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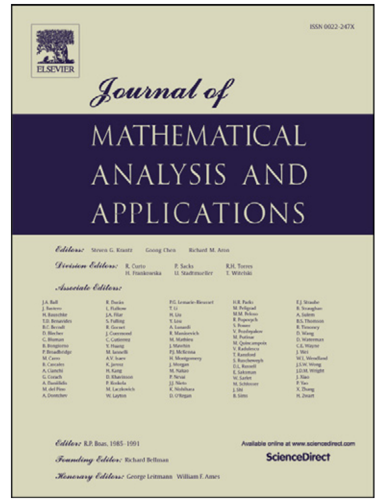

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A STUDY OF THE MATRIX CARLESON EMBEDDING THEOREM WITH APPLICATIONS TO SPARSE OPERATORS

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Abstract. In this paper, we study the dyadic Carleson Embedding Theorem in the matrix weighted setting. We provide two new proofs of this theorem, which highlight connections between the matrix Carleson Embedding Theorem and both maximal functions and H^1-BMO duality. Along the way, we establish boundedness results about maximal functions associated to matrix A_2 weights and duality results concerning H^1 and BMO sequence spaces in the matrix setting. As an application, we then use this Carleson Embedding Theorem to show that if S is a sparse operator, then the operator norm of S on $L^2(W)$ satisfies

$$\|S\|_{L^2(W)\to L^2(W)} \lesssim [W]_{A_2}^{\frac{3}{2}},$$

for every matrix A_2 weight W.

Keywords. Matrix Carleson Embedding Theorem; Matrix A_2 weights; Sparse operators

1. Introduction

The subject of this paper is the dyadic Carleson Embedding Theorem– an important tool for establishing the boundedness of paraproducts via testing conditions– and its applications and shortcomings in the matrix weighted setting. This paper is particularly motivated by interest in the matrix A_2 conjecture. To set the scene, let us briefly review the scalar situation.

1.1. **Scalar A_2 Conjecture.** Let T be a Calderón-Zygmund operator and $w(x)$ a weight, i.e. a locally integrable function on $\mathbb{R}$ that is positive almost everywhere. It is classically known that every Calderón-Zygmund operator T extends to a bounded operator on $L^2(w)$ if and only if w is an A_2 Muckenhoupt weight, namely, if and only if

$$[w]_{A_2} \equiv \sup_I \langle w \rangle_I \langle w^{-1} \rangle_I < \infty,$$

where the supremum is taken over all intervals I and $\langle w \rangle_I \equiv \frac{1}{|I|}\int_I w(x)dx$. In contrast, the question of the dependence of the operator norm of T on $[w]_{A_2}$, called the A_2 *conjecture*, remained open for decades. Mathematicians first established linear bounds for simpler operators including the martingale transforms, Hilbert transform, and more generally, dyadic shifts [12, 19, 20, 21, 25]. A key strategy for proving the linear bound reduces general Calderón-Zygmund operators to simpler operators. Using a refined method of decomposing Calderón-Zygmund operators as sums of dyadic shifts, Hytönen resolved the A_2 conjecture in 2012 in [9], showing

$$\|T\|_{L^2(w)\to L^2(w)} \lesssim [w]_{A_2}$$

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for all Calderón-Zygmund operators T and A_2 weights w. More recently, Conde-Alonso and Rey, Lerner, and Lacey [6, 11, 15, 16] developed simpler proofs of the A_2 conjecture by controlling Calderón-Zygmund operators using simple operators called *sparse operators*, which easily satisfy the linear bound.

An important tool when controlling these easier operators is the dyadic Carleson Embedding Theorem, which says:

Theorem 1.1 (Carleson Embedding Theorem)**.** *Let $\{a_I\}_{I\in\mathcal{D}}$ be a sequence of nonnegative numbers indexed by the grid of dyadic intervals $\mathcal{D}$. Then*

$$\sum_{I\in\mathcal{D}} a_I \langle w^{\frac{1}{2}} f\rangle_I^2 \leq C_1 \|f\|_{L^2}^2 \quad \forall f \in L^2(\mathbb{R}) \text{ if and only if } \frac{1}{|J|}\sum_{I:I\subseteq J} a_I \langle w\rangle_I^2 \leq C_2 \langle w\rangle_J \quad \forall J\in\mathcal{D}.$$

Moreover, $C_2 \leq C_1 \leq 4C_2$.

We are interested in the development of these ideas in the matrix setting.

1.2. **Matrix A_2 Conjecture.** The relevant definitions are as follows: a $d\times d$ matrix-valued function $W(x)$ is a *matrix weight* if its entries are locally integrable and if $W(x)$ is a positive definite matrix for almost every $x\in\mathbb{R}$. Given a matrix weight W, one can define

$$L^2(W) \equiv \left\{ f \in L^2(\mathbb{R},\mathbb{C}^d) : \|f\|_{L^2(W)}^2 \equiv \int_{\mathbb{R}} \left\| W^{\frac{1}{2}}(x) f(x)\right\|^2 dx < \infty \right\}.$$

Many scalar results have already been generalized to this setting. For example, Nazarov-Treil [18] and Volberg [24] characterized the boundedness of Calderón-Zygmund operators on these matrix weighted L^2 spaces. Indeed, they showed that for each classical scalar Calderón-Zygmund operator T, the matrix Calderón-Zygmund operator $\tilde{T}$ defined on $L^2(\mathbb{R},\mathbb{C}^d)$ by applying T component-wise is bounded on $L^2(W)$ if W is a matrix A_2 weight, namely if

$$[W]_{A_2} \equiv \sup_I \left\| \langle W\rangle_I^{\frac{1}{2}} \langle W^{-1}\rangle_I^{\frac{1}{2}} \right\|^2 < \infty.$$

Here $\|\cdot\|$ denotes the norm of the matrix acting on $\mathbb{C}^d$. In this paper, we also use $\|\cdot\|$ to denote the norm of a vector in $\mathbb{C}^d$, but it will be clear from the context whether we are considering matrices or vectors. For ease of notation, we will denote $L^2(\mathbb{R},\mathbb{C}^d)$ by simply L^2.

In the interim, the study of operators on matrix-weighted spaces has received a great deal of attention. See [1, 2, 3, 4, 8, 10, 13, 17, 22, 24]. However, the question of the sharp dependence of the operators' norms on $[W]_{A_2}$, termed the *Matrix A_2 Conjecture* is still open. In [1], the two authors with S. Petermichl showed that for the Hilbert transform H,

$$\|H\|_{L^2(W)\to L^2(W)} \lesssim [W]_{A_2}^{\frac{3}{2}} \log [W]_{A_2}, \tag{1.1}$$

for all matrix A_2 weights W. Although this is the best known estimate, the bound is unlikely to be sharp. One initial obstruction to better norm bounds was the following matrix version of Theorem 1.1. This theorem was mentioned by Isralowitz-Kwon-Pott in the final remarks of an earlier version of [10] and discussed in depth in [2]:

Theorem 1.2. *Let W be an A_2 weight and let $\{A_I\}_{I\in\mathcal{D}}$ be a sequence of positive semi-definite $d\times d$ matrices. Then*

$$\sum_{I\in\mathcal{D}} \left\langle A_I \left\langle W^{\frac{1}{2}} f\right\rangle_I, \left\langle W^{\frac{1}{2}} f\right\rangle_I \right\rangle_{\mathbb{C}^d} \leq C_1 \|f\|_{L^2}^2 \;\; \forall f\in L^2 \text{ iff } \frac{1}{|J|}\sum_{I:I\subseteq J} \langle W\rangle_I A_I \langle W\rangle_I \leq C_2 \langle W\rangle_J,$$

for all $J\in\mathcal{D}$, where $C_2 \leq C_1 \lesssim C_2[W]_{R_2}[W]_{A_2}$.

Here, the notation $A \lesssim B$ indicates $A \leq CB$, where C is a constant typically depending on the dimension d. This initial Carleson Embedding Theorem held only for A_2 weights and the guaranteed relationship between the testing constant C_2 and embedding constant C_1 was

$$C_1 \lesssim [W]_{A_2}[W]_{R_2}C_2.$$

The term $[W]_{R_2}$ is a reverse Hölder constant, which is discussed in detail in [2]. For our purposes, it suffices that $[W]_{R_2} \lesssim [W]_{A_2}$. The appearance of this (essentially) extra $[W]_{A_2}^2$ in Theorem 1.2 made it very difficult to use this result to prove sharp bounds.

However, this theorem is no longer an obstruction. Specifically, the authors recently learned that A. Culiuc and S. Treil have obtained an improved Carleson Embedding Theorem. Indeed, in [7], they established Theorem 1.2 without the A_2 dependence, for arbitrary matrix weights in a more general non-homogeneous setting. Nevertheless, the estimate (1.1) has proved resistant to improvement. The problem lies in the fact that proving the testing conditions needed to use Theorem 1.2 also constitute difficult, open problems.

Rather, the estimate (1.1) was established using the following alternate Carleson Embedding Theorem, which was proved by Treil-Volberg in [23] and extended by Isralowitz-Pott-Kwon in [10].

Theorem 1.3. *Let W be a matrix A_2 weight and let $\{A_I\}_{I\in\mathcal{D}}$ be a sequence of positive semi-definite $d \times d$ matrices. Then*

$$\sum_{I\in\mathcal{D}} \left\langle A_I \left\langle W^{\frac{1}{2}} f\right\rangle_I, \left\langle W^{\frac{1}{2}} f\right\rangle_I \right\rangle_{\mathbb{C}^d} \leq C_1 \|f\|_{L^2}^2 \ \ \forall f \in L^2 \ \ \text{if} \ \ \frac{1}{|J|}\sum_{I:I\subseteq J} \left\| \langle W\rangle_I^{\frac{1}{2}} A_I \langle W\rangle_I^{\frac{1}{2}} \right\| \leq C_2$$

for all $J \in \mathcal{D}$, where $C_1 \lesssim [W]_{R_2}C_2 \lesssim [W]_{A_2}C_2$.

1.3. **Summary of Results.** In this paper, we offer two new proofs of Theorem 1.3. The only drawback is that these proofs give $C_1 \lesssim [W]_{A_2}^2 C_2$, which is not the optimal constant. Nevertheless, these new proofs show that this matrix embedding theorem has close ties to both the boundedness of maximal functions and H^1-BMO duality. We end with an application of Theorem 1.3 to the study of sparse operators, which shows that Theorem 1.3 is a useful tool for studying operators acting on matrix weighted L^2 spaces. Here is the overview of the paper.

In Section 2, we prove Theorem 1.3 using maximal function ideas. It is worth noting that many of the ideas used in this section originate in the influential paper [4] by Christ and Goldberg. In this section, we first fix a matrix A_2 weight W and consider the maximal function M_W defined by

$$M_W f(x) \equiv \sup_{I\in\mathcal{D}:x\in I} \left\| \langle W\rangle_I^{-\frac{1}{2}} \left\langle W^{\frac{1}{2}} f\right\rangle_I \right\| \mathbf{1}_I(x).$$

This maximal function is closely related to operators studied by Christ-Goldberg in [4] and extended by Goldberg in [8]. Indeed, in Subsection 2.1, we use straightforward estimates to bound this maximal function with a related maximal function from [4]. Then, using Christ-Goldberg's arguments, we conclude in Corollary 2.2 that

$$\|M_W\|_{L^2\to L^2} \lesssim [W]_{A_2},$$

for all matrix A_2 weights W. Finally, in Subsection 2.2, we give a stopping-time argument that, when paired with the maximal function bound, yields a simple proof of Theorem 1.3.

In Section 3, we consider the following two spaces of sequences of $d \times d$ matrices indexed by the dyadic intervals $\mathcal{D}$:

$$\mathcal{S} \equiv \left\{ \{S_I\} : S^2(x) \equiv \sum_{I \in \mathcal{D}} \|S_I\|^2 \frac{1_I(x)}{|I|} \in L^1(\mathbb{R}) \text{ and } \|\{S_I\}\|_{\mathcal{S}} \equiv \|S(x)\|_{L^1(\mathbb{R})} \right\};$$

$$\mathcal{T} \equiv \left\{ \{T_I\} : \|\{T_I\}\|_{\mathcal{T}}^2 \equiv \sup_{J \in \mathcal{D}} \left\| \frac{1}{|J|} \sum_{I \subseteq J} T_I T_I^* \right\| < \infty \right\}.$$

Notice that the sequences in $\mathcal{S}$ are related to functions whose square functions are in $L^1(\mathbb{R})$. This shows $\mathcal{S}$ is an H^1-type space and similarly, $\mathcal{T}$ is a space of BMO-type sequences. These spaces are similar to matrix analogues of scalar sequence spaces studied by Lee-Lin-Lin in [14]. In Theorem 3.1, we modify arguments of Lee-Lin-Lin to establish that each $\{T_I\} \in \mathcal{T}$ induces a linear functional on $\mathcal{S}$. Then in Subsection 3.2, we use this duality relationship, paired with the maximal function bound, to provide another proof of Theorem 1.3.

Finally, in Section 4, we consider an application of Theorem 1.3. Specifically, we say that an operator $S : L^2(\mathbb{R}, \mathbb{C}^d) \to L^2(\mathbb{R}, \mathbb{C}^d)$ is *sparse* if

$$Sf(x) \equiv \sum_{I \in \mathfrak{S}} \langle f \rangle_I \mathbf{1}_1(x),$$

where $\mathfrak{S} \subseteq \mathcal{D}$ is a collection of dyadic intervals satisfying the following sparsity condition: for each $I \in \mathfrak{S}$,

$$\sum_{J \in ch_{\mathfrak{S}}(I)} |J| \leq \frac{1}{2}|I|, \tag{1.2}$$

where the sum is restricted to the $\mathfrak{S}$-children of I, namely the maximal elements of $\mathfrak{S}$ that are strictly contained in I. The constant $\frac{1}{2}$ is largely unimportant; any $0 < c < 1$ will work to define sparse families. We use Theorem 1.3 to establish the following result:

Theorem 1.4. *Let W be a $d \times d$ matrix A_2 weight and let S be a sparse operator on $L^2(\mathbb{R}, \mathbb{C}^d)$. Then*

$$\|S\|_{L^2(W) \to L^2(W)} \lesssim [W]_{A_2}^{\frac{3}{2}}.$$

It is worth noting that this dependence of a sparse operator's norm on $[W]_{A_2}$ is better than the known dependence of the Hilbert transform's norm. Also, alternate proofs of this have recently been obtained by Isralowitz-Kwon-Pott [10]. Finally, recall that in the scalar setting, sparse operators are used in [6, 11, 15] to prove bounds for general Calderón-Zgymund operators. Unfortunately, those exact reduction arguments do not appear to generalize to the matrix setting. Thus, we conclude that further study is needed to determine if sparse operators can be used to control general Calderón-Zgymund operators in the matrix setting.

2. Theorem 1.3 via Maximal Functions & A Stopping-Time Argument

2.1. The Relevant Maximal Function.

Recall the maximal function of interest:

$$M_W f(x) \equiv \sup_{I : x \in I} \left\| \langle W \rangle_I^{-\frac{1}{2}} \left\langle W^{\frac{1}{2}} f \right\rangle_I \right\| \mathbf{1}_I(x),$$

where the supremum is taken over dyadic intervals. Although this is the maximal function most closely related to the matrix Carleson Embedding Theorem, it is difficult to control directly. Instead, first notice that we have the following simple pointwise bound:

$$\begin{aligned} M_W f(x) &= \sup_{I:x\in I} \left\| \langle W\rangle_I^{-\frac{1}{2}} \left\langle W^{-1}\right\rangle_I^{-\frac{1}{2}} \left\langle W^{-1}\right\rangle_I^{\frac{1}{2}} \left\langle W^{\frac{1}{2}} f\right\rangle_I \right\| \\ &\le \sup_{I:x\in I} \left\| \langle W\rangle_I^{-\frac{1}{2}} \left\langle W^{-1}\right\rangle_I^{-\frac{1}{2}} \right\| \left\| \left\langle W^{-1}\right\rangle_I^{\frac{1}{2}} \left\langle W^{\frac{1}{2}} f\right\rangle_I \right\| \\ &\le \sup_{I:x\in I} \left\| \left\langle W^{-1}\right\rangle_I^{\frac{1}{2}} \left\langle W^{\frac{1}{2}} f\right\rangle_I \right\| \\ &= \sup_{I:x\in I} \left\| \frac{1}{|I|} \int_I \left\langle W^{-1}\right\rangle_I^{\frac{1}{2}} W^{\frac{1}{2}}(y) f(y)\, dy \right\| \\ &\le \sup_{I:x\in I} \frac{1}{|I|} \int_I \left\| \left\langle W^{-1}\right\rangle_I^{\frac{1}{2}} W^{\frac{1}{2}}(y) f(y) \right\| \, dy. \end{aligned}$$

Here, we used the fact that for each $I \in \mathcal{D}$, one can show that $\|\langle W\rangle_I^{-\frac{1}{2}} \langle W^{-1}\rangle_I^{-\frac{1}{2}}\| \le 1$. For example, this inequality can be easily deduced from Corollary 3.3 in [23].

Now, given a matrix weight V, recall the following auxiliary maximal function $\widetilde{M}_V$ of Christ-Goldberg from [4]:

$$\widetilde{M}_V f(x) \equiv \sup_{I:x\in I} \frac{1}{|I|} \int_I \left\| \langle V\rangle_I^{\frac{1}{2}} V^{-\frac{1}{2}}(y) f(y) \right\| \, dy,$$

where the supremum is again taken over dyadic intervals. Then, as long as W and W^{-1} are both matrix weights, as is the case when W is an A_2 matrix weight, our previous arguments show that pointwise

$$M_W f(x) \le \widetilde{M}_{W^{-1}} f(x). \tag{2.1}$$

In their Lemma 2.2, Christ-Goldberg showed that if V is an A_2 matrix weight, then the maximal operator $\widetilde{M}_V$ is bounded on $L^2(\mathbb{R}, \mathbb{C}^d)$. A close reading of their proof also reveals the dependence of the operator norm on the A_2 characteristic of V. We summarize their result in the following theorem:

Theorem 2.1. *If V is a $d \times d$ matrix A_2 weight, then*

$$\|\widetilde{M}_V\|_{L^2\to L^2} \lesssim [V]_{A_2},$$

where the implied constant depends on the dimension d.

For the ease of the reader, we include the following modified version of their arguments to track the exact dependence of the operator's norm on $[V]_{A_2}$:

Proof. Fix a matrix weight $V \in A_2$. We first establish the following inequality:

$$\frac{1}{|I|} \int_I \left\| V^{-\frac{1}{2}}(y) \langle V\rangle_I^{\frac{1}{2}} \right\|^{2+2\epsilon} dy \lesssim [V]_{A_2}^{1+\epsilon},$$

for some small $\epsilon > 0$ and all $I \in \mathcal{D}$. To obtain this, fix $e \in \mathbb{C}^d$ and $I \in \mathcal{D}$. Recall that [23, Lemma 3.5] says that

$$\langle V\rangle_I^{\frac{1}{2}} V^{-1}(x) \langle V\rangle_I^{\frac{1}{2}}$$

is also in A_2 with A_2 characteristic $[V]_{A_2}$. Then, as in [13, Lemma 1.5], it is not too difficult to show that the function

$$\left\langle \langle V\rangle_I^{\frac{1}{2}} V^{-1}(x) \langle V\rangle_I^{\frac{1}{2}} e, e \right\rangle_{\mathbb{C}^d}$$

is a scalar A_2 weight with A_2 characteristic at most $[V]_{A_2}$ for each $e \in \mathbb{C}^d$. Furthermore, as noted by Wittwer in [25], a careful reading of Coifman-Fefferman's proof of the reverse Hölder inequality for scalar A_2 weights in [5] shows that if v is a scalar A_2 weight and $\epsilon \equiv \frac{c}{[v]_{A_2}}$ for a small-enough constant c, then

$$\frac{1}{|I|}\int_I v^{1+\epsilon}(y)dy \leq \left(\frac{2}{|I|}\int_I v(y)dy\right)^{1+\epsilon}.$$

We will apply this to the scalar A_2 weights

$$\left\langle \langle V\rangle_I^{\frac{1}{2}} V^{-1}(x) \langle V\rangle_I^{\frac{1}{2}} e_i, e_i \right\rangle_{\mathbb{C}^d},$$

where the $\{e_i\}_{i=1}^d$ are the standard unit normal vectors in $\mathbb{C}^d$. Then, by equating norm and trace of positive definite matrices (up to a dimensional constant), one can compute

$$\begin{aligned}
\frac{1}{|I|}\int_I \left\| V^{-\frac{1}{2}}(y)\langle V\rangle_I^{\frac{1}{2}}\right\|^{2+2\epsilon} dy &= \frac{1}{|I|}\int_I \left\|\langle V\rangle_I^{\frac{1}{2}} V^{-1}(y)\langle V\rangle_I^{\frac{1}{2}}\right\|^{1+\epsilon} dy \\
&\lesssim \frac{1}{|I|}\int_I \left(\operatorname{Tr}\left(\langle V\rangle_I^{\frac{1}{2}} V^{-1}(y)\langle V\rangle_I^{\frac{1}{2}}\right)\right)^{1+\epsilon} dy \\
&\lesssim \frac{1}{|I|}\int_I \max_{1\leq i\leq d}\left\langle \langle V\rangle_I^{\frac{1}{2}} V^{-1}(y)\langle V\rangle_I^{\frac{1}{2}} e_i, e_i\right\rangle_{\mathbb{C}^d}^{1+\epsilon} dy \\
&\leq \sum_{i=1}^d \frac{1}{|I|}\int_I \left\langle \langle V\rangle_I^{\frac{1}{2}} V^{-1}(y)\langle V\rangle_I^{\frac{1}{2}} e_i, e_i\right\rangle_{\mathbb{C}^d}^{1+\epsilon} dy \\
&\leq \sum_{i=1}^d \left(\frac{2}{|I|}\int_I \left\langle \langle V\rangle_I^{\frac{1}{2}} V^{-1}(y)\langle V\rangle_I^{\frac{1}{2}} e_i, e_i\right\rangle_{\mathbb{C}^d} dy\right)^{1+\epsilon} \\
&\lesssim [V]_{A_2}^{1+\epsilon}.
\end{aligned}$$

Set $r = 2+2\epsilon$ and let $r' < 2$ be conjugate to r. Then, for $f \in L^2(\mathbb{R},\mathbb{C}^d)$,

$$\begin{aligned}
\widetilde{M}_V f(x) &= \sup_{I:x\in I}\frac{1}{|I|}\int_I \left\|\langle V\rangle_I^{\frac{1}{2}} V^{-\frac{1}{2}}(y) f(y)\right\| \, dy \\
&\leq \sup_{I:x\in I}\left(\frac{1}{|I|}\int_I \left\|V^{-\frac{1}{2}}(y)\langle V\rangle_I^{\frac{1}{2}}\right\|^r dy\right)^{\frac{1}{r}} \left(\frac{1}{|I|}\int_I \|f(y)\|^{r'} dy\right)^{\frac{1}{r'}} \\
&\lesssim [V]_{A_2}^{\frac{1}{2}}\left(M(\|f\|^{r'})(x)\right)^{\frac{1}{r'}}.
\end{aligned}$$

Define $p = \frac{2}{r'}$. Then $p > 1$ and so, the maximal function M maps $L^p(\mathbb{R},\mathbb{C}^d)$ to $L^p(\mathbb{R},\mathbb{C}^d)$. But, then

$$\begin{aligned}
\left\|\widetilde{M}_V f\right\|_{L^2}^2 &\lesssim [V]_{A_2}\left\|\left(M(\|f\|^{r'})\right)^{\frac{1}{r'}}\right\|_{L^2}^2 \\
&= [V]_{A_2}\left\|M(\|f\|^{r'})\right\|_{L^p}^p \\
&\lesssim \frac{p}{p-1}[V]_{A_2}\left\|(\|f\|^{r'})\right\|_{L^p}^p \\
&\lesssim [V]_{A_2}^2\|f\|_{L^2}^2,
\end{aligned}$$

using the maximal function bound and the fact that $\frac{p}{p-1} \approx \epsilon^{-1} \approx [V]_{A_2}$. This immediately implies that

$$\left\| \widetilde{M_V} \right\|_{L^2 \to L^2} \lesssim [V]_{A_2},$$

as desired. □

The pointwise bound (2.1) paired with Theorem 2.1 immediately gives the following corollary:

Corollary 2.2. *If V is a $d \times d$ matrix A_2 weight, then*

$$\|M_V\|_{L^2 \to L^2} \lesssim [V]_{A_2},$$

where the implied constant depends on the dimension d.

2.2. **Proof One of Theorem 1.3.** Our first proof of the matrix Carleson Embedding Theorem is motivated by a classical proof of the standard embedding result Theorem 1.1, which uses a stopping time argument and maximal function bound. Here is a stopping-time proof of Theorem 1.3 using the same types of arguments that appear in the scalar set-up:

Proof. Fix $f \in L^2(\mathbb{R}, \mathbb{C}^d)$ and for each $k \in \mathbb{Z}$, let $\mathcal{J}_k$ be the set of maximal dyadic intervals I with

$$2^{k-1} \leq \left\| \langle W \rangle_I^{-\frac{1}{2}} \left\langle W^{\frac{1}{2}} f \right\rangle_I \right\| \leq 2^k.$$

We say that $J \in \mathcal{J}_k^*$ if k is the largest integer such that $J \subseteq I$ for some $I \in \mathcal{J}_k$. Now recall that the related maximal function

$$M_W f(x) = \sup_I \left\| \langle W \rangle_I^{-\frac{1}{2}} \left\langle W^{\frac{1}{2}} f \right\rangle_I \right\| \mathbf{1}_I(x)$$

is in $L^2(\mathbb{R}, \mathbb{C}^d)$ by Corollary 2.2. Using this, it is clear that for each $J \in \mathcal{D}$, either $J \in \mathcal{J}_k^*$ for some $k \in \mathbb{Z}$ or $\langle W^{\frac{1}{2}} f \rangle_J$ is the zero vector. Now, as a way to further explore the relationship between this stopping-time set up and the maximal function, consider the function

$$g(x) \equiv \sum_{k \in \mathbb{Z}} \sum_{I \in \mathcal{J}_k} \left\| \langle W \rangle_I^{-\frac{1}{2}} \left\langle W^{\frac{1}{2}} f \right\rangle_I \right\| \mathbf{1}_I(x).$$

Again, as $M_W f \in L^2(\mathbb{R}, \mathbb{C}^d)$, we know that for almost every $x \in \mathbb{R}$, there is a largest $K \in \mathbb{Z}$ such that $x \in I \in \mathcal{J}_K$. Then, we can conclude $M_W f(x) \approx 2^K$ and further, as each $\mathcal{J}_k$ is a disjoint collection of intervals,

$$g(x) = \sum_{k \in \mathbb{Z}: k \leq K} \sum_{I \in \mathcal{J}_k} \left\| \langle W \rangle_I^{-\frac{1}{2}} \left\langle W^{\frac{1}{2}} f \right\rangle_I \right\| \mathbf{1}_I(x) \leq \sum_{k \in \mathbb{Z}: k \leq K} 2^k \lesssim 2^K \lesssim M_W f(x).$$

Then this pointwise inequality and our definition of $\mathcal{J}_k$ gives

$$\|M_W f\|_{L^2}^2 \gtrsim \|g\|_{L^2}^2 \geq \sum_{k \in \mathbb{Z}} \sum_{I \in \mathcal{J}_k} \left\| \langle W \rangle_I^{-\frac{1}{2}} \left\langle W^{\frac{1}{2}} f \right\rangle_I \right\|^2 |I| \gtrsim \sum_{k \in \mathbb{Z}} 2^{2k} \left| \bigcup_{I \in J_k} I \right|.$$

Now, assume the testing condition from Theorem 1.3:

$$\sum_{J \subseteq I} \left\| \langle W \rangle_J^{\frac{1}{2}} A_J \langle W \rangle_J^{\frac{1}{2}} \right\| \leq C_2 |I|.$$

Then, we can compute

$$\begin{aligned}\sum_{J\in\mathcal{D}}\left\langle A_J\left\langle W^{\frac{1}{2}}f\right\rangle_J,\left\langle W^{\frac{1}{2}}f\right\rangle_J\right\rangle_{\mathbb{C}^d} &= \sum_{k\in\mathbb{Z}}\sum_{J\in\mathcal{J}_k^*}\left\langle A_J\left\langle W^{\frac{1}{2}}f\right\rangle_J,\left\langle W^{\frac{1}{2}}f\right\rangle_J\right\rangle_{\mathbb{C}^d}\\
&= \sum_{k\in\mathbb{Z}}\sum_{I\in\mathcal{J}_k}\sum_{\substack{J\subseteq I\\ J\in\mathcal{J}_k^*}}\left\langle A_J\left\langle W^{\frac{1}{2}}f\right\rangle_J,\left\langle W^{\frac{1}{2}}f\right\rangle_J\right\rangle_{\mathbb{C}^d}\\
&\leq \sum_{k\in\mathbb{Z}}\sum_{I\in\mathcal{J}_k}\sum_{\substack{J\subseteq I\\ J\in\mathcal{J}_k^*}}\left\|\langle W\rangle_J^{-\frac{1}{2}}\left\langle W^{\frac{1}{2}}f\right\rangle_J\right\|^2\left\|\langle W\rangle_J^{\frac{1}{2}}A_J\langle W\rangle_J^{\frac{1}{2}}\right\|\\
&\lesssim \sum_{k\in\mathbb{Z}}2^{2k}\sum_{I\in\mathcal{J}_k}\sum_{J\subseteq I}\left\|\langle W\rangle_J^{\frac{1}{2}}A_J\langle W\rangle_J^{\frac{1}{2}}\right\|\\
&\leq C_2\sum_{k\in\mathbb{Z}}2^{2k}\sum_{I\in\mathcal{J}_k}|I|\\
&= C_2\sum_{k\in\mathbb{Z}}2^{2k}\left|\bigcup_{I\in\mathcal{J}_k}I\right|\\
&\lesssim C_2\left\|M_Wf\right\|_{L^2}^2\\
&\lesssim C_2[W]_{A_2}^2\|f\|_{L^2}^2,\end{aligned}$$

which gives the desired embedding result. □

3. Theorem 1.3 via H^1-BMO Duality and A Maximal Function

In this section, we offer an alternate proof of Theorem 1.3. The idea is to prove the result via duality using a pairing between H^1-type sequences and BMO-type sequences.

3.1. Relevant Sequence Spaces. To establish the needed duality result, we study related spaces of sequences $\mathcal{S}$ and $\mathcal{T}$, which are composed of sequences of $d\times d$ matrices indexed by the dyadic intervals. As mentioned in the introduction, we study this H^1-type space of matrix sequences

$$\mathcal{S}\equiv\left\{\{S_I\}:S^2(x)\equiv\sum_{I\in\mathcal{D}}\|S_I\|^2\frac{1_I(x)}{|I|}\in L^1(\mathbb{R})\text{ and }\|\{S_I\}\|_{\mathcal{S}}\equiv\|S(x)\|_{L^1(\mathbb{R})}\right\},$$

and this BMO-type space of matrix sequences

$$\mathcal{T}\equiv\left\{\{T_I\}:\|\{T_I\}\|_{\mathcal{T}}^2\equiv\sup_{J\in\mathcal{D}}\left\|\frac{1}{|J|}\sum_{I\subseteq J}T_IT_I^*\right\|<\infty\right\}.$$

Now, we modify the arguments of Lee-Lin-Lin from [14], which were used to study different, but related scalar sequence spaces to obtain the following result.

Theorem 3.1. *For each $\{T_I\}$ in $\mathcal{T}$, the linear functional*

$$\{S_I\}\mapsto\sum_{I\in\mathcal{D}}Tr(S_IT_I^*)$$

is continuous on $\mathcal{S}$. Namely, there is a dimensional constant $c(d)$ (not depending on $\{T_I\}$) such that for all $\{T_I\} \in \mathcal{T}$,

$$\left|\sum_{I\in\mathcal{D}} Tr\left(S_I T_I^*\right)\right| \leq c(d)\|\{T_I\}\|_{\mathcal{T}}\|\{S_I\}\|_{\mathcal{S}} \qquad \forall\ \{S_I\} \in \mathcal{S}. \tag{3.1}$$

A complete analogue of the Lee-Lin-Lin result from [14] would also show that every continuous linear functional on $\mathcal{S}$ is induced by a sequence $\{T_I\}$ in $\mathcal{T}$. Although such a result is likely true in the context, we do not prove it because we do not require that fact to obtain Theorem 1.3.

Proof. For a fixed $\{T_I\} \in \mathcal{T}$, we will establish (3.1). To this end, fix $\{S_I\} \in \mathcal{S}$ and for each $k \in \mathbb{Z}$, define the sets

$$\Omega_k \equiv \left\{x \in \mathbb{R} : S(x) > 2^k\right\};$$
$$B_k \equiv \left\{I \in \mathcal{D} : |I \cap \Omega_k| > \frac{1}{2}|I| \ \text{ and } \ |I \cap \Omega_{k+1}| \leq \frac{1}{2}|I|\right\},$$

and let $\tilde{I}$ denote the maximal intervals in B_k. Further, define the enlargement $\widetilde{\Omega}_k$ of Ω_k as follows:

$$\widetilde{\Omega}_k \equiv \left\{x \in \mathbb{R} : M(1_{\Omega_k})(x) > \frac{1}{2}\right\},$$

where M denotes the maximal function. Furthermore, notice that if $I \in B_k$ then $I \subset \widetilde{\Omega}_k$. Further, as $S(x) \in L^1(\mathbb{R})$, observe that if $I \notin B_k$ for every $k \in \mathbb{Z}$, then it must be the case that $S_I \equiv 0$. Then, we can compute the following:

$$\begin{aligned}
\sum_{I\in\mathcal{D}} \operatorname{Tr}\left(S_I T_I^*\right) &= \sum_{k\in\mathbb{Z}}\sum_{\tilde{I}\in B_k}\sum_{\substack{I\subseteq\tilde{I}\\ I\in B_k}} \operatorname{Tr}\left(S_I T_I^*\right)\\
&\leq \sum_{k\in\mathbb{Z}}\sum_{\tilde{I}\in B_k}\left(\sum_{\substack{I\subseteq\tilde{I}\\ I\in B_k}}\|S_I\|^2\right)^{\frac{1}{2}}\left(\sum_{\substack{I\subseteq\tilde{I}\\ I\in B_k}}\|T_I\|^2\right)^{\frac{1}{2}}\\
&\leq c(d)\sum_{k\in\mathbb{Z}}\sum_{\tilde{I}\in B_k}\left(\sum_{\substack{I\subseteq\tilde{I}\\ I\in B_k}}\|S_I\|^2\right)^{\frac{1}{2}}\left(\left\|\sum_{I\subseteq\tilde{I}} T_I T_I^*\right\|\right)^{\frac{1}{2}}\\
&\leq c(d)\|\{T_I\}\|_{\mathcal{T}}\sum_{k\in\mathbb{Z}}\sum_{\tilde{I}\in B_k}|\tilde{I}|^{\frac{1}{2}}\left(\sum_{\substack{I\subseteq\tilde{I}\\ I\in B_k}}\|S_I\|^2\right)^{\frac{1}{2}}\\
&\leq c(d)\|\{T_I\}\|_{\mathcal{T}}\sum_{k\in\mathbb{Z}}|\widetilde{\Omega}_k|^{\frac{1}{2}}\left(\sum_{I\in B_k}\|S_I\|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

In the above computations, we used Cauchy-Schwarz several times, the equivalence of norm and trace (up to a dimensional constant) for positive semi-definite matrices, the linearity of trace, and the fact that the $\tilde{I}$ in each B_k are disjoint. Now we show:

$$\sum_{I \in B_k} \|S_I\|^2 \leq 2^{2k+3} |\widetilde{\Omega}_k|. \tag{3.2}$$

First observe that

$$\int_{\tilde{\Omega}_k \setminus \Omega_{k+1}} S^2(x)\, dx \leq 2^{2k+2} |\widetilde{\Omega}_k|,$$

using the definition of Ω_{k+1} and similarly

$$\begin{aligned}
\int_{\tilde{\Omega}_k \setminus \Omega_{k+1}} S^2(x) dx &\geq \int_{\tilde{\Omega}_k \setminus \Omega_{k+1}} \sum_{I \in B_K} \|S_I\|^2 \frac{1_I(x)}{|I|}\, dx \\
&= \sum_{I \in B_k} \|S_I\|^2 \frac{|I \cap (\widetilde{\Omega}_k \setminus \Omega_{k+1})|}{|I|} \\
&= \sum_{I \in B_k} \|S_I\|^2 \frac{|I \setminus (I \cap \Omega_{k+1})|}{|I|} \\
&\geq \frac{1}{2} \sum_{I \in B_k} \|S_I\|^2.
\end{aligned}$$

Combining those two estimates gives (3.2). Given (3.2), our previously-calculated inequality becomes:

$$\begin{aligned}
\left| \sum_{I \in \mathcal{D}} \operatorname{Tr}(S_I T_I^*) \right| &\leq c(d) \|\{T_I\}\|_{\mathcal{T}} \sum_{k \in \mathbb{Z}} |\widetilde{\Omega}_k|^{\frac{1}{2}} \left(\sum_{I \in B_k} \|S_I\|^2 \right)^{\frac{1}{2}} \\
&\leq c(d) \|\{T_I\}\|_{\mathcal{T}} \sum_{k \in \mathbb{Z}} |\widetilde{\Omega}_k| 2^{k+2} \\
&\lesssim c(d) \|\{T_I\}\|_{\mathcal{T}} \sum_{k \in \mathbb{Z}} |\Omega_k| 2^k \\
&\lesssim c(d) \|\{T_I\}\|_{\mathcal{T}} \|S(x)\|_{L^1(\mathbb{R})} \\
&= c(d) \|\{T_I\}\|_{\mathcal{T}} \|\{S_I\}\|_{\mathcal{S}},
\end{aligned}$$

which is the desired inequality. □

3.2. **Proof Two of Theorem 1.3.**

Proof. Let W be a $d \times d$ matrix A_2 weight and assume $\{A_I\}_{I \in \mathcal{D}}$ is a sequence of positive semidefinite $d \times d$ matrices satisfying the testing condition:

$$\sum_{I \subset J} \left\| \langle W \rangle_I^{\frac{1}{2}} A_I \langle W \rangle_I^{\frac{1}{2}} \right\| \leq C_2 |J|,$$

for all $J \in \mathcal{D}$. Then we can write:

$$\begin{aligned}
\sum_{I \in \mathcal{D}} \left\langle A_I \left\langle W^{\frac{1}{2}} f \right\rangle_I, \left\langle W^{\frac{1}{2}} f \right\rangle_I \right\rangle_{\mathbb{C}^d} &= \sum_{I \in \mathcal{D}} \left\| A_I^{\frac{1}{2}} \left\langle W^{\frac{1}{2}} f \right\rangle_I \right\|^2 \\
&= \left\| \left\{ A_I^{\frac{1}{2}} \left\langle W^{\frac{1}{2}} f \right\rangle_I \right\} \right\|^2_{\ell^2(\mathcal{D}, \mathbb{C}^d)}.
\end{aligned}$$

We estimate this quantity via duality. Specifically, for each $\{b_I\} \in \ell^2(\mathcal{D}, \mathbb{C}^d)$, we compute

$$\begin{aligned}\sum_{I\in\mathcal{D}} \left\langle A_I^{\frac{1}{2}} \left\langle W^{\frac{1}{2}} f\right\rangle_I, b_I\right\rangle_{\mathbb{C}^d} &= \sum_{I\in\mathcal{D}} \mathrm{Tr}\left(A_I^{\frac{1}{2}} \left\langle W^{\frac{1}{2}} f\right\rangle_I b_I^*\right) \\ &= \sum_{I\in\mathcal{D}} \mathrm{Tr}\left(A_I^{\frac{1}{2}} \langle W\rangle_I^{\frac{1}{2}} \langle W\rangle_I^{-\frac{1}{2}} \left\langle W^{\frac{1}{2}} f\right\rangle_I b_I^*\right).\end{aligned}$$

To invoke Theorem 3.1, define the sequences $\{S_I\}$ and $\{T_I\}$ by

$$T_I = \langle W\rangle_I^{\frac{1}{2}} A_I^{\frac{1}{2}} \text{ and } S_I = \langle W\rangle_I^{-\frac{1}{2}} \left\langle W^{\frac{1}{2}} f\right\rangle_I b_I^* \qquad \forall\, I \in \mathcal{D}.$$

Notice that the testing condition implies that

$$\|\{T_I\}\|_{\mathcal{T}}^2 = \sup_{J\in\mathcal{D}} \left\| \frac{1}{|J|} \sum_{I\subseteq J} T_I T_I^* \right\| = \sup_{J\in\mathcal{D}} \left\| \frac{1}{|J|} \sum_{I\subseteq J} \langle W\rangle_I^{\frac{1}{2}} A_I \langle W\rangle_I^{\frac{1}{2}} \right\| \le c(d) C_2.$$

Then Theorem 3.1 implies that

$$\begin{aligned}\sum_{I\in\mathcal{D}} \left\langle A_I^{\frac{1}{2}} \left\langle W^{\frac{1}{2}} f\right\rangle_I, b_I\right\rangle_{\mathbb{C}^d} &= \sum_{I\in\mathcal{D}} \mathrm{Tr}\left(T_I^* S_I\right) \\ &= \sum_{I\in\mathcal{D}} \mathrm{Tr}\left(S_I T_I^*\right) \\ &\lesssim \|\{T_I\}\|_{\mathcal{T}} \|\{S_I\}\|_{\mathcal{S}} \\ &\lesssim \sqrt{C_2}\, \|\{S_I\}\|_{\mathcal{S}}.\end{aligned}$$

Now observe that

$$\begin{aligned} S^2(x) &= \sum_{I\in\mathcal{D}} \|S_I\|^2 \frac{1_I(x)}{|I|} \\ &\le \sum_{I\in\mathcal{D}} \left\| \langle W\rangle_I^{-\frac{1}{2}} \left\langle W^{\frac{1}{2}} f\right\rangle_I \right\|^2 \|b_I\|^2 \frac{1_I(x)}{|I|} \\ &\le \sup_{I: x\in I} \left\| \langle W\rangle_I^{-\frac{1}{2}} \left\langle W^{\frac{1}{2}} f\right\rangle_I \right\|^2 \sum_{I\in\mathcal{D}} \|b_I\|^2 \frac{1_I(x)}{|I|}.\end{aligned}$$

Notice that this is exactly the maximal function we studied earlier. Then Corollary 2.2 implies that

$$\begin{aligned}\|S(x)\|_{L^1(\mathbb{R})} &\le \|M_W f\|_{L^2(\mathbb{R})} \left(\int_{\mathbb{R}} \sum_{I\in\mathcal{D}} \|b_I\|^2 \frac{1_I(x)}{|I|} dx \right)^{\frac{1}{2}} \\ &\lesssim [W]_{A_2} \|f\|_{L^2(\mathbb{R})} \|\{b_I\}\|_{\ell^2(\mathcal{D},\mathbb{C}^d)},\end{aligned}$$

as desired. So we can combine this with our previous estimates to conclude

$$\sum_{I\in\mathcal{D}} \left\langle A_I^{\frac{1}{2}} \left\langle W^{\frac{1}{2}} f\right\rangle_I, b_I\right\rangle_{\mathbb{C}^d} \lesssim \sqrt{C_2} [W]_{A_2} \|f\|_{L^2(\mathbb{R})} \|\{b_I\}\|_{\ell^2(\mathcal{D},\mathbb{C}^d)}.$$

Since this estimate holds for all $\{b_I\} \in \ell^2(\mathcal{D}, \mathbb{C}^d)$, we immediately have

$$\sum_{I\in\mathcal{D}} \left\| A_I^{\frac{1}{2}} \left\langle W^{\frac{1}{2}} f\right\rangle_I \right\|^2 = \left\| \left\{ A_I^{\frac{1}{2}} \left\langle W^{\frac{1}{2}} f\right\rangle_I \right\} \right\|_{\ell^2(\mathcal{D},\mathbb{C}^d)}^2 \lesssim C_2 [W]_{A_2}^2 \|f\|_{L^2(\mathbb{R})}^2,$$

which completes the proof. □

4. Application: The Bound for Sparse Operators and the Proof of Theorem 1.4

Recall that an operator $S : L^2(\mathbb{R}, \mathbb{C}^d) \to L^2(\mathbb{R}, \mathbb{C}^d)$ is called *sparse* if

$$Sf = \sum_{I \in \mathfrak{S}} \langle f \rangle_I \mathbf{1}_I,$$

where the collection of intervals $\mathfrak{S} \subseteq \mathcal{D}$ satisfies the sparseness condition given in (1.2). Now we use Theorem 1.3 to establish Theorem 1.4, which basically says:

$$\|S\|_{L^2(W) \to L^2(W)} \lesssim [W]_{A_2}^{\frac{3}{2}},$$

for every $d \times d$ matrix A_2 weight W. Here is the proof:

Proof. Let S be a sparse operator and observe that standard arguments give

$$\|S\|_{L^2(W) \to L^2(W)} = \|SM_{W^{-1}}\|_{L^2(W^{-1}) \to L^2(W)}.$$

So, we will study the second term instead and prove the desired bound using duality. Specifically, fix $f \in L^2(W^{-1})$ and $g \in L^2(W)$. Then

$$\begin{aligned}
\langle SM_{W^{-1}} f, g \rangle_{L^2(W)} &= \sum_{I \in \mathfrak{S}} \left\langle \left\langle W^{-1} f \right\rangle_I \mathbf{1}_I, g \right\rangle_{L^2(W)} \\
&= \sum_{I \in \mathfrak{S}} \left\langle \left\langle W^{-1} f \right\rangle_I, \langle W g \rangle_I \right\rangle_{\mathbb{C}^d} |I| \\
&\leq \sum_{I \in \mathfrak{S}} \left\| \langle W \rangle_I^{\frac{1}{2}} \left\langle W^{-1} \right\rangle_I^{\frac{1}{2}} \right\| \left\| \left\langle W^{-1} \right\rangle_I^{-\frac{1}{2}} \left\langle W^{-1} f \right\rangle_I \right\| \left\| \langle W \rangle_I^{-\frac{1}{2}} \langle W g \rangle_I \right\| |I| \\
&\leq [W]_{A_2}^{\frac{1}{2}} \left(\sum_{I \in \mathfrak{S}} \left\| \left\langle W^{-1} \right\rangle_I^{-\frac{1}{2}} \left\langle W^{-1} f \right\rangle_I \right\|^2 |I| \right)^{\frac{1}{2}} \left(\sum_{I \in \mathfrak{S}} \left\| \langle W \rangle_I^{-\frac{1}{2}} \langle W g \rangle_I \right\|^2 |I| \right)^{\frac{1}{2}}.
\end{aligned}$$

We will show how to control the first sum above. The second will follow using symmetric arguments. First, observe that

$$\begin{aligned}
\sum_{I \in \mathfrak{S}} \left\| \left\langle W^{-1} \right\rangle_I^{-\frac{1}{2}} \left\langle W^{-1} f \right\rangle_I \right\|^2 |I| &= \sum_{I \in \mathfrak{S}} \left\langle \left\langle W^{-1} \right\rangle_I^{-1} |I| \left\langle W^{-1} f \right\rangle_I, \left\langle W^{-1} f \right\rangle_I \right\rangle_{\mathbb{C}^d} \\
&= \sum_{I \in \mathcal{D}} \left\langle A_I \left\langle W^{-1} f \right\rangle_I, \left\langle W^{-1} f \right\rangle_I \right\rangle_{\mathbb{C}^d},
\end{aligned}$$

where $\{A_I\}_{I \in \mathcal{D}}$ is the sequence of positive semidefinite matrices indexed by the dyadic intervals defined by:

$$A_I = \begin{cases} \left\langle W^{-1} \right\rangle_I^{-1} |I| & \text{if } I \in \mathfrak{S} \\ 0 & \text{if } I \notin \mathfrak{S}. \end{cases}$$

We wish to apply Theorem 1.3 with the weight W^{-1}. To do this, we must establish the appropriate testing conditions. Specifically, notice that if $J \in \mathcal{D}$, then

$$\begin{aligned}\frac{1}{|J|}\sum_{I:I\subseteq J}\left\|\left\langle W^{-1}\right\rangle_I^{\frac12} A_I \left\langle W^{-1}\right\rangle_I^{\frac12}\right\| &= \frac{1}{|J|}\sum_{I\subseteq J:I\in\mathfrak{S}}\left\|\left\langle W^{-1}\right\rangle_I^{\frac12}\left\langle W^{-1}\right\rangle_I^{-1}|I|\left\langle W^{-1}\right\rangle_I^{\frac12}\right\| \\ &= \frac{1}{|J|}\sum_{I\subseteq J:I\in\mathfrak{S}}|I| \\ &\leq \frac{1}{|J|}\left(|J|+\frac{|J|}{2}+\frac{|J|}{4}+\dots\right) \\ &= 2,\end{aligned}$$

where we used the sparsity condition on J, the $\mathfrak{S}$-children of J, the $\mathfrak{S}$-children of the $\mathfrak{S}$-children of J, and so on. Now, by Theorem 1.3, we can conclude that

$$\sum_{I\in\mathfrak{S}}\left\|\left\langle W^{-1}\right\rangle_I^{-\frac12}\left\langle W^{-1}f\right\rangle_I\right\|^2|I| \lesssim [W]_{A_2}\|W^{-\frac12}f\|_{L^2}^2 = [W]_{A_2}\|f\|_{L^2(W^{-1})}^2.$$

We can similarly conclude that

$$\sum_{I\in\mathfrak{S}}\left\|\left\langle W\right\rangle_I^{-\frac12}\left\langle Wg\right\rangle_I\right\|^2|I| \lesssim [W]_{A_2}\|W^{\frac12}g\|_{L^2}^2 = [W]_{A_2}\|g\|_{L^2(W)}^2.$$

It immediately follows that

$$\langle SM_{W^{-1}}f, g\rangle_{L^2(W)} \lesssim [W]_{A_2}^{\frac32}\|f\|_{L^2(W^{-1})}\|g\|_{L^2(W)}.$$

As f and g were arbitrary, we can conclude that

$$\|S\|_{L^2(W)\to L^2(W)} = \|SM_{W^{-1}}\|_{L^2(W^{-1})\to L^2(W)} \lesssim [W]_{A_2}^{\frac32},$$

as desired. □

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