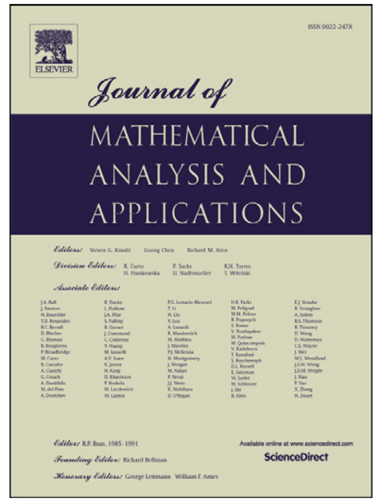

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Boundedness of solutions to a quasilinear parabolic–parabolic Keller–Segel system with logistic source

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Abstract

We study global solutions of a class of chemotaxis systems generalizing the prototype

$$\begin{cases} u_t = \nabla\cdot(\phi(u)\nabla u) - \chi\nabla\cdot(\psi(u)\nabla v) + au - bu^r, & x\in\Omega, t>0,\\ v_t = \Delta v - v + u, \quad x\in\Omega, t>0, \end{cases}$$

in a bounded domain $\Omega\subset\mathbb{R}^N(N\geq 1)$ with smooth boundary $\partial\Omega$, $\phi(u) = (u+1)^{-\alpha}$, $\psi(u) = u(u+1)^{\beta-1}$, the parameters $r>1, a\geq 0, b,\chi>0$ and $\alpha,\beta\in\mathbb{R}$. It is proved that if $0<\alpha+\beta<\max\{r-1+\alpha, \frac{2}{N}\}$, or

$$b \text{ is big enough, if } \beta = r-1,$$

then the classical solutions to the above system are uniformly-in-time bounded. Our results improve the results of Wang et al. (Discrete Contin. Dyn. Syst. Ser. A., 34(2014), 789–802) and Cao (J. Math. Anal. Appl., 412(2014), 181–188.) and also

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enlarge the range of Tao and Winkler (J. Diff. Eqns., 252(2012), 692–715) and Ishida et al. (J. Diff. Eqns., 256(2014), 2993–3010).

Key words: Boundedness; Chemotaxis; Global existence; Logistic source

2010 Mathematics Subject Classification: 92C17, 35K55, 35K59, 35K20

1 Introduction

In this paper, we consider the initial-boundary value problem for the quasilinear parabolic–parabolic Keller–Segel system with the logistic source

$$\begin{cases} u_t = \nabla \cdot (\phi(u)\nabla u) - \chi \nabla \cdot (\psi(u)\nabla v) + f(u), \quad x \in \Omega, t > 0, \\ \tau v_t = \Delta v + u - v, \quad x \in \Omega, t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega, \end{cases} \tag{1.1}$$

where $\tau = 1$, $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$, $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative on $\partial\Omega$, $\chi > 0$ is a parameter referred to as chemosensitivity, $u = u(x, t)$ denotes the density of the cells population, $v = v(x, t)$ represents the concentration of the chemoattractant, $\psi(u)$ describes the chemotactic sensitivity of cells population and the logistic source $f : [0, \infty) \mapsto \mathbb{R}$ is smooth and satisfies $f(0) = 0$ as well as

$$f(u) \leq a - bu^r \quad \text{for all} \quad u \geq 0 \tag{1.2}$$

with some $a \geq 0, b > 0$ and $r > 1$. We henceforth assume that $\phi(u)$ and $\psi(u)$ satisfy

$$\phi \in C^2([0, \infty)), \quad \text{and} \quad \psi \in C^2([0, \infty)) \quad \text{with} \quad \psi(0) = 0. \tag{1.3}$$

Moreover, in order to prove our results, we need to impose the conditions that there exist some constants $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $M_\phi > 0$ and $M_\psi > 0$ such that

$$\phi(u) \geq M_\phi(u + 1)^{-\alpha} \text{ for all } u \geq 0 \tag{1.4}$$

and

$$\psi(u) \leq M_\psi(u + 1)^{\beta} \text{ for all } u \geq 0. \tag{1.5}$$

Much attention has been paid to the properties of solutions to chemotaxis models which are kindred to (1.1), for which we refer to Murray ([18]) for a general background, Horstmann ([11]) for a survey on the Keller–Segel model. Especially, Burger et al. ([1]) proved the global existence and uniqueness of the solutions of Cauchy problem in $\mathbb{R}^N$ for linear and nonlinear diffusion with prevention of overcrowding. The model proposed herein exhibits an even higher degree of nonlinearity, and offers further possibilities to describe chemotactic movement; for example, one could imagine that the cells or bacteria are actually placed in a medium with a non-Newtonian rheology. For a detailed description of the intrinsic scaling method and some applications, we refer to the book [27].

During the past decades, the Keller–Segel models of type (1.1) have been studied extensively by many authors, where the main issue of the investigation is whether the solutions of the models are bounded or blow-up (see e.g., Cieślak et al. [7, 4, 5, 6], Calvez and Carrillo [2], Keller and Segel [15, 16], Horstmann et al. [11, 12, 13], Osaki [20, 19], Painter and Hillen [21], Perthame [22], Rascle and Ziti [23], Wang et al. [28, 29], Winkler [31, 32, 33, 34, 35, 37], Zheng [39]). Especially, in the absence of the logistic source (i.e. $f \equiv 0$) for problem (1.1), the results appear to be rather complete. In particular, if $\phi(u) \equiv 1$, Horstmann and Wang ([12]) showed that the solutions are global and bounded provided that $\psi(u) \le c(u+1)^{\frac{2}{N}-\varepsilon}$ for all $u \ge 0$ with some $\varepsilon > 0$ and $c > 0$; On the other hand, if $\psi(u) \ge c(u+1)^{\frac{2}{N}+\varepsilon}$ for all $u \ge 0$ with $\varepsilon > 0$ and $c > 0$, $\Omega \subset \mathbb{R}^N (N \ge 2)$ is a ball, and some further technical conditions are satisfied, then the solutions become unbounded in finite or infinite time. In [25], assuming that $\Omega \subset \mathbb{R}^N (N \ge 2)$ be a bounded convex domain, Tao and Winkler proved that if $\dfrac{\psi(u)}{\phi(u)} \le c(u+1)^{\frac{2}{N}+\varepsilon}$ for all $u \ge 0$ with some $\varepsilon > 0$ and $c > 0$, then the corresponding solutions are global and bounded provided that $\phi(u)$ satisfies some another technical conditions. Recently, Ishida et al. ([14]) improve the results of [25] to a bounded non-convex domain and degenerate diffusion.

On the other hand, logistic-type growth restrictions have been detected to prevent any chemotactic collapse in some systems closely related to (1.1): For example, if $\phi(u) \equiv 1, \psi(u) = u$, f satisfies (1.2) with $r = 2$, Winkler ([33]) discussed the existence of global bounded classical solutions to problem (1.1) on a smooth bounded convex domain under the assumption that either $N \le 2$, or that the logistic damping effect b is large enough. When $\phi(u) = (u+1)^{-\alpha}$, $\psi(u) = u(u+1)^{\beta-1}$ with $0 < \alpha + \beta < \frac{2}{N}$ and f satisfies (1.2), Wang et al. ([28]) obtained the unique global uniformly bounded classical solution (u, v) of problem

(1.1). Furthermore, assuming that the logistic source $f \in C^\infty([0,\infty))$ satisfies

$$f(u) \le au - bu^2 \quad \text{for all} \quad u \ge 0 \tag{1.6}$$

and ϕ, ψ fulfill

$$\phi, \psi \in C^2([0,\infty)) \quad \text{and} \quad \phi(s) \ge 0 \quad \text{for all} \quad s \ge 0.$$

$$c_1 s^p \le \phi(s) \quad \text{for all} \quad s \ge s_0,$$

$$c_1 s^\beta \le \psi(s) \le c_2 s^\beta \quad \text{for all} \quad s \ge s_0,$$

with $c_2 > c_1 > 0$, $s_0 > 1$ and $p, \beta \in \mathbb{R}$, Cao ([3]) proved that if $\beta < 1$, then the classical solution of (1.1) is global in time and bounded.

Going beyond these boundedness statements, a number of results is available which show that the interplay of chemotactic cross-diffusion and cell kinetics of logistic-type may lead to quite a colorful dynamics. For instance, the result in [34] indicates that chemotaxis models may admit finite-time blow-up solutions even in the presence of certain logistic-type growth inhibitions, provided the latter are suitably weak. If $\tau = 1, \phi(u) \equiv 1, \psi(u) = u$, $f(u) = u - bu^2$ and the ratio $\frac{b}{\chi}$ is sufficiently large, Winkler ([36]) proved that the unique nontrivial spatially homogeneous equilibrium given by $u = v \equiv \frac{1}{b}$ is globally asymptotically stable in the sense that for any choice of suitably regular nonnegative initial data (u_0, v_0) such that $u_0 \not\equiv 0$, the above problem possesses a uniquely determined global classical solution (u, v) with $(u,v)|_{t=0} = (u_0, v_0)$ which satisfies

$$\|u(\cdot,t) - \frac{1}{b}\|_{L^\infty(\Omega)} \to 0 \quad \text{and} \quad \|v(\cdot,t) - \frac{1}{b}\|_{L^\infty(\Omega)} \to 0$$

as $t \to \infty$.

Motivated by the above works, the aim of present paper is to study the boundedness of the quasilinear chemotaxis system (1.1) under the conditions (1.2)–(1.5). Our main result says that together with the nonlinear diffusion, the aggregation and the logistic dampening rule out the occurrence of blow-up whenever $0 < \alpha + \beta < \max\{r - 1 + \alpha, \frac{2}{N}\}$, or

$$b \text{ is big enough}, \quad \text{if} \quad \beta = r - 1,$$

which means the logistic source and the nonlinear diffusion benefit the boundedness of solutions in the case of $0 < \alpha + \beta < \max\{r - 1 + \alpha, \frac{2}{N}\}$. It is noted that, due to the structure of supercritical sensitivity ($\alpha + \beta > \frac{2}{N}$), we can not invoke the Gagliardo–Nirenberg interpolation inequality which is used to estimate $\|u\|_{L^k(\Omega)}(k > 1)$ in case of subcritical sensitivity ($\alpha+\beta < \frac{2}{N}$) ([25, 28]). Moreover, in order to solve the case $\beta = r-1$, we have to investigate the properties of second equation. Hence, in order to give a complete analysis of problem (1.1), we must find new techniques. Let us point out that the main difference between this work and that of [3, 14, 24] is that the nonlinearity involved in (1.1) is stronger than the one in [3, 14, 24], which makes the analysis of problem under consideration more involved.

This paper is organized as follows. In the next section, we recall some preliminary results, state the main results of this paper and the local existence of the classical solution to (1.1). Section 3 is devoted to prove the main results of this paper. More precisely, we estimate $u(x,t)$ in a higher L^p space, and then obtain the uniform-in-time boundedness for u by a iteration procedure.

2 Preliminaries and main results

Throughout this paper the Hilbert space $H = L^2(\Omega)$ is equipped with usual inner product $(\cdot,\cdot)$ and norm $|.|_2$.

Before proving our main results, we will give some preliminary lemmas, which play a crucial role in the following proofs.

Lemma 2.1. *([9, 14]) Let $s \geq 1$ and $q \geq 1$. Assume that $p > 0$ and $a \in (0,1)$ satisfy*

$$\frac{1}{2} - \frac{p}{N} = (1-a)\frac{q}{s} + a\left(\frac{1}{2} - \frac{1}{N}\right) \quad \textit{and} \quad p \leq a.$$

Then there exist $c_0, c_0' > 0$ such that for all $u \in W^{1,2}(\Omega) \cap L^{\frac{s}{q}}(\Omega)$,

$$\|u\|_{W^{p,2}(\Omega)} \leq c_0 |\nabla u|_2^a \|u\|^{1-a}_{L^{\frac{s}{q}}(\Omega)} + c_0' \|u\|_{L^{\frac{s}{q}}(\Omega)}.$$

Lemma 2.2. *([17]) Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary. If $w \in C^2(\bar{\Omega})$ satisfies $\dfrac{\partial w}{\partial \nu} = 0$, then*

$$\frac{\partial |\nabla w|^2}{\partial \nu} \leq C_\Omega |\nabla w|,$$

where $C_\Omega > 0$ is a constant depending only on the curvatures of Ω.

Lemma 2.3. *([26]) Let $y(t) \geq 0$ be a solution of problem*

$$\begin{cases} y'(t) + Ay^p \leq B \quad & t > 0, \\ y(0) = y_0 \geq 0 \end{cases} \tag{2.1}$$

with $A > 0, p > 0$ and $B \geq 0$. Then we have

$$y(t) \leq \max\left\{y_0, \left(\frac{B}{A}\right)^{\frac{1}{p}}\right\}, \quad t > 0.$$

Lemma 2.4. *Assume that $d_1, d_2 \geq 0$ and $r_0 > 0$, then*

$$(d_1 + d_2)^{r_0} \leq 2^{r_0}(d_1^{r_0} + d_2^{r_0}).$$

Theorem 2.1. *Assume that $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\theta}(\bar{\Omega})$ (with some $\theta > n$) both are nonnegative, f satisfies (1.2), ϕ and ψ satisfies (1.3)–(1.5). If $0 < \alpha+\beta < \max\{r-1+\alpha, \frac{2}{N}\}$,*

or

$$b \ \text{ is big enough}, \quad \text{if } \ \beta = r - 1,$$

then there exists a pair $(u, v) \in (C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\Omega \times (0, \infty))^2$ which solves (1.1) in the classical sense. Moreover, both u and v are bounded in $\Omega \times (0, \infty)$.

Remark 2.1. (i) We find that the coefficient b of logistic source will not be used in the case of $0 < \alpha + \beta < \max\{r - 1 + \alpha, \frac{2}{N}\}$.

(ii) Theorem 2.1 extends the results of L. Wang et al. ([28]), who proved the possibility of global, in the cases $0 < \alpha + \beta < \frac{2}{N}$, and with $\Omega \subset \mathbb{R}^N$ is a convex bounded domains.

(iii)If $f(u) \equiv 0$, Theorem 2.1 extends the results of Tao and Winkler ([25]), who proved the possibility of global, in the cases $\frac{\psi(u)}{\phi(u)} \leq cu^\alpha$ ($\alpha < \frac{2}{N}(N \geq 2)$ and $u > 1$), and with $\Omega \subset \mathbb{R}^N$ is a convex bounded domains.

(iv)Theorem 2.1 extends the results of Cao ([3]), who proved the possibility of global, in the cases $\beta = 1$, $f(u) = au - bu^2$.

(v)Theorem 2.1 enlarges the parameter range $\frac{2}{N} \leq \alpha + \beta < r - 1 + \alpha$, Ishida et al. ([14]), who proved the possibility of global, in the cases $f(u) \equiv 0$.

The following local existence result is rather standard, since a similar reasoning in [3, 7, 24, 28, 29, 30, 38]. We omit it here.

Lemma 2.5. *Assume that the nonnegative functions $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\theta}(\bar{\Omega})$ (with some $\theta > N$), f satisfies (1.2) with some $a \geq 0, b > 0$ and ϕ, ψ satisfy (1.3)–(1.5), respectively. Then problem (1.1) has a unique local-in-time non-negative classical solution $(u, v) \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\Omega \times (0, T_{max})) \cap L^\infty_{loc}((0, T_{max}); W^{1,\theta}(\Omega))$, where T_{max} denotes the maximal existence time. In addition, if $T_{max} < +\infty$, then*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\theta}(\Omega)} \to \infty \quad as \quad t \to T_{max}$$

is fulfilled. Moreover, if $T_{max} < +\infty$, then

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty \quad as \quad t \to T_{max} \tag{2.2}$$

is fulfilled.

In order to proceed, let us now pick any $s \in (0, T_{max})$ and $s \leq 1$. Then by Lemma 2.5, we can conclude that for any given $s \in (0, T_{max}), s \leq 1$, there exists $K > 0$ such that

$$\|u(\tau)\|_{L^\infty(\Omega)} \leq K, \quad \|v(\tau)\|_{L^\infty(\Omega)} \leq K \quad \text{and} \quad \|\Delta v(\tau)\|_{L^\infty(\Omega)} \leq K \quad \text{for all} \quad \tau \in [0, s]. \tag{2.3}$$

Lemma 2.6. *([14, 25, 28]) For all $l \in (1, \frac{N}{N-1})$, there exists a constant $c > 0$ such that*

$$\|v(\cdot,t)\|_{W^{1,l}} \leq c \quad for\ all \quad t \in (0, T_{max}).$$

Lemma 2.7. *([3, 10]) Suppose $\gamma \in (1, +\infty)$, $g \in L^\gamma((0,T); L^\gamma(\Omega))$ and $v_0 \in W^{2,\gamma}(\Omega)$ such that $\dfrac{\partial v_0}{\partial \nu} = 0$. Let v be a solution of the following initial boundary value*

$$\begin{cases} v_t - \Delta v = g, \\ \dfrac{\partial v}{\partial \nu} = 0, \\ v(x,0) = v_0(x). \end{cases} \tag{2.4}$$

Then there exists a positive constant δ_0 such that

$$\begin{aligned} &\int_0^T \|v(\cdot,t)\|^\gamma_{L^\gamma(\Omega)} dt + \int_0^T \|v_t(\cdot,t)\|^\gamma_{L^\gamma(\Omega)} dt + \int_0^T \|\Delta v(\cdot,t)\|^\gamma_{L^\gamma(\Omega)} dt \\ \leq\ & \delta_0 \left(\int_0^T \|g(\cdot,t)\|^\gamma_{L^\gamma(\Omega)} dt + \|v_0(\cdot,t)\|^\gamma_{L^\gamma(\Omega)} + \|\Delta v_0(\cdot,t)\|^\gamma_{L^\gamma(\Omega)} \right). \end{aligned} \tag{2.5}$$

On the other hand, assuming v is a solution of the following initial boundary value

$$\begin{cases} v_t - \Delta v + v = g, \\ \dfrac{\partial v}{\partial \nu} = 0, \\ v(x,0) = v_0(x). \end{cases} \tag{2.6}$$

Then there exists a positive constant C_γ such that if $s_0 \in [0,T)$, $v(\cdot, s_0) \in W^{2,\gamma}(\Omega)(\gamma > N)$ with $\dfrac{\partial v(\cdot, s_0)}{\partial \nu} = 0$, then

$$\begin{aligned} & \int_{s_0}^{T} e^{\gamma s} \|\Delta v(\cdot,t)\|^{\gamma}_{L^\gamma(\Omega)} ds \\ \leq \; & C_\gamma \left(\int_{s_0}^{T} e^{\gamma s} \|g(\cdot,s)\|^{\gamma}_{L^\gamma(\Omega)} ds + e^{\gamma s}(\|v_0(\cdot,s_0)\|^{\gamma}_{L^\gamma(\Omega)} + \|\Delta v_0(\cdot,s_0)\|^{\gamma}_{L^\gamma(\Omega)}) \right). \end{aligned} \tag{2.7}$$

3 The proof of main results

This section is devoted to prove Theorem 2.1. To this end, we obtain the $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ by the iteration method, which depends some a priori estimates.

Firstly, let us derive the following a-priori estimates for the solutions of model (1.1).

Lemma 3.1. *Assume that f satisfies (1.2) and (u,v) is the solution of (1.1). Then for any $T \in (s, T_{max})$, there exists $\Upsilon > 0$ such that*

$$\int_\Omega (u(x,t)+1)^\sigma dx \leq \Upsilon \quad \textit{for all} \quad t \in (s,T)$$

and

$$\int_s^T \int_\Omega (u(x,t)+1)^{r-1} dx dt \leq \Upsilon(T+1),$$

where $0 < \sigma \leq 1$.

Proof. Integrating $(1.1)_1$ (the first equation of (1.1)) over Ω and using (1.2), we obtain

$$\frac{d}{dt}\int_\Omega u(x,t)dx = \int_\Omega f(u(x,t))dx \leq a|\Omega| - b\int_\Omega u^r(x,t)dx. \tag{3.1}$$

Due to $r > 1$ and the Hölder inequality, we conclude that

$$\frac{d}{dt}\int_\Omega u(x,t)dx + b|\Omega|^{1-r}\left(\int_\Omega u(x,t)dx\right)^r \leq a|\Omega|.$$

Hence, by Lemma 2.3, we get

$$\int_\Omega u(x,t)dx \le \max\{K, (\frac{a}{b})^{\frac{1}{r}}\}|\Omega| \ \text{ for all } \ t \in (s,T), \tag{3.2}$$

that is

$$\int_\Omega (u(x,t)+1)dx \le (\max\{K, (\frac{a}{b})^{\frac{1}{r}}\}+1)|\Omega| \ \text{ for all } \ t \in (s,T). \tag{3.3}$$

If $0 < \sigma < 1$, by (3.3) and the Hölder inequality, we have

$$\int_\Omega (u(x,t)+1)^\sigma dx \le \left(\int_\Omega (u(x,t)+1)dx\right)^\sigma |\Omega|^{1-\sigma} \le (\max\{K, (\frac{a}{b})^{\frac{1}{r}}\}+1)^\sigma |\Omega|. \tag{3.4}$$

On the other hand, integrating (3.1) over (s,T) with respect to t and using (2.3), we have

$$\begin{aligned} \frac{1}{b}\int_\Omega u(x,T)dx + \int_s^T\int_\Omega u^r(x,t)dxdt \le & \ \frac{1}{b}\left(a|\Omega|T + \int_\Omega u(x,s)dx\right) \\ \le & \ \frac{|\Omega|}{b}(aT+K) \\ \le & \ \frac{|\Omega|}{b}(a+K)(T+1), \end{aligned} \tag{3.5}$$

which implies that,

$$\int_s^T\int_\Omega u^r(x,t)dxdt \le \frac{|\Omega|}{b}(a+K)(T+1). \tag{3.6}$$

Hence, by Lemma 2.4, we have

$$\begin{aligned} \int_s^T\int_\Omega (u(x,t)+1)^r dxdt \le & \ 2^r\left(\int_s^T\int_\Omega u^r(x,t)dxdt + |\Omega|T\right) \\ \le & \ 2^r[\frac{|\Omega|}{b}(a+K)(T+1) + |\Omega|T] \\ \le & \ 2^r|\Omega|(\frac{a+K}{b}+1)(T+1), \end{aligned} \tag{3.7}$$

which, combined with $r > 1$ and the Hölder inequality implies that

$$\begin{aligned} \int_s^T\int_\Omega (u(x,t)+1)^{r-1}dxdt \le & \ \int_s^T\left(\int_\Omega (u(x,t)+1)^r dx\right)^{\frac{r-1}{r}}\left(\int_\Omega 1^r dx\right)^{\frac{1}{r}} dt \\ = & \ |\Omega|^{\frac{1}{r}}\int_s^T\left(\int_\Omega (u(x,t)+1)^r dx\right)^{\frac{r-1}{r}} dt \\ \le & \ |\Omega|^{\frac{1}{r}}\left(\int_s^T\left(\int_\Omega (u(x,t)+1)^r dx\right)^{\frac{r-1}{r}\times\frac{r}{r-1}} dt\right)^{\frac{r-1}{r}}\left(\int_s^T 1^r dt\right)^{\frac{1}{r}} \\ \le & \ |\Omega|^{\frac{1}{r}}T^{\frac{1}{r}}\left(\int_s^T\left(\int_\Omega (u(x,t)+1)^r dx\right) dt\right)^{\frac{r-1}{r}} \\ \le & \ |\Omega|^{\frac{1}{r}}T^{\frac{1}{r}}\left(2^r|\Omega|(\frac{a+K}{b}+1)(T+1)\right)^{\frac{r-1}{r}} \\ \le & \ 2^{r-1}|\Omega|\left(\frac{a+K}{b}+1\right)^{\frac{r-1}{r}}(T+1). \end{aligned} \tag{3.8}$$

Finally, choosing $\Upsilon = \left[(\max\{K,(\frac{a}{b})^{\frac{1}{r}}\}+1)^{\sigma} + 2^{r-1}\left(\frac{a+K}{b}+1\right)^{\frac{r-1}{r}}\right]|\Omega|$ and using (3.4) and (3.8), we can get the results. $\square$

Next, we are in a position to improve the regularity of u in a higher L^p space, which plays important role in obtaining the L^∞-estimate of u.

Lemma 3.2. *Assume that f satisfies (1.2) with $\frac{2}{N} \le \alpha + \beta < r - 1 + \alpha$. Let (u, v) be a solution to (1.1) on $(0, T_{max})$,*

$$\kappa_r = \begin{cases} r-1 & \text{if } 1 < r < 2, \\ 1 & \text{if } r = 2, \end{cases} \tag{3.9}$$

$\delta \ge \kappa_r$ and $\mu = (r-\beta)\delta + (r-\beta-1)(r-\kappa_r)$. If

$$\int_\Omega (u+1)^\delta(x,t)dx \le C_0(T+1) \text{ for any } t \in (s,T) \tag{3.10}$$

and

$$\int_s^T \int_\Omega (u+1)^{\delta-\kappa_r+r}(x,t)dxdt \le C_0(T+1) \tag{3.11}$$

hold for some $T \in (0, T_{max})$ and some $C_0 \ge 1$, then there exist $M_0 > 0$ and $M > 0$ depending on a, b, r, K and $|\Omega|$ such that

$$\int_\Omega (u+1)^\mu(x,t)dx \le M_0 M^\mu C_0(T+1) \text{ for any } t \in (s,T)$$

and

$$\int_s^T \int_\Omega (u+1)^{\mu-\kappa_r+r}(x,t)dxdt \le M_0 M^\mu C_0(T+1).$$

Proof. Without loss of generality that $\beta \ge 0$. Multiplying $(1.1)_1$ by $(u+1)^{\mu-\kappa_r}$ and integrating over Ω, we get

$$\begin{aligned} &\frac{1}{\mu-\kappa_r+1}\frac{d}{dt}\|u+1\|^{\mu-\kappa_r+1}_{L^{\mu-\kappa_r+1}(\Omega)} + (\mu-\kappa_r)\int_\Omega (u+1)^{\mu-\kappa_r-1}\phi(u)|\nabla u|^2dx \\ = & -\chi\int_\Omega \nabla\cdot(\psi(u)\nabla v)(u+1)^{\mu-\kappa_r}dx + \int_\Omega (u+1)^{\mu-\kappa_r}f(u)dx, \end{aligned} \tag{3.12}$$

which, together with (1.4), implies that

$$\begin{aligned} &\frac{1}{\mu-\kappa_r+1}\frac{d}{dt}\|u+1\|^{\mu-\kappa_r+1}_{L^{\mu-\kappa_r+1}(\Omega)} + M_\phi(\mu-\kappa_r)\int_\Omega (u+1)^{\mu-\kappa_r-\alpha-1}|\nabla u|^2dx \\ \le & -\chi\int_\Omega \nabla\cdot(\psi(u)\nabla v)(u+1)^{\mu-\kappa_r}dx + \int_\Omega (u+1)^{\mu-\kappa_r}f(u)dx. \end{aligned} \tag{3.13}$$

Integrating by parts to the first term on the right hand side of (3.13) and using $\beta + 1 < r$ and the Young inequality, we obtain from the second equation in (1.1)

$$\begin{aligned}
& -\chi \int_\Omega \nabla \cdot (\psi(u)\nabla v)(u+1)^{\mu-\kappa_r} dx \\
= \ & (\mu-\kappa_r)\chi \int_\Omega \psi(u)(u+1)^{\mu-\kappa_r-1}\nabla(u+1)\cdot\nabla v dx \\
= \ & (\mu-\kappa_r)\chi \int_\Omega \nabla\Psi(u)\cdot\nabla v dx \\
= \ & -(\mu-\kappa_r)\chi \int_\Omega \Psi(u)\Delta v dx \\
= \ & -(\mu-\kappa_r)\chi \int_\Omega \Psi(u)(v_t+v-u) dx \\
\le \ & (\mu-\kappa_r)\chi \int_\Omega (\Psi(u)|v_t| + \Psi(u)u)\, dx \\
\le \ & M_\psi \frac{\mu-\kappa_r}{\mu-\kappa_r+\beta}\chi \int_\Omega \left((u+1)^{\mu-\kappa_r+\beta}|v_t| + (u+1)^{\mu-\kappa_r+\beta+1}\right) dx \\
\le \ & \chi M_\psi \int_\Omega (u+1)^{\mu-\kappa_r+\beta+1} dx + \chi M_\psi \int_\Omega (u+1)^{\mu-\kappa_r+\beta}|v_t| dx \\
\le \ & \frac{b}{2^{r+1}} \int_\Omega (u+1)^{\mu-\kappa_r+r} dx + C_1(\mu) + C_2(\mu)\int_\Omega |v_t|^{\frac{\mu-\kappa_r+r}{r-\beta}} dx,
\end{aligned} \tag{3.14}$$

where

$$\Psi(u) = \int_0^u \psi(\tau)(\tau+1)^{\mu-\kappa_r-1} d\tau, \tag{3.15}$$

$$\begin{aligned}
C_1(\mu) := \ & \frac{r-\beta-1}{\mu-\kappa_r+r}\left(\frac{b}{2^{r+2}} \times \frac{\mu-\kappa_r+r}{\mu-\kappa_r+\beta+1}\right)^{-\frac{\mu-\kappa_r+\beta+1}{r-\beta-1}} (\chi M_\psi)^{\frac{\mu-\kappa_r+r}{r-\beta-1}} |\Omega| \\
= \ & \frac{r-\beta-1}{\mu-\kappa_r+r}\left(\frac{b}{2^{r+2}}\right)^{-\frac{\beta}{r-\beta-1}} (\chi M_\psi)^{\frac{r-1}{r-\beta-1}} \\
& \times\left(1+\frac{r-\beta-1}{\mu-\kappa_r+\beta+1}\right)^{-\frac{\mu-\kappa_r+\beta+1}{r-\beta-1}} \left[\left(\frac{2^{r+2}\chi M_\psi}{b}\right)^{\frac{1}{r-\beta-1}}\right]^{(\mu-\kappa_r+1)} |\Omega| \\
\le \ & (r-\beta-1)\left(\frac{b}{2^{r+2}}\right)^{-\frac{\beta}{r-\beta-1}} (\chi M_\psi)^{\frac{r-1}{r-\beta-1}} \\
& \times\left(1+\frac{r-\beta-1}{\mu-\kappa_r+\beta+1}\right)^{-\frac{\mu-\kappa_r+\beta+1}{r-\beta-1}} |\Omega| \frac{\left[\left(\frac{2^{r+2}\chi M_\psi}{b}\right)^{\frac{1}{r-\beta-1}}\right]^{(\mu-\kappa_r+1)}}{\mu-\kappa_r+1}
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
C_2(\mu) := \ & \frac{r-\beta}{\mu-\kappa_r+r}\left(\frac{b}{2^{r+2}} \times \frac{\mu-\kappa_r+r}{\mu-\kappa_r+\beta}\right)^{-\frac{\mu-\kappa_r+\beta}{r-\beta}} (\chi M_\psi)^{\frac{\mu-\kappa_r+r}{r-\beta}} \\
= \ & \frac{r-\beta}{\mu-\kappa_r+r}\left(\frac{b}{2^{r+2}}\right)^{-\frac{\beta-1}{r-\beta}} (\chi M_\psi)^{\frac{r-1}{r-\beta}} \left(1+\frac{r-\beta}{\mu-\kappa_r+\beta}\right)^{-\frac{\mu-\kappa_r+\beta}{r-\beta}} \left[\left(\frac{2^{r+2}\chi M_\psi}{b}\right)^{\frac{1}{r-\beta}}\right]^{(\mu-\kappa_r+1)} \\
\le \ & (r-\beta)\left(\frac{b}{2^{r+2}}\right)^{-\frac{\beta-1}{r-\beta}} (\chi M_\psi)^{\frac{r-1}{r-\beta}} \left(1+\frac{r-\beta}{\mu-\kappa_r+\beta}\right)^{-\frac{\mu-\kappa_r+\beta}{r-\beta}} \frac{\left[\left(\frac{2^{r+2}\chi M_\psi}{b}\right)^{\frac{1}{r-\beta}}\right]^{(\mu-\kappa_r+1)}}{\mu-\kappa_r+1}.
\end{aligned} \tag{3.17}$$

Here we have used the fact that u and v are nonnegative functions.

Due to (1.2), we have

$$\begin{aligned}
&\int_\Omega (u+1)^{\mu-\kappa_r} f(u)dx \\
\leq\ &\int_\Omega (u+1)^{\mu-\kappa_r}(a-bu^r)dx \\
=\ & a\int_\Omega (u+1)^{\mu-\kappa_r}dx - b\int_\Omega (u+1)^{\mu-\kappa_r}u^r dx.
\end{aligned} \tag{3.18}$$

On the other hand, due to

$$\int_\Omega (u+1)^{\mu-\kappa_r}2^r(u^r+1)dx \geq \int_\Omega (u+1)^{\mu-\kappa_r}(u+1)^r dx = \int_\Omega (u+1)^{\mu-\kappa_r+r}dx,$$

we have

$$b\int_\Omega (u+1)^{\mu-\kappa_r}u^r dx \geq \frac{b}{2^r}\int_\Omega (u+1)^{\mu-\kappa_r+r}dx - b\int_\Omega (u+1)^{\mu-\kappa_r}dx.$$

Inserting the above inequality into (3.18), we get

$$\int_\Omega (u+1)^{\mu-\kappa_r} f(u)dx \leq (a+b)\int_\Omega (u+1)^{\mu-\kappa_r}dx - \frac{b}{2^r}\int_\Omega (u+1)^{\mu-\kappa_r+r}dx. \tag{3.19}$$

By (3.12), (3.14), and (3.19), we obtain

$$\begin{aligned}
&\frac{1}{\mu-\kappa_r+1}\frac{d}{dt}\|u+1\|^{\mu-\kappa_r+1}_{L^{\mu-\kappa_r+1}(\Omega)} + M_\psi(\mu-\kappa_r)\int_\Omega (u+1)^{\mu-\kappa_r-\alpha-1}|\nabla u|^2 dx \\
\leq\ & -\frac{b}{2^{r+1}}\int_\Omega (u+1)^{\mu-\kappa_r+r}dx + (a+b)\int_\Omega (u+1)^{\mu-\kappa_r}dx + C_1(\mu) + C_2(\mu)\int_\Omega |v_t|^{\frac{\mu-\kappa_r+r}{r-\beta}}dx.
\end{aligned} \tag{3.20}$$

Since $r>1$, and with the help of the Young inequality, we see that

$$\begin{aligned}
&\frac{1}{\mu-\kappa_r+1}\frac{d}{dt}\|u+1\|^{\mu-\kappa_r+1}_{L^{\mu-\kappa_r+1}(\Omega)} + M_\phi(\mu-\kappa_r)\int_\Omega (u+1)^{\mu-\kappa_r-\alpha-1}|\nabla u|^2 dx \\
\leq\ & -\frac{b}{2^{r+2}}\int_\Omega (u+1)^{\mu-\kappa_r+r}dx + C_1(\mu) + C_3(\mu) + C_2(\mu)\int_\Omega |v_t|^{\frac{\mu-\kappa_r+r}{r-\beta}}dx,
\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}
C_3(\mu) :=\ & \frac{r}{\mu-\kappa_r+r}\left(\frac{b}{2^{r+2}}\times\frac{\mu-\kappa_r+r}{\mu-\kappa_r}\right)^{-\frac{\mu-\kappa_r}{r}}(a+b)^{\frac{\mu-\kappa_r+r}{r}}|\Omega| \\
=\ & \frac{r}{\mu-\kappa_r+r}\left(\frac{b}{2^{r+2}}\right)^{\frac{1}{r}}(a+b)^{\frac{r-1}{r}}\left(1+\frac{r}{\mu-\kappa_r}\right)^{-\frac{\mu-\kappa_r}{r}}\left[\left(\frac{2^{r+2}(a+b)}{b}\right)^{\frac{1}{r}}\right]^{(\mu-\kappa_r+1)}|\Omega| \\
\leq\ & r\left(\frac{b}{2^{r+2}}\right)^{\frac{1}{r}}(a+b)^{\frac{r-1}{r}}\left(1+\frac{r}{\mu-\kappa_r}\right)^{-\frac{\mu-\kappa_r}{r}}|\Omega|\frac{\left[\left(\frac{2^{r+2}(a+b)}{b}\right)^{\frac{1}{r}}\right]^{(\mu-\kappa_r+1)}}{\mu-\kappa_r+1}.
\end{aligned} \tag{3.22}$$

Next, let

$$C_4 = (r-\beta-1)\left(\frac{b}{2}\right)^{-\frac{\beta}{r-\beta-1}}(\chi M_\psi)^{\frac{r-1}{r-\beta-1}}\left(1+\frac{r-\beta-1}{r(r-\beta-1)+\beta+1}\right)^{-\frac{r(r-\beta-1)+\beta+1}{r-\beta-1}}|\Omega|,$$

$$C_5 = (r-\beta)(\frac{b}{2^{r+2}})^{-\frac{\beta-1}{r-\beta}}(\chi M_\psi)^{\frac{r-1}{r-\beta}}(1+\frac{1}{r-1})^{-(r-1)}$$

and

$$C_6 = r(\frac{b}{2^{r+2}})^{\frac{1}{r}}(a+b)^{\frac{r-1}{r}}(1+\frac{1}{r-\beta-1})^{-(r-\beta-1)}|\Omega|$$

and choose

$$M_1 = \max\{C_4, C_5, C_6\},$$
$$M_2 = \max\{1+(\frac{2^{r+2}\chi M_\psi}{b})^{\frac{1}{r-\beta-1}}, 1+(\frac{2^{r+2}\chi M_\psi}{b})^{\frac{1}{r-\beta}}, 1+(\frac{2^{r+2}(a+b)}{b})^{\frac{1}{r}}\},$$

then by (3.16), (3.21) and (3.22), we have

$$\frac{1}{\mu-\kappa_r+1}\frac{d}{dt}\|u+1\|^{\mu-\kappa_r+1}_{L^{\mu-\kappa_r+1}(\Omega)} + \frac{1}{2^{r+2}}b\int_\Omega (u+1)^{\mu-\kappa_r+r}dx \le M_1\frac{M_2^{\mu-\kappa_r+1}}{\mu-\kappa_r+1}\int_\Omega |v_t|^{\frac{\mu-\kappa_r+r}{r-\beta}}dx. \tag{3.23}$$

Here we have used the fact that $f(x) = (1+\frac{1}{x})^x$ is a strictly increasing function on $(r(r-\beta-1),+\infty)$, $\lim_{x\to+\infty}(1+\frac{1}{x})^x = e$ and $\mu-\kappa_r \ge r(r-\beta-1) > 0$.

Now, inserting (3.23) over (s,T) and using (2.3) and $\mu-\kappa_r+1 \ge 1$, we have

$$\begin{aligned}
&\int_\Omega (u+1)^{\mu-\kappa_r+1}(x,t)dx \\
\le\ & \int_\Omega (u+1)^{\mu-\kappa_r+1}(x,s)dx + M_1M_2^{\mu-\kappa_r+1}\int_s^T\int_\Omega |v_t|^{\frac{\mu-\kappa_r+r}{r-\beta}}dxdt \\
\le\ & K^{\mu-\kappa_r+1}|\Omega| + M_1M_2^{\mu-\kappa_r+1}\int_s^T\int_\Omega |v_t|^{\frac{\mu-\kappa_r+r}{r-\beta}}dxdt \quad \text{for all} \quad t\in(s,T)
\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}
&\int_s^T\int_\Omega (u+1)^{\mu-\kappa_r+r}dxdt \\
\le\ & \frac{2^{r+2}}{b(\mu-\kappa_r+1)}\int_\Omega (u+1)^{\mu-\kappa_r+1}(x,s)dx + M_1\frac{2^{r+2}M_2^{\mu-\kappa_r+1}}{b(\mu-\kappa_r+1)}\int_s^T\int_\Omega |v_t|^{\frac{\mu-\kappa_r+r}{r-\beta}}dxdt \\
\le\ & \frac{2^{r+2}}{b}\left(K^{\mu-\kappa_r+1}|\Omega| + M_1M_2^{\mu-\kappa_r+1}\int_s^T\int_\Omega |v_t|^{\frac{\mu-\kappa_r+r}{r-\beta}}dxdt\right).
\end{aligned} \tag{3.25}$$

On the other hand, with the help of the Young inequality, multiplying $(1.1)_2$ by $v^{\frac{\mu-\kappa_r+r}{r-\beta}-1}$

and integrating by parts, we obtain

$$\begin{aligned}\frac{r-\beta}{\mu-\kappa_r+r}\frac{d}{dt}\|v\|^{\frac{\mu-\kappa_r+r}{r-\beta}}_{L^{\frac{\mu-\kappa_r+r}{r-\beta}}} &= -(\frac{\mu-\kappa_r+r}{r-\beta}-1)\int_\Omega v^{\frac{\mu-\kappa_r+r}{r-\beta}-2}|\nabla v|^2dx+\int_\Omega(uv^{\frac{\mu-\kappa_r+r}{r-\beta}-1}-v^{\frac{\mu-\kappa_r+r}{r-\beta}})dx\\ &\le \int_\Omega(uv^{\frac{\mu-\kappa_r+r}{r-\beta}-1}-v^{\frac{\mu-\kappa_r+r}{r-\beta}})dx\\ &\le \frac{r-\beta}{\mu-\kappa_r+r}\int_\Omega\left(u^{\frac{\mu-\kappa_r+r}{r-\beta}}-v^{\frac{\mu-\kappa_r+r}{r-\beta}}\right)dx\\ &\le \frac{r-\beta}{\mu-\kappa_r+r}\int_\Omega\left((u+1)^{\frac{\mu-\kappa_r+r}{r-\beta}}-v^{\frac{\mu-\kappa_r+r}{r-\beta}}\right)dx.\end{aligned} \tag{3.26}$$

Integrating (3.26) over (s,T), using (3.10)–(3.11) and (2.3), then we have

$$\begin{aligned}\int_s^T\int_\Omega v^{\frac{\mu-\kappa_r+r}{r-\beta}}(x,t)dxdt &\le \int_\Omega v^{\frac{\mu-\kappa_r+r}{r-\beta}}(x,s)dx+\int_s^T\int_\Omega(u+1)^{\frac{\mu-\kappa_r+r}{r-\beta}}(x,t)dxdt\\ &\le K^{\frac{\mu-\kappa_r+r}{r-\beta}}|\Omega|+C_0(T+1)\\ &\le C_7M_3^{\frac{\mu-\kappa_r+r}{r-\beta}-1}C_0(T+1),\end{aligned} \tag{3.27}$$

where $C_7=K|\Omega|+1$ and $M_3=\max\{1,K\}$. Here we have used the fact that $\mu=(r-\beta)\delta+(r-\beta-1)(r-\kappa_r)$. Therefore, Lemma 2.7, (3.11) and (3.27) yield

$$\begin{aligned}&\int_s^T\int_\Omega|v_t|^{\frac{\mu-\kappa_r+r}{r-\beta}}(x,t)dxdt\\ \le\;& \delta_0\int_s^T\int_\Omega\left(u^{\frac{\mu-\kappa_r+r}{r-\beta}}(x,t)+v^{\frac{\mu-\kappa_r+r}{r-\beta}}(x,t)\right)dxdt\\ &+\delta_0\int_\Omega\left(|\Delta v|^{\frac{\mu-\kappa_r+r}{r-\beta}}(x,s)+v^{\frac{\mu-\kappa_r+r}{r-\beta}}(x,s)\right)dx\\ \le\;& \delta_0\int_s^T\int_\Omega\left((u+1)^{\frac{\mu-\kappa_r+r}{r-\beta}}(x,t)+v^{\frac{\mu-\kappa_r+r}{r-\beta}}(x,t)\right)dxdt\\ &+\delta_0\int_\Omega\left(|\Delta v|^{\frac{\mu-\kappa_r+r}{r-\beta}}(x,s)+v^{\frac{\mu-\kappa_r+r}{r-\beta}}(x,s)\right)dx\\ \le\;& \delta_0M_3^{\frac{\mu-\kappa_r+r}{r-\beta}-1}C_0(T+1)+\delta_0C_0(T+1)+2K^{\frac{\mu-\kappa_r+r}{r-\beta}}|\Omega|\delta_0(T+1)\\ \le\;& C_8M_3^{\frac{\mu-\kappa_r+r}{r-\beta}-1}C_0(T+1),\end{aligned} \tag{3.28}$$

where $C_8=2\delta_0(1+M_2|\Omega|),\delta_0$ is the constant given in Lemma 2.7.

Now, inserting (3.28) into (3.24) and (3.25), respectively, and using $\beta+1<r$, we have

$$\begin{aligned}&\int_\Omega(u+1)^{\mu-\kappa_r+1}(x,t)dx\\ \le\;& K^{\mu-\kappa_r+1}|\Omega|+M_1M_2^{\mu-\kappa_r+1}C_8M_3^{\frac{\mu-\kappa_r+r}{r-\beta}-1}C_0(T+1)\\ \le\;& M_3^\mu K^2|\Omega|+M_1M_2^\mu M_2^2C_8M_3^\mu M_3^\beta C_0(T+1)\\ \le\;& C_9M_4^\mu C_0(T+1)\end{aligned} \tag{3.29}$$

and

$$
\begin{aligned}
&\int_s^T\int_\Omega (u+1)^{\mu-\kappa_r+r}dxdt\\
\leq\ & \frac{2^{r+2}}{b}\left(K^{\mu-\kappa_r+1}|\Omega|+M_1M_2^{\mu-\kappa_r+1}C_8M_3^{\frac{\mu-\kappa_r+r}{r-\beta}-1}C_0(T+1)\right)\\
\leq\ & \frac{2^{r+2}}{b}\left(M_2^\mu K^2|\Omega|+M_1M_2^\mu M_2^2C_8M_3^\mu M_3^\beta C_0(T+1)\right)\\
\leq\ & C_9M_4^\mu C_0(T+1),
\end{aligned}
\tag{3.30}
$$

where $C_9=(1+\frac{2^{r+2}}{b})(K^2|\Omega|+C_8M_1M_2^2M_3^\beta)$, $M_4=M_2M_3$. Finally, it follows from the Hölder inequality and (3.29) that

$$
\begin{aligned}
&\int_\Omega (u+1)^\mu(x,t)dx\\
\leq\ & (\int_\Omega (u+1)^{\mu-\kappa_r+1}(x,t)dx)^{\frac{\mu}{\mu-\kappa_r+1}}|\Omega|^{\frac{1-\kappa_r}{\mu-\kappa_r+1}}\\
\leq\ & (C_9M_4^\mu C_0(T+1)))^{\frac{\mu}{\mu-\kappa_r+1}}\,|\Omega|^{\frac{1-\kappa_r}{\mu-\kappa_r+1}}\\
\leq\ & M_0M^\mu C_0(T+1),
\end{aligned}
\tag{3.31}
$$

where $M_0=(C_9)^{\frac{\mu}{\mu-\kappa_r+1}}\,|\Omega|^{\frac{1-\kappa_r}{\mu-\kappa_r+1}}+C_9$, $M=M_4$. Combining (3.30) with (3.31), we can get the results. This completes the proof of Lemma 3.2. □

Lemma 3.3. *Assume that $0<\alpha+\beta<\dfrac{2}{N}$. Let (u,v) be a solution to (1.1) on $(0,T_{max})$. Then for any $T\in(s,T_{max})$, $k\geq 1$, $m\geq 1$, there exist a positive constant C such that*

$$
\|u(\cdot,t)\|_{L^k(\Omega)}\leq C \quad \textit{and} \quad \|\nabla v(\cdot,t)\|_{L^{2m}(\Omega)}\leq C \tag{3.32}
$$

hold for all $t\in(s,T)$.

Proof. Let $l\in[1,\frac{N}{N-1})$. Throughout this proof, $C_i(i\geq 1)$ are positive constants depending only on some of $c_0, c_0', c_\Omega, \alpha, \beta, N, \Omega, a, b, r, \chi, u_0$ and v_0. From L^1-boundedness of u, it suffices to prove the assertion for sufficiently large k. Now for $k>1$, using L^1-boundedness of u and following the same procedure as in ([28], (30) and (45)), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega (u+1)^kdx+\frac{1}{m}\int_\Omega|\nabla v|^{2m}dx\right)+\frac{2kM_\phi(k-1)}{(k-\alpha)^2}\int_\Omega|\nabla(u+1)^{\frac{k-\alpha}{2}}|^2dx\\
&+\frac{2(m-1)}{m^2}\int_\Omega|\nabla|\nabla v|^m|^2dx\\
\leq\ & C_1\left(\int_\Omega|\nabla(u+1)^{\frac{k-\alpha}{2}}|^2dx\right)^{\theta_1}\left(\int_\Omega|\nabla|\nabla v|^m|^2dx\right)^{\sigma_1}\\
&+C_2\left(\int_\Omega|\nabla(u+1)^{\frac{k-\alpha}{2}}|^2dx\right)^{\theta_2}\left(\int_\Omega|\nabla|\nabla v|^m|^2dx\right)^{\sigma_2}\\
&+C_3\int_{\partial\Omega}\frac{\partial|\nabla v|^2}{\partial\nu}|\nabla v|^{2m-2}dx+C_4,
\end{aligned}
\tag{3.33}
$$

where θ_1, θ_2, σ_1 and σ_2 depend only on N, α, β and k and satisfy

$$0 < \theta_i + \sigma_i < 1 \quad (i = 1, 2).$$

Next we deal with the integration on $\partial\Omega$. We see from Lemma 2.2 that

$$\begin{aligned} & \int_{\partial\Omega} \frac{\partial |\nabla v|^2}{\partial \nu} |\nabla v|^{2m-2} dx \\ \leq \;& C_\Omega \int_{\partial\Omega} |\nabla v|^{2m} dx \\ = \;& C_\Omega ||\nabla v|^m|^2_{L^2(\partial\Omega)}. \end{aligned} \tag{3.34}$$

Let us take $r \in (0, \frac{1}{2})$. By the embedding $W^{r+\frac{1}{2},2}(\Omega) \hookrightarrow L^2(\partial\Omega)$ is compact (see e.g. Haroske and Triebel [8]), we have

$$||\nabla v|^m|^2_{L^2(\partial\Omega)} \leq C_5 ||\nabla v|^m|^2_{W^{r+\frac{1}{2},2}(\Omega)}. \tag{3.35}$$

In order to apply Lemma 2.1 to the right-hand side of (3.35), let us pick $a \in (0, 1)$ satisfying

$$a = \frac{\frac{1}{2} - \frac{1}{2N} - \frac{m}{l} - \frac{m}{N}}{\frac{1}{2} - \frac{1}{N} - \frac{m}{l}}.$$

Noting that $r \in (0, \frac{1}{2})$ and $m > 1$ imply that $r + \frac{1}{2} \leq a < 1$, we see from the fractional Gagliardo–Nirenberg inequality (Lemma 2.1) and boundedness of $|\nabla v|^l$ (see Lemma 2.6) that

$$\begin{aligned} & ||\nabla v|^m|^2_{W^{r+\frac{1}{2},2}(\Omega)} \\ \leq \;& c_0 |\nabla |\nabla v|^m|_2^a \| |\nabla v|^m \|^{1-a}_{L^{\frac{s}{m}}(\Omega)} + c_0' \| |\nabla v|^m \|_{L^{\frac{s}{m}}(\Omega)} \\ \leq \;& C_6 |\nabla |\nabla v|^m|_2^a + C_6. \end{aligned} \tag{3.36}$$

Combining (3.34) and (3.35) with (3.36), we obtain

$$\int_{\partial\Omega} \frac{\partial |\nabla v|^2}{\partial \nu} |\nabla v|^{2m-2} dx \leq C_7 |\nabla |\nabla v|^m|_2^a + C_7. \tag{3.37}$$

Inserting (3.37) into (3.33) and using $a \in (0, 1)$ and the Young inequality, we have

$$\begin{aligned} & \frac{d}{dt}\left(\int_\Omega (u+1)^k dx + \int_\Omega |\nabla v|^{2m} dx\right) + \frac{2kM_\phi(k-1)}{(k-\alpha)^2} \int_\Omega |\nabla (u+1)^{\frac{k-\alpha}{2}}|^2 dx \\ & + \frac{2(m-1)}{m} \int_\Omega |\nabla |\nabla v|^m|^2 dx \\ \leq \;& C_8. \end{aligned} \tag{3.38}$$

Finally, let $y := \int_\Omega (u+1)^k dx + \int_\Omega |\nabla v|^{2m} dx$, we see from the same way as in [28] that (24) yields

$$\frac{d}{dt}y(t) + C_9 y^h(t) \le C_{10}$$

with some positive constant h. Thus a standard ODE comparison argument implies boundedness of $y(t)$ for all $t \in (0, T_{max})$. Clearly, $\|u(\cdot,t)\|_{L^k(\Omega)}$ and $\|\nabla v(\cdot,t)\|_{L^{2m}(\Omega)}$ are bounded for all $t \in (0, T_{max})$. The proof of Lemma 3.3 is complete. $\square$

Lemma 3.4. *For any fixed* $r > 1, \delta \ge 1, \chi M_\psi, C_{r+\delta-1} > 0$, *let*

$$H(y) = y + \frac{1}{r+\delta-1}\left[\frac{r+\delta-1}{r+\delta-2}\right]^{-(r+\delta-2)} y^{-(r+\delta-2)} \left(\frac{\chi M_\psi}{r+\delta-2}\right)^{r+\delta-1} C_{r+\delta-1} (y>0).$$

Then

$$\min_{y>0} H(y) = \frac{\chi M_\psi C_{r+\delta-1}^{\frac{1}{r+\delta-1}}}{r+\delta-2}.$$

Proof. Let $\rho_0 = r+\delta-2$ and

$$A_1 = \frac{1}{r+\delta-1}\left[\frac{r+\delta-1}{r+\delta-2}\right]^{-(r+\delta-2)}. \tag{3.39}$$

Then $H(y)$ will be

$$H(y) = y + A_1 y^{-\rho_0}\left(\frac{\chi M_\psi}{\rho_0}\right)^{\rho_0+1} C_{\rho_0+1}.$$

Hence,

$$H'(y) = 1 - A_1\rho_0\left(\frac{\chi M_\psi}{\rho_0}\right)^{\rho_0+1} C_{\rho_0+1} y^{-\rho_0-1}.$$

Let $H'(y) = 0$, we have

$$y_0 = (A_1 C_{\rho_0+1}\rho_0)^{\frac{1}{\rho_0+1}} \frac{M_\psi}{\rho_0}\chi.$$

On the other hand, by $\lim_{y\to 0^+} H(y) = +\infty$ and $\lim_{y\to+\infty} H(y) = +\infty$, we have

$$\min_{y>0} H(y) = H(y_0) = \frac{\chi M_\psi C_{r+\delta-1}^{\frac{1}{r+\delta-1}}}{r+\delta-2}.$$

$\square$

Lemma 3.5. *Assume that* $\beta + 1 = r$ *and* b *is big enough. Let* (u,v) *be a solution to (1.1) on* $(0, T_{max})$. *Then for any* $T \in (s, T_{max})$, *for all* $\delta > 1$, *there exists a positive constant* $C = C(\delta, |\Omega|, b, \chi, M_\psi, u_0, v_0)$ *such that*

$$\int_\Omega (u(x,t)+1)^\delta dx \le C \quad \textit{for all} \;\; t \in (s,T) \tag{3.40}$$

holds.

Proof. Multiplying $(1.1)_1$ by $(u+1)^{\delta-1}$ and integrating over Ω, we get

$$\begin{aligned} &\frac{1}{\delta}\frac{d}{dt}\|u+1\|^{\delta}_{L^{\delta}(\Omega)} + (\delta-1)\int_{\Omega}(u+1)^{\delta-2}\phi(u)|\nabla u|^2dx \\ = &-\chi\int_{\Omega}\nabla\cdot(\psi(u)\nabla v)(u+1)^{\delta-1}dx + \int_{\Omega}(u+1)^{\delta-1}f(u)dx, \end{aligned} \tag{3.41}$$

which, together with (1.4) implies that

$$\begin{aligned} &\frac{1}{\delta}\frac{d}{dt}\|u+1\|^{\delta}_{L^{\delta}(\Omega)} + M_{\phi}(\delta-1)\int_{\Omega}(u+1)^{\delta-\alpha-2}|\nabla u|^2dx \\ \leq &-\chi\int_{\Omega}\nabla\cdot(\psi(u)\nabla v)(u+1)^{\delta-1}dx + \int_{\Omega}(u+1)^{\delta-1}f(u)dx, \end{aligned}$$

that is,

$$\begin{aligned} &\frac{1}{\delta}\frac{d}{dt}\|u+1\|^{\delta}_{L^{\delta}(\Omega)} \\ \leq &-\frac{r+\delta-1}{\delta}\int_{\Omega}(u+1)^{\delta}dx - \chi\int_{\Omega}\nabla\cdot(\psi(u)\nabla v)(u+1)^{\delta-1}dx \\ &+\int_{\Omega}\left(\frac{r+\delta-1}{\delta}(u+1)^{\delta} + (u+1)^{\delta-1}f(u)\right)dx. \end{aligned} \tag{3.42}$$

Due to (1.2), we have

$$\begin{aligned} &\int_{\Omega}(u+1)^{\delta-1}f(u)dx \\ \leq &\int_{\Omega}(u+1)^{\delta-1}(a-bu^r)dx \\ = &\ a\int_{\Omega}(u+1)^{\delta-1}dx - b\int_{\Omega}(u+1)^{\delta-1}u^rdx. \end{aligned} \tag{3.43}$$

On the other hand, due to

$$\int_{\Omega}(u+1)^{\delta-1}2^r(u^r+1)dx \geq \int_{\Omega}(u+1)^{\delta-1}(u+1)^rdx = \int_{\Omega}(u+1)^{r+\delta-1}dx,$$

we have

$$b\int_{\Omega}(u+1)^{\delta-1}u^rdx \geq \frac{b}{2^r}\int_{\Omega}(u+1)^{r+\delta-1}dx - b\int_{\Omega}(u+1)^{\delta-1}dx.$$

Inserting the above inequality into (3.43), we get

$$\int_{\Omega}(u+1)^{\delta-1}f(u)dx \leq (a+b)\int_{\Omega}(u+1)^{\delta-1}dx - \frac{b}{2^r}\int_{\Omega}(u+1)^{r+\delta-1}dx. \tag{3.44}$$

Hence, by Young inequality, we have

$$\begin{aligned} &\int_{\Omega}\left(\frac{r+\delta-1}{\delta}(u+1)^{\delta} + (u+1)^{\delta-1}f(u)\right)dx \\ \leq &\ \frac{r+\delta-1}{\delta}\int_{\Omega}(u+1)^{\delta}dx + (a+b)\int_{\Omega}(u+1)^{\delta-1}dx - \frac{b}{2^r}\int_{\Omega}(u+1)^{r+\delta-1}dx \\ \leq &\ (\varepsilon_1+\varepsilon_2-\frac{b}{2^r})\int_{\Omega}(u+1)^{r+\delta-1}dx + C_1(\varepsilon_1,\delta) + C_2(\varepsilon_2,\delta), \end{aligned} \tag{3.45}$$

where

$$C_1(\varepsilon_1,\delta)=\frac{r-1}{\delta+r-1}\left(\varepsilon_1\frac{\delta+r-1}{\delta}\right)^{-\frac{\delta}{r-1}}\left(\frac{r+\delta-1}{\delta}\right)^{\frac{\delta+r-1}{r-1}}|\Omega|$$

and

$$C_2(\varepsilon_2,\delta)=\frac{r}{\delta+r-1}\left(\varepsilon_2\frac{\delta+r-1}{\delta-1}\right)^{-\frac{\delta-1}{r}}(a+b)^{\frac{\delta+r-1}{r}}|\Omega|.$$

Next, integrating by parts to the first term on the right hand side of (3.41), using $\beta+1=r$ and the Young inequality, we obtain

$$\begin{aligned}
&-\chi\int_\Omega\nabla\cdot(\psi(u)\nabla v)(u+1)^{\delta-1}dx\\
=\;&(\delta-1)\chi\int_\Omega\psi(u)(u+1)^{\delta-2}\nabla(u+1)\cdot\nabla v dx\\
=\;&-(\delta-1)\chi\int_\Omega\tilde{\Psi}(u)\Delta v dx\\
\le\;&(\delta-1)\chi\int_\Omega\tilde{\Psi}(u)|\Delta v|dx\\
\le\;&\frac{\delta-1}{\beta+\delta-1}\chi M_\psi\int_\Omega(u+1)^{\beta+\delta-1}|\Delta v|dx\\
=\;&\frac{\delta-1}{r+\delta-2}\chi M_\psi\int_\Omega(u+1)^{r+\delta-2}|\Delta v|dx\\
\le\;&\chi M_\psi\int_\Omega(u+1)^{r+\delta-2}|\Delta v|dx,
\end{aligned}\tag{3.46}$$

where $\tilde{\Psi}(u)=\int_0^u\psi(\tau)(\tau+1)^{\delta-2}d\tau$. Here, we have used the fact that $r>1$. For any fixed $\lambda_0>0$, applying the Young inequality to (3.46), we have

$$\begin{aligned}
&-\chi\int_\Omega\nabla\cdot(\psi(u)\nabla v)(u+1)^{\delta-1}dx\\
\le\;&\lambda_0\int_\Omega(u+1)^{r+\delta-1}dx+\frac{1}{r+\delta-1}\left[\lambda_0\frac{r+\delta-1}{r+\delta-2}\right]^{-(r+\delta-2)}\left(\frac{\chi M_\psi}{r+\delta-2}\right)^{r+\delta-1}\int_\Omega|\Delta v|^{r+\delta-1}dx\\
=\;&\lambda_0\int_\Omega(u+1)^{r+\delta-1}dx+A_1\lambda_0^{-(r+\delta-2)}\left(\frac{\chi M_\psi}{r+\delta-2}\right)^{r+\delta-1}\int_\Omega|\Delta v|^{r+\delta-1}dx,
\end{aligned}\tag{3.47}$$

where A_1 is given by (3.39). Thus, inserting (3.45) and (3.47) into (3.42), we get

$$\begin{aligned}
&\frac{1}{\delta}\frac{d}{dt}\|u+1\|^\delta_{L^\delta(\Omega)}\\
\le\;&(\varepsilon_1+\varepsilon_2+\lambda_0-\frac{b}{2^r})\int_\Omega(u+1)^{r+\delta-1}dx-\frac{r+\delta-1}{\delta}\int_\Omega(u+1)^\delta dx\\
&+A_1\lambda_0^{-(r+\delta-2)}\left(\frac{\chi M_\psi}{r+\delta-2}\right)^{r+\delta-1}\int_\Omega|\Delta v|^{r+\delta-1}dx+C_1(\varepsilon_1,\delta)+C_2(\varepsilon_2,\delta).
\end{aligned}$$

Employing the variation-of-constants formula to the above inequality, we obtain

$$
\begin{aligned}
&\frac{1}{\delta}\|u(t)+1\|^{\delta}_{L^{\delta}(\Omega)} \\
\leq\ & \frac{1}{\delta}e^{-(r+\delta-1)(t-s_0)}\|u(s_0)+1\|^{\delta}_{L^{\delta}(\Omega)} + (\varepsilon_1+\varepsilon_2+\lambda_0-\frac{b}{2^r})\int_{s_0}^{t} e^{-(r+\delta-1)(t-s)}\int_{\Omega}(u+1)^{r+\delta-1}dxds \\
& + A_1\lambda_0^{-(r+\delta-2)}\left(\frac{\chi M_{\psi}}{r+\delta-2}\right)^{r+\delta-1}\int_{s_0}^{t} e^{-(r+\delta-1)(t-s)}\int_{\Omega}|\Delta v|^{r+\delta-1}dxds \\
& + (C_1(\varepsilon_1,\delta)+C_2(\varepsilon_2,\delta))\int_{s_0}^{t} e^{-(r+\delta-1)(t-s)}ds \\
\leq\ & (\varepsilon_1+\varepsilon_2+\lambda_0-\frac{b}{2^r})\int_{s_0}^{t} e^{-(r+\delta-1)(t-s)}\int_{\Omega}(u+1)^{r+\delta-1}dxds \\
& + A_1\lambda_0^{-(r+\delta-2)}\left(\frac{\chi M_{\psi}}{r+\delta-2}\right)^{r+\delta-1}\int_{s_0}^{t} e^{-(r+\delta-1)(t-s)}\int_{\Omega}|\Delta v|^{r+\delta-1}dxds + C_3(\delta,\varepsilon_1,\varepsilon_2)
\end{aligned} \tag{3.48}
$$

for all $t \in (s_0, T)$ and $s_0 \leq 1$, where

$$
C_3 := C_3(\delta,\varepsilon_1,\varepsilon_2) = \frac{1}{\delta}\|u(s_0)+1\|^{\delta}_{L^{\delta}(\Omega)} + (C_1(\varepsilon_1,\delta)+C_2(\varepsilon_2,\delta))\int_{s_0}^{T} e^{-(r+\delta-1)(t-s)}ds.
$$

Now, by Lemma 2.7, we have

$$
\begin{aligned}
& A_1\lambda_0^{-(r+\delta-2)}\left(\frac{\chi M_{\psi}}{r+\delta-2}\right)^{r+\delta-1}\int_{s_0}^{t} e^{-(r+\delta-1)(t-s)}\int_{\Omega}|\Delta v|^{r+\delta-1}dxds \\
=\ & A_1\lambda_0^{-(r+\delta-2)}\left(\frac{\chi M_{\psi}}{r+\delta-2}\right)^{r+\delta-1} e^{-(r+\delta-1)t}\int_{s_0}^{t} e^{(r+\delta-1)s}\int_{\Omega}|\Delta v|^{r+\delta-1}dxds \\
\leq\ & A_1\lambda_0^{-(r+\delta-2)}\left(\frac{\chi M_{\psi}}{r+\delta-2}\right)^{r+\delta-1} e^{-(r+\delta-1)t}C_{r+\delta-1}(\int_{s_0}^{t}\int_{\Omega} e^{(r+\delta-1)s}u^{r+\delta-1}dxds \\
& + e^{(r+\delta-1)s_0}\|v_0(s_0,t)\|^{r+\delta-1}_{W^{2,r+\delta-1}}) \\
=\ & A_1\lambda_0^{-(r+\delta-2)}\left(\frac{\chi M_{\psi}}{r+\delta-2}\right)^{r+\delta-1} C_{r+\delta-1}(\int_{s_0}^{t}\int_{\Omega} e^{-(r+\delta-1)(t-s)}(u+1)^{r+\delta-1}dxds \\
& + e^{(r+\delta-1)s_0}\|v_0(s_0,t)\|^{r+\delta-1}_{W^{2,r+\delta-1}})
\end{aligned} \tag{3.49}
$$

for all $t \in (s_0, T)$. By substituting (3.49) into (3.48), we get

$$
\begin{aligned}
&\frac{1}{\delta}\|u(t)+1\|^{\delta}_{L^{\delta}(\Omega)} \\
\leq\ & (\varepsilon_1+\varepsilon_2+\lambda_0+A_1\lambda_0^{-(r+\delta-2)}\left(\frac{\chi M_{\psi}}{r+\delta-2}\right)^{r+\delta-1}C_{r+\delta-1}-\frac{b}{2^r})\int_{s_0}^{t} e^{-(r+\delta-1)(t-s)}\int_{\Omega}(u+1)^{r+\delta-1}dxds \\
& + A_1\lambda_0^{-(r+\delta-2)}\left(\frac{\chi M_{\psi}}{r+\delta-2}\right)^{r+\delta-1} e^{-(r+\delta-1)(t-s_0)}C_{r+\delta-1}\|v_0(s_0,t)\|^{r+\delta-1}_{W^{2,r+\delta-1}} + C_3(\delta,\varepsilon_1,\varepsilon_2).
\end{aligned} \tag{3.50}
$$

Let $\sigma_{r,\delta,\lambda_0,\chi,M_{\psi}} = \lambda_0 + A_1\lambda_0^{-(r+\delta-2)}\left(\frac{\chi M_{\psi}}{r+\delta-2}\right)^{r+\delta-1}C_{r+\delta-1}$. Then by Lemma 3.4 and $\delta \geq 1$, we have

$$
\min_{\lambda_0>0}\sigma_{r,\delta,\lambda_0,\chi,M_{\psi}} = \frac{\chi M_{\psi}C_{r+\delta-1}^{\frac{1}{r+\delta-1}}}{r+\delta-2}.
$$

Choosing b big enough such that $b > \frac{2^r(r+\delta-2)}{\chi M_\psi C_{r+\delta-1}^{\frac{1}{r+\delta-1}}}$, then there exist ε_1 and $\varepsilon_2 >$ satisfying

$$\varepsilon_1 = \varepsilon_2 \in (0, \frac{\frac{b}{2^r} - \sigma_{r,\delta,\lambda_0,\chi,M_\psi}}{2}),$$

such that

$$\varepsilon_1 + \varepsilon_2 + \lambda_0 + A_1\lambda_0^{-(r+\delta-2)}\left(\frac{\chi M_\psi}{r+\delta-2}\right)^{r+\delta-1} C_{r+\delta-1} - \frac{b}{2^r} \le 0.$$

Therefore, (3.50) implies that

$$\int_\Omega (u(x,t)+1)^\delta dx \le C \tag{3.51}$$

for all $t \in (s_0, T)$, where

$$C = \delta A_1 \lambda_0^{-(r+\delta-2)}\left(\frac{\chi M_\psi}{r+\delta-2}\right)^{r+\delta-1} e^{-(r+\delta-1)(t-s_0)} C_{r+\delta-1}\|v_0(s_0,t)\|_{W^{2,r+\delta-1}}^{r+\delta-1} + \delta C_3.$$

The proof Lemma 3.5 is complete. □

In the following, we will set up the iteration procedure to derive the main results.

Lemma 3.6. *Let $\beta + 1 < r$. Then for any $T \in (0, T_{max})$, there exists a constant $C > 0$ independent on T such that $\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C$ for all $t \in (0,T)$.*

Proof. Let $\mu_0 = \kappa_r$, $\mu_k = (r-\beta)\mu_{k-1} + (r-\beta-1)(r-\kappa_r)$, where κ_r is given in (3.9). Then we have $\dfrac{\mu_k + r - \kappa_r}{\mu_{k-1} + r - \kappa_r} = (r-\beta)$. By a simple computation, we obtain

$$\mu_k = r(r-\beta)^k - r + \kappa_r \quad \text{for all} \quad k \ge 0 \tag{3.52}$$

and

$$\sum_{j=0}^{k} \mu_j = r \times \frac{(r-\beta)^{k+1}-1}{r-\beta-1} - (k+1)r + (k+1)\kappa_r \quad \text{for all} \quad k \ge 0. \tag{3.53}$$

On the other hand, Lemma 3.1 and Lemma 3.2 give us

$$\int_\Omega u^{\mu_k}(x,t)dx \le M_0^k M^{\sum_{j=0}^k \mu_j} C_0(T+1) \quad \text{for all} \quad t \in (s,T) \text{ and } k \ge 0, \tag{3.54}$$

that is,

$$\|u(\cdot,t)\|_{L^{\mu_k}(\Omega)} \le M_0^{\frac{k}{\mu_k}} M^{\frac{\sum_{j=0}^k \mu_j}{\mu_k}} C_0^{\frac{1}{\mu_k}} (T+1)^{\frac{1}{\mu_k}} \quad \text{for all} \quad t \in (s,T) \text{ and } k \ge 0. \tag{3.55}$$

Because $\beta + 1 < r$ implies that $\mu_k \to \infty$ as $k \to \infty$. Thus letting $k \to \infty$ on both sides of (3.55), we have

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le M^{\frac{r-\beta}{r-\beta-1}} \quad \text{for all} \quad t \in (s,T). \tag{3.56}$$

Here we have used the fact

$$\lim_{k\to\infty} \frac{k}{\mu_k} = 0.$$

On the other hand, it follows from (2.3) that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le K, \quad \text{for all} \quad t \in (0,s]. \tag{3.57}$$

Now, choosing $C := \max\{K, M^{\frac{r-\beta}{r-\beta-1}}\}$, we complete the proof. $\square$

Based on Lemma 3.3 and Lemma 3.6, we can prove Theorem 2.1.

The proof of Theorem 2.1 Theorem 2.1 will be proved if we can show $T_{max} = \infty$. Suppose on contrary that $T_{max} < \infty$. From Lemma A.1 in [25], we know that there is $\gamma_0 > 0$ such that if

$$\|u(\cdot,t)\|_{L^\gamma(\Omega)} < +\infty \tag{3.58}$$

for all $\gamma \ge \gamma_0$ and $t \in (0, T_{max})$, then there exists $C_1 > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C_1$$

for all $t \in (0, T_{max})$. In view of Lemma 3.5, we have (3.58) holds. Thus, by Lemma A.1 in [25], we have

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C_1.$$

Now, using Lemma 3.3 and Lemma 3.6, we have $\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C$ for all $t \in (0, T_{max})$, where constant C is independent of T_{max}. This contradicts with Lemma 2.5. Hence the classical solution (u,v) of (1.1) is global in time and bounded.

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