



## On functional equations of the multiplicative type

Esteban A. Chávez<sup>b</sup>, Prasanna K. Sahoo<sup>a,\*</sup><sup>a</sup> Department of Mathematics, University of Louisville, Louisville, KY 40292, USA<sup>b</sup> Department of Applied Probability and Statistics, University of California, Santa Barbara, Santa Barbara, CA 93106, USA

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## ABSTRACT

This paper aims to determine the general solution  $f : \mathbb{F}^2 \rightarrow S$  of the equation  $f(\phi(x, y, u, v)) = f(x, y) f(u, v)$  for suitable conditions on the function  $\phi : \mathbb{F}^4 \rightarrow \mathbb{F}^2$ , where  $\mathbb{F}$  will denote either  $\mathbb{R}$  or  $\mathbb{C}$ , and  $S$  is a multiplicative semigroup. Using this result, we determine the general solution of several functional equations studied earlier, namely  $f(ux + vy, uy + vx) = f(x, y) f(u, v)$ ;  $f(ux + (\lambda - 1)vy, vx + uy + (\lambda - 2)vy) = f(x, y) f(u, v)$ ;  $f(ux - vy, uy - vx) = f(x, y) f(u, v)$ ;  $f(ux + vy, uy - vx) = f(x, y) f(u, v)$ ;  $f(ux + vy, uy + v(x + y)) = f(x, y) f(u, v)$ ; and  $f(ux - vy, uy + v(x + y)) = f(x, y) f(u, v)$ .

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## 1. Introduction

In [3], Blecksmith and Broudno gave a simple proof of the following result: *For every positive integer  $m$ , there is an integer  $n$  with at least  $m$  proper representations as the sum of three fourth powers.* If one lets the quadratic form on the right hand side of the *Proth identity*

$$x^4 + y^4 + (x + y)^4 = 2(x^2 + xy + y^2)^2 \quad (1.1)$$

be

$$f(x, y) = x^2 + xy + y^2 \quad (1.2)$$

then it can be easily verified that

$$f(ux - vy, uy + v(x + y)) = f(x, y) f(u, v) \quad (1.3)$$

\* Corresponding author.

E-mail addresses: ealejandrochavez@yahoo.com.mx (E.A. Chávez), saho@louisville.edu (P.K. Sahoo).

for all  $x, y, u, v \in \mathbb{R}$ . It is obvious that  $f$  given in (1.2) is a solution of the above functional equation (1.3). To prove the above mentioned result, Blecksmith and Broudno [3] used the Proth identity and the above functional equation (1.3). However, the general solution of the above equation was not given in [3].

The question that arises is whether  $f(x, y) = x^2 + xy + y^2$  is the only solution of (1.3) or there are other solutions. This question was addressed by Chávez and Sahoo [4] (and subsequently fixed an error in [4] by Chung and Sahoo [5]) in the following theorem.

**Theorem 1.1.** *The general solution  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  of the functional equation*

$$f(ux - vy, uy + v(x + y)) = f(x, y) f(u, v) \quad (1.4)$$

for all  $u, v, x, y \in \mathbb{R}$  is given by  $f \equiv 1$  or

$$f(x, y) = M(x^2 + xy + y^2) e^{A\left(\arctan\left(\frac{\sqrt{3}y}{2x+y}\right)\right)}, \quad f(0, 0) = 0 \quad (1.5)$$

for all  $x, y \in \mathbb{R} \setminus \{0\}$ , where  $M : (0, \infty) \rightarrow \mathbb{R}$  is a multiplicative function and  $A : \mathbb{R} \rightarrow \mathbb{R}$  is an additive function satisfying  $A(2\pi) = 0$ .

One of the well known properties of matrices is the following: The determinant of the product of two square matrices is the product of their determinants. If

$$\begin{pmatrix} x & y \\ y & x \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u & v \\ v & u \end{pmatrix}$$

are any two symmetric matrices, then

$$\det\left(\begin{pmatrix} x & y \\ y & x \end{pmatrix} \begin{pmatrix} u & v \\ v & u \end{pmatrix}\right) = \det\begin{pmatrix} x & y \\ y & x \end{pmatrix} \det\begin{pmatrix} u & v \\ v & u \end{pmatrix},$$

which is

$$\det\begin{pmatrix} ux + vy & uy + vx \\ uy + vx & ux + vy \end{pmatrix} = \det\begin{pmatrix} x & y \\ y & x \end{pmatrix} \det\begin{pmatrix} u & v \\ v & u \end{pmatrix}. \quad (1.6)$$

If we define a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$f(x, y) = \det\begin{pmatrix} x & y \\ y & x \end{pmatrix} \quad (1.7)$$

for all  $x, y \in \mathbb{R}$ , then we arrive at the functional equation

$$f(ux + vy, uy + vx) = f(x, y) f(u, v) \quad (1.8)$$

for all  $x, y, u, v \in \mathbb{R}$ . Since

$$\det\begin{pmatrix} x & y \\ y & x \end{pmatrix} = x^2 - y^2,$$

it is clear that  $f(x, y) = x^2 - y^2$  is a solution of (1.8). The question that arises is whether  $f(x, y) = x^2 - y^2$  is the only solution or there are other solutions. Chung and Sahoo in [6] have shown that there are other solutions besides  $f(x, y) = x^2 - y^2$ . Chung and Sahoo in [6] proved the following theorem:

**Theorem 1.2.** *The function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfies the functional equation (1.8) for all  $x, y, u, v \in \mathbb{R}$  if and only if*

$$f(x, y) = M_1(x + y) M_2(x - y) \quad (1.9)$$

where  $M_1, M_2 : \mathbb{R} \rightarrow \mathbb{R}$  are multiplicative functions.

A more general (pexiderized) version of the functional equation (1.8),

$$f(ux + vy, uy + vx) = g(x, y) h(u, v) \quad (1.10)$$

for all  $x, y, u, v \in \mathbb{R}$  was also solved in [6].

Recently, Akkouchi and Rhali in [2] expanded the work of Chung and Sahoo [6]. They examined a functional equation that was related to the determinant of an  $n \times n$  matrix. Consider the matrix  $A_n(x, y) = [a_{ij}(x, y)]$  where  $1 \leq i, j \leq n$  such that  $a_{ii}(x, y) = x$  and  $a_{ij}(x, y) = y$  for  $i \neq j$ . Then it can be shown that

$$A_n(x, y) A_n(u, v) = A_n(ux + (n-1)vy, vx + uy + (n-2)vy).$$

Hence we have,

$$\det(A_n(x, y)) \det(A_n(u, v)) = \det(A_n(ux + (n-1)vy, vx + uy + (n-2)vy)).$$

If we define a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$f(x, y) = \det(A_n(x, y))$$

then we arrive at the functional equation

$$f(ux + (n-1)vy, vx + uy + (n-2)vy) = f(x, y) f(u, v) \quad (1.11)$$

for all  $x, y, u, v \in \mathbb{R}$ . Notice that

$$f(x, y) = \det(A_n(x, y)) = (x + (n-1)y)(x - y)^{n-1} \quad (1.12)$$

is a solution to (1.11). Also, when  $n = 2$ , (1.12) reduces to  $f(x, y) = x^2 - y^2$ .

Let  $\mathbb{K}$  be the real or complex field and  $\lambda \in \mathbb{K}^* = \mathbb{K} \setminus \{0\}$ . In [2], Akkouchi and Rhali, using a variation of the method proposed in [6], proved the following theorem:

**Theorem 1.3.** *The function  $f : \mathbb{K}^2 \rightarrow \mathbb{K}$  satisfies the functional equation*

$$f(ux + (\lambda - 1)vy, vx + uy + (\lambda - 2)vy) = f(x, y) f(u, v) \quad (1.13)$$

for all  $x, y, u, v \in \mathbb{K}$  and  $\lambda \in \mathbb{K}^*$  if and only if

$$f(x, y) = M_1(x + (\lambda - 1)y) M_2(x - y) \quad (1.14)$$

for all  $x, y \in \mathbb{K}$ , where  $M_1$  and  $M_2$  are multiplicative functions defined on the field  $\mathbb{K}$ .

Notice that when  $\lambda = 2$ , Theorem 1.3 yields Theorem 1.2.

Another functional equation arises when one substitutes  $-v$  for  $v$  in (1.6). It is easy to see that

$$\det \begin{pmatrix} u & v \\ v & u \end{pmatrix} = u^2 - v^2 = \det \begin{pmatrix} u & -v \\ -v & u \end{pmatrix}.$$

Thus we have

$$\det \begin{pmatrix} ux - vy & uy - vx \\ uy - vx & ux - vy \end{pmatrix} = \det \begin{pmatrix} x & y \\ y & x \end{pmatrix} \det \begin{pmatrix} u & v \\ v & u \end{pmatrix},$$

which leads us to the functional equation

$$f(ux - vy, uy - vx) = f(x, y) f(u, v). \quad (1.15)$$

Houston and Sahoo [8] proved the following theorem concerning (1.15).

**Theorem 1.4.** *The function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfies the functional equation (1.15) for all  $x, y, u, v \in \mathbb{R}$  if and only if*

$$f(x, y) = M(x^2 - y^2), \quad (1.16)$$

where  $M : \mathbb{R} \rightarrow \mathbb{R}$  is a multiplicative function.

A variant of the above functional equation (1.15) is the following:

$$f(ux - vy, uy - vx) = f(x, y) + f(u, v) + f(x, y) f(u, v). \quad (1.17)$$

In the solved and unsolved problems column of the *Newsletters* of the European Mathematical Society, Sahoo [9] asked to determine the general solution of (1.17) without assuming any regularity condition on the unknown function  $f$ . This functional equation was solved in [8] and also in [10].

A generalization of the functional equation (1.17) is

$$f(ux - vy, uy - vx) = g(x, y) + h(u, v) + \ell(x, y) m(u, v), \quad (1.18)$$

where  $f, g, h, \ell, m : \mathbb{R}^2 \rightarrow \mathbb{R}$  are unknown functions. The general solution of this functional equation was determined in a recent paper by Houston and Sahoo [7].

Like determinant, the permanent of a matrix is also an important concept. For a  $2 \times 2$  symmetric matrix, it is given by

$$\text{per} \begin{pmatrix} x & y \\ y & x \end{pmatrix} = x^2 + y^2.$$

It is easy to check that

$$\text{per} \begin{pmatrix} ux + vy & uy - vx \\ uy - vx & ux + vy \end{pmatrix} = \text{per} \begin{pmatrix} x & y \\ y & x \end{pmatrix} \text{per} \begin{pmatrix} u & v \\ v & u \end{pmatrix}.$$

If one defines  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$f(x, y) = \text{per} \begin{pmatrix} x & y \\ y & x \end{pmatrix}, \quad (1.19)$$

then one obtains the following functional equation:

$$f(ux + vy, uy - vx) = f(x, y) f(u, v) \quad (1.20)$$

for all  $x, y, u, v \in \mathbb{R}$ . Houston and Sahoo [8] proved the following theorem concerning the above functional equation.

**Theorem 1.5.** *The function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfies the functional equation (1.20) for all  $x, y, u, v \in \mathbb{R}$  if and only if*

$$f(x, y) = M(x^2 + y^2), \quad (1.21)$$

where  $M : \mathbb{R} \rightarrow \mathbb{R}$  is a multiplicative function.

A variant of (1.20) is

$$f(ux + vy, uy - vx) = f(x, y) + f(u, v) + f(x, y) f(u, v) \quad (1.22)$$

for all  $x, y, u, v \in \mathbb{R}$ . The general solution of (1.22) was determined in [8]. To find the general solution of a more general equation than (1.22), the solution of the following functional equation was needed:

$$f(ux + vy, uy - vx) = f(x, y) f(v, u). \quad (1.23)$$

Although this equation is quite similar to (1.20), yet this equation cannot be solved using the technique used in Theorem 1.5. It was this equation that motivated us to look for a new method that will allow us to solve all the above mentioned functional equations in a common framework.

The main goal of this paper is to determine the general solution  $f : \mathbb{F}^2 \rightarrow S$  of the functional equation  $f(\phi(x, y, u, v)) = f(x, y) f(u, v)$  for suitable conditions on the function  $\phi : \mathbb{F}^4 \rightarrow \mathbb{F}^2$ , where  $\mathbb{F}$  will denote either  $\mathbb{R}$  or  $\mathbb{C}$ , and  $S$  is an abelian semigroup. Using this general result, we also determine the general solution, without assuming any regularity condition, of the several other functional equations, namely (1.8), (1.11), (1.15), (1.20), (1.3), and (1.23). The interested reader should refer to [1,11–14] for an account on functional equations.

## 2. Main result

In this sequel, all semigroups and groups will be denoted using multiplicative notation. A function  $M$  defined on a semigroup  $H$  and taking values on a semigroup  $K$  is said to be a semigroup morphism if and only if  $M(xy) = M(x)M(y)$  for all  $x, y \in H$ . If the semigroup operations in  $H$  and  $K$  are both multiplication, then the semigroup morphism  $M$  is said to be a multiplicative function. If the semigroup operations in  $H$  and  $K$  are both addition, then the semigroup morphism  $M$  is said to be an additive function. If the semigroup operation of  $H$  is addition and the semigroup operation of  $K$  is multiplication, then the semigroup morphism  $M$  is said to be an exponential function. If the semigroup operation of  $H$  is multiplicative and the semigroup operation of  $K$  is addition, then the semigroup morphism  $M$  is said to be a logarithmic function. For more on additive, multiplicative, exponential and logarithmic functions the interested reader is referred to [1,12–14]. Let  $m$  and  $n$  be two positive integers. If  $G$  is a set closed under multiplication, the product of  $x, y \in G^n$  will be defined by

$$xy = (x_1, x_2, \dots, x_n)(y_1, y_2, \dots, y_n) = (x_1y_1, x_2y_2, \dots, x_ny_n).$$

If  $f : G^n \rightarrow G^m$ , then we define the coordinate functions  $f_i : G^n \rightarrow G$ ,  $i = 1, 2, \dots, m$  by means of the expression

$$f(t) = (f_1(t), f_2(t), \dots, f_m(t)), \quad t \in G^n.$$

$\mathbb{F}$  will denote either  $\mathbb{R}$  or  $\mathbb{C}$  and  $S, T$  will be commutative multiplicative semigroups.

**Theorem 2.1.** *Let  $G$  be a multiplicative semigroup with unity 1, and let  $S$  and  $T$  be semigroups. Let  $\phi : G^4 \rightarrow G^2$  be a function such that there exists a bijective function  $\alpha : G^2 \rightarrow T^2$  satisfying*

$$\alpha(\phi(x, y, u, v)) = \alpha(x, y) \alpha(u, v) \quad (2.1)$$

for all  $x, y, u, v \in G$ . Then, the general solution  $f : G^2 \rightarrow S$  of the functional equation

$$f(\phi(x, y, u, v)) = f(x, y) f(u, v) \quad (2.2)$$

for all  $u, v, x, y \in G$  is given by

$$f(x, y) = M_1(\alpha_1(x, y)) M_2(\alpha_2(x, y)) \quad (2.3)$$

where  $M_1, M_2 : T \rightarrow S$  are semigroup morphisms,  $\alpha_1, \alpha_2$  are coordinates functions of  $\alpha : G^2 \rightarrow T^2$ .

**Proof.** Define the function  $g : T^2 \rightarrow S$  as

$$g(s, t) = f(\alpha^{-1}(s, t)) \quad (2.4)$$

for all  $s, t \in T$ . Then

$$f(x, y) = g(\alpha(x, y)) \quad (2.5)$$

for all  $x, y \in G$ . Now use (2.5) to rewrite (2.2) in terms of  $g$  as

$$g(\alpha(\phi(x, y, u, v))) = g(\alpha(x, y)) g(\alpha(u, v))$$

for all  $u, v, x, y \in G$ . However, by (2.1) this equation is just

$$g(\alpha(x, y) \alpha(u, v)) = g(\alpha(x, y)) g(\alpha(u, v))$$

for all  $u, v, x, y \in G$ . Since  $\alpha$  is surjective, we let  $(s_1, t_1) = \alpha(x, y)$  and  $(s_2, t_2) = \alpha(u, v)$  to obtain that

$$g(s_1 s_2, t_1 t_2) = g(s_1, t_1) g(s_2, t_2) \quad (2.6)$$

for all  $s_1, s_2, t_1, t_2 \in T$ . Next, using (2.6) it is easy to demonstrate that the functions  $M_1, M_2 : T \rightarrow S$  defined by

$$\begin{aligned} M_1(s) &= g(s, 1) \\ M_2(t) &= g(1, t) \end{aligned}$$

are semigroup morphisms and that

$$g(s, t) = M_1(s) M_2(t) \quad (2.7)$$

for all  $s, t \in T$ . Finally, we use (2.7) in (2.5) to obtain the asserted solution (2.3).  $\square$

**Remark 2.2.** If  $G$  is a commutative semigroup without unity 1, and  $T$  is a commutative semigroup and  $S$  is a commutative group, then the assertion of [Theorem 2.1](#) still holds. For a proof see [\[4\]](#).

**Remark 2.3.** Obviously [Theorem 2.1](#) holds if we replace semigroup  $G$  by a field.

### 3. Some examples

The following theorem follows from [Theorem 2.1](#).

**Theorem 3.1.** *The function  $f : \mathbb{C}^2 \rightarrow \mathbb{C}$  satisfies the functional equation*

$$f(ux - vy, uy + v(x + y)) = f(x, y) f(u, v) \quad (3.1)$$

for all  $u, v, x, y \in \mathbb{C}$  if and only if

$$f(x, y) = M_1 \left( x + \frac{1 + i\sqrt{3}}{2} y \right) M_2 \left( x + \frac{1 - i\sqrt{3}}{2} y \right) \quad (3.2)$$

where  $M_1, M_2 : \mathbb{C} \rightarrow \mathbb{C}$  are multiplicative functions.

**Proof.** Let  $\phi : \mathbb{C}^4 \rightarrow \mathbb{C}^2$  be defined by

$$\phi(x, y, u, v) = (ux - vy, uy + v(x + y)) \quad (3.3)$$

for all  $x, y, u, v \in \mathbb{C}$ . Let  $\alpha : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  be given by

$$\alpha(x, y) = \left( x + \frac{1 + i\sqrt{3}}{2} y, x + \frac{1 - i\sqrt{3}}{2} y \right). \quad (3.4)$$

The clearly  $\alpha : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  is a bijection and

$$\alpha^{-1}(x, y) = \left( \frac{i\sqrt{3}}{3}(y - x), \frac{3 + i\sqrt{3}}{6}x + \frac{3 - i\sqrt{3}}{6}y \right). \quad (3.5)$$

Using [\(3.3\)](#) and [\(3.4\)](#), it is easy to check that

$$\alpha(\phi(x, y, u, v)) = \alpha(x, y) \alpha(u, v).$$

Hence applying [Theorem 2.1](#) with  $G = S = T = \mathbb{C}$  we have the asserted solution  $f(x, y)$  as stated in [\(3.2\)](#).  $\square$

Now, it will be assumed that  $\phi$  is a linear transformation of  $X = (x, y)^T$ , that is,  $\phi(X, U) = A_U X$  where  $A_U \in M_{2 \times 2}(\mathbb{F})$  has entries that depend on  $U = (u, v)^T$ . Furthermore, suppose that  $A_U$  is diagonalizable, that is, there exist  $Q, \Lambda_U \in M_{2 \times 2}(\mathbb{F})$  satisfying that  $Q$  is invertible,  $\Lambda_U$  is diagonal and  $A_U = Q\Lambda_U Q^{-1}$ . Thus, equation [\(2.2\)](#) is transformed into

$$f(Q\Lambda_U Q^{-1}X) = f(X) f(U) \quad (3.6)$$

for all  $X, U \in \mathbb{F}^2$ . Let  $K = \{Q^{-1}X \mid X \in \mathbb{F}^2\}$ . Notice that  $Y \in K$  if and only if  $QY \in \mathbb{F}^2$ . Define  $g : K \rightarrow S$  by

$$g(Y) = f(QY) \quad (3.7)$$

for all  $Y \in K$ . Then

$$f(X) = g(Q^{-1}X) \quad (3.8)$$

and equation (3.6) can be written in terms of  $g$  as

$$g(\Lambda_U Q^{-1}X) = g(Q^{-1}X) g(Q^{-1}U) \quad (3.9)$$

for all  $X, U \in \mathbb{F}^2$ . Set  $Y = Q^{-1}X$  and  $V = Q^{-1}U$  to express equation (3.9) as

$$g(\Lambda_V Y) = g(Y) g(V) \quad (3.10)$$

for all  $V, Y \in K$ , where  $\Lambda_V$  corresponds to the diagonal matrix  $\Lambda_U$  expressed in terms of  $V = (u, v)^T$ . Thus, (3.6) has been transformed into the simplest equation (3.10) which, in most cases, is easier to solve.

Next, we determine solution of some functional equations illustrating this technique.

**Theorem 3.2.** *Let  $\mathbb{F}$  denote the real or complex field, and  $S$  be a semigroup. The function  $f : \mathbb{F}^2 \rightarrow S$  satisfies the functional equation*

$$f(ux + vy, uy + vx) = f(x, y) f(u, v) \quad (3.11)$$

for all  $x, y, u, v \in \mathbb{F}$  if and only if

$$f(x, y) = M_1(x + y) M_2(x - y) \quad (3.12)$$

where  $M_1, M_2 : \mathbb{F} \rightarrow S$  are multiplicative functions.

**Proof.** In matrix terminology, equation (3.11) can be written as

$$f\left(\begin{pmatrix} u & v \\ v & u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\right) = f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) f\left(\begin{pmatrix} u \\ v \end{pmatrix}\right). \quad (3.13)$$

Now, diagonalize the linear operator depending on  $U = (u, v)^T$  to obtain

$$f\left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u+v & 0 \\ 0 & u-v \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\right) = f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) f\left(\begin{pmatrix} u \\ v \end{pmatrix}\right). \quad (3.14)$$

Define  $g : \mathbb{F}^2 \rightarrow S$  by

$$g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = f\left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\right) \quad (3.15)$$

for all  $Y = (x, y)^T \in \mathbb{F}^2$  to rewrite equation (3.14) in terms of  $g$  as

$$g\left(\begin{pmatrix} u+v & 0 \\ 0 & u-v \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\right) = g\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\right) g\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}\right). \quad (3.16)$$

Making the change of variables

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (3.17)$$



and

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad (3.18)$$

and noticing that

$$\begin{pmatrix} u+v & 0 \\ 0 & u-v \end{pmatrix} = \begin{pmatrix} 2\tilde{u} & 0 \\ 0 & 2\tilde{v} \end{pmatrix} \quad (3.19)$$

we reduce equation (3.16) into the simpler form

$$g\left(\begin{pmatrix} 2\tilde{u} & 0 \\ 0 & 2\tilde{v} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}\right) = g\left(\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}\right) g\left(\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}\right). \quad (3.20)$$

Thus, we reduced equation (3.11) into equation (3.20) which can be written as

$$g(2ux, 2vy) = g(x, y) g(u, v) \quad (3.21)$$

for all  $(x, y), (u, v) \in K = \mathbb{F}^2$ , which is easier to solve. Let  $h : \mathbb{F} \rightarrow S$  be defined as

$$h(x, y) = g\left(\frac{x}{2}, \frac{y}{2}\right). \quad (3.22)$$

Then

$$g(x, y) = h(2x, 2y). \quad (3.23)$$

Using (3.23) in (3.21), we obtain

$$h(2x \, 2u, 2y \, 2v) = h(2x, 2y) h(2u, 2v).$$

The last equation can be rewritten as

$$h(xu, yv) = h(x, y) h(u, v) \quad (3.24)$$

for all  $x, y, u, v \in \mathbb{F}$ . Hence

$$h(x, y) = M_1(x) M_2(y) \quad (3.25)$$

where  $M_1, M_2 : \mathbb{F} \rightarrow S$  are multiplicative functions. Using (3.23) and (3.25) we see that

$$g(x, y) = M_1(2x) M_2(2y). \quad (3.26)$$

From (3.15) and (3.26)

$$f(x+y, x-y) = M_1(2x) M_2(2y). \quad (3.27)$$

Therefore

$$f(x, y) = M_1(x+y) M_2(x-y) \quad (3.28)$$

for all  $x, y \in \mathbb{F}$ . This completes the proof of the theorem.  $\square$

**Theorem 3.3.** Let  $\mathbb{F}$  denote the real or complex field and let  $\lambda \in \mathbb{F}^*$ . Let  $S$  be a semigroup. The function  $f : \mathbb{F}^2 \rightarrow S$  satisfies the functional equation

$$f(ux + (\lambda - 1)vy, uy + vx + (\lambda - 2)vy) = f(x, y) f(u, v) \quad (3.29)$$

for all  $x, y, u, v \in \mathbb{F}$  if and only if

$$f(x, y) = M_1(x + (\lambda - 1)y) M_2(x - y) \quad (3.30)$$

where  $M_1, M_2 : \mathbb{F} \rightarrow S$  are multiplicative functions.

**Proof.** In matrix terminology, equation (3.29) can be written as

$$f\left(\begin{pmatrix} u & (\lambda - 1)v \\ v & u + (\lambda - 2)v \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\right) = f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) f\left(\begin{pmatrix} u \\ v \end{pmatrix}\right). \quad (3.31)$$

Diagonalize the linear operator depending on  $U = (u, v)^T$  to obtain

$$f\left(\begin{pmatrix} 1 & \lambda - 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u + (\lambda - 1)v & 0 \\ 0 & u - v \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda} & \frac{\lambda - 1}{\lambda} \\ \frac{1}{\lambda} & -\frac{1}{\lambda} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\right) = f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) f\left(\begin{pmatrix} u \\ v \end{pmatrix}\right). \quad (3.32)$$

Define  $g : \mathbb{F}^2 \rightarrow S$  by

$$g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = f\left(\begin{pmatrix} 1 & \lambda - 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\right) \quad (3.33)$$

for all  $Y = (x, y)^T \in \mathbb{F}^2$  to rewrite equation (3.32) in terms of  $g$  as

$$g\left(\begin{pmatrix} u + (\lambda - 1)v & 0 \\ 0 & u - v \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda} & \frac{\lambda - 1}{\lambda} \\ \frac{1}{\lambda} & -\frac{1}{\lambda} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\right) = g\left(\begin{pmatrix} \frac{1}{\lambda} & \frac{\lambda - 1}{\lambda} \\ \frac{1}{\lambda} & -\frac{1}{\lambda} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\right) g\left(\begin{pmatrix} \frac{1}{\lambda} & \frac{\lambda - 1}{\lambda} \\ \frac{1}{\lambda} & -\frac{1}{\lambda} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}\right). \quad (3.34)$$

Making the change of variables

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda} & \frac{\lambda - 1}{\lambda} \\ \frac{1}{\lambda} & -\frac{1}{\lambda} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (3.35)$$

and

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda} & \frac{\lambda - 1}{\lambda} \\ \frac{1}{\lambda} & -\frac{1}{\lambda} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad (3.36)$$

and noticing that

$$\begin{pmatrix} u + (\lambda - 1)v & 0 \\ 0 & u - v \end{pmatrix} = \begin{pmatrix} \lambda \tilde{u} & 0 \\ 0 & \lambda \tilde{v} \end{pmatrix} \quad (3.37)$$

we reduce equation (3.34) into the simpler form

$$g\left(\begin{pmatrix} \lambda \tilde{u} & 0 \\ 0 & \lambda \tilde{v} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}\right) = g\left(\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}\right) g\left(\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}\right). \quad (3.38)$$

Thus, we reduced equation (3.29) into equation (3.38) which can be written as

$$g(\lambda ux, \lambda vy) = g(x, y) g(u, v) \quad (3.39)$$

for all  $(x, y), (u, v) \in K = \mathbb{F}^2$ , since  $\lambda \in \mathbb{F}$ . Let  $h : \mathbb{F} \rightarrow S$  be defined as

$$h(x, y) = g\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right). \quad (3.40)$$

Then

$$g(x, y) = h(\lambda x, \lambda y). \quad (3.41)$$

Using (3.41) in (3.39), we obtain

$$h(\lambda x \lambda u, \lambda v \lambda y) = h(\lambda x, \lambda y) h(\lambda u, \lambda v).$$

The last equation can be rewritten as

$$h(xu, yv) = h(x, y) h(u, v) \quad (3.42)$$

for all  $x, y, u, v \in \mathbb{F}$ . Hence

$$h(x, y) = M_1(x) M_2(y) \quad (3.43)$$

where  $M_1, M_2 : \mathbb{F} \rightarrow S$  are multiplicative functions. Using (3.41) and (3.43) we see that

$$g(x, y) = M_1(\lambda x) M_2(\lambda y). \quad (3.44)$$

From (3.33) and (3.44)

$$f(x + (\lambda - 1)y, x - y) = M_1(\lambda x) M_2(\lambda y). \quad (3.45)$$

Therefore

$$f(x, y) = M_1(x + (\lambda - 1)y) M_2(x - y) \quad (3.46)$$

for all  $x, y \in \mathbb{F}$  and  $\lambda \neq 0$ . This completes the proof of the theorem.  $\square$

**Theorem 3.4.** *Let  $S$  be a semigroup. The function  $f : \mathbb{C}^2 \rightarrow S$  satisfies the functional equation*

$$f(ux + vy, uy - vx) = f(x, y) f(u, v) \quad (3.47)$$

*for all  $x, y, u, v \in \mathbb{C}$  if and only if*

$$f(x, y) = M(x^2 + y^2) \quad (3.48)$$

*where  $M : \mathbb{C} \rightarrow S$  is a multiplicative function.*

**Proof.** The if part of the proof follows by direct substitution of (3.66) into the functional equation (3.47).

Next we prove the only if part. For the sake of readability, the previous discussion will be applied step-by-step to this particular theorem. In matrix terminology, equation (3.47) can be written as

$$f\left(\begin{pmatrix} u & v \\ -v & u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\right) = f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) f\left(\begin{pmatrix} u \\ v \end{pmatrix}\right). \quad (3.49)$$

Now, diagonalize the linear operator depending on  $U = (u, v)^T$  to obtain

$$f\left(\begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u-vi & 0 \\ 0 & u+vi \end{pmatrix} \begin{pmatrix} -\frac{1}{2}i & \frac{1}{2} \\ -\frac{1}{2}i & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\right) = f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) f\left(\begin{pmatrix} u \\ v \end{pmatrix}\right). \quad (3.50)$$

Define  $g : \mathbb{C}^2 \rightarrow S$  by

$$g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = f\left(\begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\right) \quad (3.51)$$

for all  $Y = (x, y)^T \in \mathbb{C}^2$  to rewrite equation (3.50) in terms of  $g$  as

$$g\left(\begin{pmatrix} u-vi & 0 \\ 0 & u+vi \end{pmatrix} \begin{pmatrix} -\frac{1}{2}i & \frac{1}{2} \\ -\frac{1}{2}i & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\right) = g\left(\begin{pmatrix} -\frac{1}{2}i & \frac{1}{2} \\ -\frac{1}{2}i & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\right) g\left(\begin{pmatrix} -\frac{1}{2}i & \frac{1}{2} \\ -\frac{1}{2}i & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}\right). \quad (3.52)$$

Making the change of variables

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}i & \frac{1}{2} \\ -\frac{1}{2}i & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (3.53)$$

and

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}i & \frac{1}{2} \\ -\frac{1}{2}i & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad (3.54)$$

and noticing that

$$\begin{pmatrix} u-vi & 0 \\ 0 & u+vi \end{pmatrix} = \begin{pmatrix} 2\tilde{v}i & 0 \\ 0 & 2\tilde{u}i \end{pmatrix} \quad (3.55)$$

we reduce equation (3.52) into the simpler form

$$g\left(\begin{pmatrix} 2\tilde{v}i & 0 \\ 0 & 2\tilde{u}i \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}\right) = g\left(\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}\right) g\left(\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}\right). \quad (3.56)$$

Using the terminology of the previous discussion, we have that

$$\left\{ \begin{array}{l} A_U = \begin{pmatrix} u & v \\ -v & u \end{pmatrix}, \quad Q = \begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix}, \\ \Lambda_U = \begin{pmatrix} u-vi & 0 \\ 0 & u+vi \end{pmatrix}, \quad Q^{-1} = \frac{1}{2} \begin{pmatrix} -i & 1 \\ -i & -1 \end{pmatrix}, \\ \Lambda_V = \begin{pmatrix} 2\tilde{v}i & 0 \\ 0 & 2\tilde{u}i \end{pmatrix} \end{array} \right. \quad (3.57)$$

and  $K = \mathbb{C}^2$ . Thus, we reduced equation (3.47) into equation (3.56) which can be written as

$$g(2vxi, 2uyi) = g(x, y) g(u, v) \quad (3.58)$$

for all  $x, y, u, v \in \mathbb{C}$ , which is easier to solve. Let  $h : \mathbb{C} \rightarrow S$  be defined as

$$h(x, y) = g\left(-\frac{x}{2}i, -\frac{y}{2}i\right). \quad (3.59)$$

Then

$$g(x, y) = h(2xi, 2yi). \quad (3.60)$$

Using (3.60) in (3.58), we obtain

$$h(2xi \, 2vi, 2ui \, 2yi) = h(2xi, 2yi) h(2ui, 2vi).$$

The last equation can be rewritten as

$$h(xv, yu) = h(x, y) h(u, v) \quad (3.61)$$

for all  $x, y, u, v \in \mathbb{C}$ . Notice that  $f \equiv 0$  if and only if  $h \equiv 0$ , so let's assume that  $h \not\equiv 0$ . Then, by setting  $u = v = 1$  in (3.61) it is easily seen that  $h(1, 1) = 1$ . Using this fact and setting  $x = y = 1$  in (3.61) it follows that  $h(u, v) = h(v, u)$  for all  $u, v \in \mathbb{C}$ . Now, put  $u = y = 1$  in (3.61) and use the symmetry of  $h$  to notice that the function  $M : \mathbb{C} \rightarrow S$  defined by

$$M(x) = h(x, 1) \quad (3.62)$$

is multiplicative. Finally, put  $y = v = 1$  to obtain that

$$h(x, u) = M(xu) \quad (3.63)$$

for all  $x, u \in \mathbb{C}$ . Therefore, we recover  $f$  by first finding

$$g(x, y) = h(2ix, 2iy) = M(-4xy) \quad (3.64)$$

so that

$$f(x, y) = g(Q^{-1}X) = g\left(\frac{1}{2}y - \frac{1}{2}xi, -\frac{1}{2}y - \frac{1}{2}xi\right) = M(x^2 + y^2). \quad (3.65)$$

This completes the proof of the theorem.  $\square$

**Theorem 3.5.** *Let  $S$  be a semigroup. The function  $f : \mathbb{R}^2 \rightarrow S$  satisfies the functional equation (3.47) for all  $x, y, u, v \in \mathbb{R}$  if and only if*

$$f(x, y) = M(x^2 + y^2) \quad (3.66)$$

where  $M : [0, \infty) \rightarrow S$  is a multiplicative function.

**Proof.** The proof is similar to the one in Theorem 3.4. However, in this case, the functions  $f : \mathbb{R}^2 \rightarrow S$ ,  $g : K \rightarrow S$  and  $h : L \rightarrow S$  have different domains, where  $K = \{(a + bi, -a + bi) \mid a, b \in \mathbb{R}\}$  and

$L = \{(z, \bar{z}) \mid z \in \mathbb{C}\}$ . Equations (3.57) through (3.61) also follow. As in the proof of Theorem 3.4, it may be assumed that  $h(1, 1) = 1$  and that  $h(u, v) = h(v, u)$ . Then it can be easily shown that the function  $M : [0, \infty) \rightarrow S$  defined by

$$M(z\bar{z}) = h(z, \bar{z}) \quad \forall z \in \mathbb{C} \quad (3.67)$$

is multiplicative. Therefore,

$$\begin{aligned} f(x, y) &= g\left(\frac{1}{2}y - \frac{1}{2}xi, -\frac{1}{2}y - \frac{1}{2}xi\right) \\ &= h(x + yi, x - yi) \\ &= M(x^2 + y^2) \end{aligned} \quad (3.68)$$

and we are done.  $\square$

**Theorem 3.6.** Let  $S$  be a semigroup and  $\mathbb{C}$  be the field of complex numbers. The function  $f : \mathbb{C}^2 \rightarrow S$  satisfies the equation

$$f(ux + vy, uy - vx) = f(x, y) f(v, u) \quad (3.69)$$

for all  $x, y, u, v \in \mathbb{C}$  if and only if

$$f(x, y) = M_1(y - xi) M_2(y + xi) \quad (3.70)$$

where  $M_1, M_2 : \mathbb{C} \rightarrow S$  are multiplicative functions.

**Proof.** Notice that  $A_U$ ,  $\Lambda_U$  and  $Q$  are the same as in Theorem 3.4, and  $U$  is the only different vector. However, in this theorem, equation (3.52) takes the form

$$g\left(\begin{pmatrix} u - vi & 0 \\ 0 & u + vi \end{pmatrix} \begin{pmatrix} -\frac{1}{2}i & \frac{1}{2} \\ -\frac{1}{2}i & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\right) = g\left(\begin{pmatrix} -\frac{1}{2}i & \frac{1}{2} \\ -\frac{1}{2}i & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\right) g\left(\begin{pmatrix} -\frac{1}{2}i & \frac{1}{2} \\ -\frac{1}{2}i & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix}\right). \quad (3.71)$$

Next, make the change of variables (3.53) and

$$\begin{pmatrix} \tilde{v} \\ \tilde{u} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}i & \frac{1}{2} \\ -\frac{1}{2}i & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix} \quad (3.72)$$

and notice that

$$\begin{pmatrix} u - vi & 0 \\ 0 & u + vi \end{pmatrix} = \begin{pmatrix} 2\tilde{u} & 0 \\ 0 & -2\tilde{v} \end{pmatrix} \quad (3.73)$$

to reduce equation (3.71) into the simpler form

$$g\left(\begin{pmatrix} 2\tilde{u} & 0 \\ 0 & -2\tilde{v} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}\right) = g\left(\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}\right) g\left(\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}\right) \quad (3.74)$$

where

$$\Lambda_V = \begin{pmatrix} 2u & 0 \\ 0 & -2v \end{pmatrix} \quad (3.75)$$

for  $V = (u, v)$ . Then,  $g : \mathbb{C}^2 \rightarrow S$  satisfies

$$g(2ux, -2vy) = g(x, y) g(u, v). \quad (3.76)$$

Set  $x = \frac{1}{2}x$ ,  $y = \frac{1}{2}y$ ,  $u = \frac{1}{2}u$  and  $v = \frac{1}{2}v$  in (3.76) to notice that the function  $h : \mathbb{C}^2 \rightarrow S$  defined by

$$h(x, y) = g\left(\frac{1}{2}x, \frac{1}{2}y\right) \quad (3.77)$$

satisfies

$$h(ux, -vy) = h(x, y) h(u, v). \quad (3.78)$$

Put  $y = v = -1$  in (3.78) to obtain that  $M_1 : \mathbb{C} \rightarrow S$  defined by

$$M_1(x) = h(x, -1) \quad (3.79)$$

is multiplicative. Also, put  $x = u = 1$ ,  $y = -y$  and  $v = -v$  in (3.78) to obtain that  $M_2 : \mathbb{C} \rightarrow S$  defined by

$$M_2(y) = h(1, -y) \quad (3.80)$$

is multiplicative. Thus, putting  $u = 1$  and  $y = -1$  in (3.78) we obtain that

$$h(x, v) = M_1(x) M_2(-v) \quad (3.81)$$

for all  $x, v \in \mathbb{C}$ . Now, we recover the function  $g$  as follows:

$$g(x, y) = h(2x, 2y) = M_1(2x) M_2(-2y) \quad (3.82)$$

for all  $x, y \in \mathbb{C}$ . Finally,

$$f(x, y) = g\left(\frac{1}{2}y - \frac{1}{2}xi, -\frac{1}{2}y - \frac{1}{2}xi\right) = M_1(y - xi) M_2(y + xi), \quad (3.83)$$

and it is easy to check that any function  $f$  given by (3.83) satisfies (3.69).  $\square$

**Theorem 3.7.** *Let  $S$  be a semigroup. The function  $f : \mathbb{R}^2 \rightarrow S$  satisfies the equation (3.69) for all  $x, y, u, v \in \mathbb{R}$  if and only if*

$$f(x, y) = M(y - xi) \quad (3.84)$$

where  $M : \mathbb{C} \rightarrow S$  is a multiplicative function.

**Proof.** It is easy to check that equations (3.75) through (3.78) are also true in this theorem, except that  $g, h : K \rightarrow S$  where  $K = \{(a + bi, -a + bi) \mid a, b \in \mathbb{R}\}$ . Nevertheless, the function  $M : \mathbb{C} \rightarrow S$  defined by

$$M(a + bi) = h(a + bi, -a + bi) \quad (3.85)$$

is multiplicative and

$$g(x, y) = h(2x, 2y) = M(2x). \quad (3.86)$$

Finally,

$$\begin{aligned} f(x, y) &= g\left(\frac{1}{2}y - \frac{1}{2}xi, -\frac{1}{2}y - \frac{1}{2}xi\right) \\ &= M(y - xi). \quad \square \end{aligned} \quad (3.87)$$

**Theorem 3.8.** *Let  $S$  be a semigroup and  $\mathbb{C}$  be the field of complex numbers. Every multiplicative function  $m : \mathbb{C} \rightarrow S$  satisfies*

$$m(z) = M(|z|) E(\arg z) \quad (3.88)$$

for all  $z \in \mathbb{C}$  where  $M : \mathbb{R} \rightarrow S$  is multiplicative,  $E : \mathbb{R} \rightarrow S$  is exponential and  $m(0) = M(0) E(\theta)$  for all  $\theta \in \mathbb{R}$ . In particular, if  $S = \mathbb{F}$ , then either  $m \equiv 1$  or  $m$  satisfies

$$m(x + yi) = M(x^2 + y^2) E\left(\arctan\left(\frac{y}{x}\right)\right) \quad (3.89)$$

with  $M(0) = 0$ .

**Proof.** The proof is basically to write the complex numbers in polar form and apply straightforward computations. We leave the details to the reader. Notice that even though the function  $e$  has a natural domain which is  $[0, 2\pi)$ , this exponential function can be uniquely extended to an exponential function whose domain is  $\mathbb{R}$ .  $\square$

**Theorem 3.9.** *The function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfies the equation (3.69) for all  $x, y, u, v \in \mathbb{R}$  if and only if*

$$f(x, y) = M(x^2 + y^2) E\left(\arctan\left(-\frac{x}{y}\right)\right) \quad (3.90)$$

where  $M : \mathbb{C} \rightarrow S$  is a multiplicative function.

**Proof.** The proof of this theorem follows from Theorems 3.7 and 3.8.  $\square$

**Theorem 3.10.** *The function  $f : \mathbb{R}^2 \rightarrow \mathbb{C}$  satisfies the functional equation (3.1) for all  $u, v, x, y \in \mathbb{R}$  if and only if*

$$f(x, y) = M\left(x + \frac{1 + i\sqrt{3}}{2}y\right) \quad (3.91)$$

where  $M : \mathbb{C} \rightarrow \mathbb{C}$  is a multiplicative function.

**Proof.** A slight variation of Theorem 2.1 can be used to solve this problem. Let  $T = \{(z, \bar{z}) \mid z \in \mathbb{C}\}$ . Then, the function  $\alpha : \mathbb{R}^2 \rightarrow T$  defined by (3.4) is clearly bijective. Moreover, following the proof of Theorem 2.1, it is easy to demonstrate that the function  $g : T \rightarrow \mathbb{C}$  defined by (2.4) satisfies

$$g(zw, \overline{zw}) = g(z, \bar{z}) g(w, \bar{w}), \quad (3.92)$$

similar to equation (2.6), for all  $w, z \in \mathbb{C}$ . Thus, the function  $M : \mathbb{C} \rightarrow \mathbb{C}$  defined by

$$M(z) = g(z, \bar{z}) \quad (3.93)$$



is multiplicative. Now, recover  $f$  using (2.5) and obtain that

$$f(x, y) = M \left( x + \frac{1 + i\sqrt{3}}{2} y \right). \quad \square \quad (3.94)$$

#### 4. Concluding remarks

We solved six different equations, under the assumptions that the domain is  $\mathbb{R}$  and  $\mathbb{C}$ . Notice that the form of the solutions may differ, even when the equation is the same but the domain changes. It would be interesting if one can find an answer for at least one of the following questions:

- (1) Given  $\phi$ , when is it possible to find  $\alpha$  in Theorem 2.1? That is, is it possible to have some algebraic (or even analytical) conditions for which we can guarantee the existence of  $\alpha$ ?
- (2) In case  $\alpha$  does not exist for a given  $\phi$ , is it possible to find a way of solving (2.2)?
- (3) If  $\alpha$  does not exist for  $f : G^2 \rightarrow K$ , is it possible to show its existence for some  $S \supset G$ ?

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