) &= \frac{t(2+\varepsilon t)}{2(1+\varepsilon t)} g(\nabla_Y E, X)^2 \\ &\quad - \frac{\varepsilon t^2}{2(1+\varepsilon t)} g(\nabla_X E, Y) g(\nabla_Y E, X). \end{aligned}$$

Using [Corollary 2.4](#), the third one vanishes and the last one is given by  $g(D^t(Y, D^t(X, Y)), X) = \frac{\varepsilon t^2}{4(1+\varepsilon t)} (g(\nabla_Y E, X)^2 - g(Y, \nabla_X E)^2)$ .  $\square$

**Example 3.7.** A Riemannian manifold  $(M, g_R)$  is called of quasi-constant sectional curvature if there exists a unitary vector field  $E \in \mathfrak{X}(M)$  such that the sectional curvature of any plane only depends on the basepoint and the angle between the plane and  $E$ , [7, 11]. On the other hand, a Lorentzian manifold  $(M, g_L)$  is called infinitesimal null isotropic if there exists a timelike unitary vector field  $E \in \mathfrak{X}(M)$  such that the lightlike sectional curvature with respect to  $E$  only depends on the basepoint, [14, 16]. After Theorems 3.2 and 3.6 we can easily check both definitions are equivalent in their respective settings.

Indeed. Suppose that  $(M, g_R)$  is a Riemannian manifold of quasi-constant sectional curvature. If we call  $\omega$  and  $\theta$  the one forms metrically equivalent to  $E$  and  $\nabla_E E$  respectively, then it is known that  $\nabla_X^R E = \lambda X$ ,  $d\omega(X, Y) = 0$  and  $(\nabla_X^R \theta)(Y) - \theta(X)\theta(Y) = \gamma g_R(X, Y)$ , where  $X, Y \in E^\perp$  and  $\lambda, \gamma$  are functions on  $M$ , [11]. Take  $g_L$  the standard canonical variation along  $E$ . Using above formulas and Theorems 3.2 and 3.6, it is easy to show that, in  $(M, g_L)$ , the sectional curvature of planes orthogonal to  $E$  and planes containing  $E$  only depends on the basepoint, which implies that it is infinitesimal null isotropic. In the same manner, we can check that if  $(M, g_L)$  is a Lorentzian manifold infinitesimal null isotropic, then the standard canonical variation is a Riemannian manifold of quasi-constant sectional curvature.

**Corollary 3.8.** Let  $(M, g)$  be a Riemannian or a Lorentzian manifold and  $g_t$  the canonical variation along a (timelike) unitary vector field  $E$  with  $A_E$  orthogonally normal. Given  $X \in E^\perp$ ,

$$\begin{aligned} Ric_t(X, X) &= Ric(X, X) - \frac{t}{1 + \varepsilon t} g(R_{XE} E, X) + \frac{t}{1 + \varepsilon t} g(A_E(X), X) \operatorname{div} E \\ &\quad + \frac{\varepsilon t^2}{1 + \varepsilon t} g(A_E^2(X), X) - tg(A_E(X), A_E(X)) \\ &\quad + \frac{t}{1 + \varepsilon t} (g(\nabla_X A_E(E), X) - \varepsilon g(A_E(E), X)^2). \end{aligned}$$

**Proof.** Use Theorems 3.2, 3.6 and observe that

$$\sum_{i=1}^{n-1} d\omega(X, e_i)^2 = 2g(A_E(X), A_E(X)) - 2g(X, A_E^2(X)),$$

because  $A_E$  is orthogonally normal.  $\square$

**Corollary 3.9.** Let  $(M, g)$  be a Riemannian or a Lorentzian manifold and  $g_t$  the canonical variation along a (timelike) unitary vector field  $E$  with  $A_E$  orthogonally normal. The scalar curvatures  $S_t$  and  $S$  are related by

$$\begin{aligned} S_t &= S - \frac{2t}{1 + \varepsilon t} Ric(E, E) + \frac{2t}{1 + \varepsilon t} \operatorname{div} \nabla_E E - tg(\nabla_E E, \nabla_E E) \\ &\quad + \frac{t}{1 + \varepsilon t} (tr(A'_E)^2 - tr(A_E'^2)) + \frac{\varepsilon t^2}{2(1 + \varepsilon t)} (tr(A_E'^2) - \|A'_E\|^2), \end{aligned}$$

where  $A'_E$  is the restriction to  $E^\perp$  of  $A_E$ .

**Example 3.10.** We can easily construct a Riemannian odd sphere with constant negative scalar curvature. Indeed, if we take  $g_t$  the canonical variation along the Hopf vector field, since it is unitary and Killing, using the above corollary,  $S_t = 2n(2n + 1 - t)$ , which is negative for  $t$  large enough.

Finally, we compute  $g_t(R_{EX}^t X, Y)$ , where  $X, Y \in E^\perp$ , which will be useful later.

**Proposition 3.11.** *Let  $(M, g)$  be a Riemannian or a Lorentzian manifold and  $g_t$  the canonical variation along a (timelike) unitary vector field  $E$ . If  $X, Y \in E^\perp$ , then*

$$\begin{aligned} g_t(R_{EX}^t X, Y) &= g(R_{EX} X, Y) + \frac{t}{2}(-\varepsilon(\nabla_X d\omega)(X, Y) \\ &\quad + g(A_E(X), X)g(A_E(E), Y) - 2g(X, A_E(Y))g(A_E(E), X) \\ &\quad + g(A_E(X), Y)g(A_E(E), X)). \end{aligned}$$

**Proof.** The proof is as in [Theorems 3.2 and 3.6](#).  $\square$

#### 4. Standard canonical variation along a Killing vector field

Suppose that  $E$  is a Killing unitary vector field in a Riemannian manifold  $(M, g_R)$  and consider  $g_L = g_R - 2\omega \otimes \omega$  the standard canonical variation along it. In this case, from formula [\(4\)](#), we have

$$\nabla_U^L V = \nabla_U^R V - 2(\omega(U)\nabla_V^R E + \omega(V)\nabla_U^R E).$$

Moreover, from [Corollary 2.4](#),  $E$  is also Killing for  $g_L$ .

The symmetric with respect to  $E$  of a vector  $v = \alpha E + Y$ , where  $Y \perp E$ , is  $v^* = \alpha E - Y$ . The symmetric with respect to  $E$  of a plane  $\Pi = \text{span}(X, v)$ , where  $X \perp E$ , is the plane given by  $\Pi^* = \text{span}(X, v^*)$ .

We denote  $\mathcal{K}_L^E(\Pi)$  the lightlike sectional curvature of a lightlike plane  $\Pi$  of  $(M, g_L)$  with respect to  $E$ .

**Proposition 4.1.** *Let  $(M, g_R)$  be a Riemannian manifold,  $E \in \mathfrak{X}(M)$  a Killing unitary vector field and  $g_L$  the standard canonical variation along  $E$ .*

1. *If  $\Pi$  is a nondegenerate plane for  $g_L$ , then  $K_R(\Pi^*) \leq -\cos(2\theta)K_L(\Pi)$ ,  $\theta$  being the angle between  $\Pi$  and  $E$ . Moreover, the equality holds if and only if  $E \in \Pi$ .*
2. *If  $\Pi$  is a degenerate plane for  $g_L$ , then  $2K_R(\Pi^*) \leq \mathcal{K}_L^E(\Pi)$ .*
3. *Given  $v \in TM$  it holds  $\text{Ric}_R(v, v) \leq \text{Ric}_L(v^*, v^*)$  and the equality holds if and only if  $v$  is proportional to  $E$ .*
4. *The scalar curvatures  $S_R$  and  $S_L$  hold  $S_R \leq S_L$  and the equality holds if and only if  $E$  is parallel.*

**Proof.** (1) Suppose  $\Pi = \text{span}(X, V)$  where  $X$  and  $V$  are  $g_R$ -unitary,  $V = \alpha E + Y$  and  $X \perp Y \perp E$ . Since  $E$  is Killing,

$$\nabla_X^R(d\omega)(X, Y) = 2(g_R(\nabla_X^R \nabla_X^R E, Y) - g_R(\nabla_{\nabla_X^R E}^R X, Y)) = -2g_R(R_{EX}^R X, Y). \quad (7)$$

Now, recall that  $g_L$  corresponds to the values  $t = -2$  and  $\varepsilon = 1$  in formula [\(3\)](#) and therefore, applying [Theorems 3.2, 3.6](#) and [Proposition 3.11](#), we get

$$g_L(R_{VX}^L X, V) = g_R(R_{V^*X}^R X, V^*) + 6g_R(\nabla_X^R E, Y)^2. \quad (8)$$

(2) It follows from Eq. [\(8\)](#).

(3) Suppose  $v = \alpha E + X$  with  $X \in E^\perp$ . Using [Corollary 3.5](#),  $\text{Ric}_L(E, E) = \text{Ric}_R(E, E)$ . From [Proposition 3.11](#) and formula [\(7\)](#) we have  $\text{Ric}_L(E, X) = -\text{Ric}_R(E, X)$  and by [Corollary 3.8](#),

$$\text{Ric}_L(X, X) = \text{Ric}_R(X, X) - 2g_R(R_{XE}^R E, X) + 6g_R(\nabla_X^R E, \nabla_X^R E).$$



Since  $E$  is Killing,  $g_R(\nabla_X^R E, \nabla_X^R E) = g_R(R_{XE}^R, X)$  and therefore

$$Ric_L(v, v) = Ric_R(v^*, v^*) + 4g_R(\nabla_X^R E, \nabla_X^R E).$$

(4) Since  $E$  is Killing,  $\|A_E\|^2 = Ric_R(E, E)$  and thus, [Corollary 3.9](#) gives us  $S_R + 2 Ric_R(E, E) = S_L$ . The statement holds now trivially.  $\square$

The existence of a Killing vector field on a Lorentzian manifold with an isolated zero implies that  $M$  has even dimension, [\[2\]](#). We can prove the following related result.

**Theorem 4.2.** *Let  $(M, g_L)$  be a compact Lorentzian manifold with negative sectional curvature on timelike planes. If there exists a timelike unitary Killing vector field, then  $M$  has odd dimension.*

**Proof.** Take  $g_R = g_L + 2\omega \otimes \omega$ . Applying the above proposition, if  $\Pi$  is a plane containing  $E$ , then  $K_R(\Pi) = -K_L(\Pi) > 0$ . Since  $E$  is also unitary and Killing for  $g_R$  (see [Corollary 2.4](#)), using Berger's theorem [\[5,4\]](#),  $M$  has odd dimension.  $\square$

**Example 4.3.** Since the Lorentzian Berger sphere  $(\mathbb{S}^{2n+1}, g_L)$  is obtained as the standard canonical variation of the Euclidean sphere along the Hopf vector field  $E$ , we have  $2 \leq K_E(\Pi)$  for all degenerate planes and the scalar curvature is  $S_L = 2n(2n + 3)$ .

Now we give an application using Bochner techniques. Given a Killing vector field  $U$  in a compact semi-Riemannian manifold  $(M, g)$ , it holds

$$\int_M \|A_U\|^2 dg = \int_M Ric(U, U) dg.$$

Therefore, in the Riemannian case, we have

$$0 \leq \int_M Ric_R(U, U) dg_R. \quad (9)$$

Moreover, if  $U$  is nonzero everywhere and we take  $\{e_1, \dots, e_{n-1}, \frac{U}{|U|}\}$  an orthonormal basis, then

$$\|A_U\|^2 = \sum_{i=1}^{n-1} g_R(\nabla_{e_i}^R U, \nabla_{e_i}^R U) + \frac{1}{g_R(U, U)} g_R(\nabla_U^R U, \nabla_U^R U)$$

and so we can refine the inequality [\(9\)](#) obtaining

$$0 \leq \int_M \left( Ric_R(U, U) - \frac{1}{g_R(U, U)} g_R(\nabla_U^R U, \nabla_U^R U) \right) dg_R. \quad (10)$$

In the Lorentzian case, even if  $U$  is timelike,  $\|A_U\|^2$  and  $\sum_{i=1}^{n-1} g_L(\nabla_{e_i}^L U, \nabla_{e_i}^L U)$  do not have sign and we cannot obtain the inequalities [\(9\)](#) nor [\(10\)](#). Nevertheless, in [\[21\]](#) it is observed that  $\|A_E\|^2 \geq 0$ , where  $E$  is the unitary of  $U$ , and thus if  $U$  is timelike and Killing it holds

$$\int_M \frac{1}{g_L(U, U)} Ric_L(U, U) dg_L \leq 0.$$

Using the canonical variation we can prove a similar inequality to (10) in the Lorentzian case. First, we need some preliminaries lemmas.

**Lemma 4.4.** *Let  $(M, g)$  be a semi-Riemannian manifold and  $E \in \mathfrak{X}(M)$  a vector field with  $g(E, E) = c \in \mathbb{R} - \{0\}$ . Suppose that  $E$  is orthogonally conformal, i.e.,  $(L_E g)(X, Y) = 2\rho g(X, Y)$  for all  $X, Y \in E^\perp$ , and take  $\lambda \in C^\infty(M)$ . Then  $U = \lambda E$  is conformal if and only if*

$$\begin{aligned} E(\lambda) &= \lambda\rho, \\ cX(\lambda) &= -\lambda g(\nabla_E E, X) \quad \text{for all } X \in E^\perp. \end{aligned}$$

**Proof.** It is enough to use the formula  $L_U = \lambda L_E + d\lambda \otimes \omega + \omega \otimes d\lambda$ .  $\square$

**Lemma 4.5.** *Let  $(M, g_L)$  be a Lorentzian manifold and  $U$  a timelike conformal/Killing vector field with unitary  $E$ . Then  $U$  is also conformal/Killing for the canonical variation along  $E$ .*

**Proof.** Just apply Corollary 2.4 and the above lemma.  $\square$

**Theorem 4.6.** *Let  $(M, g_L)$  be a compact Lorentzian manifold and  $U$  a timelike Killing vector field. Then*

$$0 \leq \int_M \left( Ric_L(U, U) - \frac{2}{g_L(U, U)} g_L(\nabla_U^L U, \nabla_U^L U) \right) dg_L.$$

**Proof.** Call  $E$  the unitary of  $U$  ( $U = \lambda E$ ) and consider  $g_t$  the canonical variation along  $E$ . For  $1 < t < 2$ , we have

$$0 \leq \int_M Ric_t(U, U) dg_t,$$

since  $g_t$  is Riemannian and  $U$  is Killing. Using Corollary 3.5,

$$\begin{aligned} Ric_t(U, U) &= \lambda^2 Ric_t(E, E) = \lambda^2 (Ric_L(E, E) - t \operatorname{div} \nabla_E^L E + t(t-2) \|A'_E\|^2) \\ &= Ric_L(U, U) + \frac{2t}{\lambda^2} g_L(\nabla_U^L U, \nabla_U^L U) - t \operatorname{div} \nabla_U^L U + t(t-2) \lambda^2 \|A'_E\|^2 \end{aligned}$$

But  $\|A'_E\|^2 \geq 0$  and  $dg_t = \sqrt{|1-t|} dg_L$ . Therefore

$$0 \leq \int_M \left( Ric_L(U, U) - \frac{2t}{g_L(U, U)} g_L(\nabla_U^L U, \nabla_U^L U) \right) dg_L,$$

for  $1 < t < 2$ . Taking  $t \rightarrow 1$  we get the result.  $\square$

**Example 4.7.** In general, the integral (10) is not positive in the Lorentzian case. In fact, let  $(N, g_0)$  be a compact Riemannian manifold,  $f \in C^\infty(N)$  a positive function and consider  $(M, g_L) = (N \times \mathbb{S}^1, g_0 - f^2 dt^2)$ . The vector field  $U = \partial_t$  is timelike and Killing and it is easy to show that

$$\int_M \left( Ric_L(U, U) - \frac{1}{g_L(U, U)} g_L(\nabla_U^L U, \nabla_U^L U) \right) dg_L = - \int_M g_L(\nabla f, \nabla f) dg_L \leq 0.$$

## 5. Standard canonical variation along a closed vector field

Suppose that  $E$  is a closed unitary vector field in a Riemannian manifold  $(M, g_R)$  and consider  $g_L = g_R - 2\omega \otimes \omega$  the standard canonical variation along it. From formula (4), we have

$$\nabla_U^L V = \nabla_U^R V + 2g_R(\nabla_U^R E, V)E.$$

In particular, the shape operators of the orthogonal leaves of  $E$  in  $(M, g_R)$  and  $(M, g_L)$  coincide.

**Proposition 5.1.** *Let  $(M, g_R)$  be a Riemannian manifold,  $E \in \mathfrak{X}(M)$  a closed unitary vector field and  $g_L$  the standard canonical variation along  $E$ .*

1. *If  $\Pi$  is a nondegenerate plane for  $g_L$ , then*

$$-\cos(2\theta)K_L(\Pi) = K_R(\Pi) + 2\sin^2(\theta)(\hat{K}_R(\mathfrak{p}(\Pi)) - K_R(\mathfrak{p}(\Pi))),$$

where  $\theta$  is the angle between  $E$  and  $\Pi$ ,  $\hat{K}_R$  is the induced curvature on the orthogonal leaves of  $E$  and  $\mathfrak{p}$  is the orthogonal projection onto  $E^\perp$ .

In particular,  $K_L(\Pi) = -K_R(\Pi)$  for any plane containing  $E$ .

2. *If  $\Pi$  is a degenerate plane for  $g_L$ , then*

$$\mathcal{K}_L^E(\Pi) = 2K_R(\Pi) + 2(\hat{K}_R(\mathfrak{p}(\Pi)) - K_R(\mathfrak{p}(\Pi))).$$

3. *For any  $v \in TM$ , it holds*

$$\begin{aligned} Ric_L(v, v) &= Ric_R(v, v) + 2g_R(A_E(v), v)div_R E \\ &\quad - 2g_R(A_E(v), A_E(v)) - 2g_R(R_{vE}^R E, v). \end{aligned}$$

4.  $S_L = S_R + 4E(div_R E) + 2(\|A_E\|^2 + (div_R E)^2)$ .

**Proof.** Suppose  $\Pi = span(X, V)$  where  $X$  and  $V$  are  $g_R$ -unitary,  $V = \alpha E + Y$  and  $X \perp Y \perp E$ . Applying Theorems 3.2, 3.6 and Proposition 3.11, we get

$$g_L(R_{VX}^L X, V) = g_R(R_{VX}^R X, V) + 2(g_R(A_E(X), X)g_R(A_E(Y), Y) - g_R(A_E(X), Y)^2)$$

and we can easily deduce points 1 and 2.

(3) It follows from Corollaries 3.5, 3.8 and Proposition 3.11.

(4) Use that  $Ric_R(E) + E(div_R E) + \|A_E\|^2 = 0$  and Corollary 3.9.  $\square$

**Theorem 5.2.** *Let  $(M, g_R)$  be a Riemannian manifold,  $E \in \mathfrak{X}(M)$  a closed unitary vector field and  $g_L$  the standard canonical variation along  $E$ . If  $E$  is complete and nonparallel, then there exists a point  $p \in M$  such that  $S_R(p) < S_L(p)$ .*

**Proof.** Suppose that  $S_L \leq S_R$  for all points in  $M$ . Using the above proposition,

$$2E(div_R E) + (div_R E)^2 \leq 2E(div_R E) + \|A_E\|^2 + (div_R E)^2 \leq 0.$$

Since  $E$  is complete,  $div_R E = 0$  and  $S_L = S_R + 2\|A_E\|^2 \geq S_R \geq S_L$ . Therefore  $\|A_E\|^2 = 0$  and  $E$  is parallel.  $\square$

Now, we are interested in knowing whether the completeness is preserved by the canonical variation. In general, this does not hold, as the following examples show.

**Example 5.3.** Take  $(L, g_0)$  a complete Riemann manifold and the warped product  $(\mathbb{R} \times L, -dt^2 + f^2(t)g_0)$ . The standard canonical variation along  $\partial_t$  is  $g_R = dt^2 + f^2(t)g_0$ , which is always complete. However, we can choose  $f$  such that  $-dt^2 + f^2(t)g_0$  is not complete, [22].

**Example 5.4.** Consider the Lorentzian plane  $(\mathbb{R}^2, g_L = -dx^2 + dy^2)$ . Call

$$f(x, y) = e^{-(x+y)}, \quad a(x, y) = \frac{1 + f(x, y)^2}{2f(x, y)}, \quad b(x, y) = \frac{1 - f(x, y)^2}{2f(x, y)}.$$

Now,  $E = a\partial_x + b\partial_y$  is a timelike unitary vector field and the standard canonical variation along it is  $g_R = (2a^2 - 1)dx^2 + (2b^2 + 1)dy^2 - 4ab dx dy$ . Take  $\gamma : [0, \infty) \rightarrow \mathbb{R}^2$  given by  $\gamma(t) = (t, t)$ . It escapes any compact subset of  $\mathbb{R}^2$ , but  $\lim_{s \rightarrow \infty} \int_0^s \sqrt{g_R(\gamma', \gamma')} = \frac{\sqrt{2}}{2}$ . Therefore,  $(\mathbb{R}^2, g_R)$  is incomplete.

An important case when completeness is preserved is when  $E$  is a Killing vector field. In this case, if  $(M, g_R)$  is complete, then it is also  $(M, g_L)$ . Using this, it follows that a compact Lorentzian manifold furnished with a timelike Killing vector field is complete, [20].

We can prove another case when completeness is assured. For this, we give the following definition.

**Definition 5.5.** Let  $\bar{M}$  be a nondegenerate hypersurface of a semi-Riemannian manifold  $(M, g)$ . We say that  $\bar{M}$  is strongly curved if its shape operator is semi-definite.

**Proposition 5.6.** Let  $(M, g_R)$  be a Riemannian manifold and  $E$  a complete, unitary and closed vector field. If the orthogonal leaves of  $E$  are complete and strongly curved, then  $(M, g_R)$  is complete.

**Proof.** We can suppose that  $M$  is simply connected. Hence, it splits as  $(\mathbb{R} \times L, ds^2 + g_s)$ , where  $\partial_s$  is identified with  $E$ ,  $L$  is an orthogonal leaf and  $g_s$  is a Riemannian metric on  $L$  for each  $s \in \mathbb{R}$ , [12].

Suppose that the shape operator of the orthogonal leaves is semi-definite negative. Let  $v \in T_x L$  and take  $V(s) = (0_s, v_x)$ . Then, since  $[V, E] = 0$ , we have  $\frac{d}{ds} g_R(V(s), V(s)) = 2g_R(\nabla_V^R E, V) \geq 0$ . Thus  $g_s(v, v)$  is a nondecreasing function.

Call  $d$  the distance induced by  $g_R$  in  $M$ ,  $d_s$  the distance induced by  $g_s$  in  $L$  and take  $\{p_n = (t_n, x_n)\}$  a Cauchy sequence. Since  $d(p_n, p_m) \geq |t_n - t_m|$  we have that  $\{t_n\}$  converges to, say,  $t_0$  and we can suppose that  $|t_n - t_0| \leq \delta$  for all  $n \in \mathbb{N}$  and certain  $\delta \in \mathbb{R}$ . Given  $0 < \varepsilon < \delta$  there is  $n_0 \in \mathbb{N}$  such that  $d(p_m, p_n) < \varepsilon$  for  $m, n \geq n_0$ . Let  $\gamma(s) = (s(t), x(t))$  be a curve with  $\gamma(0) = p_m$  and  $\gamma(1) = p_n$ . We can suppose that  $|s(t) - t_0| \leq 2\delta$  since on the contrary case,  $L(\gamma) > \delta > d(p_m, p_n)$ . Now,

$$\begin{aligned} L(\gamma) &= \int_0^1 \sqrt{s'(t)^2 + g_{s(t)}(x'(t), x'(t))} \geq \int_0^1 \sqrt{g_{s^*}(x'(t), x'(t))} \\ &\geq d_{s^*}(x_n, x_m), \end{aligned}$$

where  $s^* = t_0 - 2\delta$ , and therefore  $d_{s^*}(x_n, x_m) \leq d(p_n, p_m) < \varepsilon$ . Hence  $\{x_n\}$  is a Cauchy sequence in  $(L, g_{s^*})$  and so it converges.  $\square$

**Theorem 5.7.** Let  $(M, g_L)$  a Lorentzian manifold and  $E$  a complete timelike unitary and closed vector field. If the orthogonal leaves of  $E$  are complete and strongly curved, then the standard canonical variation  $g_R$  along  $E$  is also complete.

**Proof.** As it was said at the beginning of this section, the shape operators coincide in  $(M, g_L)$  and  $(M, g_R)$ , thus orthogonal leaves of  $E$  are strongly curved in  $(M, g_R)$  too. Applying the above proposition we obtain that  $g_R$  is complete.  $\square$

## 6. Lightlike hypersurfaces

Let  $(M, g_L)$  be a Lorentzian manifold,  $E \in \mathfrak{X}(M)$  a timelike unitary vector field and  $\overline{M}$  a lightlike hypersurface. We can fix a lightlike vector field  $\xi \in \mathfrak{X}(\overline{M})$  with  $g_L(E, \xi) = \frac{1}{\sqrt{2}}$  and consider the screen distribution given by  $\mathcal{S} = T\overline{M} \cap E^\perp$ . Given  $U, V \in \mathfrak{X}(\overline{M})$ , the second fundamental form of  $\overline{M}$  is  $B(U, V) = -g_L(\nabla_U^L \xi, V)$ . The hypersurface  $\overline{M}$  is totally geodesic if  $B = 0$  and totally umbilic if  $B = \rho g$  for certain  $\rho \in C^\infty(\overline{M})$ .

We can decompose

$$\begin{aligned}\nabla_U^L V &= \overline{\nabla}_U^L V + B(U, V)N, \\ \nabla_U^L \xi &= -\tau(U)\xi - A^*(U),\end{aligned}$$

where  $\overline{\nabla}_U^L V \in T\overline{M}$ ,  $A^*(U) \in \mathcal{S}$ ,  $\tau$  is a one form and  $N = \sqrt{2}E + \xi$  is the transverse vector field to  $\overline{M}$ . Recall that  $\nabla_\xi^L \xi = -\tau(\xi)\xi$  and  $N$  is lightlike and orthogonal to  $\mathcal{S}$ . Moreover,  $\tau(\xi) = \sqrt{2}g_L(\nabla_\xi^L E, \xi)$ .

Observe that  $X_0 = \frac{1}{\sqrt{2}}E + \xi$  is orthogonal to  $E$  and therefore  $\xi = -\frac{1}{\sqrt{2}}E + X_0$  and  $N = \frac{1}{\sqrt{2}}E + X_0$ , i.e.,  $N$  is the symmetric of  $-\xi$  with respect to  $E$ .

If we consider  $g_R$  the standard canonical variation, then we have  $g_R(\xi, \xi) = g_R(N, N) = 1$  and  $N$  is  $g_R$ -normal to  $\overline{M}$ . Call  $\mathbb{I}$  the second fundamental form of  $\overline{M}$  in  $(M, g_R)$ .

**Proposition 6.1.** *Given  $X, Y \in \mathcal{S}$ , it holds*

$$\begin{aligned}\mathbb{I}(X, Y) &= \left( B(X, Y) - \frac{1}{\sqrt{2}}(L_E g_L)(X, Y) \right) N, \\ \mathbb{I}(X, \xi) &= -g_L(\nabla_{E+\sqrt{2}\xi}^L E, X)N, \\ \mathbb{I}(\xi, \xi) &= -(2g_L(\xi, \nabla_E^L E) + \tau(\xi))N.\end{aligned}$$

**Proof.** By Proposition 2.3,  $g_R(D(X, Y), N) = -\frac{1}{\sqrt{2}}(L_E g)(X, Y)$  and thus we have

$$\begin{aligned}g_R(\mathbb{I}(X, Y), N) &= g_R(\nabla_X^L Y, N) - \frac{1}{\sqrt{2}}(L_E g)(X, Y) \\ &= B(X, Y) - \frac{1}{\sqrt{2}}(L_E g)(X, Y).\end{aligned}$$

On the other hand  $g_R(D(X, \xi), N) = -\sqrt{2}g_L(\nabla_\xi^L E, X) - g_L(\nabla_E^L E, X)$  and since  $g_R(\nabla_X^L \xi, N) = 0$ , we get  $\mathbb{I}(X, \xi) = -g_L(\nabla_{E+\sqrt{2}\xi}^L E, X)N$ .

Finally,  $g_R(D(\xi, \xi), N) = -\sqrt{2}g(\xi, \nabla_N^L E) = -(2g_L(\xi, \nabla_E^L E) + \tau(\xi))$ , and since  $g_R(\nabla_\xi^L \xi, N) = 0$ , we have  $\mathbb{I}(\xi, \xi) = -(2g_L(\xi, \nabla_E^L E) + \tau(\xi))N$ .  $\square$

If  $E$  is parallel and  $\overline{M}$  is totally geodesic, then it is also totally geodesic in  $(M, g_R)$ , but however, if  $\overline{M}$  is umbilic, then it is not umbilic in  $(M, g_R)$ .

Recall that the lightlike mean curvature of  $\overline{M}$  is  $H_L = \sum_{i=1}^{n-2} B(e_i, e_i)$ , where  $\{e_1, \dots, e_{n-2}\}$  is an orthonormal basis of  $\mathcal{S}$ , and the mean curvature of  $\overline{M}$  as hypersurface of  $(M, g_R)$  is  $H_R = \sum_{i=1}^{n-1} g_R(\mathbb{I}(v_i, v_i), N)$ ,  $\{v_1, \dots, v_{n-1}\}$  being a  $g_R$  orthonormal basis of  $T\overline{M}$ .

**Proposition 6.2.** *Let  $(M, g_L)$  be a Lorentzian manifold,  $E \in \mathfrak{X}(M)$  a timelike unitary vector field and  $g_R$  the standard canonical variation along it. Take  $\overline{M}$  a lightlike hypersurface.*

- $H_R = H_L - \sqrt{2} \operatorname{div}_L E + \tau(\xi)$ .
- If  $\overline{M}$  is compact and orientable, then  $\int_{\overline{M}} H_L dg_R = 0$ .

**Proof.** If  $\{e_1, \dots, e_{n-2}\}$  is an orthonormal basis in  $\mathcal{S}$ , then  $\{e_1, \dots, e_{n-2}, \xi\}$  is a  $g_R$ -orthonormal basis of  $\overline{M}$ . Therefore, using Proposition 6.1,

$$H_R = H_L - \sqrt{2} \sum_{i=1}^{n-2} g_L(\nabla_{e_i}^L E, e_i) - 2g_L(\xi, \nabla_E^L E) - \tau(\xi).$$

But now, observe that  $\{e_1, \dots, e_{n-2}, \sqrt{2}\xi + E, E\}$  is an orthonormal basis and so

$$\begin{aligned} \operatorname{div}_L E &= \sum_{i=1}^{n-2} g_L(\nabla_{e_i}^L E, e_i) + g_L(\nabla_{\sqrt{2}\xi+E}^L E, \sqrt{2}\xi + E) \\ &= \sum_{i=1}^{n-2} g_L(\nabla_{e_i}^L E, e_i) + \sqrt{2}\tau(\xi) + \sqrt{2}g_L(\nabla_E^L E, \xi). \end{aligned}$$

For the second point, since  $g_R(D(e_i, \xi), e_i) = 0$  (Corollary 2.4), we have  $\operatorname{div}_R^{\overline{M}} \xi = -H_L$ .  $\square$

We can easily obtain the following generalization of Theorem 8 in [3].

**Corollary 6.3.** *Let  $(M, g_L)$  be a Lorentzian manifold,  $E$  a timelike unitary Killing vector field and  $\overline{M}$  a lightlike hypersurface. Consider  $g_R$  the standard canonical variation along  $E$ . The lightlike mean curvature of  $\overline{M}$  vanishes if and only if the mean curvature of  $\overline{M}$  as hypersurface of  $(M, g_R)$  vanishes.*

In a time-orientable and orientable Lorentzian manifold satisfying the null convergence condition ( $\operatorname{Ric}(u, u) \geq 0$  for all lightlike vector  $u$ ) any compact lightlike hypersurface with nonpositive (or non-negative) mean curvature is totally geodesic. In fact, we only have to take into account second point of Proposition 6.2 and the well-known Raychaudhuri equation (see for example [8,10,17]). On the other hand, a standard application of the Raychaudhuri equation also gives us that a lightlike hypersurface is totally geodesic if the null convergence condition and the completeness of lightlike geodesics of the hypersurface are assumed. Observe that this result is not applicable to above situation because the compactness does not ensure, in general, the completeness of the lightlike geodesics of the hypersurface.

We can obtain other results ensuring that a compact lightlike hypersurface is totally geodesic under curvature hypotheses. First, we need the following lemma.

**Lemma 6.4.** *Let  $(M, g_L)$  be a Lorentzian manifold and  $U \in \mathfrak{X}(M)$  a timelike Killing vector field with unitary  $E$ . If  $\lambda = |U|$ , then for any  $X \perp E$  we have*

$$g_L(R_{XE}^L E, X) = \frac{1}{\lambda} g_L(\nabla_X^L \nabla \lambda, X) + g_L(\nabla_X^L E, \nabla_X^L E). \quad (11)$$

Recall that if  $U$  is a Killing vector field and  $\overline{M}$  a compact hypersurface of a Riemannian manifold  $(M, g_R)$  with normal unitary  $N$  and mean curvature  $H_R$ , then

$$\int_{\overline{M}} g_R(U, N) H_R dg_R = 0. \quad (12)$$

Moreover, if  $\overline{M}$  has constant mean curvature, then

$$\int_{\overline{M}} g_R(U, N) \|S\|^2 - Ric_R(N, \overline{U}) dg_R = 0, \quad (13)$$

where  $S(X) = -\nabla_X^R N$  is the shape operator of  $\overline{M}$  and  $\overline{U}$  is the  $g_R$ -projection of  $U$  onto  $\overline{M}$ , [24].

**Theorem 6.5.** *Let  $(M, g_L)$  be a Lorentzian manifold,  $U \in \mathfrak{X}(M)$  a timelike Killing vector field and  $\overline{M}$  a compact lightlike hypersurface. Take  $E$  the unitary of  $U$ ,  $\lambda = |U|$  and  $\xi \in \mathcal{X}(\overline{M})$  a lightlike vector field with  $g_L(\xi, E) = \frac{1}{\sqrt{2}}$ . Call  $X_0$  the orthogonal projection of  $\xi$  onto  $E^\perp$  and  $N$  the symmetric of  $-\xi$  with respect to  $E$ . If:*

- $H_L - g_L(\nabla \ln \lambda, \xi)$  has sign.
- $0 \leq Ric_L(N, \xi) + \Delta \ln \lambda - 4g_L(\nabla_{X_0}^L E, \nabla_{X_0}^L E)$ .

Then,  $\overline{M}$  is totally geodesic.

**Proof.** First, observe that  $\nabla_E^L E = \nabla \ln \lambda$  and  $\tau(\xi) = -g_L(\xi, \nabla \ln \lambda)$ . Take  $g_R$  the canonical variation along  $E = \frac{U}{\lambda}$ . We know that  $U$  is also Killing for  $g_R$  (Lemma 4.5) and it is easy to show that  $g_R(U, N) = \frac{\lambda}{\sqrt{2}}$ . Applying formula (12), we have  $\int_{\overline{M}} \lambda H_R dg_R = 0$ . But, from Proposition 6.2,  $H_R = H_L - g_L(\nabla \ln \lambda, \xi)$  which has sign. Therefore,  $H_R = 0$  and since the projection of  $U$  onto  $\overline{M}$  is given by  $-\frac{\lambda}{\sqrt{2}}\xi$ , formula (13) gives us

$$\int_{\overline{M}} \lambda \|S\|^2 dg_R = - \int_{\overline{M}} \lambda Ric_R(N, \xi) dg_R.$$

Using Corollaries 3.5 and 3.8 and Lemma 6.4,

$$\begin{aligned} Ric_R(N, \xi) &= -\frac{1}{2} Ric_R(E, E) + Ric_R(X_0, X_0) \\ &= Ric_L(N, \xi) + \Delta \ln \lambda - 6g_L(\nabla_{X_0}^L E, \nabla_{X_0}^L E) \\ &\quad + 2 \left( g_L(R_{X_0 E}^L E, X_0) - \frac{1}{\lambda} g_L(\nabla_{X_0}^L \nabla \lambda, X_0) \right) \\ &= Ric_L(N, \xi) + \Delta \ln \lambda - 4g_L(\nabla_{X_0}^L E, \nabla_{X_0}^L E). \end{aligned}$$

Thus,  $\|S\|^2 = 0$  and from Proposition 6.1,  $\overline{M}$  is a totally geodesic lightlike hypersurface of  $(M, g_L)$ .  $\square$

**Corollary 6.6.** *Let  $(M, g_L)$  be a Lorentzian manifold,  $E \in \mathfrak{X}(M)$  a timelike unitary Killing vector field and  $\overline{M}$  a compact lightlike hypersurface. Take  $\xi \in \mathcal{X}(\overline{M})$  a lightlike vector field with  $g_L(\xi, E) = \frac{1}{\sqrt{2}}$  and  $N$  the symmetric of  $-\xi$  with respect to  $E$ . If:*

- $H_L$  has sign.
- $0 \leq Ric_L(N, \xi) + 2K_L(\text{span}(\xi, N))$ .

Then  $\overline{M}$  is totally geodesic.

**Proof.** From Lemma 6.4,

$$g_L(\nabla_{X_0}^L E, \nabla_{X_0}^L E) = g_L(R_{X_0 E}^L E, X_0) = -\frac{1}{2} K^L(\text{span}(\xi, N)). \quad \square$$

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