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Jinghao Huang, Yongjin Li

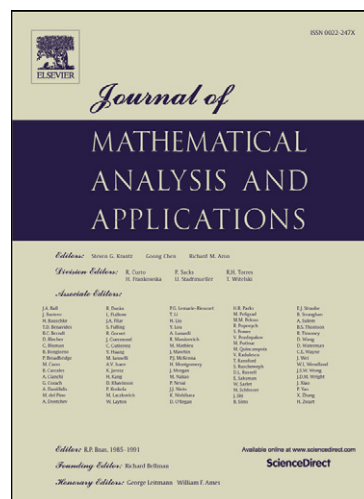
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HYERS-ULAM STABILITY OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

JINGHAO HUANG AND YONGJIN LI

ABSTRACT. In this paper, the stability of some classes of linear functional differential equations was discussed by direct method, iteration method, fixed point method and open mapping theorem. It is shown that the Hyers-Ulam stability holds true for $y^{(n)} = g(t)y(t - \tau) + h(t)$. The stability of functional differential equations with multiple delays of first order and general delay differential equations also have been discussed.

1. INTRODUCTION AND PRELIMINARIES

In 1940, S.M. Ulam [26] raised a problem whether an approximate solution of a functional equation can be approximated by a solution of the corresponding equation. This question was first answered by D.H. Hyers [12] one year later. Thereafter, T. Aoki [2], D.G. Bourgin [4] and Th.M. Rassias [24] improved the result of D.H. Hyers. For more details and further discussions, we refer the readers to the book by S.-M. Jung [13]. In recent years, accompanied by the development of the Hyers-Ulam stability of functional equations e.g. [5–7], many mathematicians paid attention to the problem of differential equations and for the papers concerning the stability of ordinary differential equations we refer to [14, 16, 18, 19, 22]. Further, the interested readers can see [1, 9, 15] concerning the stability of partial differential equations.

Delay differential equations (DDEs) are a special type of ordinary differential equations in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times (Advanced differential equations (ADEs) for the case of future times), which are very important in life science and engineering. DDEs are also called functional differential equations (FDEs) with delays [27].

To the best of our knowledge, the first mathematicians who investigated the stability of DDEs are S.-M. Jung and J. Brzdęk [16]. They proved the Hyers-Ulam stability of $y'(t) = \lambda y(t - \tau)$ with an initial condition, where $\lambda > 0$ and $\tau > 0$ are

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Corresponding author: Yongjin Li, E-mail address: stslyj@mail.sysu.edu.cn.

real constants. Motivated by their work, we investigate the Hyers-Ulam stability of more general linear FDEs with constant delay(s) on a bounded interval by some other methods. (The case for advanced functional differential equations is absolutely the same.)

$\Gamma^n[a, b; \tau]$ denotes the family of all continuous functions mapping the real interval $[a - \tau, b]$ into $\mathbb{R}$ which are n times continuously differentiable on $[a, b]$. Without loss of generality, we focus our attention on $\Gamma^n[0, b; \tau]$ in this paper.

2. STABILITY OF $y^{(n)} = g(t)y(t - \tau) + h(t)$

In this section, we study the Hyers-Ulam stability of delay differential equations of the form $y^{(n)} = g(t)y(t - \tau) + h(t)$ on $[0, \tau]$ by some different methods, where $\tau > 0$ is a constant.

Now, we spend some time presenting in details some possibilities for the proof of Hyers-Ulam stability of $y^{(n)} = g(t)y(t - \tau) + h(t)$, to see the advantages and the disadvantages.

2.1. Open mapping theorem method.

In 2003, H. Takagi, T. Miura and S.-E. Takahasi [20] showed that open mapping theorem is closely linked with the Hyers-Ulam stability, which makes the stability problem of linear bounded operator much more explicit.

We need some terminologies before displaying their result. Let $\ker T$ denote the kernel of the bounded linear operator $T : A \rightarrow B$. Define the induced one-to-one operator $\tilde{T}$ from the quotient space $A/\ker T$ into B by

$$\tilde{T}(u + \ker T) := Tu \quad \forall u \in A.$$

DEFINITION 2.1. *We say an operator T has the Hyers-Ulam stability, if there is a constant K with the following properties:*

For any $g \in T(A)$, $\varepsilon \geq 0$ and $f \in A$ satisfying $\|Tf - g\| \leq \varepsilon$, there exists $f_0 \in A$ such that $Tf_0 = g$ and $\|f - f_0\| \leq K\varepsilon$.

We call K a Hyers-Ulam stability constant of operator T .

DEFINITION 2.2. *We say an equation $Tf = g$ has the Hyers-Ulam stability, if there is a constant K with the following properties:*

For any $\varepsilon \geq 0$ and $f \in A$ satisfying $\|Tf - g\| \leq \varepsilon$, there exists $f_0 \in A$ such that $Tf_0 = g$ and $\|f - f_0\| \leq K\varepsilon$.

We call K a Hyers-Ulam stability constant of equation $Tf = g$.

REMARK 2.3. *Obviously, if operator T has the Hyers-Ulam stability and $g \in T(A)$, then equation $Tf = g$ has the Hyers-Ulam stability.*

THEOREM 2.4. *Let A and B be Banach spaces and T be a bounded linear operator from A into B . Then the following statements are equivalent:*

- (a) T has the Hyers-Ulam stability;
- (b) The range of T is closed;
- (c) $\tilde{T}^{-1}$ is a bounded linear operator.

Moreover, if one of the conditions is true, then the infimum of the set of Hyers-Ulam stability constants is $\|\tilde{T}^{-1}\|$.

Next, we give the definition of the Hyers-Ulam stability of Eq.(1) in the classic sense.

DEFINITION 2.5. *We say that Eq.(1) has the Hyers-Ulam stability on $[-\tau, b]$ if there exists a constant $K > 0$ with the following property: for every $\varepsilon > 0$, $y \in \Gamma^n[0, b; \tau]$, if*

$$|y^{(n)}(t) - g(t)y(t - \tau)| \leq \varepsilon$$

for all $t \in [0, b]$, then there exists some $y_0 \in \Gamma^n[0, b; \tau]$ satisfying

$$y_0^{(n)}(t) = g(t)y_0(t - \tau)$$

for all $t \in [0, b]$ with $|y(t) - y_0(t)| \leq K\varepsilon$ for all $t \in [-\tau, b]$.

THEOREM 2.6. *Given $g(t) \in C[0, b]$, then*

$$y^{(n)}(t) = g(t)y(t - \tau) \tag{1}$$

has the Hyers-Ulam stability on $[-\tau, b]$.

Proof. Firstly, for every $y \in \Gamma^n[0, b; \tau]$, we equip it with the norm

$$\|y\| = \max \left(\max_{1 \leq i \leq n} \sup_{x \in [0, b]} |y^{(i)}(x)|, \sup_{x \in [-\tau, b]} |y(x)| \right).$$

It is easy to verify that $(\Gamma^n[0, b; \tau], \|\cdot\|)$ and $C[0, b]$ are Banach space, we omit the proof here. (For the details of the proof, we refer the readers to [10] and [11]).

Define an operator $T : (\Gamma^n[0, b; \tau], \|\cdot\|) \rightarrow C[0, b]$ by

$$(Tf)(t) = f^{(n)}(t) - g(t)f(t - \tau)$$

for all $f(t) \in \Gamma^n[0, b; \tau]$.

Obviously, T is well-defined and is a linear operator. Moreover,

$$\begin{aligned} \|T\| &= \sup_{\|f\|=1} \|Tf\| \\ &= \sup_{\|f\|=1} \sup_{t \in [0, b]} \left| f^{(n)}(t) - g(t)f(t - \tau) \right| \\ &\leq \sup_{\|f\|=1} \sup_{t \in [0, b]} \left(\left| f^{(n)}(t) \right| + |g(t)f(t - \tau)| \right) \\ &\leq 1 + \sup_{t \in [0, b]} |g(t)| < \infty \end{aligned}$$

since $g(t)$ is continuous on $[0, b]$. That is to say, T is a linear bounded operator.

Next, we shall show that the range of T is closed. Obviously, $Tf \in C[0, b]$ for every $f \in \Gamma^n[0, b; \tau]$. Conversely, from the basic theory of functional differential equations [27] (see also subsection 2.2), for every $y \in C[0, b]$, there are some $f \in \Gamma^n[0, b; \tau]$ such that $Tf = y$. Moreover, $C[0, b]$ is a Banach space, from which it follows that the range of T is closed.

It follows from Theorem 2.4 that operator T has the Hyers-Ulam stability. By noticing that $0 \in T(\Gamma^n[0, b; \tau])$, we obtain that: for every given $\varepsilon \geq 0$, there is a constant $K > 0$ such that, if $f \in \Gamma^n[0, b; \tau]$ satisfies

$$|f^{(n)}(t) - g(t)f(t - \tau)| \leq \varepsilon,$$

for every $t \in [0, b]$, i.e., $\|Tf\| \leq \varepsilon$ for all $t \in [0, b]$, then there exists a solution $f_0 \in \Gamma^n[0, b; \tau]$ of Eq. (1) with

$$\|f_0 - f\| \leq K\varepsilon.$$

Moreover, according to the definition of the norm in $\Gamma^n[0, b; \tau]$, we have $\sup_{t \in [-\tau, b]} |f_0(t) - f(t)| \leq \|f_0 - f\| \leq K\varepsilon$, which implies that the Hyers-Ulam stability in the classic sense also holds true. $\square$

COROLLARY 2.7. *The nonhomogeneous equation $y^{(n)}(t) = g(t)y(t - \tau) + h(t)$ also has the Hyers-Ulam stability, where $h(t) \in C[0, b]$.*

Proof. It is suffice to show that operator

$$(Hy)(t) := y^{(n)}(t) - g(t)y(t - \tau) - h(t)$$

has the Hyers-Ulam stability, since $0 \in T(\Gamma^n[0, b; \tau])$.

Since $h \in C[0, b]$ and the operator T has the Hyers-Ulam stability, namely: for any $g_0 \in C[0, b]$, $\varepsilon \geq 0$ and $f \in \Gamma^n[0, b; \tau]$ with $\|Tf - g_0 - h\| \leq \varepsilon$, there exists an $f_0 \in \Gamma^n[0, b; \tau]$ such that $Tf_0 - g_0 - h = 0$ with a Hyers-Ulam constant K .

The above statement is equivalent to the following:

for any $g_0 \in C[0, b]$, $\varepsilon \geq 0$ and $f \in \Gamma^n[0, b; \tau]$ with $\|Hf - g_0\| \leq \varepsilon$, there exists an $f_0 \in \Gamma^n[0, b; \tau]$ such that $Hf_0 - g_0 = 0$ with a Hyers-Ulam constant K . Namely: H has the Hyers-Ulam stability.

Obviously, the Hyers-Ulam stability in the classic sense also holds true for the corresponding nonhomogeneous equation. $\square$

It is a question whether $\|\tilde{T}^{-1}\|$ is a Hyers-Ulam constant of T ? Namely, can we find an approximate element such that a Hyers-Ulam constant of Eq.(1) is $\|\tilde{T}^{-1}\|$? For more results concerning the best Hyers-Ulam constant, we refer the readers to [21].

2.2. Direct method.

In this part, we figure out a Hyers-Ulam stability constant of Eq.(1) by direct method [14] and iteration method [18, 19, 22], and prove the Hyers-Ulam stability holds true for it incidentally.

As we have seen, the theorem proved by H. Takagi et al. [20] is a very useful tool to know whether an operator has the Hyers-Ulam stability. But it is unknown that whether the infimum $\|\tilde{T}^{-1}\|$ of the set of all Hyers-Ulam stability constants is a Hyers-Ulam stability constant [21], and $\|\tilde{T}^{-1}\|$ is not easy to be worked out in most cases (see subsection 2.1). In fact, it is much easier to figure out a Hyers-Ulam constant by direct method.

THEOREM 2.8. *If a function $f \in \Gamma^n[0, m \cdot \tau; \tau]$ satisfies*

$$|f^{(n)}(t) - g(t)f(t - \tau) - h(t)| \leq \varepsilon$$

for all $t \in [0, m\tau]$, where m is an positive integer. Then there exists a solution $f_0 \in \Gamma^n[0, m\tau; \tau]$ of $y^{(n)}(t) = g(t)y(t - \tau) + h(t)$ with

$$|f(t) - f_0(t)| \leq K\varepsilon$$

for all $t \in [-\tau, m\tau]$, where K is a positive constant only depending on $g(t)$ and the interval $[-\tau, m\tau]$.

Proof. Define a function $f_0 \in \Gamma^n[0, \tau; \tau]$ by

$$\begin{aligned} f_0(t) &= f(t), \quad t \in [-\tau, 0] \\ f_0(t) &= \int_{k\tau}^t \left[\cdots \int_{k\tau}^{v_3} \left[\int_{k\tau}^{v_2} \left(g(v_1)f_0(v_1 - \tau) + h(v_1) \right) dv_1 + f_0^{(n-1)}(k\tau) \right] dv_2 \cdots \right. \\ &\quad \left. + f_0'(k\tau) \right] dv_n + f_0(k\tau), \quad t \in [k\tau, (k+1)\tau] \end{aligned}$$

for every $k \in \{0, 1, \dots, m-1\}$, where $f_0^{(i)}(0) = f^{(i)}(0^+)$ for all $0 \leq i \leq n-1$ and $f_0^{(i)}(k\tau) = f_0^{(i)}(k\tau^-)$ for all $k \geq 1, 0 \leq i \leq n-1$.

Obviously, f_0 is well-defined and it is a solution of Eq.(1). Notice that

$$|f_0(t) - f(t)| = 0$$

on $[-\tau, 0]$,

$$\begin{aligned} & |f_0^{(n)}(t) - f^{(n)}(t)| \\ & \leq |g(t)f_0(t - \tau) + h(t) - g(t)f(t - \tau) - h(t)| \\ & + |f^{(n)}(t) - g(t)f(t - \tau) - h(t)| \\ & \leq |g(t)f_0(t - \tau) - g(t)f(t - \tau)| + \varepsilon \\ & \leq |f_0(t - \tau) - f(t - \tau)| \cdot \sup_{t \in [(j-1)\tau, j\tau]} |g(t)| + \varepsilon \end{aligned}$$

on $[(j-1)\tau, j\tau]$ for all $1 \leq j \leq m$, and

$$\begin{aligned} \sup_{t \in [(j-1)\tau, j\tau]} |f_0^{(i)}(t) - f^{(i)}(t)| & \leq \sup_{t \in [(j-1)\tau, j\tau]} |f_0^{(i+1)}(t) - f^{(i+1)}(t)| \cdot \tau \\ & + |f_0^{(i)}((j-1)\tau) - f^{(i)}((j-1)\tau)| \end{aligned}$$

for all $0 \leq i \leq n-1$ and $1 \leq j \leq m$. We define a sequence $\{a_{i,j}\}_{0 \leq i \leq n, 0 \leq j \leq m}$ by

$$\begin{aligned} a_{i,0} &= 0, & \forall 0 \leq i \leq n; \\ a_{n,j} &= a_{0,j-1} \cdot \sup_{t \in [(j-1)\tau, j\tau]} |g(t)| + 1, & \forall 1 \leq j \leq m; \\ a_{i,j} &= \tau a_{i+1,j} + a_{i,j-1}, & \forall 0 \leq i < n-1. \end{aligned}$$

It is easy to verify that

$$|f_0^{(i)}(t) - f^{(i)}(t)| \leq a_{i,j} \cdot \varepsilon$$

on $[(j-1)\tau, j\tau]$ for each $0 \leq i \leq n-1$ and each $1 \leq j \leq m$.

Obviously, $a_{0,m} \geq a_{0,j}$ for all $0 \leq j \leq m$, since every $a_{i,j}$ is nonnegative. Then

$$|f_0(t) - f(t)| \leq a_{0,m} \varepsilon$$

for all $t \in [-\tau, m\tau]$. Moreover, $\{a_{i,j}\}$ only depends on $g(t)$ and the interval. Let $K = a_{0,m}$ and this completes our proof. $\square$

REMARK 2.9. For an given interval $[-\tau, b]$, there is a minimum integer m such that $m\tau \geq b$, which implies that Eq.(1) has the Hyers-Ulam stability on $[-\tau, b]$.

REMARK 2.10. We note that f_0 is not necessarily the best approximate element. For example:

Given an equation $y'(t) = y(t-1)$ and $f = 1$ on $[-1, 1]$. We construct a solution f_0 as the above theorem, then

$$|f(t) - f_0(t)| \leq 1$$

for all $t \in [-1, 1]$, but we can find a function $f_1(t) \in \Gamma^1[0, 1; 1]$ defined by

$$f_1(t) := \begin{cases} \frac{2}{3} & \forall t \in [-1, 0] \\ \frac{2}{3}(t+1) & \forall t \in [0, 1], \end{cases}$$

which is a solution of $y'(t) = y(t-1)$ with

$$|f(t) - f_1(t)| \leq \frac{1}{3}$$

for all $t \in [-1, 1]$.

2.3. Fixed point method.

Fixed point method is the second most popular technique of proving the Hyers-Ulam stability of functional equations [8, 25], and some kinds of ordinary differential equations were also investigated [10, 17] by this method. Fixed point method is very convenient, but not accurate in most cases. We can prove the Hyers-Ulam stability of Eq.(1) on a local interval by fixed point method, and extend the solution as we did in [10].

In fact, using fixed point method to investigate the stability of the linear functional differential equations (1) makes heavy work for simple problems, it is just a complex version of direct method.

For example, given

$$|f^{(n)}(t) - g(t)f(t-\tau) - h(t)| \leq \varepsilon$$

for all $t \in [-\tau, \tau]$. We define an operator $T : \Gamma^n[0, \tau; \tau] \rightarrow \Gamma^n[0, \tau; \tau]$ by

$$Tf(t) := \begin{cases} f(t), & [-\tau, 0]; \\ \int_0^t [\cdots \int_0^{v_3} [\int_0^{v_2} (g(v_1)f(v_1-\tau) + h(v_1))dv_1 \\ + f^{(n-1)}(0)]dv_2 \cdots + f'(0)]dv_n + f(0), & [0, \tau]. \end{cases}$$

Then we can use the Banach's contraction mapping principle [3] on $[0, \mu]$ to find out a solution of the corresponding equation, where the positive integer μ is small enough. However, as we have seen, Tf is a solution of Eq.(1), which implies that the iteration process of the contraction mapping is redundant.

2.4. Example.

EXAMPLE 2.11. Given $x(t) \in \Gamma^2[0, 10; 2] \cap \Gamma^1[-1, 10; 1]$ satisfying the following inequality

$$|x''(t) - (a+b)x'(t-1) + ab \cdot x(t-2)| \leq \varepsilon$$

for all $t \in [0, 10]$. Let $y(t) = x'(t) - ax(t-1)$ for all $t \in [-1, 10]$, then $y(t) \in \Gamma^1[0, 10; 1]$ with

$$|y'(t) - by(t-1)| \leq \varepsilon$$

for all $t \in [0, 10]$. There exists a function $y_0(t) \in \Gamma^1[0, 10; 1]$ satisfying $y'_0(t) = by_0(t-1)$ for all $t \in [0, 10]$ with $|y_0(t) - y(t)| \leq m_1\varepsilon$ for all $t \in [-1, 10]$, where m_1 is a nonnegative constant. That is,

$$|y_0(t) - [x'(t) - ax(t-1)]| \leq m_1\varepsilon$$

for all $t \in [-1, 10]$. By an argument similar to the above, there exists a function $x_0(t) \in \Gamma^1[-1, 10; 1]$ satisfying

$$y_0(t) - [x'_0(t) - ax_0(t-1)] = 0$$

for all $t \in [-1, 10]$ with $|x_0(t) - x(t)| \leq m_1 \cdot m_2 \varepsilon$ for all $t \in [-2, 10]$, where m_2 is a nonnegative constant.

Moreover, since $x'_0(t) = y_0(t) + ax_0(t-1)$ and $y_0 \in \Gamma^1[0, 10; 1]$, $x(t) \in \Gamma^2[0, 10; 2] \cap \Gamma^1[-1, 10; 1]$ with

$$x''_0(t) - (a+b)x'_0(t-1) + ab \cdot x_0(t-2) = 0$$

for $t \in [0, 10]$. Namely, $x''(t) = (a+b)x'(t-1) - ab \cdot x(t-2)$ has the Hyers-Ulam stability on $[-2, 10]$.

3. FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MULTIPLE DELAYS

Assume that $0 < \tau_1 \leq \tau_2 \leq \dots \leq \tau_m$. Using the open mapping method, we prove the Hyers-Ulam stability of functional differential equations of first order with multiple delays of the form

$$y'(t) = \sum_{i=1}^m g_i(t)y(t-\tau_i) + h(t), \quad (2)$$

where $g_1, g_2, \dots, g_m, h \in C[0, b]$. We remark here that the direct method is also available to figure out a Hyers-Ulam stability constant of Eq.(2).

THEOREM 3.1.

$$y'(t) = \sum_{i=1}^m g_i(t)y(t-\tau_i) \quad (3)$$

has the Hyers-Ulam stability on $[-\tau_m, b]$.

Proof. Let every $y \in \Gamma^1[0, b; \tau_m]$ equip with the norm

$$\|y\| = \max \left(\sup_{x \in [0, b]} |y'(x)|, \sup_{x \in [-\tau_m, b]} |y(x)| \right),$$

it follows from theorem 2.6 that $(\Gamma^1[0, b; \tau_m], \|\cdot\|)$ is a Banach space.

Define an operator $T : (\Gamma^1[0, b; \tau_m], \|\cdot\|) \rightarrow C[0, b]$ by

$$(Ty)(t) = y'(t) - \sum_{i=1}^m g_i(t)y(t-\tau_i)$$

for every $y(t) \in \Gamma^1[0, b; \tau_m]$.

By an argument similar to Theorem 2.6, T is a linear bounded operator with a closed range.

So by the Theorem 2.4, we prove the Hyers-Ulam stability of T , hence the Hyers-Ulam stability of Eq.(3). $\square$

In a similar way to the proof of Corollary 2.7, we obtain:

COROLLARY 3.2. *The nonhomogeneous equation (2) has the Hyers-Ulam stability on $[-\tau_m, b]$.*

REMARK 3.3. *The Hyers-Ulam stability in the classic sense also holds true for Eq.(2) and (3).*

4. STABILITY OF GENERAL LINEAR FDE WITH CONSTANT DELAY

In this part, we investigate the Hyers-Ulam stability of

$$y^{(n)}(t) = \sum_{j=0}^{n-1} \sum_{i=1}^{n_j} g_{ji}(t) y^{(j)}(t - \tau_{ji}) + h(t), \quad (4)$$

where $n_j \in \mathbb{N}^+$ for all $0 \leq j \leq n-1$, $\{\tau_{ji}\}$ is a family of positive constants, $h(t) \in C[0, b]$ and $\{g_{ji}(t)\}$ is a family of continuous functions defined on $[0, b]$.

For each $0 \leq j \leq n-1$, let $\tau_j = \max_{i \in \{1, \dots, n_j\}} \{\tau_{ji}\}$ and τ denote $\max\{\tau_0, \dots, \tau_{n-1}\}$. We will prove the Hyers-Ulam stability holds true for Eq.(4) on $[-\tau, b]$.

Let $\Gamma = \left(\bigcap_{j=0}^n \Gamma^j[-\tau_j, b; \tau - \tau_j] \right)$, where $\tau_n = 0$. Next, we shall construct a norm $\|\cdot\|$ such that $(\Gamma, \|\cdot\|)$ is a Banach space. For every $y \in \Gamma$, define

$$\|y\| = \max_{\substack{0 \leq i \leq j, \\ 0 \leq j \leq n}} \sup_{x \in [-\tau_j, b]} |y^{(i)}(x)|.$$

We assert that $(\Gamma, \|\cdot\|)$ is a Banach space.

Define an linear operator $T : (\Gamma, \|\cdot\|) \rightarrow C[0, b]$ by

$$(Ty)(t) = y^{(n)}(t) - \sum_{j=0}^{n-1} \sum_{i=1}^{n_j} g_{ji}(t) y^{(j)}(t - \tau_{ji}), \quad y(t) \in (\Gamma, \|\cdot\|)$$

which is bounded and has a closed range $C[0, b]$. By noticing $0 \in C[0, b]$, according to Theorem 2.4 and Remark 2.3, we obtain the following result:

THEOREM 4.1.

$$y^{(n)}(t) = \sum_{j=0}^{n-1} \sum_{i=1}^{n_j} g_{ji}(t) y^{(j)}(t - \tau_{ji}) \quad (5)$$

has the Hyers-Ulam stability on $[-\tau, b]$.

COROLLARY 4.2. *The nonhomogeneous equation (4) has the Hyers-Ulam stability on $[-\tau, b]$.*

REMARK 4.3. *The Hyers-Ulam stability in the classic sense also holds true for Eq. (4) and (5).*

REMARK 4.4. *We assert here that a Hyers-Ulam constant of Eq. (4) can also be figured out by direct method.*

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(Jinghao Huang) DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275 P. R. CHINA *E-mail* : hjinghao@mail2.sysu.edu.cn

(Yongjin Li) DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275 P. R. CHINA *E-mail* : stslyj@mail.sysu.edu.cn