J. Math. Anal. Appl. 441 (2016) 763–800

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On stochastic evolution equations for nonlinear bipolar fluids: Well-posedness and some properties of the solution

Erika Hausenblas, Paul André Razafimandimby *

Department of Mathematics and Information Technology, Montanuniversität Leoben, Franz Josef Str. 18, 8700 Leoben, Austria

ARTICLE INFO

Article history:
Received 19 September 2012
Available online 22 April 2016
Submitted by U. Stadtmueller

Keywords:
Stochastic evolution equations
Strong solution
Ergodicity
Invariant measure
Bipolar fluids
Poisson random measure

ABSTRACT

We investigate the stochastic evolution equations describing the motion of a non-Newtonian fluids excited by multiplicative noise of Lévy type. We show that the system we consider has a unique global strong solution. We also give some results concerning the properties of the solution. We mainly prove that the unique solution satisfies the Markov–Feller property. This enables us to prove by means of some results from ergodic theory that the semigroup associated to the unique solution admits at least an invariant measure which is ergodic and tight on a subspace of the Lebesgue space L^2.

1. Introduction

Turbulence in Hydrodynamics is one of the most fascinating and difficult problems in Mathematics and in applied sciences in general. Many scientists believe that Newtonian fluid or the Navier–Stokes Equations (NSE for short) can accurately describe the most intricate complexities of turbulence in fluids flows. However, there are mainly two major obstacles for the mathematical study of turbulent flows. First, it is well known that the question of whether the three dimensional NSE admits or not a unique weak solution for all time still remains open. As it is not always easy to prove the existence of a global attractor in the case of lack of uniqueness of solution, this becomes a daunting obstacle for the investigation of the long-time behavior of the Navier–Stokes equations which is very important for a better understanding of turbulence and some physical features of the fluids. We refer, for instance, to [3,22,23,51], and [57] for some results in this direction. Second, there are a lot of fluid models exhibiting turbulent behavior that cannot be described by the Navier–Stokes equations. To overcome these problems one generally has to use other models of fluids or some regularizations, which might be of mathematical nature, of the Newtonian fluid. This has

* Corresponding author.
E-mail addresses: erika.hausenblas@unileoben.ac.at (E. Hausenblas), paulrazafi@gmail.com (P.A. Razafimandimby).

http://dx.doi.org/10.1016/j.jmaa.2016.04.044

motivated many scientists to consider fluids such that their stress tensors are a nonlinear function of the strain rate. This class of fluids forms the family of non-Newtonian fluids. One example of such fluids is the nonlinear bipolar fluids which are themselves contained in the class of multipolar fluids. The theory of viscous multipolar fluids was initiated by Necas and Silhavy [43], and developed later on in numerous works of prominent scientists such as Necas, Novotny and Silhavy [42], Bellout, Bloom and Necas [4]. Although bipolar fluids resemble the models that Ladyzhenskaya considered in [37] and [38] they differ in two aspects. First both bipolar fluids and Ladyzhenskaya models allow for a nonlinear velocity dependent viscosity, but in contrast to the bipolar fluids the Ladyzhenskaya models do not incorporate a higher-order velocity gradients. Second, in contrary to the Ladyzhenskaya models the theory of multipolar fluids is compatible with the basic principles of thermodynamics such as the Clausius–Duhem inequality and the principle of frame indifference. Moreover, results up to date indicate that the theory of multipolar fluids may lead to a better understanding of hydrodynamic turbulence, see for example [7].

Around the 70s Bensoussan and Temam [10] started the investigation of stochastic version of dynamical equations for Newtonian turbulent fluids. The Stochastic Partial Differential Equations they analyzed are obtained by adding noise terms to the deterministic NSE. This approach is basically motivated by Reynolds' work which stipulates that the velocity of a fluid particle in turbulent regime is composed of slow (deterministic) and fast (stochastic) components. While this belief was based on empirical and experimental data, Rozovskii and Mikulevicius were able to derive the models rigorously in their recent work [41], thereby confirming the importance of this approach in hydrodynamic turbulence. It is also pointed out in some recent articles such as [29] and [36] that some rigorous information on questions in Turbulence might be obtained from stochastic versions of the equations of fluid dynamics. Since the pioneering work of Bensoussan and Temam [10] on stochastic Navier–Stokes equations, stochastic models for Newtonian fluid dynamics and SPDEs in general have been the object of intense investigations which have generated several important results. We refer, for instance, to [1,9,12,13,16,15,17,19,18,25,24,26,27,41,44,45,54,53,55]. However, there are only very few results for the dynamical behavior of stochastic models for non-Newtonian fluids (see [31,47,46,48,34]).

In this paper, we are interested in the Lévy driven SPDEs for the nonlinear bipolar fluids. More precisely, let $d = 2, 3$, and $\mathcal{O} \subset \mathbb{R}^d$ be a smooth bounded open domain, we consider

$$\begin{cases} d\mathbf{u} + [\mathbf{u}\cdot\nabla\mathbf{u} - \nabla\cdot\mathbf{T}(\mathcal{E}(\mathbf{u})) + \nabla\pi]\,dt = \int_Z \sigma(t,\mathbf{u},z)\widetilde{\eta}(dz,dt),\ x\in\mathcal{O}, t\in(0,T],\\ \mathbf{u}(x,0) = \mathbf{u}_0,\ x\in\mathcal{O},\\ \nabla\cdot\mathbf{u} = 0,\ x\in\mathcal{O}, t\in[0,T],\\ \mathbf{u}(x,t) = \tau_{ijl}n_jn_l = 0,\ x\in\partial\mathcal{O}, t\in(0,T) \end{cases} \tag{1}$$

where $\mathbf{u}$ is the velocity of the fluids, π its pressure, $\mathbf{n}$ denotes the normal exterior to the boundary and

$$\mathcal{E}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial \mathbf{u}_i}{\partial x_j} + \frac{\partial \mathbf{u}_j}{\partial x_i}\right), |\mathcal{E}(\mathbf{u})|^2 = \sum_{i,j=1}^{n} |\mathcal{E}_{ij}(\mathbf{u})|^2,$$

$$\mathbf{T}(\mathcal{E}(\mathbf{u})) = 2\kappa_0(\varkappa + |\mathcal{E}(\mathbf{u})|^2)^{\frac{p-2}{2}}\mathcal{E}(\mathbf{u}) - 2\kappa_1\Delta\mathcal{E}(\mathbf{u}).$$

The quantities κ_0, κ_1 and $\varkappa$ denote positive constants. Here $\widetilde{\eta}$ is a compensated Poisson random measure defined on a prescribed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking its values in a separable Hilbert space H to be defined later. The system (1) describes the equations of motion of isothermal incompressible nonlinear bipolar fluids excited by random forces.

For $p = 2$, $\kappa_1 = 0$, $\sigma \equiv 0$, (1) is the Navier–Stokes equations which has been extensively studied, see, for instance, [56]. If $1 < p < 2$ then the fluid is shear thinning, and it is shear thickening when $2 < p$. The problem (1) is as interesting as the Navier–Stokes equations. It contains two nonlinear terms which makes the problem as difficult as any nonlinear evolution equations. During the last two decades, the deterministic version of (1) has been the object of intense mathematical investigation which has generated several important results. We refer to [5,6,8,39,40] for relevant examples. Despite these numerous results there are still a lot of open problems related to the mathematical theory of multipolar fluids. Some examples are the existence of weak solution for all values of p, the uniqueness of such weak solutions and many more. We refer, for instance, to [6,30] and [40] for some discussions about these challenges.

For $1 < p$ and the noise is replaced by a cylindrical Wiener process, the existence of martingale and stationary solution of (1) was established in [31]. In [50] Razafimandimby and Sango studied the exponential stability and some stabilization of (1) with $1 < p \leq 2$ and with a Wiener noise. It seems that this article is the first work studying the Lévy driven SPDEs (1). Our first main goal is to prove the existence and uniqueness of strong solution which should be understood in the sense of stochastic differential equations. To achieve this goal we mainly follow the idea initially developed by Breckner in [12] (see also [11]) and used later in many articles such as [19,26,49]. This method is based on Galerkin approximation and it allows to prove that the whole sequence of the Galerkin approximation converges in mean square to the exact solution. The second goal of the present paper is to give some partial results concerning the properties of the solution. We concentrate on proving that the solution satisfies the Markov–Feller property which enables us to prove that (1) admits at least an invariant measure which is ergodic and tight on a subspace of the Lebesgue space $L^2(\mathcal{O})$. Unfortunately, we could not proceed further and prove the uniqueness of the invariant measure. The investigation of the uniqueness of ergodic SPDEs driven by pure jump noise seems to be very difficult and out of reach of the most recent methods used to prove the uniqueness of invariant measure of SPDEs. We postpone this investigation in future work.

To close this introduction we give the outline of the article. In Section 2 we give most of the notations and necessary preliminary used throughout the work. By means of Galerkin approximation we show the existence of strong solution in Section 3. The pathwise uniqueness of the solution and the convergence of the whole sequence of Galerkin approximate solution to the exact solution are proved in Section 4. Section 5 is devoted to the investigation of some properties of the strong solution.

Notations. By $\mathbb{N}$ we denote the set of nonnegative integers, i.e. $\mathbb{N} = \{0, 1, 2, \cdots\}$ and by $\bar{\mathbb{N}}$ we denote the set $\mathbb{N} \cup \{+\infty\}$. Whenever we speak about $\mathbb{N}$ (or $\bar{\mathbb{N}}$)-valued measurable functions we implicitly assume that the set is equipped with the trivial σ-field $2^{\mathbb{N}}$ (or $2^{\bar{\mathbb{N}}}$). By $\mathbb{R}_+$ we will denote the interval $[0, \infty)$ and by $\mathbb{R}_*$ the set $\mathbb{R} \setminus \{0\}$. If X is a topological space, then by $\mathcal{B}(X)$ we will denote the Borel σ-field on X. By λ_d we will denote the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, by λ the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

If $(S, \mathcal{S})$ is a measurable space then by $M(S)$ we denote the set of all real valued measures on $(S, \mathcal{S})$, and by $\mathcal{M}(S)$ the σ-field on $M(S)$ generated by the functions $i_B : M(S) \ni \mu \mapsto \mu(B) \in \mathbb{R}$, $B \in \mathcal{S}$. By $M_+(S)$ we denote the set of all nonnegative measures on S, and by $\mathcal{M}(S)$ the σ-field on $M_+(S)$ generated by the functions $i_B : M_+(S) \ni \mu \mapsto \mu(B) \in \mathbb{R}_+$, $B \in \mathcal{S}$. Finally, by $M_I(S)$ we denote the family of all $\overline{\mathbb{N}}$-valued measures on $(S, \mathcal{S})$, and by $\mathcal{M}_I(S)$ the σ-field on $M_I(S)$ generated by functions $i_B : M(S) \ni \mu \mapsto \mu(B) \in \bar{\mathbb{N}}$, $B \in \mathcal{S}$. If $(S, \mathcal{S})$ is a measurable space then we will denote by $\mathcal{S} \otimes \mathcal{B}(\mathbb{R}_+)$ the product σ-field on $S \times \mathbb{R}_+$ and by $\nu \otimes \lambda$ the product measure of ν and the Lebesgue measure λ.

2. Mathematical settings of the problem (1)

Throughout this paper we mainly use the same notations as in [31]. By $L^q(\mathcal{O})$ we denote the Lebesgue space of q-th integrable functions with norm $\|\cdot\|_{L^q}$. For the particular case $q = 2$, we denote its norm by $\|\cdot\|$. For $q = \infty$ the norm is defined by $\|\mathbf{u}\|_{L^\infty} = \operatorname{ess\,sup}_{x\in\mathcal{O}} |\mathbf{u}(x)|$, where $|x|$ is the Euclidean norm of the vector $x \in \mathbb{R}^n$. The Sobolev space $\{\mathbf{u} \in L^q(\mathcal{O}) : D^k\mathbf{u} \in L^q(\mathcal{O}), k \leq \sigma\}$ with norm $\|\cdot\|_{q,\sigma}$ is denoted by $W^{q,\sigma}(\mathcal{O})$. $C(I, X)$ is the space of continuous functions from the interval $I = [0, T]$ to X, and $L^q(I, X)$ is the space of all measurable functions $u : [0, T] \to X$, with the norm defined by

$$\|\mathbf{u}\|^q_{L^q(I,X)} = \int_0^T \|\mathbf{u}(t)\|^q_X dt,\ q \in [1,\infty),$$

and when $q = \infty$, $\|\mathbf{u}\|_{L^\infty(I,X)} = \operatorname{ess\,sup}_{t\in[0,T]} \|\mathbf{u}(t)\|_X$.

The mathematical expectation associated to the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is denoted by $\mathbb{E}$ and as above we also define the space $L^q(\Omega, X)$.

We proceed with the definitions of some additional spaces frequently used in this work. We define a space of smooth functions with support strictly contained in $\mathcal{O}$ and satisfying the divergence free condition:

$$\mathcal{V} = \{\mathbf{u} \in \mathcal{C}^\infty_c(\mathcal{O}) : \nabla \cdot \mathbf{u} = 0\}.$$

We denote by H the closure of $\mathcal{V}$ with norm $|\cdot|$ in $L^2(\mathcal{O})$. It is a Hilbert space when equipped with the $L^2(\mathcal{O})$-inner product $(.,.)$. $\mathcal{H}^\sigma$ is the closure of $\mathcal{V}$ in $W^{2,\sigma}(\mathcal{O})$ with the norm $\|\cdot\|_{2,\sigma}$. We denote by $\|\mathbf{u}\|_\sigma$ the norm induced by $\|\mathbf{u}\|_{2,\sigma}$ on $\mathcal{H}^\sigma$. We denote by $\mathcal{H}^{-\sigma}$ the dual space of $\mathcal{H}^\sigma$ ($\sigma \geq 1$) wrt the norm $\|\mathbf{u}\|_\sigma$. If $\sigma = 2$, then $V = \mathcal{H}^2$ and V^* is the dual space of V. The duality product between V and V^* is denoted by $\langle .,. \rangle$. It should be noted that V is not the usual space of divergence-free functions of $W^{2,1}(\mathcal{O})$ used for the Navier–Stokes equations. Here it is a space of divergence-free functions of $W^{2,2}(\mathcal{O})$. We assume throughout the paper that there exists a positive constant λ_1 such that the Poincaré inequalities type

$$\|\mathbf{u}\|^2_\sigma \leq \frac{1}{\lambda_1}\|\mathbf{u}\|^2_{\sigma+1},\ \forall \mathbf{u} \in \mathcal{H}^{\sigma+1},\ \sigma \geq 0, \tag{2}$$

hold.

As mentioned in the introduction we will study a stochastic model for a nonlinear bipolar fluids excited by random forces. In the following lines we describe the forces acting on the fluids. Let $(Z, \mathcal{Z})$ be a separable metric space and let ν be a σ-finite positive measure on it. Suppose that $\mathfrak{P} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions, and $\eta : \Omega \times \mathcal{B}(\mathbb{R}_+) \times \mathcal{Z} \to \bar{\mathbb{N}}$ is a time homogeneous Poisson random measure, with intensity measure ν, defined over the filtered probability space $\mathfrak{P}$. A time homogeneous Poisson random measure defined over $\mathfrak{P}$ is given in the following definition.

Definition 2.1. Let Z be a metric space and $\mathcal{Z}$ its Borel σ-algebra, ν a positive σ-finite measure on $(Z, \mathcal{Z})$.

A *Poisson random measure*, with intensity measure ν, η defined on $(Z, \mathcal{Z})$ over $\mathfrak{P}$ is a measurable map $\eta : (\Omega, \mathcal{F}) \to (M_I(Z \times \mathbb{R}_+), \mathcal{M}_I(Z \times \mathbb{R}_+))$ satisfying the following conditions:

(i) for all $B \in \mathcal{B}(Z \otimes \mathbb{R}_+)$, $\eta(B) : \Omega \to \overline{\mathbb{N}}$ is a Poisson random measure with parameter $\mathbb{E}[\eta(B)]$;
(ii) η is independently scattered, i.e., if the sets $B_j \in \mathcal{B}(Z \otimes \mathbb{R}_+)$, $j = 1, \dots, n$, are disjoint then the random variables $\eta(B_j)$, $j = 1, \dots, n$, are independent;
(iii) for all $U \in \mathcal{Z}$ and $I \in \mathcal{B}(\mathbb{R}_+)$

$$\mathbb{E}[\eta(U \times I)] = \lambda(I)\nu(U);$$

(iv) for all $U \in \mathcal{Z}$ the $\bar{\mathbb{N}}$-valued process $(N(U,t))_{t\geq 0}$ defined by $N(U,t) := \eta(U \times (0,t])$, $t \geq 0$, is $\mathbb{F}$-adapted and its increments are independent of the past, i.e., if $t > s \geq 0$, then the random variable $N(U,t) - N(U,s) = \eta(U \times (s,t])$ is independent of $\mathcal{F}_s$.

We will denote by $\tilde{\eta}$ the *compensated Poisson random measure* defined by $\tilde{\eta} := \eta - \gamma$, where *the compensator* $\gamma : \mathcal{B}(\mathcal{Z} \times \mathbb{R}_+) \to \mathbb{R}_+$ is defined by

$$\gamma(A \times I) = \lambda(I)\nu(A), \qquad I \in \mathcal{B}(\mathbb{R}_+), \quad A \in \mathcal{Z}.$$

While items (i) and (ii) are the classical definition, see for e.g. [45, Definition 6.1], of a Poisson Random measure η, the remaining items implicitly indicate that our η is associated to a certain Lévy process $\tilde{L}$, see, for instance [45, Proposition 4.16].

Let $\mathcal{M}^2(\mathbb{R}_+, L^2(Z,\nu,H))$ be the class of all progressively measurable processes $\xi : \mathbb{R}_+ \times Z \times \Omega \to V$ satisfying the condition

$$\mathbb{E}\int_0^T\int_Z |\xi(r,z)|_H^2 \nu(dz)\, dr < \infty, \quad \forall T > 0. \tag{3}$$

If $T > 0$, the class of all progressively measurable processes $\xi : [0,T] \times Z \times \Omega \to V$ satisfying the condition (3) just for this one T, will be denoted by $\mathcal{M}^2(0,T,L^2(Z,\nu,H))$. Also, let $\mathcal{M}^2_{step}(\mathbb{R}_+, L^2(Z,\nu,H))$ be the space of all processes $\xi \in \mathcal{M}^2(\mathbb{R}_+, L^2(Z,\nu,H))$ such that

$$\xi(r) = \sum_{j=1}^n 1_{(t_{j-1},t_j]}(r)\xi_j, \quad 0 \le r,$$

where $\{0 = t_0 < t_1 < \ldots < t_n < \infty\}$ is a partition of $[0,\infty)$, and for all j, ξ_j is an $\mathcal{F}_{t_{j-1}}$ measurable random variable. For any $\xi \in \mathcal{M}^2_{step}(\mathbb{R}_+, L^2(Z,\nu,H))$ we set

$$\tilde{I}(\xi) = \sum_{j=1}^n \int_Z \xi_j(z)\tilde{\eta}\left(dz,(t_{j-1},t_j]\right). \tag{4}$$

This is basically the definition of stochastic integral of a random step process ξ with respect to the compound random Poisson measure $\tilde{\eta}$. The extension of this integral on $\mathcal{M}^2(\mathbb{R}_+, L^2(Z,\nu,H))$ is possible thanks to the following result which is taken from [14, Theorem C.1].

Theorem 2.2. *There exists a unique bounded linear operator*

$$I : \mathcal{M}^2(\mathbb{R}_+, L^2(Z,\nu;H)) \to L^2(\Omega,\mathcal{F};H)$$

such that for $\xi \in \mathcal{M}^2_{step}(\mathbb{R}_+, L^2(Z,\nu;H))$ we have $I(\xi) = \tilde{I}(\xi)$. In particular, there exists a constant $C = C(H)$ such that for any $\xi \in \mathcal{M}^2(\mathbb{R}_+, L^2(Z,\nu,H))$,

$$\mathbb{E}\Big|\int_0^t\int_Z \xi(r,z)\tilde{\eta}(dz,dr)\Big|_H^2 \le C\,\mathbb{E}\int_0^t\int_Z |\xi(r,z)|_H^2\,\nu(dz)\,dr,\ t \ge 0. \tag{5}$$

Moreover, for each $\xi \in \mathcal{M}^2(\mathbb{R}_+, L^2(Z,\nu,H))$, the process $I(1_{[0,t]}\xi)$, $t \ge 0$, is an H-valued càdlàg martingale. The process $1_{[0,t]}\xi$ is defined by $[1_{[0,t]}\xi](r,z,\omega) := 1_{[0,t]}(r)\xi(r,z,\omega)$, $t \ge 0$, $r \in \mathbb{R}_+$, $z \in Z$ and $\omega \in \Omega$.

As usual we will write

$$\int_0^t\int_Z \xi(r,z)\tilde{\eta}(dz,dr) := I(\xi)(t), \quad t \ge 0.$$

Now, we assume that $p \in (1,2]$ which will be fixed in the whole section. We will rewrite (1) in the following equivalent form

$$\begin{cases} d\mathbf{u} + [\kappa_1 \mathcal{A}\mathbf{u} + 2\kappa_0 \mathcal{A}_p \mathbf{u} + B(\mathbf{u},\mathbf{u})]\, dt = \int_Z \sigma(t,\mathbf{u},z)\widetilde{\eta}(dz,dt), \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{6}$$

where the operator $\mathcal{A}$ is defined through the relation

$$\langle \mathcal{A}\mathbf{u}, \mathbf{v}\rangle = a(\mathbf{u},\mathbf{v}) = \int_{\mathcal{O}} \frac{\partial \mathcal{E}_{ij}(\mathbf{u})}{\partial x_k}\frac{\partial \mathcal{E}_{ij}(\mathbf{v})}{\partial x_k} dx,\ \ \mathbf{u} \in D(A), \mathbf{v} \in V.$$

Here and after the summation over repeated indices is enforced.

Note that

$$D(\mathcal{A}) = \{\mathbf{u} \in V : \exists f \in H \subset V^* \text{ for which } a(\mathbf{u},\mathbf{v}) = (f,\mathbf{v}), \forall \mathbf{v} \in V\},$$

$\mathcal{A} = \mathbf{P}\Delta^2$, where $\mathbf{P}$ is the orthogonal projection defined on $L^2(\mathcal{O})$ onto H.

Remark 2.3. It is shown in [8] that there exist two positive constants k_1 and k_2 depending only on $\mathcal{O}$ such that

$$k_1 \|\mathbf{u}\|_2^2 \leq \langle \mathcal{A}\mathbf{u}, \mathbf{u}\rangle \leq k_2 \|\mathbf{u}\|_2^2, \tag{7}$$

for any $\mathbf{u} \in V$. Thanks to this we will just write $\|\mathbf{u}\|_2^2$ in place of $\langle \mathcal{A}\mathbf{u}, \mathbf{u}\rangle$, $\mathbf{u} \in V$. Also, it is not difficult to see that $\mathcal{A}$ is symmetric. This fact together with (7) yields that $\mathcal{A}$ is self-adjoint.

The bilinear form $B(\mathbf{u},\mathbf{v}) : \mathcal{H}^1 \times \mathcal{H}^1 \to \mathcal{H}^{-1}$ is defined as follows:

$$\langle B(\mathbf{u},\mathbf{v}), \mathbf{w}\rangle = b(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_{\mathcal{O}} \mathbf{u}_i \frac{\partial \mathbf{v}_j}{\partial x_i}\mathbf{w}_j dx,\ \ \mathbf{u},\mathbf{v},\mathbf{w} \in \mathcal{H}^1,$$

where $b(.,.,.)$ is the well-known trilinear form used in the mathematical analysis of Navier–Stokes equations (see for instance [56]). The bilinear form $B(\cdot,\cdot)$ enjoys the following properties:

- for any $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathcal{H}^1$, we have

$$\langle B(\mathbf{u},\mathbf{v}), \mathbf{w}\rangle = -\langle B(\mathbf{u},\mathbf{w}), \mathbf{v}\rangle \text{ and } \langle B(\mathbf{u},\mathbf{v}), \mathbf{v}\rangle = 0. \tag{8}$$

- There exists a constant C_0 such that

$$\langle B(\mathbf{u},\mathbf{v}), \mathbf{w}\rangle \leq C_0 |\mathbf{u}| \|\mathbf{v}\|_1 \|\mathbf{v}\|_2 \tag{9}$$

  for any $\mathbf{u} \in \mathcal{H}^1$, $\mathbf{v} \in V$, $\mathbf{w} \in V$.

The inequality (9) can be proved by using Hölder's and Sobolev inequalities (see [56]).

The nonlinear term $\mathcal{A}_p : V \to V^*$ is defined as follows:

$$(\mathcal{A}_p\mathbf{u}, \mathbf{v}) = \int_{\mathcal{O}} \Gamma(\mathbf{u})\mathcal{E}_{ij}(\mathbf{u})\mathcal{E}_{ij}(\mathbf{v}) dx, \mathbf{u},\mathbf{v} \in V,$$

where $\Gamma(\mathbf{u}) = (\varkappa + |\mathcal{E}(\mathbf{u})|^2)^{\frac{p-2}{2}}$. Some of the properties of $\mathcal{A}_p$ is given below.

Lemma 2.4.

(i) *There exists a positive constant* $C(\varkappa, p)$ *such that*

$$\|\mathcal{A}_p \mathbf{u} - \mathcal{A}_p \mathbf{v}\|_{V^*} \leq C\|\mathbf{u}-\mathbf{v}\|_1,\ \ \mathbf{u},\mathbf{v} \in V. \tag{10}$$

(ii) *For any* $\mathbf{u},\mathbf{v} \in V$

$$\langle \mathcal{A}_p \mathbf{u} - \mathcal{A}_p \mathbf{v}, \mathbf{u}-\mathbf{v}\rangle \geq 0. \tag{11}$$

To check the results in the above lemma we need to recall the following results whose proofs can be found in [30].

Lemma 2.5 *(Korn's inequalities). Let* $1 < p < \infty$ *and let* $\mathcal{O} \subset \mathbb{R}^d$ *be of class* C^1*. Then there exist two positive constants* $k_p^i = k^i(\mathcal{O}, p)$*,* $i = 1, 2$ *such that*

$$k_p^1\|\mathbf{u}\|_1 \leq \left(\int\limits_{\mathcal{O}} |\mathcal{E}(\mathbf{u})|^2 dx\right)^{\frac{1}{2}} \leq k_p^2\|\mathbf{u}\|_1,$$

for any $\mathbf{u} \in \mathcal{H}^1$*.*

Proof of Lemma 2.4. Let $\mathbf{w}$ be an arbitrary element of V. Let us set $\delta = \frac{|\mathbf{e}|}{\sqrt{\varkappa}}$ and $\tilde{\varkappa} = \max(\varkappa^{\frac{p-2}{2}}, \varkappa^{\frac{p-5}{2}})$. Let us first note that

$$\left|\frac{\partial \mathbf{T}}{\partial e_{ij}}(\mathbf{e})\right| \leq \varkappa^{\frac{p-2}{2}}(1+\delta^2)^{\frac{p-2}{2}} + \varkappa^{\frac{p-5}{2}}\delta(1+\delta^2)^{\frac{p-4}{2}},$$

which implies that

$$\left|\frac{\partial \mathbf{T}}{\partial e_{ij}}(\mathbf{e})\right| \leq 3\tilde{\varkappa}(1+\delta^2)^{\frac{p-2}{2}}.$$

Since $p \in (1, 2]$ we have that

$$\left|\frac{\partial \mathbf{T}}{\partial e_{ij}}(\mathbf{e})\right| \leq 3\tilde{\varkappa}. \tag{12}$$

Secondly, we have

$$\begin{aligned}
\langle \mathbf{T}(\mathcal{E}(\mathbf{u})) - \mathbf{T}(\mathcal{E}(\mathbf{v})), \mathbf{w}\rangle &= 2\kappa_0 \int\limits_{\mathcal{O}} [\mathbf{T}(\mathcal{E}(\mathbf{u})) - \mathbf{T}(\mathcal{E}(\mathbf{v}))] \cdot \mathcal{E}(\mathbf{w}) dx,\\
&= 2\kappa_0 \int\limits_{\mathcal{O}} \mathcal{E}(\mathbf{w}) \cdot \int\limits_0^1 \frac{\partial \mathbf{T}(\mathcal{E}(\mathbf{v}) + s(\mathcal{E}(\mathbf{u}) - \mathcal{E}(\mathbf{v})))}{\partial s} ds dx,\\
&= 2\kappa_0 \int\limits_{\mathcal{O}} \mathcal{E}(\mathbf{w}) \cdot (\mathcal{E}(\mathbf{u}) - \mathcal{E}(\mathbf{v})) \int\limits_0^1 \frac{\partial \mathbf{T}(\mathcal{E}(\mathbf{v}) + s(\mathcal{E}(\mathbf{u}) - \mathcal{E}(\mathbf{v})))}{\partial e_{ij}} ds dx.
\end{aligned}$$

By using (12) in the last equation yields

$$|\langle \mathbf{T}(\mathcal{E}(\mathbf{u})) - \mathbf{T}(\mathcal{E}(\mathbf{v})), \mathbf{w}\rangle| \leq 6\kappa_0 \int\limits_{\mathcal{O}} |\mathcal{E}(\mathbf{w})| \cdot |\mathcal{E}(\mathbf{u}) - \mathcal{E}(\mathbf{v})| dx.$$

Invoking Hölder's and Korn's inequalities implies the existence of a positive constant K such that

$$|\langle \mathbf{T}(\mathcal{E}(\mathbf{u})) - \mathbf{T}(\mathcal{E}(\mathbf{v})), \mathbf{w}\rangle| \leq 6\tilde{\varkappa}\kappa_0 K \|\mathbf{u}-\mathbf{v}\|_1 \|\mathbf{w}\|_1,$$

for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. We easily conclude from this the proof of (i).

It is known from [40] that for any $p \in (1, \infty)$ and for all $\mathbf{D}, \mathbf{E} \in \mathbb{R}^{d\times d}_{sym}$:

$$(\mathbf{T}(\mathbf{D}) - \mathbf{T}(\mathbf{E})) \cdot (\mathbf{D} - \mathbf{E}) \geq 0,$$

where

$$\mathbb{R}^{d\times d}_{sym} = \{\mathbf{D} \in \mathbb{R}^{d\times d} : D_{ij} = D_{ji}, i, j = 1, 2, \dots, d\}.$$

Therefore, we see that for any $\mathbf{u}, \mathbf{v} \in V$

$$\langle \mathcal{A}_p\mathbf{u} - \mathcal{A}_p\mathbf{v}, \mathbf{u} - \mathbf{v}\rangle = \int_{\mathcal{O}} [\mathbf{T}(\mathcal{E}(\mathbf{u})) - \mathbf{T}(\mathcal{E}(\mathbf{v}))] \cdot [\mathcal{E}(\mathbf{u}) - \mathcal{E}(\mathbf{v})]\, dx \geq 0,$$

which proves (ii). $\square$

To close this section we introduce the main set of hypotheses used in this article. Throughout this work we suppose that we are given a function σ satisfying the following set of constraints:

Condition 1. *There exist nonnegative constants ℓ_0, ℓ_1, ℓ_2, ℓ_3 such that, for any $t \in [0, T]$ and all $\mathbf{u}, \mathbf{v} \in H$, we have*

(1) $|\sigma(t, \mathbf{u})|^2_{L^2(Z,\nu;H)} \leq \ell_0 + \ell_1|\mathbf{u}|^2$;
(2) $|\sigma(t, \mathbf{u}) - \sigma(t, \mathbf{v})|^2_{L^2(Z,\nu;H)} \leq \ell_2|\mathbf{u} - \mathbf{v}|^2$.
(3) $|\sigma(t, \mathbf{u})|^4_{L^4(Z,\nu;H)} \leq \ell_3(1 + |\mathbf{u}|^4)$.

3. Existence of a strong solution

In this section, we will show that (1) admits at least one strong solution. The proof is based on Galerkin approximation and idea borrowed from [12]. But before we proceed further we define explicitly what we mean by strong solution of (1) or (6).

Definition 3.1. Let $(Z, \mathcal{Z})$ be a separable metric space on which is defined a σ-finite measure ν and $\mathbf{u}_0 \in H$. A strong solution to the problem (6) is a stochastic process $\mathbf{u}$ such that:

(1) $\mathbf{u} = \{\mathbf{u}(t); t \geq 0\}$ is a $\mathbb{F}$-progressively measurable process such that

$$\mathbb{E} \sup_{s\in[0,T]} |\mathbf{u}(s)|^4 + \mathbb{E}\int_0^T \|\mathbf{u}(t)\|_2^2 dt < \infty;$$

(2) the following holds

$$
\begin{aligned}
(\mathbf{u}(t), \mathbf{w}) = (\mathbf{u}_0, \mathbf{w}) - \kappa_1 \int_0^t \left(\langle \mathcal{A}\mathbf{u}(s), \mathbf{w} \rangle - \langle B(\mathbf{u}(s), \mathbf{u}(s)), \mathbf{w} \rangle \right) ds \\
- 2\kappa_0 \int_0^t \langle \mathcal{A}_p \mathbf{u}(s), \mathbf{w} \rangle ds + \int_0^t \int_Z (\sigma(s, \mathbf{u}(s), z), \mathbf{w}) \widetilde{\eta}(dz, ds),
\end{aligned}
\tag{13}
$$

for any $\mathbf{w} \in V$, for almost all $t \in [0, T]$ and $\mathbb{P}$-almost surely.

Theorem 3.2. *Let the set of constraints in Condition 1 be satisfied and $r = 1, 2$. Then, for any initial value ξ with $\mathbb{E}|\xi|^{2r} < \infty$, there exists a solution $\mathbf{u}$ to problem (6) which satisfies*

$$
\mathbb{E} \left(\sup_{0 \le t \le T} |\mathbf{u}(t)|^{2r} + \int_0^T \|\mathbf{u}(s)\|_2^2 \, |\mathbf{u}(s)|^{2r-2} \, ds \right) \le C \left(\mathbb{E} \, |\xi|^{2r} + 1 \right), \ T \ge 0. \tag{14}
$$

Before we prove this result let us recall an important statement which is borrowed from [21].

Lemma 3.3. *Let X, Y, I and ϕ be non-negative processes and Z be a nonnegative integrable random variable. Assume that I is non-decreasing and that there exist nonnegative constants C, α, β, γ, δ and T satisfying first*

$$
\int_0^T \phi(s) \, ds \le C, \quad a.s., \quad 2\beta e^C \le 1, \quad 2\delta e^C \le \alpha,
$$

and secondly for all $t \in [0, T]$ there exists a constant $\tilde{C} > 0$ such that

$$
X(t) + \alpha Y(t) \le Z + \int_0^t \phi(r) X(r) \, dr + I(t), \quad a.s.,
$$

$$
\mathbb{E} I(t) \le \beta \mathbb{E} X(t) + \gamma \int_0^t \mathbb{E} X(s) \, ds + \delta \mathbb{E} Y(t) + \tilde{C}.
$$

If $X \in L^\infty([0, T] \times \Omega)$, then we have

$$
\mathbb{E} \left[X(t) + \alpha Y(t) \right] \le 2 \exp \left(C + 2t\gamma e^C \right) \left(\mathbb{E} Z + \tilde{C} \right), \quad t \in [0, T].
$$

The proof of Theorem 3.2 will be split into five steps.

A priori uniform estimates:

The operator $\mathcal{A}$ is self-adjoint and it follows from Rellich's theorem that it is compact on H. Therefore, there exists a sequence of positive numbers $\{\tilde{\lambda}_i : i = 1, 2, 3 \ldots\}$ and a family of smooth function $\{\phi_i : i = 1, 2, 3, \ldots\}$ satisfying

$$
\mathcal{A}\phi_i = \tilde{\lambda}_i \phi_i, \tag{15}
$$

for any $i \in \mathbb{N}$. We can assume that the family $\{\phi_i : i = 1, 2, 3, \ldots\}$ is an orthonormal basis of H which is orthogonal and dense in V.

Let Π_m denote the projection of V^* onto $H_m := span\{\phi_1, \cdots, \phi_m\}$. That is

$$\Pi_m x = \sum_{i=1}^{m} \langle x, \phi_i \rangle \phi_i, x \in V^*.$$

Also, $\Pi_{m|_H}$ is the orthogonal projection of H onto H_m.

For every $m \in \mathbb{N}$, we consider the finite dimensional system of SDEs on H_m given by

$$\begin{aligned} d\mathbf{u}^m(t) &= \Pi_m F(\mathbf{u}^m(t))\, dt + \int_Z \Pi_m \sigma(t, \mathbf{u}^m(t), z)\, \tilde{\eta}(dz, dt),\ t \geq 0, \\ \mathbf{u}^m(0) &= \Pi_m \xi, \end{aligned} \tag{16}$$

where $F(\mathbf{u}^m(s)) = -A\mathbf{u}^m(s) - \mathcal{A}_p \mathbf{u}^m(s) + B(\mathbf{u}^m(s), \mathbf{u}^m(s))$. To shorten notation we set

$$B_m(\cdot,\cdot) = \Pi_m B(\cdot,\cdot) \text{ and } \sigma_m(\cdot,\cdot,\cdot) = \Pi_m \sigma(\cdot,\cdot,\cdot).$$

We note that since Π_m is a contraction of V^*, we infer from (8), (9) and item (2) of Condition 1 that F is locally Lipschitz and $\sigma_m := \Pi_m \sigma$ is globally Lipschitz. As we know from e.g. Albeverio, Brzeźniak and Wu [1], on the basis of Condition 1, equation (16) has a unique H_m-valued càdlàg local strong solution $\mathbf{u}^m$. The following proposition implies that it is in fact a global solution.

Proposition 3.4. *Let the assumptions be as in Theorem 3.2. Then there exists a constant $C > 0$ such that for $r = 1, 2$ we have*

$$\sup_m \mathbb{E} \left(\sup_{t \in [0,T]} |\mathbf{u}^m(t)|^{2r} + \int_0^T \|\mathbf{u}^m(s)\|_2^2 \, |\mathbf{u}^m(s)|^{2r-2} \, ds \right) \leq C \left(\mathbb{E}\, |\xi|^{2r} + 1 \right).$$

Proof of Proposition 3.4. As mentioned above, it follows from [1, Theorem 2.8] that Equation (16) has a unique càdlàg local strong solution $\mathbf{u}^m$ in H_m. That means that for any $m \in \mathbb{N}$ there exists a unique solution on a short interval $[0, T_m]$ satisfying

$$\mathbf{u}^m(t) = \Pi_m \xi + \int_0^t \Pi_m F(\mathbf{u}^m(s))\, ds + \int_0^t \int_Z \sigma_m(s, \mathbf{u}^m(s), z)\, \tilde{\eta}(dz, ds), \qquad t \in [0, T_m].$$

We begin by checking the estimate in the proposition with the case $r = 1$. We argue as in [1, Proof of Theorem 3.1]. Let $(\tau_M)_M$ be an increasing sequence of stopping times defined by

$$\tau_M = \inf\{t \in [0, T] : |\mathbf{u}^m(t)|^2 + \int_0^t \|\mathbf{u}^m(s)\|^2 ds \geq M^2\} \wedge T,$$

for any integer M. Throughout, we fix $r \in [0, T]$ and $0 \leq t \leq r \wedge \tau_M$. Since we can identify the space H_m with $\mathbb{R}^m$ then we can apply the finite dimensional Itô's formula (see, for example, [35, Chapter II, Theorem 5.1]) to the function $|\cdot|^{2r}$ and the process $\mathbf{u}^m$. This procedure along with (8) yields

$$
\begin{aligned}
|\mathbf{u}^m(t)|^2 = |\Pi_m\xi|^2 - 2\kappa_1 \int_0^t \|\mathbf{u}^m(s)\|_2^2\, ds - 2\kappa_0 \int_0^t \langle \mathcal{A}_p \mathbf{u}^m(s), \mathbf{u}^m(s)\rangle ds \\
+ 2\int_0^t \int_Z (\mathbf{u}^m(s-), \sigma_m(s, \mathbf{u}^m(s), z))\tilde{\eta}(dz, ds) \\
+ \int_0^t \int_Z \Psi(s,z)\eta(dz, ds),
\end{aligned} \tag{17}
$$

where

$$
\Psi(s,z) = |\mathbf{u}^m(s-) + \sigma_m(s, \mathbf{u}^m(s), z)|^2 - |\mathbf{u}^m(s-)|^2 - (\mathbf{u}^m(s-), \sigma(s, \mathbf{u}^m(s), z))\,.
$$

From the fact that $|x|^2 - |y|^2 + |x-y|^2 = 2\langle x-y, x\rangle$, $x, y \in H$ and (11), we derive from (17) that

$$
\begin{aligned}
|\mathbf{u}^m(t)|^2 + 2\int_0^t \kappa_1 \|\mathbf{u}^m(s)\|_2^2 ds \leq |\Pi_m\xi|^2 + 2\int_0^t \int_Z (\mathbf{u}^m(s-), \sigma(s, \mathbf{u}^m(s), z))\tilde{\eta}(dz, ds) \\
+ \int_0^t \int_Z |\sigma(s, \mathbf{u}^m(s), z)|^2 \eta(dz, ds)
\end{aligned} \tag{18}
$$

for any $t \in [0, r \wedge \tau_M]$, $r \in [0, T]$. For any $r \in [0, T]$ and $t \in [0, r \wedge \tau_m]$ we define the following stochastic processes

$$
\begin{aligned}
X(t) &:= \sup_{0\leq s\leq t} |\mathbf{u}^m(s)|^2; \\
Y(t) &:= \int_0^t \|\mathbf{u}^m(s)\|_2^2\, ds; \\
I(t) &:= \sup_{0\leq s\leq t} \Big(2\Big|\int_0^s \int_Z (\mathbf{u}^m(\tau-), \sigma(\tau, \mathbf{u}^m(\tau), z))\tilde{\eta}(dz, d\tau)\Big| + \int_0^s \int_Z |\sigma(\tau, \mathbf{u}^m(\tau), z)|^2 \eta(dz, d\tau)\Big), \\
&:= \sup_{s\in[0,t]} |I_1(s)| + I_2(t),
\end{aligned}
$$

where

$$
I_1(t) = \int_0^t \int_Z (\sigma(s, \mathbf{u}^m(s-), z), \mathbf{u}^m(s-))\widetilde{\eta}(dz, ds),
$$

and

$$
I_2(t) = \sup_{0\leq s\leq t} \int_0^s \int_Z |\sigma(\tau, \mathbf{u}^m(\tau-), z)|^2 \eta(dz, d\tau).
$$

Since $I_1(t)$ is a local martingale we can apply Burkholder–Davis–Gundy's inequality and get

$$\mathbb{E}\sup_{s\in[0,r\wedge\tau_M]}|I_1(s)|\leq C\mathbb{E}\left(\int_0^{r\wedge\tau_m}\int_Z(\mathbf{u}^m(s-),\sigma(s,\mathbf{u}^m(s),z))^2\nu(dz)ds\right)^{\frac{1}{2}}.$$

Thanks to Hölder's and Young's inequalities we have

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]}|I_1(s)|&\leq C\left[\varepsilon\mathbb{E}\sup_{s\in[0,t]}|\mathbf{u}^m(s)|^2\right]^{\frac{1}{2}}\left[\frac{1}{\varepsilon}\mathbb{E}\int_0^t\int_Z|\sigma(s,\mathbf{u}^m(s),z)|^2\nu(dz)ds\right]^{\frac{1}{2}}\\&\leq C\varepsilon\mathbb{E}\sup_{s\in[0,t]}|\mathbf{u}^m(s)|^2+\frac{C}{\varepsilon}\mathbb{E}\int_0^t\int_Z|\sigma(s,\mathbf{u}^m(s),z)|^2\nu(dz)ds.\end{aligned}$$

Invoking item (2) of Condition 1 we see that

$$\mathbb{E}\sup_{s\in[0,t]}|I_1(s)|\leq C\varepsilon\mathbb{E}X(t)+\frac{C}{\varepsilon}\ell_0t+\frac{C}{\varepsilon}\ell_1\int_0^t\mathbb{E}X(s)ds.\tag{19}$$

Next, we will deal with the second term of $I(t)$. Taking into account that the process

$$\int_0^t\int_Z|\sigma_m(r,\mathbf{u}^m(r))|^2\eta(dz,dr)$$

has only positive jumps, we obtain

$$\mathbb{E}I_2(t)\leq\mathbb{E}\int_0^t\int_Z|\sigma(s,\mathbf{u}^m,z)|^2\nu(dz)ds.$$

Thanks to the item (1) of Condition 1 we see that

$$\mathbb{E}I_2(t)\leq\ell_0t+\ell_1\int_0^t\mathbb{E}X(s)ds.\tag{20}$$

Thanks to (18) along with (19) and (20) we apply Lemma 3.3 and derive that there exists a positive constant C such that

$$\mathbb{E}\left[\sup_{0\leq s\leq t}|\mathbf{u}^m(s)|^2+2\kappa_1\int_0^t\|\mathbf{u}^m(s)\|_2ds\right]\leq C(\mathbb{E}|\xi|^2+1),$$

for any $m\in\mathbb{N}$ and $t\in[0,r\wedge\tau_m]$, $r\in[0,T]$. We have just shown that

$$\mathbb{E}\left[\sup_{0\leq s\leq t\wedge\tau_M}|\mathbf{u}^m(s)|^2+2\kappa_1\int_0^t\|\mathbf{u}^m(s)\|_2ds\right]\leq C(\mathbb{E}|\xi|^2+1)\forall t\in[0,T],\tag{21}$$

from which we can infer that

$$\mathbb{P}(\tau_M<t)\leq CM^{-2},\forall t\in[0,T],\forall M>0.$$

Hence, $\lim_{M\to\infty} \mathbb{P}(\tau_M < t) = 0$, for all $t \in [0, T]$. That is, $\tau_M \to \infty$ in probability. Therefore, there exists a subsequence τ_{M_k} such that $\tau_{M_k} \to \infty$, a.s. Since the sequence $(\tau_M)_M$ is increasing, we infer that $\tau_{M_k} \nearrow \infty$ a.s. Now we use Fatou's lemma and pass to the limit in (21) and derive that

$$\mathbb{E}\left[\sup_{0\le s\le t} |\mathbf{u}^m(s)|^2 + 2\kappa_1 \int_0^t \|\mathbf{u}^m(s)\|_2 ds\right] \le C(\mathbb{E}|\xi|^2 + 1).$$

The proposition is then proved for $r = 1$. Thus, it remains to show that it is true for the case $r = 2$. We again apply Itô's formula to obtain

$$\begin{aligned}|\mathbf{u}^m(t)|^{2r} = 2r \int_0^t \int_Z |\mathbf{u}^m(s-)|^{2(r-1)}(\mathbf{u}^m(s-), \sigma_m(s, \mathbf{u}^m(s), z))\widetilde{\eta}(dz, ds)\\ + |\Pi_m\xi|^{2r} + 2r \int_0^t |\mathbf{u}^m(s)|^{2(r-1)}(F_m(\mathbf{u}^m(s)), \mathbf{u}^m(s))ds\\ + \int_0^t \int_Z \Phi(s, z)\eta(dz, ds),\end{aligned} \tag{22}$$

where

$$\Phi(s, z) = |\mathbf{u}^m(s-) + \sigma_m(s, \mathbf{u}^m(s), z)|^{2r} - |\mathbf{u}^m(s-)|^{2r} - 2r|\mathbf{u}^m(s-)|^{2(r-1)}(\mathbf{u}^m(s-), \sigma(s, \mathbf{u}^m(s), z)).$$

Thanks to (8) and (11) the estimate (22) becomes

$$\begin{aligned}|\mathbf{u}^m(t)|^{2r} + 2r\kappa_1 \int_0^t |\mathbf{u}^m(s)|^{2(r-1)}\|\mathbf{u}^m(s)\|_2^2 ds - |\Pi_m\xi|^{2r} - \int_0^t \int_Z \Phi(s, z)\eta(dz, ds)\\ \le 2r \int_0^t \int_Z |\mathbf{u}^m(s-)|^{2(r-1)}(\mathbf{u}^m(s-), \sigma_m(s, \mathbf{u}^m(s), z))\widetilde{\eta}(dz, ds).\end{aligned}$$

Taking the supremum over $[0, t]$ on both sides of the above estimate leads to

$$\sup_{s\in[0,t]} |\mathbf{u}^m(s)| + 2r\kappa_1 \int_0^t |\mathbf{u}^m(s)|^{2(r-1)}\|\mathbf{u}^m(s)\|_2^2 ds \le |\Pi_m\xi|^{2r} + J(t), \tag{23}$$

where $J(t) = J_1(t) + J_2(t)$ with

$$\begin{aligned}J_1(t) &= 2r \sup_{s\in[0,t]} \left|\int_0^s \int_Z |\mathbf{u}^m(s-)|^{2(r-1)}(\mathbf{u}(\tau-), \sigma_m(s, \mathbf{u}^m(\tau), z))\widetilde{\eta}(dz, d\tau)\right|,\\ J_2(t) &= \sup_{s\in[0,t]} \left|\int_0^s \int_Z \Phi(\tau, z)\eta(dz, d\tau)\right|.\end{aligned}$$

First, we apply the Burkholder–Davis–Gundy inequality

$$\mathbb{E}J_1(t) \le 2rC\mathbb{E}\left(\int_0^t\int_Z |\mathbf{u}^m(s)|^{4(q-1)}|\mathbf{u}^m(s)|^2|\sigma_m(s,\mathbf{u}^m(s),z))|^2\,\nu(dz)ds\right)^{\frac{1}{2}}.$$

Then using item (1) of Condition 1 and Hölder's inequality implies

$$\mathbb{E}J_1(t) \le \left(\frac{1}{\varepsilon}\mathbb{E}\int_0^t |\mathbf{u}^m(s)|^{2r-2}\Big(\ell_0+\ell_1|\mathbf{u}^m(s)|^2\Big)ds\right)^{\frac{1}{2}}$$

$$\times\, 2rC\left(\varepsilon\mathbb{E}\sup_{s\in[0,t]}|\mathbf{u}^m(s)|^{2r}\right)^{\frac{1}{2}}.$$

Invoking Young's inequality yields

$$\mathbb{E}J_1(t) \le \frac{2rC}{2}\varepsilon\mathbb{E}\sup_{s\in[0,t]}|\mathbf{u}^m(s)|^{2r} + \frac{rC\ell_0}{2\varepsilon}\mathbb{E}\int_0^t |\mathbf{u}^m(s)|^{2r-2}ds + \frac{2rC\ell_1}{2\varepsilon}\mathbb{E}\int_0^t |\mathbf{u}^m(s)|^{2r}ds.$$

Using the fact that for $r \ge 2$, $|x|^{2r-2} \le C(1+|x|^{2r})$, we deduce from the last inequality that

$$\mathbb{E}J_1(t) \le \frac{2rC\ell_0 T}{2\varepsilon} + \frac{2rC}{2}\varepsilon\mathbb{E}\sup_{s\in[0,t]}|\mathbf{u}^m(s)|^{2r} + \frac{2rC(\ell_0+\ell_1)}{2\varepsilon}\mathbb{E}\int_0^t |\mathbf{u}^m(s)|^{2r}ds. \tag{24}$$

Now we deal with $J_2(t)$. First, note that

$$\mathbb{E}J_2(t) \le \left(\frac{r^2+r}{2}\right)\mathbb{E}\int_0^t\int_Z \Big(|\mathbf{u}^m(s-)|^{2(r-1)}|\sigma_m(s,\mathbf{u}^m(s),z)|^2 + |\sigma_m(s,\mathbf{u}^m(s),z)|^{2r}\Big)\nu(dz)ds, \tag{25}$$

where we have used the fact that

$$\Big||x+h|^{2r} - |x|^{2r} - 2r|x|^{2(r-1)}(x,h)\Big| \le \frac{r^2+r}{2}(|x|^{2(r-1)}|h|^2 + |h|^{2r}), \tag{26}$$

for all $x, h \in H$. Let us set $C_r = \frac{r^2+r}{2}$. Now thanks to items (1) and (3) of Condition 1 we derive from (25) that there exist positive constants ℓ_r and C_r such that

$$\begin{aligned}\mathbb{E}J_2(t) &\le C_r\ell_r\mathbb{E}\int_0^t(1+|\mathbf{u}^m(s)|^{2r} + |\mathbf{u}^m(s)|^{2(r-1)}\|\mathbf{u}^m(s)\|_2^{2r})ds\\ &\le C_r\ell_r T + C_r\ell_r\int_0^t |\mathbf{u}^m(s)|^{2r}ds + C_r\ell_r\int_0^t |\mathbf{u}^m(s)|^{2(r-1)}\|\mathbf{u}^m(s)\|_2^2 ds.\end{aligned} \tag{27}$$

Therefore, we see from (24) and (27) that there exist positive constants C_r', M_r, ℓ_r', and L_r' such that

$$\mathbb{E}J(t) \le C_r'T + M_r\varepsilon\mathbb{E}\sup_{s\in[0,t]}|\mathbf{u}^m(s)|^{2r} + \frac{\ell_r'}{2\varepsilon}\int_0^t \mathbb{E}|\mathbf{u}^m(s)|^{2r}ds + L_r'\mathbb{E}\int_0^t |\mathbf{u}^m(s)|^{2(r-1)}\|\mathbf{u}^m(s)\|_1^2 ds, \tag{28}$$

for any $m \in \mathbb{N}$ and $t \in [0,T]$. Set

$$X(t) = \sup_{s\in[0,t]} |\mathbf{u}^m(s)|^{2r},$$

and

$$Y(t) = \int_0^t |\mathbf{u}^m(s)|^{2(r-1)} \|\mathbf{u}^m(s)\|_2^2 ds.$$

Thanks to (23), (28) and an appropriate choice of $\varepsilon > 0$ we find that $X(\cdot)$ and $Y(\cdot)$ verify the conditions in Lemma 3.3. Therefore we infer the existence of a positive constant C such that

$$\mathbb{E} \sup_{s\in[0,t]} |\mathbf{u}^m(s)|^{2r} + C(q,\kappa_1) \int_0^t |\mathbf{u}^m(s)|^{2(r-1)} \|\mathbf{u}^m(s)\|_2^2 ds \le C(\mathbb{E}|\xi|^{2r} + 1),\ r = 2,$$

for any $t \in [0,T]$ and $m \in \mathbb{N}$. This completes the proof of the proposition. □

Passage to the limit:

To prove the existence of the solution of (6) we need to pass to the limit in the terms of (16) and in the estimate of Proposition 3.4. Before we do so we recall that there exists a constant $C > 0$ such that

$$\sup_{m\in\mathbb{N}} \left(\mathbb{E} \sup_{s\in[0,T]} |\mathbf{u}^m(s)|^{2r} + \mathbb{E} \int_0^T |\mathbf{u}^m(s)|^{2(r-1)} \|\mathbf{u}^m(s)\|_2^2 ds \right) < C. \tag{29}$$

We have the following weak compactness result.

Proposition 3.5. *We can find a subsequence* $\mathbf{u}^m$ *which is not relabeled and a stochastic process* $\mathbf{u}$ *such that*

$$\mathbf{u}^m \rightharpoonup \mathbf{u} \text{(weak star) in } L^4(\Omega; L^\infty([0,T]; H)), \tag{30}$$

$$\mathbf{u}^m \rightharpoonup \mathbf{u} \text{ in } L^2(\Omega \times [0,T]; V). \tag{31}$$

Moreover, there exist three elements $\mathbf{B}, \Sigma, \mathbf{A}$ *such that*

$$B_m(\mathbf{u}^m, \mathbf{u}^m) \rightharpoonup \mathbf{B} \text{ in } L^2(\Omega \times [0,T]; V^*), \tag{32}$$

$$\mathcal{A}_p \mathbf{u}^m \rightharpoonup \mathbf{A} \text{ in } L^2(\Omega \times [0,T]; V^*), \tag{33}$$

$$\sigma_m(t, \mathbf{u}^m, \cdot) \rightharpoonup \Sigma \text{ in } L^2(\Omega \times [0,T]; L^2(Z,\nu; H)). \tag{34}$$

Proof. Since $L^2(\Omega \times [0,T]; V)$ is a Hilbert space and $L^4(\Omega; L^\infty(0,T; H))$ is a Banach space and the dual of $L^{\frac{4}{3}}(\Omega; L^1(0,T; H))$, we easily infer from Banach-Alaoglu's theorem and the uniqueness of weak limit that there exist a subsequence of $\mathbf{u}^m$ (which is denoted with the same fashion) and a stochastic process $\mathbf{u}$ belonging to $L^4(\Omega; L^\infty(0,T; H)) \cap L^2(\Omega \times [0,T]; V)$ such that (30) and (31) hold true.

It remains to show (32)-(34). To prove (32) we first recall that there exists a positive constant C_0 such that

$$|\langle B_m(\Phi, \mathbf{v}), \mathbf{w}\rangle| \le C_0 |\Phi| \|\mathbf{v}\|_1 \|\mathbf{w}\|_2,$$

for any $\Phi \in H$, $\mathbf{v} \in V$ and $\mathbf{w} \in V$. This inequality implies that

$$\mathbb{E}\int_0^T \|B_m(\mathbf{u}^m(s),\mathbf{u}^m(s))\|^2_{V^*} \le C\mathbb{E}\int_0^T |\mathbf{u}^m(s)|^2\|\mathbf{u}^m(s)\|_1^2 ds,$$

from which and (29) we get that $B_m(\mathbf{u}^m,\mathbf{u}^m)$ is a bounded sequence in the Hilbert space $L^2(\Omega\times[0,T];V^*)$. Thus, there exists an element of $L^2(\Omega\times[0,T];V^*)$ that we denote by $\mathbf{B}$ such that $B_m(\mathbf{u}^m,\mathbf{u}^m)$ converges weakly to $\mathbf{B}$ in $L^2(\Omega\times[0,T];V^*)$.

By invoking (10) and (29) we see that the following uniform estimate holds

$$\sup_{m\in\mathbb{N}}\mathbb{E}\int_0^T \|\mathcal{A}_p\mathbf{u}^m(s)\|^2_{V^*}ds \le C.$$

Therefore, the proof of (33) follows the same lines as for the proof of (32).

From item (1) of Condition 1 and estimate (29) we easily obtain the uniform estimate

$$\sup_{m\in\mathbb{N}}\mathbb{E}\int_0^T \|\sigma_m(s,\mathbf{u}^m(s),z)\|^2_{L^2(Z,\nu;H)} \le K_0T + K_1\mathbb{E}\int_0^T |\mathbf{u}^m(s)|^2 ds + K_1'\mathbb{E}\int_0^T \|\mathbf{u}^m(s)\|_2 ds \le C,$$

which implies that $\sigma_m(s,\mathbf{u}^m(s),z)$ is a bounded sequence in $L^2(\Omega\times[0,T];L^2(Z,\nu;H))$. Therefore, by Banach-Alaoglu we deduce the existence of Σ belonging to $L^2(\Omega\times[0,T];L^2(Z,\nu;H))$ such that (34) holds. This completes the proof of the proposition. $\square$

With the convergences in Proposition 3.5 we can pass to the limit in each term of (16) and obtain that

$$\mathbf{u}(t) + \kappa_1\int_0^t \mathcal{A}\mathbf{u}(s)ds + \kappa_0\int_0^t \mathbf{A}(s)ds = \mathbf{u}_0 + \int_0^t\int_Z \Sigma(s,z)\widetilde{\eta}(dz,ds), \tag{35}$$

$\mathbb{P}$-a.s. and for any $t\in[0,T]$ as an equality in V^*. Also, passing to the limit in (29) gives the estimate in Theorem 3.2. Thanks to (32) and (34) we can deduce from [32] that the stochastic process $\mathbf{u}$ has a càdlàg modification taking values in H. From now on we will identify $\mathbf{u}$ with its càdlàg modification. Henceforth, we need to show the following identities to complete the proof of Theorem 3.2.

Proposition 3.6. *We have the following identities*

$$\mathbf{B} = B(\mathbf{u},\mathbf{u}) \text{ in } L^2(\Omega\times[0,T];V^*), \tag{36}$$

$$\mathbf{A} = \mathcal{A}_p\mathbf{u} \text{ in } L^2(\Omega\times[0,T];V^*), \tag{37}$$

$$\Sigma = \sigma(t,\mathbf{u},\cdot) \text{ in } L^2(\Omega\times[0,T];L^2(Z,\nu;H)). \tag{38}$$

For any integer $M\ge 1$ we consider the sequence of stopping times $\{\tau_M : M\ge 1\}$ defined by

$$\tau_M = \inf\{t\in[0,T] : |\mathbf{u}(t)|^2 + \int_0^T \|\mathbf{u}(s)\|_2^2 \ge M^2\}\wedge T.$$

The proof of Proposition 3.6 requires the following convergences.

Lemma 3.7. *For any $M \geq 1$ we have that, as $m \to \infty$,*

$$1_{[0,\tau_M]}(\mathbf{u}^m - \mathbf{u}) \to 0 \text{ in } L^2(\Omega \times [0,T]; V), \tag{39}$$

and

$$\mathbb{E}|\mathbf{u}^m(\tau_M) - \mathbf{u}(\tau_M)| \to 0. \tag{40}$$

Proof of Lemma 3.7. Let $\tilde{\mathbf{u}}^m$ be the orthogonal projection of $\mathbf{u}$ onto $\operatorname{Span}\{\phi_1, \dots, \phi_m\}$, that is

$$\tilde{\mathbf{u}}^m = \sum_{i=1}^m (\mathbf{u}, \phi_i)\phi_i.$$

It is clear that as $m \to \infty$

$$\tilde{\mathbf{u}}^m \to \mathbf{u} \text{ in } L^2(\Omega \times [0,T]; H). \tag{41}$$

We can also check that

$$\mathbb{E}|\tilde{\mathbf{u}}^m(\tau_M) - \mathbf{u}(\tau_M)|^2 \to 0, \tag{42}$$

as $m \to \infty$.

First we should note that

$$\langle \mathcal{A}\tilde{\mathbf{u}}^m(t), \tilde{\mathbf{u}}^m(t)\rangle = \langle \sum_j \mathcal{A}\phi_j(\tilde{\mathbf{u}}^m(t), \phi_j), \sum_i (\tilde{\mathbf{u}}^m(t), \phi_i)\phi_i\rangle.$$

Thanks to (15) we have

$$\begin{aligned}\langle \mathcal{A}\tilde{\mathbf{u}}^m(t), \tilde{\mathbf{u}}^m(t)\rangle &= \sum_{i,j} (\tilde{\mathbf{u}}^m(t), \phi_j)(\tilde{\mathbf{u}}^m(t), \phi_i)(\lambda_j\phi_j, \phi_i), \\ &= \sum_j (\tilde{\mathbf{u}}^m(t), \phi_j)^2(\lambda_j\phi_j, \phi_j).\end{aligned}$$

Thanks to (15) again we have

$$\langle \mathcal{A}\tilde{\mathbf{u}}^m(t), \tilde{\mathbf{u}}^m(t)\rangle = \sum_j (\tilde{\mathbf{u}}^m(t), \phi_j)^2 \langle \mathcal{A}\phi_j, \phi_j\rangle.$$

From this, we can easily derive that

$$\|\tilde{\mathbf{u}}^m(t)\|_2^2 \leq \|\mathcal{A}\| \; |\mathbf{u}(t)|^2, \tag{43}$$

for almost all $(\omega, t) \in \Omega \times [0,T]$. Also,

$$\begin{aligned}\|\tilde{\mathbf{u}}^m(s) - \mathbf{u}(s)\|_2^2 &\leq \langle \mathcal{A}\tilde{\mathbf{u}}^m(s) - \mathcal{A}\mathbf{u}(s), \tilde{\mathbf{u}}^m(s) - \mathbf{u}(s)\rangle \\ &\leq \langle \sum_{i=m+1}^\infty (\mathbf{u}(s), \mathcal{A}\phi_i)\phi_i, \sum_{j=m+1}^\infty (\mathbf{u}(s), \phi_j)\phi_j\rangle, \\ &\leq \langle \sum_{i=m+1}^\infty (\mathbf{u}(s), \phi_i)\tilde{\lambda}_i\phi_i, \sum_{j=m+1}^\infty (\mathbf{u}(s), \phi_j)\phi_j\rangle,\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j=m+1}^{\infty} (\mathbf{u}(s), \phi_j)^2 \langle \mathcal{A}\phi_j, \phi_j \rangle \\
&\le \|\mathcal{A}\| \sum_{j=m+1}^{\infty} (\mathbf{u}(s), \phi_j)^2, \qquad (44)
\end{aligned}
$$

for any m. Since $\mathbf{u} \in H$ for almost all $(\omega, t) \in \Omega \times [0, T]$, we see that the right hand side of the last inequality converges to 0 as $m \to \infty$. Therefore

$$\tilde{\mathbf{u}}^m(s) \to \mathbf{u} \text{ in } V \text{ for almost all } (\omega, t) \in \Omega \times [0, T]. \qquad (45)$$

Furthermore, owing to (43) and the dominated convergence theorem we can state that

$$\tilde{\mathbf{u}}^m \to \mathbf{u} \text{ in } L^2(\Omega \times [0, T]; V). \qquad (46)$$

Next, it is not difficult to see that $\tilde{\mathbf{u}}^m$ satisfies the following equations

$$\tilde{\mathbf{u}}^m(t) + \kappa_1 \int_0^t \mathcal{A}\tilde{\mathbf{u}}^m(s)ds + \kappa_0 \int_0^t \Pi_m \mathbf{A}(s)ds + \int_0^t \Pi_m B(s) = \Pi_m \xi + \int_0^t \int_Z \Pi_m \Sigma(s, z)\widetilde{\eta}(dz, ds).$$

Let X^m be the stochastic processes defined by $X^m = \mathbf{u}^m - \tilde{\mathbf{u}}^m$. From the last line and (16) we obtain

$$
\begin{aligned}
X^m(t) + \kappa_0 \int_0^t [\mathcal{A}_p \mathbf{u}^m(s) - \Pi_m \mathbf{A}(s)]ds + \int_0^t [B_m(\mathbf{u}^m(s), \mathbf{u}^m(s)) - \Pi_m \mathbf{B}(s)]ds \\
= \int_0^t \int_Z [\sigma_m(s, \mathbf{u}^m(s), z) - \Pi_m \Sigma(s, z)]\widetilde{\eta}(dz, ds) - \kappa_1 \int_0^t \mathcal{A}X^m(s)ds.
\end{aligned}
$$

Applying Itô's formula to the function $\Phi(x) = |x|^2$ and $X^m(t)$ yields

$$
\begin{aligned}
|X^m(t)|^2 + 2\kappa_1 \int_0^t \langle \mathcal{A}X^m(s), X^m(s)\rangle ds + 2\kappa_0 \int_0^t \langle \mathcal{A}_p \mathbf{u}^m(s) - \Pi_m \mathbf{A}(s), X^m(s)\rangle ds \\
= 2\int_0^t \langle \Pi_m \mathbf{B}(s) - B_m(\mathbf{u}^m(s), \mathbf{u}^m(s)), X^m(s)\rangle ds + \int_0^t \int_Z \Psi(s, z)\eta(dz, ds) \\
+ 2\int_0^t \int_Z (\sigma_m(s, \mathbf{u}^m(s), z) - \Pi_m \Sigma(s, z), X^m(s))\widetilde{\eta}(dz, ds),
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi(s, z) &= |X^m(s-) + \sigma_m(s, \mathbf{u}^m(s), z) - \Pi_m \Sigma(s, z)|^2 - |X^m(s-)|^2 \\
&\quad - 2(\sigma_m(s, \mathbf{u}^m(s), z) - \Pi_m \Sigma(s, z), X^m(s-)) \\
&= |\sigma_m(s, \mathbf{u}^m(s), z) - \Pi_m \Sigma(s, z)|^2.
\end{aligned}
$$

Let $r(t)$ be the real valued stochastic process defined by $r(t) = K_1 t + \frac{C_0^2}{4\kappa_1} \int_0^t \|\mathbf{u}(s)\|_2^2 ds$. Applying Itô's formula to $e^{-r(t)}|X^m(t)|^2$ leads to

$$\begin{aligned}
&e^{-r(t)}|X^m(t)|^2 + 2\kappa_1 \int_0^t e^{-r(s)}\|X^m(s)\|_2^2 ds + 2\kappa_0 \int_0^t e^{-r(s)}\langle \mathcal{A}_p \mathbf{u}^m(s) - \Pi_m \mathbf{A}(s), X^m(s)\rangle ds \\
&= 2\int_0^t e^{-r(s)}\langle \Pi_m \mathbf{B}(s) - B_m(\mathbf{u}^m(s), \mathbf{u}^m(s)), X^m(s)\rangle ds - \frac{C_0^2}{4\kappa_1}\int_0^t e^{-r(s)}|X^m(s)|^2\|X^m(s)\|_2^2 ds \\
&\quad - K_1 \int_0^t e^{-r(s)}|X^m(s)|^2 ds + \int_0^t \int_Z e^{-r(s)}|\sigma_m(s, \mathbf{u}^m(s), z) - \Pi_m \Sigma(s,z)|^2 \eta(dz, ds) \\
&\quad + 2\int_0^t e^{-r(s)} \int_Z (\sigma_m(s, \mathbf{u}^m(s), z) - \Pi_m \Sigma(s,z), X^m(s))\widetilde{\eta}(dz, ds).
\end{aligned} \tag{47}$$

Let us study each term of (47). For the nonlinear term involving B_m and $\mathbf{B}$ we have that

$$B_m(\tilde{\mathbf{u}}^m, \tilde{\mathbf{u}}^m) - B_m(\mathbf{u}^m, \mathbf{u}^m) = B_m(X^m, \tilde{\mathbf{u}}^m) + B_m(\mathbf{u}^m, X^m). \tag{48}$$

Out of this and (8) we obtain that

$$\begin{aligned}
\langle \Pi_m \mathbf{B}(s) - B_m(\mathbf{u}^m(s), \mathbf{u}^m(s)), X^m(s)\rangle &= \langle \Pi_m \mathbf{B}(s) - B_m(\tilde{\mathbf{u}}^m(s), \tilde{\mathbf{u}}^m(s)), X^m(s)\rangle \\
&\quad + \langle B_m(X^m(s), \tilde{\mathbf{u}}^m(s)), X^m(s)\rangle,
\end{aligned}$$

which along with (9) and Young's inequality imply that

$$\begin{aligned}
\langle \Pi_m \mathbf{B}(s) - B_m(\mathbf{u}^m(s), \mathbf{u}^m(s)), X^m(s)\rangle &\le \langle \Pi_m \mathbf{B}(s) - B_m(\tilde{\mathbf{u}}^m(s), \tilde{\mathbf{u}}^m(s)), X^m(s)\rangle \\
&\quad \frac{C_0^2}{4\kappa_1}|X^m(s)|^2\|\tilde{\mathbf{u}}^m(s)\|_2^2 + \kappa_1\|X^m(s)\|_2^2.
\end{aligned} \tag{49}$$

Next, we have

$$\begin{aligned}
\langle \mathcal{A}_p \mathbf{u}^m(s) - \mathbf{A}(s), X^m(s)\rangle &= \langle \mathcal{A}_p \mathbf{u}^m(s) - \mathcal{A}_p \tilde{\mathbf{u}}^m(s), X^m(s)\rangle \\
&\quad + \langle \mathcal{A}_p \tilde{\mathbf{u}}^m(s) - \mathbf{A}(s), X^m(s)\rangle.
\end{aligned} \tag{50}$$

Invoking the item (ii) of Lemma 2.4 we see that

$$\langle \mathcal{A}_p \mathbf{u}^m(s) - \mathcal{A}_p \tilde{\mathbf{u}}^m(s), X^m(s)\rangle \ge 0. \tag{51}$$

Setting $S = |\sigma_m(s, \mathbf{u}^m(s), z) - \Pi_m \Sigma(s,z)|^2$ we see that

$$\begin{aligned}
S = {} & |\Pi_m[\sigma(s, \mathbf{u}^m(s), z) - \sigma(s, \mathbf{u}(s), z)]|^2 - |\Pi_m[\sigma(s, \mathbf{u}(s), z) - \Sigma(s,z)]|^2 \\
& 2\left(\Pi_m[\sigma(s, \mathbf{u}^m(s), z) - \Sigma(s,z)], \Pi_m[\sigma(s, \mathbf{u}(s), z) - \Sigma(s,z)]\right).
\end{aligned}$$

Owing to item (1) of Condition 1 we have that

$$\begin{aligned}
S \le {} & \ell_2|X^m(s)|^2 + \ell_2|\tilde{\mathbf{u}}^m(s) - \mathbf{u}(s)|^2 - |\Pi_m[\sigma(s, \mathbf{u}(s), z) - \Sigma(s,z)]|^2 \\
& 2\left(\Pi_m[\sigma(s, \mathbf{u}^m(s), z) - \Sigma(s,z)], \Pi_m[\sigma(s, \mathbf{u}(s), z) - \Sigma(s,z)]\right).
\end{aligned} \tag{52}$$

Inserting (49), (50), (51) and (52) into (47), replacing t by τ_M and taking the mathematical expectation lead to

$$\mathbb{E}e^{-r(\tau_M)}|X^m(\tau_M)|^2 + \mathbb{E}\int_0^{\tau_M}\int_Z e^{-r(s)}|\Pi_m[\sigma(s,\mathbf{u}(s),z)-\Sigma(s,z)]|^2\eta(dz,ds)$$
$$\le -\kappa_1\mathbb{E}\int_0^{\tau_M} e^{-r(s)}\|X^m(s)\|_2^2 ds + 2\kappa_0\mathbb{E}\int_0^{\tau_M} e^{-r(s)}\langle \Pi_m\mathbf{A}(s)-\mathcal{A}_p\tilde{\mathbf{u}}^m(s), X^m(s)\rangle ds$$
$$+ 2\mathbb{E}\int_0^{\tau_M}\int_Z e^{-r(s)}\left(\Pi_m[\sigma(s,\mathbf{u}^m(s),z)-\Sigma(s,z)], \Pi_m[\sigma(s,\mathbf{u}(s),z)-\Sigma(s,z)]\right)\eta(dz,ds) \tag{53}$$
$$+\mathbb{E}\int_0^{\tau_M} e^{-r(s)}\langle \Pi_m\mathbf{B}(s)-B_m(\tilde{\mathbf{u}}^m(s),\tilde{\mathbf{u}}^m(s)), X^m(s)\rangle ds$$
$$+K_1\mathbb{E}\int_0^{\tau_M}|\tilde{\mathbf{u}}^m(s)-\mathbf{u}(s)|^2 e^{-r(s)}ds.$$

Now we will show that the last four terms of the right hand side of (53) will tend to 0 as $m\to 0$. Thanks to (41) we have

$$\mathbb{E}\int_0^T 1_{[0,\tau_M]}(s)e^{-r(s)}|\tilde{\mathbf{u}}^m(s)-\mathbf{u}(s)|^2 ds \to 0. \tag{54}$$

Owing to (48) and (9) we see that

$$\left\|1_{[0,\tau_M]}(t)e^{-r(t)}[B(\tilde{\mathbf{u}}^m(t),\tilde{\mathbf{u}}^m(t))-B(\mathbf{u}(t),\mathbf{u}(t))]\right\|_{V^*} \le 1_{[0,\tau_M]}(t)C_0\|\tilde{\mathbf{u}}^m(t)\|_1\;|\tilde{\mathbf{u}}^m(t)-\mathbf{u}(t)|$$
$$+1_{[0,\tau_M]}(t)C_0|\mathbf{u}(t)|\;\|\tilde{\mathbf{u}}^m(t)-\mathbf{u}(t)\|_2, \tag{55}$$

which with (45) implies that

$$\left\|1_{[0,\tau_M]}(t)e^{-r(t)}[B(\tilde{\mathbf{u}}^m(t),\tilde{\mathbf{u}}^m(t))-B(\mathbf{u}(t),\mathbf{u}(t))]\right\|_{V^*}\to 0 \text{ a.e. } (\omega,t)\in\Omega\times[0,T],$$

as $m\to\infty$. Furthermore, owing to (43) and (44) we see from (55) that

$$\left\|1_{[0,\tau_M]}(t)e^{-r(t)}[B(\tilde{\mathbf{u}}^m(t),\tilde{\mathbf{u}}^m(t))-B(\mathbf{u}(t),\mathbf{u}(t))]\right\|_{V^*}\le 2C_0M\|A\|^{\frac{1}{2}}|\mathbf{u}(t)|. \tag{56}$$

Note that $|\mathbf{u}(t)|$ is bounded in $L^2(\Omega\times[0,T],\mathbb{R})$. Thus, the Dominated Convergence Theorem implies that

$$\left\|1_{[0,\tau_M]}(t)e^{-r(t)}[B(\tilde{\mathbf{u}}^m(t),\tilde{\mathbf{u}}^m(t))-B(\mathbf{u}(t),\mathbf{u}(t))]\right\|_{V^*}\to 0 \text{ in } L^2(\Omega\times[0,T];\mathbb{R}) \tag{57}$$

By the convergences (31) and (46) we have

$$\tilde{\mathbf{u}}^m-\mathbf{u}^m\rightharpoonup 0 \text{ in } L^2(\Omega;L^2(0,T;V)). \tag{58}$$

We derive from this, (56) and (57) that

$$E\int_0^{\tau_M} e^{-r(s)}\langle B(\tilde{\mathbf{u}}^m(s),\tilde{\mathbf{u}}^m(s)) - B(\mathbf{u}(s),\mathbf{u}(s)),\tilde{\mathbf{u}}^m(s)-\mathbf{u}^m(s)\rangle ds \to 0$$

as $m\to\infty$. Hence

$$\begin{aligned}
&\lim_{m\to\infty} E\int_0^{\tau_M} e^{-r(s)}\langle B(\tilde{\mathbf{u}}^m(s),\tilde{\mathbf{u}}^m(s)) - B^*(s),\tilde{\mathbf{u}}^m(s)-\mathbf{u}^m(s)\rangle ds\\
&= \lim_{m\to\infty} E\int_0^{\tau_M} e^{-r(s)}\langle B(\tilde{\mathbf{u}}^m(s),\tilde{\mathbf{u}}^m(s)) - B(\mathbf{u}(s),\mathbf{u}(s)),\tilde{\mathbf{u}}^m(s)-\mathbf{u}^m(s)\rangle ds\\
&\quad + \lim_{m\to\infty} E\int_0^{\tau_M} e^{-r(s)}\langle B(\mathbf{u},\mathbf{u}) - B^*(s),\tilde{\mathbf{u}}^m(s)-\mathbf{u}^m(s)\rangle ds\\
&= 0.
\end{aligned}$$

Since $\Pi_m\circ\Pi_m=\Pi_m$ and $\|\Pi_m\|\le 1$, it follows that $1_{[0,\tau_M]}e^{-r(s)}\Pi_m[\sigma(s,\mathbf{u}(s),z)-\Sigma(s,z)]$ is bounded in $L^2(\Omega\times[0,T];L^2(Z,\nu;H))$. Therefore we see from (34) that

$$2\mathbb{E}\int_0^{\tau_M}\int_Z e^{-r(s)}\left(\Pi_m[\sigma(s,\mathbf{u}^m(s),z)-\Sigma(s,z)],\Pi_m[\sigma(s,\mathbf{u}(s),z)-\Sigma(s,z)]\right)\eta(dz,ds)\to 0$$

as $m\to\infty$.

Now it is not difficult to check that

$$\begin{aligned}
\mathbb{E}\int_0^t e^{-r(s)}\langle\Pi_m[\mathbf{A}(s)-\mathcal{A}_p\tilde{\mathbf{u}}^m(s)],X^m(s)\rangle ds &= \mathbb{E}\int_0^t e^{-r(s)}\langle\Pi_m[\mathbf{A}(s)-\mathcal{A}_p\mathbf{u}(s)],X^m(s)\rangle ds\\
&\quad + \mathbb{E}\int_0^t e^{-r(s)}\langle\Pi_m[\mathcal{A}_p\mathbf{u}(s)-\mathcal{A}\tilde{\mathbf{u}}^m(s)],X^m(s)\rangle ds.
\end{aligned}$$

Since $\langle\Pi_m\mathbf{v},X^m\rangle=\langle\mathbf{v},X^m\rangle$ for any $\mathbf{v}\in V^*$ and $e^{-r(s)}(\mathbf{A}(s)-\mathcal{A}_p\mathbf{u}(s))$ is a bounded element of $L^2(\Omega\times[0,T];V^*)$, we derive from (58) that the first term of the right hand side of the above equation tends to zero as $m\to\infty$. Owing to item (i) of Lemma 2.4, the strong convergence (46) and the weak convergence (58) we see that the second term of the right hand side converges to zero as well. Thus, we have just proved that

$$\mathbb{E}\int_0^t e^{-r(s)}\langle\Pi_m\mathbf{A}(s)-\mathcal{A}\tilde{\mathbf{u}}^m(s),X^m(s)\rangle ds\to 0,$$

as $m\to\infty$. With this we have just shown that the last four terms of (53) converges to zero as $m\to\infty$. Then, we can conclude that

$$\mathbb{E}e^{-r(\tau_M)}|X^m(\tau_M)|^2+\kappa_1\mathbb{E}\int_0^{\tau_M} e^{-r(s)}\|X^m(s)\|_2^2 ds\to 0, \tag{59}$$

$$\mathbb{E}\int_0^{\tau_M}\int_Z e^{-r(s)}|\Pi_m[\sigma(s,\mathbf{u}(s),z)-\Sigma(s,z)]|^2\eta(dz,ds)\to 0, \tag{60}$$

as $m\to\infty$. We easily terminate the proof of the lemma by plugging equations (42) and (46) into (59). □

Now, we give the promised proof of the proposition.

Proof of Proposition 3.6. First note that for any $\mathbf{w}\in V$

$$\begin{aligned} S &= \langle B(\mathbf{u}^m,\mathbf{u}^m)-B(\mathbf{u},\mathbf{u}),\mathbf{w}\rangle \\ &= \langle B(\mathbf{u}^m-\mathbf{u},\mathbf{u}^m),\mathbf{w}\rangle+\langle B(\mathbf{u},\mathbf{u}^m-\mathbf{u}),\mathbf{w}\rangle. \end{aligned} \tag{61}$$

We also have the following equations

$$\begin{aligned} \langle B(\mathbf{u}^m-\mathbf{u},\mathbf{u}^m),\mathbf{w}\rangle &= \langle B(\mathbf{u}^m,\mathbf{u}^m),\mathbf{w}\rangle-\langle B(\mathbf{u},\mathbf{u}^m),\mathbf{w}\rangle, \\ \langle B(\mathbf{u}^m,\mathbf{u}-\mathbf{u}^m),\mathbf{w}\rangle &= \langle B(\mathbf{u}^m,\mathbf{u}),\mathbf{w}\rangle-\langle B(\mathbf{u}^m,\mathbf{u}^m),\mathbf{w}\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} S = \langle B(\mathbf{u},\mathbf{u}-\mathbf{u}^m),\mathbf{w}\rangle-\langle B(\mathbf{u}^m,\mathbf{u}-\mathbf{u}^m),\mathbf{w}\rangle+\langle B(\mathbf{u}^m,\mathbf{u}),\mathbf{w}\rangle \\ -\langle B(\mathbf{u},\mathbf{u}^m),\mathbf{w}\rangle. \end{aligned} \tag{62}$$

The operator

$$\begin{aligned} B_{\mathbf{a},\cdot}&: V\to V^* \\ &\mathbf{v}\mapsto B_{\mathbf{a},\cdot}(\mathbf{v})=B(\mathbf{a},\mathbf{v}) \end{aligned}$$

is linear continuous for any fixed $a\in V$. Due to this fact and (31), it is true that

$$B(\mathbf{u},\mathbf{u}^m)\rightharpoonup B(\mathbf{u},\mathbf{u}) \text{ weakly in } L^2(\Omega\times[0,T];V^*). \tag{63}$$

By a similar argument, we also prove the following convergence

$$B(\mathbf{u}^m,\mathbf{u})\rightharpoonup B(\mathbf{u},\mathbf{u}) \text{ weakly in } L^2(\Omega\times[0,T];V^*). \tag{64}$$

Now let $\mathbf{w}$ be an element of $L^\infty(\Omega\times[0,T];V)$. We deduce from the property (9) that

$$\begin{aligned} &\left|\mathbb{E}\int_0^T 1_{[0,\tau_M]}\langle B(\mathbf{u}(s),\mathbf{u}(s)-\mathbf{u}^m(s)),\mathbf{w}(s)\rangle-\langle B(\mathbf{u}^m(s),\mathbf{u}(s)-\mathbf{u}^m(s)),\mathbf{w}(s)\rangle ds\right| \\ &\le C\mathbb{E}\int_0^{\tau_M}|\mathbf{u}(s)|\|\mathbf{u}^m(s)-\mathbf{u}(s)\|_2 ds+C\mathbb{E}\int_0^{\tau_M}|\mathbf{u}^m(s)|\,\|\mathbf{u}^m(s)-\mathbf{u}(s)\|_2 ds, \end{aligned}$$

from which and (39) we derive that

$$\lim_{m\to\infty}\mathbb{E}\int_0^T 1_{[0,\tau_M]}\langle B(\mathbf{u}(s),\mathbf{u}(s)-\mathbf{u}^m(s)),\mathbf{w}(s)\rangle-\langle B(\mathbf{u}^m(s),\mathbf{u}(s)-\mathbf{u}^m(s)),\mathbf{w}(s)\rangle ds=0 \tag{65}$$

Since $\tau_M \nearrow T$ almost surely and $L^\infty(\Omega \times [0,T]; V)$ is dense in $L^2(\Omega \times [0,T]; V)$, we deduce from (62)–(65) that the identity (36) holds.

Next, thanks to the property (10) of $\mathcal{A}_p$ we see that

$$\mathbb{E}\int_0^{\tau_M} \|\mathcal{A}_p(\mathbf{u}^m(s)) - \mathcal{A}_p(\mathbf{u})\|_{V^*}^2 ds \le C\mathbb{E}\int_0^{\tau_M} \|\mathbf{u}^m - \mathbf{u}\|_2^2 ds.$$

Owing to (39) and the fact that $\tau_M \nearrow T$ almost surely as $M \to \infty$, we obtain the equation (37).

The identity (38) easily follows from (60). This completes the proof of the Proposition 3.6. $\square$

4. Pathwise uniqueness and convergence of the whole sequence of Galerkin approximation

In this section we show the pathwise uniqueness of the solution and some (strong) convergences of the Galerkin approximate solution to the exact solution of (1). For a $\mathcal{F}_0$-measurable and square integrable H-valued random variable ξ, we denote by $\mathbf{u}(\cdot,\xi)$ the solution to (1) with initial data ξ.

Theorem 4.1. *Let ξ_1 and ξ_2 be two $\mathcal{F}_0$-measurable and square integrable H-valued random variables. Let $\mathbf{u}_1(\cdot,\xi_1)$ and $\mathbf{u}_2(\cdot,\xi_2)$ be the strong solutions to (6) corresponding to ξ_1 and ξ_2, respectively. Then, for any $t \in [0,T]$ there exists a constant $C > 0$ such that*

$$\mathbb{E}\left(e^{-\frac{C_0^2}{\kappa_1}\int_0^t \|\mathbf{u}_1(s,\xi_1)\|_2^2 ds}|\mathbf{u}_1(t,\xi_1) - \mathbf{u}_2(t,\xi_2)|^2\right) \le C\mathbb{E}|\xi_1 - \xi_2|^2.$$

Moreover, if $\xi_1 = \xi_2$ almost surely, then for any $t \in [0,T]$

$$\mathbb{P}(\mathbf{u}_1(t,\xi_1) = \mathbf{u}_2(t,\xi_1)) = 1.$$

Proof. Let $\mathbf{u}_1$ (resp., $\mathbf{u}_2$) be a strong solution to (6) with initial condition ξ_1 (resp., ξ_2). Let $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$ and $\xi = \xi_1 - \xi_2$. It is not hard to see that

$$\begin{aligned}
&\mathbf{w}(t) + \kappa_1 \int_0^t A\mathbf{w}(s)ds + \kappa_0 \int_0^t (\mathcal{A}_p\mathbf{u}_1(s) - \mathcal{A}_p\mathbf{u}_2(s))\, ds \\
&= \xi + \int_0^t \int_Z (\sigma(s,\mathbf{u}_1(s),z) - \sigma(s,\mathbf{u}_2(s),z))\, \widetilde{\eta}(dz,ds) \\
&\quad + \int_0^t (B(\mathbf{u}_2(s),\mathbf{u}_2(s)) - B(\mathbf{u}_1(s),\mathbf{u}_1(s)))\, ds.
\end{aligned}$$

Applying Itô's formula to the function $\Phi(x) = |x|^2$ and $\mathbf{w}(t)$ implies that

$$\begin{aligned}
&|\mathbf{w}(t)|^2 + 2\kappa_1 \int_0^t \|\mathbf{w}(s)\|_2^2 ds + 2\kappa_0 \int_0^t \langle \mathcal{A}_p\mathbf{u}_1(s) - \mathcal{A}_p\mathbf{u}_2(s), \mathbf{w}(s)\rangle ds \\
&= |\xi|^2 + 2\int_0^t \int_Z (\sigma(s,\mathbf{u}_1(s),z) - \sigma(s,\mathbf{u}_2(s),z), \mathbf{w}(s))\, \widetilde{\eta}(dz,ds)
\end{aligned}$$

$$
\begin{aligned}
&- 2 \int_0^t \langle B(\mathbf{u}_1(s), \mathbf{u}_1(s)) - B(\mathbf{u}_2(s), \mathbf{u}_2(s)), \mathbf{w}(s) \rangle ds \\
&+ \int_0^t \int_Z |\sigma(s, \mathbf{u}_1(s), z) - \sigma(s, \mathbf{u}_2(s), z)|^2 \eta(dz, ds).
\end{aligned}
$$

Next we introduce the real valued process

$$
\rho(t) = e^{-\frac{C_0^2}{\kappa_1} \int_0^t \|\mathbf{u}_1(s)\|_2^2 ds}.
$$

Now we apply Itô's formula to $\rho(t)|\mathbf{w}(t)|^2$ and we get

$$
\begin{aligned}
&\rho(t)|\mathbf{w}(t)|^2 + 2\kappa_1 \int_0^t \rho(s)\|\mathbf{w}(s)\|_2^2 ds + 2\kappa_0 \int_0^t \rho(s)\langle \mathcal{A}_p \mathbf{u}_1(s) - \mathcal{A}_p \mathbf{u}_2(s), \mathbf{w}(s) \rangle ds \\
&= -2 \int_0^t \rho(s)\langle B(\mathbf{u}_1(s), \mathbf{u}_1(s)) - B(\mathbf{u}_2(s), \mathbf{u}_2(s)), \mathbf{w}(s) \rangle ds - \frac{C_0^2}{\kappa_1} \int_0^t \rho(s)|\mathbf{w}(s)|^2 \|\mathbf{u}_1(s)\|_2^2 ds \\
&\quad + 2 \int_0^t \int_Z \rho(s) \left(\sigma(s, \mathbf{u}_1(s), z) - \sigma(s, \mathbf{u}_2(s), z), \mathbf{w}(s) \right) \widetilde{\eta}(dz, ds) \\
&\quad + \int_0^t \int_Z \rho(s)|\sigma(s, \mathbf{u}_1(s), z) - \sigma(s, \mathbf{u}_2(s), z)|^2 \eta(dz, ds) + |\xi|^2.
\end{aligned}
$$

By making use of (8), (9), (11), (61) and Young's inequality with $\varepsilon = \kappa_1$ in the above estimate and by taking the mathematical expectation to both sides of the resulting estimate yield

$$
\mathbb{E}\rho(t)|\mathbf{w}(t)|^2 + 2\kappa_1 \mathbb{E} \int_0^t \rho(s)\|\mathbf{w}(s)\|_2^2 ds \leq \mathbb{E} \int_0^t \int_Z \rho(s)|\sigma(s, \mathbf{u}_1(s), z) - \sigma(s, \mathbf{u}_2(s), z)|^2 \nu(dz) ds + |\xi|^2.
$$

Using item (1) of Condition 1 yields that

$$
\mathbb{E}\rho(t)|\mathbf{w}(t)|^2 \leq \mathbb{E}|\xi|^2 + \mathbb{E} \int_0^t \rho(s)|\mathbf{w}(s)|^2 ds,
$$

from which and Gronwall's lemma we deduce the existence of a constant $C > 0$ such that

$$
\mathbb{E}\rho(t)|\mathbf{w}(t)|^2 \leq C\mathbb{E}|\xi|^2,
$$

for any $t \in [0, T]$, which completes the first part of the theorem.

Since $\rho(t)$ is bounded and positive $\mathbb{P}$-a.s, we conclude easily the second part of the theorem from the last estimate. □

Next we will show that the whole sequence of solutions to the Galerkin approximation system (16) converges in mean square to the exact strong solution of (1).

Theorem 4.2. *The whole sequence of Galerkin approximation* $\{\mathbf{u}^m : m \in \mathbb{N}\}$ *defined by* (16) *satisfies*

$$\lim_{m\to\infty} \mathbb{E}|\mathbf{u}^m(T-) - \mathbf{u}(T-)|^2 = 0, \tag{66}$$

$$\lim_{m\to\infty} \mathbb{E}\int_0^T \|\mathbf{u}^m(s) - \mathbf{u}(s)\|_2^2 ds = 0. \tag{67}$$

The main ingredient of the proof of this result is the following lemma, its proof follows a very small modification of the proof of [11, Proposition B.3].

Lemma 4.3. *Let* $\{Q_m; m \geq 1\} \subset L^2(\Omega \times [0,T]; \mathbb{R})$ *be a sequence of càdlàg real-valued process, and let* $\{T_M; M \geq 1\}$ *be a sequence of* $\mathcal{F}^t$*-stopping times such that* T_M *is increasing to* T, $\sup_{m\geq 1} \mathbb{E}|Q_m(T)|^2 < \infty$, *and* $\lim_{m\to\infty} \mathbb{E}|Q_m(T_M)| = 0$ *for all* $M \geq 1$. *Then* $\lim_{m\to\infty} \mathbb{E}|Q_m(T-)| = 0$.

Proof of Theorem 4.2. It follows from Lemma 3.7 that

$$\lim_{m\to\infty} \mathbb{E}\int_0^{\tau_M} \|\mathbf{u}^m(t) - \mathbf{u}(t)\|_2^2 dt = 0, \tag{68}$$

and

$$\lim_{m\to\infty} \mathbb{E}|\mathbf{u}^m(\tau_M) - \mathbf{u}(\tau_M)|^2 = 0, \tag{69}$$

for any $M \geq 1$. So by applying the preceding lemma to $Q_m(t) = |\mathbf{u}^m(t) - \mathbf{u}(t)|^2$, $T_M = \tau_M$ and taking into account (69), the estimates in Proposition 3.4 and the uniqueness of $\mathbf{u}$, we see that the whole sequence $\mathbf{u}^m$ defined by (16) satisfies (66). To prove (67) we need an extra estimate for the sequence $\{\mathbf{u}^m : m \in \mathbb{N}\}$. Since

$$\int_0^t \int_Z \Psi(s,z)\tilde{\eta}(dz,ds) = \int_0^t \int_Z \Psi(s,z)\eta(ds,dz) - \int_0^t \int_Z \Psi(s,z)\nu(dz)ds,$$

and $|\mathbf{v}|^2 + 2(\mathbf{v},\mathbf{w}) = |\mathbf{v}+\mathbf{w}|^2 - |\mathbf{w}|^2$ we deduce from (18) that

$$\begin{aligned}4\kappa_1^2\mathbb{E}\left(\int_0^t \|\mathbf{u}^m(s)\|_2^2 ds\right)^2 &\leq 2\mathbb{E}|\xi|^4 + 4\mathbb{E}\left(\int_0^t \int_Z |\sigma(s,\mathbf{u}^m(s),z)|\nu(dz)ds\right)^2 \\ &\quad + 4\mathbb{E}\left(\int_0^t \int_Z \Big[|\sigma(s,\mathbf{u}^m(s),z) + \mathbf{u}^m(s-)|^2 - |\mathbf{u}^m(s-)|^2\Big]\tilde{\eta}(dz,ds)\right)^2.\end{aligned}$$

Using item (1) of Condition 1 and the estimate in Proposition 3.4 we infer from the last inequality that

$$\begin{aligned}4\kappa_1^2\mathbb{E}\left(\int_0^t \|\mathbf{u}^m(s)\|_2^2 ds\right)^2 &\leq 4\mathbb{E}\left(\int_0^t \int_Z \Big[|\sigma(s,\mathbf{u}^m(s),z) + \mathbf{u}^m(s-)|^2 - |\mathbf{u}^m(s-)|^2\Big]\tilde{\eta}(dz,ds)\right)^2 \\ &\quad + 2\mathbb{E}|\xi|^4 + 4\ell_0^2 T^2 + 4C\ell_1^2 T^2(\mathbb{E}|\xi|^4 + 1).\end{aligned}$$

Now invoking [52, Theorem 4.14] we see that

$$4\kappa_1^2\mathbb{E}\Big(\int_0^t \|\mathbf{u}^m(s)\|_2^2 ds\Big)^2 \leq 4\mathbb{E}\int_0^t\int_Z \Big[\Big||\sigma(s,\mathbf{u}^m(s),z)+\mathbf{u}^m(s-)|^2-|\mathbf{u}^m(s-)|^2\Big|^2\Big]\nu(dz)ds$$
$$+2\mathbb{E}|\xi|^4+4\ell_0^2T^2+4C\ell_1^2T^2(\mathbb{E}|\xi|^4+1),$$

from which with item (3) of Condition 1 and Proposition 3.4 we derive that

$$4\kappa_1^2\mathbb{E}\Big(\int_0^t \|\mathbf{u}^m(s)\|_2^2 ds\Big)^2 \leq \big(2+4C\ell_1^2T^2+C\ell_3T+CT\big)\,\mathbb{E}|\xi|^4+4\ell_0^2T^2+4C\ell_1^2T^2+CT.$$

This also implies that

$$4\kappa_1^2\mathbb{E}\Big(\int_0^t \|\mathbf{u}(s)\|_2^2 ds\Big)^2 \leq \big(2+4C\ell_1^2T^2+C\ell_3T+CT\big)\,\mathbb{E}|\xi|^4+4\ell_0^2T^2+4C\ell_1^2T^2+CT.$$

We see easily from the last two estimates and (68) that $Q_m(t)=\int_0^t\|\mathbf{u}^m(s)-\mathbf{u}(s)\|_2^2ds$, $T_M=\tau_M$ satisfy the hypotheses of the above lemma, therefore we can deduce that (67) holds. This ends the proof of Theorem 4.2. □

5. Existence and ergodicity of invariant measure

In this section we are interested in the study of some qualitative properties of the solution of (1). We will mainly analyze the Markov, Fellerian properties of the solution. We will also derive the existence of ergodic invariant measures. For these goals we will assume that the noise coefficient σ is time independent, i.e.,

$$\sigma(t,\mathbf{v},z)=\sigma(\mathbf{v},z),\ \text{ for any } t\geq 0, \mathbf{v}\in H, z\in Z.$$

To start with our investigation we denote by $\mathbf{u}(t;\xi)$ the solution of (1) with initial condition $\xi\in H$, and by $C_b(H)$ we describe the space of all continuous real-valued functionals defined on H. Next, we define the family of linear mappings $\{\mathcal{P}_t, t\geq 0\}$ ($\mathcal{P}_t$ for short) defined on $C_b(H)$ by

$$\mathcal{P}_t\phi(\xi)=\mathbb{E}\phi(\mathbf{u}(t;\xi)),$$

for any $\phi\in C_b(H)$, $\xi\in H$, and $t\geq 0$. Some properties of the solution $\mathbf{u}(t;\xi)$ and the semigroup $\mathcal{P}_t$ are given in the following results.

Theorem 5.1. *If σ is time independent and satisfies Condition 1, then the solution $\mathbf{u}(t;\xi)$ defines a Markov process and $\mathcal{P}_t$ defines a semigroup satisfying $\mathcal{P}_{t+s}=\mathcal{P}_t\mathcal{P}_s$ for any $t,s\geq 0$. Moreover, $\mathcal{P}_t$ has the Feller property, i.e, the semigroup $\mathcal{P}_t$ satisfies*

$$\mathcal{P}_t\left(C_b(H)\right)\subset C_b(H),$$

for any $t\geq 0$.

Remark 5.2. Note that all, but the property $\mathcal{P}_{t+s}=\mathcal{P}_t\mathcal{P}_s$, properties in the above theorem are still true even if σ is time-dependent.

Before we proceed to the proof of these statements let us give an auxiliary result.

Lemma 5.3. *Let ξ_1, ξ_2 be two distinct initial conditions satisfying the assumption of Theorem 4.1, and $\mathbf{u}(t;\xi_1)$ and $\mathbf{u}(t;\xi_2)$ be two solutions of (1) associated to them. Let*

$$\tau_R^{\xi} = \inf\{t : |\mathbf{u}(t;\xi)| > R\}, \forall R > 0, \xi \in H. \tag{70}$$

Let us also set $\tau_R^{\xi_1,\xi_2} = \tau_R^{\xi_1} \wedge \tau_R^{\xi_2}$, $t_R = t \wedge \tau_R^{\xi_1,\xi_2}$ and $\mathbf{w}(t) = \mathbf{u}(t;\xi_1) - \mathbf{u}(t;\xi_2)$, $t \in [0,\infty)$. Then, for any $R > 0$ and $t \in [0,\infty)$ there exists a positive constant C such that

$$\mathbb{E}|\mathbf{w}(t_R)|^2 \leq C\mathbb{E}|\xi_1 - \xi_2|^2. \tag{71}$$

Proof of Lemma 5.3. As in the proof of Theorem 4.1 we can check by making use of Itô's formula that $|\mathbf{w}(t_R)|^2$ satisfies

$$\begin{aligned}
|\mathbf{w}(t_R)|^2 + 2\kappa_1 \int_0^{t_R} \|\mathbf{u}(s;\xi)\|_2^2 ds \leq |\xi_1 - \xi_2|^2 + 2\int_0^{t_R} \langle B(\mathbf{w}(s), \mathbf{u}(s;\xi_1), \mathbf{w}(s)\rangle ds \\
+ \int_0^{t_R}\int_Z |\sigma(\mathbf{u}(s;\xi_1), z) - \sigma(\mathbf{u}(s;\xi_2), z)|^2 \eta(dz, ds) \\
+ 2\int_0^{t_R}\int_Z (\sigma(\mathbf{u}(s;\xi_1), z) - \sigma(\mathbf{u}(s;\xi_2), z), \mathbf{w}(s-))\, \tilde{\eta}(dz, ds).
\end{aligned}$$

Using the skew-symmetricity of B and Hölder's inequality we derive from the last inequality that

$$\begin{aligned}
|\mathbf{w}(t_R)|^2 + 2\kappa_1 \int_0^{t_R} \|\mathbf{u}(s;\xi)\|_2^2 ds \leq |\xi_1 - \xi_2|^2 + 2C\int_0^{t_R} \Big(|\mathbf{w}(s) \cdot \nabla \mathbf{w}(s)| \times |\mathbf{u}(s;\xi_1)| \Big) ds \\
+ \int_0^{t_R}\int_Z |\sigma(\mathbf{u}(s;\xi_1), z) - \sigma(\mathbf{u}(s;\xi_2), z)|^2 \eta(dz, ds) \\
+ 2\int_0^{t_R}\int_Z (\sigma(\mathbf{u}(s;\xi_1), z) - \sigma(\mathbf{u}(s;\xi_2), z), \mathbf{w}(s-))\, \tilde{\eta}(dz, ds).
\end{aligned}$$

Owing to Hölder's inequality and the fact $|\mathbf{u}(s;\xi_1)| \geq R$ on $[0, t_R]$ we infer the existence of a constant $C_R = C(R) > 0$ such that

$$\begin{aligned}
|\mathbf{w}(t_R)|^2 + 2\kappa_1 \int_0^{t_R} \|\mathbf{u}(s;\xi)\|_2^2 ds \leq |\xi_1 - \xi_2|^2 + 2C_R\int_0^{t_R} \Big(|\mathbf{w}(s)| \times |\nabla \mathbf{w}(s)|_{L^q} \Big) ds \\
+ \int_0^{t_R}\int_Z |\sigma(\mathbf{u}(s;\xi_1), z) - \sigma(\mathbf{u}(s;\xi_2), z)|^2 \eta(dz, ds) \\
+ 2\int_0^{t_R}\int_Z (\sigma(\mathbf{u}(s;\xi_1), z) - \sigma(\mathbf{u}(s;\xi_2), z), \mathbf{w}(s-))\, \tilde{\eta}(dz, ds),
\end{aligned}$$

where $2 < q \le \frac{2n}{n-2}$. Thanks to Young's inequality and the continuous embedding $\mathcal{H}^1 \subset L^q$ we easily see that

$$|\mathbf{w}(t_R)|^2 + 2\kappa_1 \int_0^{t_R} \|\mathbf{u}(s;\xi)\|_2^2 ds \le |\xi_1 - \xi_2|^2 + \frac{2C_R}{\varepsilon} \int_0^{t_R} |\mathbf{w}(s)|^2 ds + \varepsilon \int_0^{t_R} \|\mathbf{w}(s)\|_2^2 ds$$
$$+ \int_0^{t_R} \int_Z |\sigma(\mathbf{u}(s;\xi_1), z) - \sigma(\mathbf{u}(s;\xi_2), z)|^2 \eta(dz, ds)$$
$$+ 2 \int_0^{t_R} \int_Z (\sigma(\mathbf{u}(s;\xi_1), z) - \sigma(\mathbf{u}(s;\xi_2), z), \mathbf{w}(s-))\, \tilde{\eta}(dz, ds).$$

Choosing $\varepsilon = \kappa_1$, using item (2) of Condition 1 and taking the mathematical expectation yield that

$$\mathbb{E}|\mathbf{w}(t_R)|^2 + \kappa_1 \mathbb{E} \int_0^{t_R} \|\mathbf{u}(s;\xi)\|_2^2 ds \le \mathbb{E}|\xi_1 - \xi_2|^2 + \left(\frac{2C_R}{\kappa_1} + L_1 \right) \int_0^{t_R} \mathbb{E}|\mathbf{w}(s)|^2 ds, \tag{72}$$

where we have used the fact that

$$\mathbb{E} \int_0^{t_R} \int_Z |\sigma(\mathbf{u}(s;\xi_1), z) - \sigma(\mathbf{u}(s;\xi_2), z)|^2 \eta(dz, ds)$$
$$= \mathbb{E} \int_0^{t_R} |\sigma(\mathbf{u}(s;\xi_1), z) - \sigma(\mathbf{u}(s;\xi_2), z)|^2 \nu(dz) ds,$$

and

$$2\mathbb{E} \int_0^{t_R} \int_Z (\sigma(\mathbf{u}(s;\xi_1), z) - \sigma(\mathbf{u}(s;\xi_2), z), \mathbf{w}(s-))\, \tilde{\eta}(dz, ds) = 0.$$

Notice that (72) can be rewritten in the following form

$$\mathbb{E}|\mathbf{w}(t_R)|^2 + \kappa_1 \mathbb{E} \int_0^{t_R} \|\mathbf{u}(s;\xi)\|_2^2 ds \le \mathbb{E}|\xi_1 - \xi_2|^2 + \left(\frac{2C_R}{\kappa_1} + L_1 \right) \int_0^{t} \mathbb{E}|\mathbf{w}(s \wedge \tau_R)|^2 ds,$$

from which along with the application Gronwall's lemma we deduce the existence of a positive constant $C = C(t, R)$ such that

$$\mathbb{E}|\mathbf{w}(t_R)|^2 \le C\mathbb{E}|\xi_1 - \xi_2|^2.$$

The proof of the lemma is now finished. □

Now we continue with the proof of Theorem 5.1.

Proof of Theorem 5.1. Owing to the Theorem 4.1 and the fact that $\tilde{\eta}(A \times [0,t])$, $A \times [0,t] \in \mathcal{B}(Z \times \mathbb{R}_+)$ is time homogeneous, the Markovian property of $\mathbf{u}(t;\xi)$, $\xi \in H$, can be checked using the same argument as in [25, Theorem 9.14] (see also [2, Theorem 6.1] or [45, Theorem 9.30]). More precisely, we will show that for arbitrary $\phi \in B_b(H)$, $\xi \in H$ and $0 \le r \le s \le t \le T$ we have

$$\mathbb{E}[\phi(\mathbf{u}(t,s,\xi))|\mathcal{F}_s] = \mathbb{E}(\phi(\mathbf{u}(t,s,\theta))), \quad \mathbb{P}\text{-a.s.}, \tag{73}$$

where $\theta = \mathbf{u}(s,u,\xi)$ and $\mathbf{u}(t,s,\xi)$, $s \le t \le T$, is the solution of (6) on the time interval $[s,T]$ with the initial data ξ at the initial time s. Thanks to the uniqueness in Theorem 4.1 we have

$$\mathbf{u}(t,s,x) = \mathbf{u}(t,s,\mathbf{u}(s,u,x)), \qquad \text{for any } \mathbb{P}\text{-a.s.}$$

Thus, (73) is equivalent to

$$\mathbb{E}[\phi(\mathbf{u}(t,s,\theta))|\mathcal{F}_s] = \mathbb{E}(\phi(\mathbf{u}(t,s,\theta))), \quad \mathbb{P}\text{-a.s.} \tag{74}$$

Hence, as in [25, Theorem 9.14] (see also [2, Theorem 6.1] or [45, Theorem 9.30]) it is enough to prove (74) for any $\phi \in C_b(H)$ and any square integrable $\mathcal{F}_s$-measurable random variable $\theta \in H$. Since the argument in the above three references do not use the coefficients' structure of the stochastic system, we can exactly argue as in these references to establish (74) when θ is a simple $\mathcal{F}_s$-measurable random variable. For a general $\mathcal{F}_s$-random variable θ satisfying $\mathbb{E}|\theta|^2 < \infty$, we can find a sequence of simple $\mathcal{F}_s$-measurable random variables $(\theta_n)_{n\in\mathbb{N}}$ such that $\mathbb{E}|\theta - \theta_n|^2 \to 0$ as $n \to \infty$. Moreover, (74) holds for θ_n, $n \in \mathbb{N}$. Owing to Theorem 4.1 the sequence of process Δ_n defined by

$$\Delta_n(t) = \mathbb{E}\left(e^{-\frac{C_0^2}{2\kappa_1}\int_0^t \|\mathbf{u}(r,s,\theta)\|_2^2 dr}|\mathbf{u}(t,s,\theta) - \mathbf{u}(t,s,\theta_n)|\right),$$

converges to zero as $n \to \infty$. This implies that one can find a subsequence, still denoted by θ_n, such that $\theta_n \to \theta$ and $\mathbf{u}(t,s,\theta) \to \mathbf{u}(t,s,\theta_n)$ in H almost surely. Thanks to (14) we can pass to the limit and infer that (74) is verified for any square integrable $\mathcal{F}_s$-measurable random variable $\theta \in H$. This completes the proof of the first claim of the theorem.

To prove that $\mathcal{P}_t$ is a semigroup satisfying $\mathcal{P}_{t+s} = \mathcal{P}_t\mathcal{P}_s$, it is sufficient to check that for any $s \ge 0$ the process $\mathbf{u}(s,0,\xi)$ and $\mathbf{u}(t+s,t,\xi)$ are identical in law. In fact, if this is the case, then, by the Markov property above and equality of laws, we have

$$\begin{aligned}
\mathbb{E}\phi(\mathbf{u}(t,0,\xi)) &= \mathbb{E}\left(\mathbb{E}[\phi(t,0,\xi)|\mathcal{F}_s]\right) \\
&= \mathbb{E}\left(\mathbb{E}\phi(\mathbf{u}(t,s,\mathbf{u}(s,0,\xi)))\right) = \mathbb{E}\left(\mathbb{E}\phi(\mathbf{u}(t-s,0,\mathbf{u}(s,0,\xi)))\right) \\
&= \mathbb{E}\mathcal{P}_{t-s}\phi(\mathbf{u}(s,0,x)) = \mathcal{P}_s\mathcal{P}_{t-s}\phi(x), \text{ for any } 0 \le s \le t,
\end{aligned}$$

where we have set $\mathbf{u}(t,0,\xi) = \mathbf{u}(t,\xi)$. Now, let us prove that for $\mathbf{u}(s,0,\xi)$ and $\mathbf{u}(t+s,t,\xi)$ have the same distribution. Since $\mathbf{u}(t+s,t,\xi)$ is solution of (6) with initial data ξ at time t, we have

$$\begin{aligned}\mathbf{u}(t+s,t,\xi) &= \xi - \int_t^{t+s} \Big[(A+\mathcal{A}_p)\left(\mathbf{u}(r,t,\xi)\right) + B(\mathbf{u}(r,t,\xi),\mathbf{u}(r,t,\xi)) \Big] dr \\ &\quad + \int_t^{t+s}\int_{\mathcal{Z}} \sigma(\mathbf{u}(r,t,\xi))\tilde{\eta}(dz,dr) \\ &= \xi - \int_0^{s} \Big[(A+\mathcal{A}_p)\left(\mathbf{u}(r+t,t,\xi)\right) + B(\mathbf{u}(r+t,t,\xi),\mathbf{u}(r+t,t,\xi)) \Big] \\ &\quad + \int_0^{s}\int_{\mathcal{Z}} \sigma(\mathbf{u}(r+t,t,\xi))\hat{\tilde{\eta}}(dz,dr) \\ &= \xi - \int_0^{s} \Big[(A+\mathcal{A}_p)\left(\mathbf{u}(r+t,t,\xi)\right) + B(\mathbf{u}(r+t,t,\xi),\mathbf{u}(r+t,t,\xi)) \Big] \\ &\quad + \int_0^{s}\int_{\mathcal{Z}} \sigma(\mathbf{u}(r+t,t,\xi))\tilde{\eta}(dz,dr)\end{aligned}$$

where we used the fact that the compensated Poisson random measure $\hat{\tilde{\eta}}$, defined by $\hat{\tilde{\eta}}(K,(r,\tau]) = \tilde{\eta}(K,(r+t,\tau+t])$ for $K\times(r,\tau]\in\mathcal{Z}\times\mathcal{B}(\mathbb{R}_0)$, and $\tilde{\eta}$ are equally distributed. The last line of the chain of identities above and Theorem 4.1 imply that $\mathbf{u}(t+s,t,\xi)$ and $\mathbf{u}(s,0,\xi)$ are identical in law.

Now, we show that $\mathcal{P}_t$ has the Feller, *i.e.*, we prove that $\mathcal{P}_t\left(C_b(H)\right)\subset C_b(H)$. For this purpose, let us consider $\xi\in H$ and a sequence $\{\xi_m : m\in\mathbb{N}\}\subset H$ such that $\xi_m\to\xi$ as $m\to\infty$. Let us prove that

$$\mathcal{P}_t\phi(\xi_m)\to\mathcal{P}_t\phi(\xi), \forall\phi\in C_b(H),$$

as m tends to infinity. To shorten notation we set $\tau_R=\tau_R^{\xi_m}\wedge\tau_R^{\xi}$ where the stopping time τ_R^{ξ} is defined as in (70). For any $t\in[0,T]$, $T\geq 0$ and $\phi\in C_b(H)$, we have

$$\begin{aligned}|\mathcal{P}_t\phi(\xi_m)-\mathcal{P}_t\phi(\xi) &= \left|\mathbb{E}\left(\Big[\phi(\mathbf{u}(t;\xi_m))-\phi(\mathbf{u}(t;\xi))\Big]\mathbf{1}_{[t<\tau_R]\cup[t\geq\tau_R]}\right)\right|, \\ &\leq \left|\mathbb{E}\left(\Big[\phi(\mathbf{u}(t;\xi_m))-\phi(\mathbf{u}(t;\xi))\Big]\left(\mathbf{1}_{[t\geq\tau_R^{\xi_m}]}+\mathbf{1}_{[t\geq\tau_R^{\xi}]}\right)\right)\right| \\ &\quad + \left|\mathbb{E}\left(\Big[\phi(\mathbf{u}(t;\xi_m))-\phi(\mathbf{u}(t;\xi))\Big]\mathbf{1}_{[t<\tau_R]}\right)\right|.\end{aligned}$$

Thanks to the fact that $\mathbb{E}|\mathbf{u}(t;\xi)|^2<C(\xi),\forall\xi\in H$ (see the estimate in Theorem 3.2), we obtain that for any $\varepsilon>0$ there exists m_1 such that for any $R>m_1$

$$\mathbb{P}\left(\tau_R^{\xi_m}\geq t\right)+\mathbb{P}\left(\tau_R^{\xi}\geq t\right)\leq\frac{\varepsilon}{4\|\phi\|_\infty},$$

where

$$\|\phi\|_\infty=\sup_{\xi\in H}|\phi(x)|.$$

Thus

$$|\mathcal{P}_t\phi(\xi_m) - \mathcal{P}_t\phi(\xi)| \le \left|\mathbb{E}\left(\left[\phi(\mathbf{u}(t;\xi_m)) - \phi(\mathbf{u}(t;\xi))\right]\mathbf{1}_{[t<\tau_R]}\right)\right| + 2\|\phi\|_\infty \frac{\varepsilon}{4\|\phi\|_\infty}.$$

That is,

$$|\mathcal{P}_t\phi(\xi_m) - \mathcal{P}_t\phi(\xi)| \le \left|\mathbb{E}\left(\left[\phi(\mathbf{u}(t;\xi_m)) - \phi(\mathbf{u}(t;\xi))\right]\mathbf{1}_{[t<\tau_R]}\right)\right| + \frac{\varepsilon}{2}.$$

Since $\mathbf{1}_{[t<\tau_R]} \le 1$ and $t \wedge \tau_R = t$ when $t < \tau_R$, we readily have that

$$|\mathcal{P}_t\phi(\xi_m) - \mathcal{P}_t\phi(\xi)| \le \left|\mathbb{E}\left(\left[\phi(\mathbf{u}(t_R;\xi_m)) - \phi(\mathbf{u}(t_R;\xi))\right]\right)\right| + \frac{\varepsilon}{2}, \tag{75}$$

where we have put $t_R = t \wedge \tau_R$. By the continuity of ϕ, for the same $\varepsilon > 0$ as above we can find $\kappa > 0$ such that if $|\mathbf{u}(t_R;\xi_m) - \mathbf{u}(t_R;\xi)| < \kappa$ we have

$$|\phi(\mathbf{u}(t_R;\xi_m)) - \phi(\mathbf{u}(t_R;\xi))| < \frac{\varepsilon}{4}. \tag{76}$$

Note that from (75) we derive that

$$\begin{aligned}|\mathcal{P}_t\phi(\xi_m) - \mathcal{P}_t\phi(\xi)| \le &\left|\mathbb{E}\left(\left[\phi(\mathbf{u}(t_R;\xi_m)) - \phi(\mathbf{u}(t_R;\xi))\right]\mathbf{1}_{\{|\mathbf{u}(t_R;\xi_m)-\mathbf{u}(t_R;\xi)|\ge\kappa\}}\right)\right| \\ &+ \mathbb{E}\left(\left[\phi(\mathbf{u}(t_R;\xi_m)) - \phi(\mathbf{u}(t_R;\xi))\right]\mathbf{1}_{\{|\mathbf{u}(t_R;\xi_m)-\mathbf{u}(t_R;\xi)|<\kappa\}}\right)\Bigg| + \frac{\varepsilon}{2},\end{aligned}$$

from which all together with (76) we derive that

$$\begin{aligned}|\mathcal{P}_t\phi(\xi_m) - \mathcal{P}_t\phi(\xi)| \le 2\|\phi\|_\infty \mathbb{P}\left(|\mathbf{u}(t_R;\xi_m) - \mathbf{u}(t_R;\xi)| \ge \kappa\right) \\ + \mathbb{E}\left(\left[\phi(\mathbf{u}(t_R;\xi_m)) - \phi(\mathbf{u}(t_R;\xi))\right]\mathbf{1}_{\{|\mathbf{u}(t_R;\xi_m)-\mathbf{u}(t_R;\xi)|<\kappa\}}\right)\Bigg| + \frac{\varepsilon}{2}.\end{aligned} \tag{77}$$

Invoking the estimate (71) and Chebychev's inequality we obtain that

$$2\|\phi\|_\infty \mathbb{P}\left(|\mathbf{u}(t_R;\xi_m) - \mathbf{u}(t_R;\xi)| \ge \kappa\right) \le \frac{2\|\phi\|_\infty C}{\kappa^2}|\xi_m - \xi|^2. \tag{78}$$

But as $\xi_m \to \xi$ as $m \to \infty$ we have that for any $\delta > 0$ there exists $m_2 > 0$ such that if $m > m_2$ we have $|\xi_m - \xi|^2 < \delta$. Choosing $\delta = \frac{\varepsilon\kappa^2}{8C\|\phi\|_\infty}$ we can derive from (78) that

$$2\|\phi\|_\infty \mathbb{P}\left(|\mathbf{u}(t_R;\xi_m) - \mathbf{u}(t_R;\xi)| \ge \kappa\right) \le \frac{\varepsilon}{4}. \tag{79}$$

So combining (76), (77) and (79) we see that for any $\varepsilon > 0$ there exists $m_0 > 0$ such that if $m > m_0$ then

$$|\mathcal{P}_t\phi(\xi_m) - \mathcal{P}_t\phi(\xi)| < \varepsilon,$$

which shows that $\mathcal{P}_t$ is a Fellerian semigroup. □

Owing to Theorem 5.1 we can discuss about the existence of the invariant measure associated to the semigroup $\mathcal{P}_t$.

Theorem 5.4. *The Markovian semigroup $\mathcal{P}_t$ has at least one invariant measure μ. Moreover, μ is concentrated on V, i.e., $\mu(V) = 1$.*

Proof. Let $\{T_n; n \in \mathbb{N}\} \subset [0, \infty)$ be a sequence such that $T_n \nearrow \infty$ as $n \to \infty$. For any $A \in \mathcal{B}(H)$ let us set

$$\mu_n(A) = \frac{1}{T_n} \int_0^{T_n} \mathbb{P}\left(\mathbf{u}(t;\xi) \in A\right) dt.$$

It is clear that μ_n defines a measure on $(H, \mathcal{B}(H))$. Let $R > 0$ and $A_R = \{\mathbf{u} : \|\mathbf{u}\|_2 > R\}$. Using Chebychev's inequality and Fubini's Theorem we see that

$$\mu_n(A_R) \leq \frac{1}{R^2} \frac{1}{T_n} \mathbb{E} \int_0^{T_n} \|\mathbf{u}(s;\xi)\|_2^2 ds.$$

Owing to the estimate in Theorem 3.2 we have that

$$\mu_n(A_R) \leq \frac{C(1 + |\xi|^2)}{R^2}.$$

This implies that $\mu_n(A_R) \to 0$ uniformly in n as $R \to \infty$. Since the ball $B_R = V \backslash A_R$ is compact in H, we conclude that the family of measures μ_n is tight on H. This yields that there exists a subsequence μ_{n_k} and a measure μ defined on $(H, \mathcal{B}(H))$ such that

$$\int_H \phi(x) \mu_{n_k}(dx) \to \int_H \phi(x) \mu(dx), \forall \phi \in C_b(H).$$

Since $\mathcal{P}_t$ satisfies the Markov–Feller property, we can infer from Krylov–Bogoluibov's theorem that it admits an invariant measure which is equal to μ.

It remains to show that μ is concentrated on V. For this purpose it is sufficient to show that $\mu(H \backslash V) = 0$. To do so we will first show that

$$\mu_n(H \backslash V) = 0, \forall n.$$

Thanks to the estimate in Theorem 3.2 we can find a set $I \times \Omega_0 \subset \Omega_{T_n}, T_n \geq 0$ $(\Omega_{T_n} = [0, T_n] \times \Omega)$ with $\lambda \otimes \mathbb{P}(\Omega_t \backslash I \times \Omega_0) = 0$ and $\mathbf{u}(t;\xi)(\omega) \in V$ for any $(t, \omega) \in I \times \Omega_0$. This fact implies that

$$\mathbb{P}\left(\int_0^{T_n} \mathbf{1}_N(t, \omega) dt\right) = 0,$$

where

$$N = \{(t, \omega) \in \Omega_{T_n} : \mathbf{u}(t;\xi)(\omega) \in H \backslash V\}.$$

Owing to Fubini's theorem we infer the existence of $J \subset [0, T_n]$ with $\lambda([0, T_n] \backslash J) = 0$ and

$$\mathbb{P}\left(\{\omega \in \Omega : \mathbf{u}(t;\xi) \in H \backslash V\}\right) = 0,$$

for any $t \in J$. Setting $N_t = \{\omega \in \Omega; \mathbf{u}(t;\xi) \in H \backslash V\}$ for any $t \in J$, we find that

$$
\begin{aligned}
\mu_n(H\backslash V) &= \frac{1}{T_n}\int_0^{T_n} \mathbb{P}(N_t)dt,\\
&= \frac{1}{T_n}\int_0^{T_n} \mathbf{1}_J(t)\mathbb{P}(N_t)dt,\\
&= 0.
\end{aligned}
$$

This means that the support of μ_n is included in V. Since μ is the weak limit of μ_n, we derive from [20, Theorem 2.2] that the support of μ is included in V. $\square$

Our next concern is to check whether the invariant measure μ is ergodic or not. In fact we will find that it is ergodic provided that κ_1 is large enough. We will make our claim clearer later on, but for now let us prove an important fact about the invariant measure μ.

Proposition 5.5. *If $2\kappa_1\lambda_1^2 - \ell_1 > 0$, then there exists a constant $\tilde{L} > 0$ depending only on $\kappa_1, \lambda_1, \ell_0, \ell_1$ such that*

$$
\int_H \left(|\xi|^2 + \|\xi\|_2^2\right)\mu(dx) < \tilde{L}. \tag{80}
$$

Proof. First we should notice that by Itô's formula we have

$$
\begin{aligned}
&|\mathbf{u}(t;\xi)|^2 + 2\kappa_1\int_0^t \|\mathbf{u}(s;\xi)\|_2^2 ds + 2\int_0^t \langle \mathcal{A}_p\mathbf{u}(s;\xi), \mathbf{u}(s,\xi)\rangle ds\\
&= |\xi|^2 + \int_0^t\int_Z |\sigma(\mathbf{u}(s;\xi), z)|^2\eta(dz,ds) + 2\int_0^t\int_Z \left(\sigma(\mathbf{u}(s;\xi), z), \mathbf{u}(s-;\xi)\right)\tilde{\eta}(dz,ds).
\end{aligned} \tag{81}
$$

Now for any $\varepsilon > 0$ let $\Phi(y) = \frac{y}{1+\varepsilon y}$, $y \in \mathbb{R}_+$. It is clear that

$$
\begin{aligned}
\Phi'(y) &= \frac{1}{(1+\varepsilon y)^2},\\
\Phi''(y) &= \frac{-2\varepsilon}{(1+\varepsilon y)^3},
\end{aligned}
$$

for any $y \geq 0$. It is clear from the last equality that $\Phi''(y) < 0$, and $|\Phi''| \leq 2\varepsilon$ for any $y \geq 0$. Notice also that $\eta(dz,ds) = \tilde{\eta}(dz,ds) + \nu(dz)ds$ and

$$
|\sigma(\mathbf{u}(s,\xi), z)|^2 + 2\left(\sigma(\mathbf{u}(s,\xi), z), \mathbf{u}(s-;\xi)\right) = |\sigma(\mathbf{u}(s,\xi), z) + \mathbf{u}(s-;\xi)|^2 - |\mathbf{u}(s-;\xi)|^2.
$$

By setting $Y(t) = |\mathbf{u}(t;\xi)|^2$ and $\Psi = |\sigma(\mathbf{u}(s,\xi), z) + \mathbf{u}(s-;\xi)|^2 - |\mathbf{u}(s-;\xi)|^2$ we can rewrite (81) in the following form

$$
\begin{aligned}
Y(t) + 2\kappa_1\int_0^t \|\mathbf{u}(s;\xi)\|_2^2 ds + 2\int_0^t \langle \mathcal{A}_p\mathbf{u}(s;\xi), \mathbf{u}(s;\xi)\rangle ds &= |\xi|^2 + \int_0^t\int_Z |\sigma(\mathbf{u}(s;\xi), z)|^2\nu(dz)ds\\
&\quad + \int_0^t\int_Z \Psi\tilde{\eta}(dz,ds).
\end{aligned}
$$

Applying Itô's formula to $\Phi(Y)$ we obtain that

$$\Phi(Y(t)) + 2\kappa_1 \int_0^t \Phi'(Y(s))\|\mathbf{u}(s;\xi)\|_2^2 ds + 2\int_0^t \Phi'(Y(s))\langle \mathcal{A}_p \mathbf{u}(s;\xi), \mathbf{u}(s;\xi)\rangle ds$$
$$= \Phi(|\xi|^2) + \int_0^t \int_Z \Big(\Phi(Y(s-)+\Psi) - \Phi(Y(s-)) - \Phi'(Y(s-))\Psi \Big) \eta(dz, ds)$$
$$+ \int_0^t \Phi'(Y(s)) \int_Z |\sigma(\mathbf{u}(s;\xi), z)|^2 \nu(dz) ds + \int_0^t \int_Z \Big(\Phi(Y(s-)+\Psi) - \Phi(Y(s-)) \Big) \tilde{\eta}(dz, ds).$$

Since $\langle \mathcal{A}_p \mathbf{u}(s;\xi), \mathbf{u}(s;\xi)\rangle \geq 0$ and $\Phi'(y) > 0$ for any $y \geq 0$, we can drop out the third term from the left-hand side of the last equation. Therefore we obtain that

$$\Phi(Y(t)) + 2\kappa_1 \int_0^t \Phi'(Y(s))\|\mathbf{u}(s;\xi)\|_2^2 ds \leq \Phi(|\xi|^2) + \int_0^t \Phi'(Y(s)) \Big(\int_Z |\sigma(\mathbf{u}(s;\xi), z)|^2 \nu(dz) \Big) ds$$
$$+ \int_0^t \int_Z \Big(\int_0^1 \Phi''(Y(s-) + \theta\Psi)\Psi^2 d\theta \Big) \left(\tilde{\eta}(dz, ds) + \nu(dz)ds\right)$$
$$+ \int_0^t \int_Z \Big(\int_0^1 \Phi'(Y(s-) + \theta\Psi)\Psi d\theta \Big) \tilde{\eta}(dz, ds),$$

where we have used the identities

$$\Phi(y+\psi) - \Phi(y) = \int_0^1 \Phi'(y + \theta\psi)\psi d\theta,$$
$$\Phi(y+\psi) - \Phi(y) - \Phi'(y)\psi = \int_0^1 \Phi''(y + \theta\psi)\psi^2 d\theta.$$

Since $|\Phi'(\cdot)| < 1$ and $|\Phi''| < 2\varepsilon$ and

$$\mathbb{E}\Psi^r \leq C\mathbb{E}(1 + |\mathbf{u}(s;\xi)|^{2r}) < C,$$

with $r = 1, 2$, the stochastic integrals

$$\int_0^t \int_Z \Big(\int_0^1 \Phi'(Y(s) + \theta\Psi)\Psi d\theta \Big) \tilde{\eta}(dz, ds),$$
$$\int_0^t \int_Z \Big(\int_0^1 \Phi''(Y(s-) + \theta\Psi)\Psi^2 d\theta \Big) \tilde{\eta}(dz, ds),$$

are martingales with zero mean. Hence taking the mathematical expectation yields

$$\begin{aligned}\mathbb{E}\Phi(Y(t)-\Phi(|\xi|^2)) \le \mathbb{E}\int_0^t \Phi'(Y(s))\Big(\int_Z |\sigma(\mathbf{u}(s;\xi),z)|^2\nu(dz)\Big)ds \\ +\mathbb{E}\int_0^t\int_Z\Big(\int_0^1 \Phi''(Y(s-)+\theta\Psi)\Psi^2 d\theta\Big)\nu(dz)ds \\ -2\kappa_1\mathbb{E}\int_0^t \Phi'(Y(s))\|\mathbf{u}(s;\xi)\|_2^2 ds.\end{aligned} \tag{82}$$

Since

$$\begin{aligned}\Phi''(Y(s-)+\theta\Psi)\Psi^2 &= \frac{-2\varepsilon\Psi^2}{(1+\varepsilon Y(s)+\varepsilon\theta\Psi)^3} \\ &= \frac{-2\varepsilon\Psi^2}{(1+\varepsilon\theta|\sigma(\mathbf{u}(s,\xi),z)+\mathbf{u}(s-;\xi)|^2-\varepsilon(1-\theta)|\mathbf{u}(s-;\xi)|^2)},\end{aligned}$$

we see that $\Phi''(Y(s-)+\theta\Psi)\Psi^2 \le 0$ for any $\theta\in[0,1]$. Therefore we can drop out the second term in the right-hand side of (82), use item (2) in Condition 1 to obtain

$$\begin{aligned}\mathbb{E}\Phi(Y(t))+2\kappa_1\mathbb{E}\int_0^t \Phi'(Y(s))\|\mathbf{u}(s;\xi)\|_2^2 ds \le \Phi(|\xi|^2)+\ell_1\mathbb{E}\int_0^t \Phi'(Y(s))|\mathbf{u}(s;\xi)|^2 ds \\ +\ell_0\mathbb{E}\int_0^t \Phi'(Y(s))ds.\end{aligned} \tag{83}$$

By using Poincaré's inequality (see (2)) the last estimate becomes

$$\begin{aligned}\mathbb{E}\Phi(Y(t))+2\kappa_1\lambda_1^2\mathbb{E}\int_0^t \Phi'(Y(s))|\mathbf{u}(s;\xi)|_2^2 ds \le \Phi(|\xi|^2)+\ell_1\mathbb{E}\int_0^t \Phi'(Y(s))|\mathbf{u}(s;\xi)|^2 ds \\ +\ell_0\mathbb{E}\int_0^t \Phi'(Y(s))ds.\end{aligned} \tag{84}$$

By integrating both side of this last inequality wrt μ on H and using the fact that

$$\int_H \mathbb{E}\phi(\mathbf{u}(s,\xi))\mu(dx) = \int_H \phi(\xi)\mu(dx), \forall\phi\in C_b(H),\ (\mu \text{ is an invariant measure}) \tag{85}$$

we obtain from (84) that

$$\big(2\kappa_1\lambda_1^2-\ell_1\big)\int_H \frac{|\xi|^2}{(1+\varepsilon|\xi|^2)^2}\mu(dx) \le \ell_0\int_H \frac{1}{(1+\varepsilon|\xi|^2)^2}\mu(dx).$$

From this inequality we obtain that

$$\int_H \frac{|\xi|^2}{(1+\varepsilon|\xi|^2)^2}\mu(dx) \le \frac{\ell_0}{2\kappa_1\lambda_1^2-\ell_1}, \tag{86}$$

where we have used the facts that $\frac{1}{(1+\varepsilon|\xi|^2)^2}\ge 1$, $2\kappa_1\lambda_1^2-\ell_1>0$ and $\mu(V)+\mu(H\backslash V)=1$.

From (83) and (86) we derive that

$$2\kappa_1 \int_H \mathbb{E} \int_0^t \Phi'(Y(s))\|\mathbf{u}(s;\xi)\|_2^2 ds \leq \frac{\ell_0}{2\kappa_1\lambda_1^2 - \ell_1}(\ell_1 + 1) + \ell_0. \tag{87}$$

Choosing $\phi(\mathbf{u}(s;\xi)) = \int_0^t \Phi'(Y(s))\|\mathbf{u}(s;\xi)\|_2^2 ds$ and using (85) we see that

$$\int_H \frac{\|\xi\|_2^2}{(1+\varepsilon|\xi|^2)^2}\mu(dx) \leq \frac{\ell_0}{2\kappa_1(2\kappa_1\lambda_1^2 - \ell_1)}(\ell_1 + 1) + \frac{\ell_0}{2\kappa_1}. \tag{88}$$

Adding up (86) and (88) side by side, letting $\varepsilon \to 0$ and using Fatou's lemma imply that

$$\int_H \left(|\xi|^2 + \|\xi\|_2^2\right)\mu(dx) \leq \frac{\ell_0}{2\kappa_1\lambda_1^2 - \ell_1}\left(\frac{\ell_1 + 1}{2\kappa_1} + 1\right) + \frac{\ell_0}{2\kappa_1}, \tag{89}$$

which terminates the proof of the proposition. □

We can prove the ergodicity of the invariant measure under the condition that κ_1 is large enough.

Theorem 5.6. *Assume that $2\kappa_1\lambda_1^2 > \ell_1$. Then, the Markovian semigroup $\mathcal{P}_t$ has an invariant measure μ which is tight and ergodic on H.*

Proof. Let $\mathcal{M} \subset \mathcal{M}_1(H)$ be the set of invariant measure of $\mathcal{P}_t$ and

$$\tilde{\ell} = \frac{\ell_0}{2\kappa_1\lambda_1^2 - \ell_1}\left(\frac{\ell_0}{2\kappa_1} + 1\right) + \frac{\ell_0}{2\kappa_1}.$$

It is not difficult to show that $\mathcal{M}$ is convex (see for example [33, page 296]). As before let $R > 0$ and $A_R = \{\mathbf{u} \in H : \|\mathbf{u}\|_2 > R\}$. We see from Chebychev–Markov's inequality that

$$\sup_{\mu\in\mathcal{M}} \mu(A_R) \leq \frac{1}{R^2}\int_H \|\xi\|_2^2 \mu(dx).$$

Owing to (88) we have that

$$\sup_{\mu\in\mathcal{M}} \mu(A_R) \leq \frac{\tilde{\ell}}{R^2},$$

which implies that for any $\varepsilon > 0$

$$\mu(B_V(\frac{1}{\sqrt{\varepsilon}})) \geq 1 - \varepsilon,$$

where $B_V(\frac{1}{\sqrt{\varepsilon}}) = V \backslash A_{\frac{1}{\sqrt{\varepsilon}}}$. Since $B_V(\frac{1}{\sqrt{\varepsilon}})$ is compact in H we infer that the set $\mathcal{M}$ is tight on H. Since $\mathcal{M}$ is non-empty, convex and tight, by Krein–Millman's theorem (see, for instance, [28, Theorem 3.65, p. 110]) it has extrema which are ergodic. We deduce from the above argument that $\mathcal{P}_t$ has at least one invariant measure which is ergodic. □

Acknowledgment

The authors are very grateful to the anonymous Reviewer whose insightful comments and suggestions greatly improved the manuscript. Razafimandimby's research is partially supported by the Austrian Science Fund (FWF) through the Lise Meitner project M1487.

References

[1] S. Albeverio, Z. Brzeźniak, J.L. Wu, Existence of global solutions and invariant measures for stochastic differential equations driven by Poisson type noise with non-Lipschitz coefficients, J. Math. Anal. Appl. 371 (1) (2010) 309–322.
[2] S. Albeverio, V. Mandrekar, B. Rüdiger, Existence of mild solutions for stochastic differential equations and semilinear equations with non-Gaussian Lévy noise, Stochastic Process. Appl. 119 (3) (2009) 835–863.
[3] A.V. Babin, M.I. Vishik, Attractors of Evolution Equations, Stud. Math. Appl., vol. 25, North-Holland Publishing Co., Amsterdam, 1992.
[4] H. Bellout, F. Bloom, J. Necas, Phenomenological behavior of multipolar viscous fluids, Quart. Appl. Math. 50 (1992) 559–583.
[5] H. Bellout, F. Bloom, J. Necas, Solutions for incompressible non-Newtonian fluids, C. R. Math. Acad. Sci. Paris Sér I. Math. 317 (1993) 795–800.
[6] H. Bellout, F. Bloom, J. Necas, Young measure-valued solutions for non-Newtonian incompressible fluids, Comm. Partial Differential Equations 19 (11&12) (1994) 1763–1803.
[7] H. Bellout, F. Bloom, J. Necas, Bounds for the dimensions of the attractors of nonlinear bipolar viscous fluids, Asymptot. Anal. 11 (2) (1995) 131–167.
[8] H. Bellout, F. Bloom, J. Necas, Existence, uniqueness and stability of solutions to initial boundary value problems for bipolar fluids, Differential Integral Equations 8 (1995) 453–464.
[9] A. Bensoussan, Stochastic Navier–Stokes equations, Acta Appl. Math. 38 (1995) 267–304.
[10] A. Bensoussan, R. Temam, Equations stochastiques du type Navier–Stokes, J. Funct. Anal. 13 (1973) 195–222.
[11] H. Breckner, Approximation and optimal control of the stochastic Navier–Stokes equation, Dissertation, Martin-Luther University, Halle-Wittenberg, 1999.
[12] H. Breckner, Galerkin approximation and the strong solution of the Navier–Stokes equation, J. Appl. Math. Stoch. Anal. 13 (3) (2000) 239–259.
[13] Z. Brzezniak, L. Debbi, On stochastic Burgers equation driven by a fractional Laplacian and space–time white noise, in: Stochastic Differential Equations: Theory and Applications, in: Interdiscip. Math. Sci., vol. 2, World Sci. Publ., 2007, pp. 135–167.
[14] Z. Brzeźniak, E. Hausenblas, Maximal regularity for stochastic convolutions driven by Lévy processes, Probab. Theory Related Fields 145 (3–4) (2009) 615–637.
[15] Z. Brzeźniak, E. Hausenblas, J. Zhu, 2D stochastic Navier–Stokes equations driven by jump noise, Nonlinear Anal. Theory Methods Appl. 79 (2013) 122–139.
[16] Z. Brzezniak, B. Maslowski, J. Seidler, Stochastic nonlinear beam equation, Probab. Theory Related Fields 132 (2) (2005) 119–144.
[17] T. Caraballo, J.A. Langa, T. Taniguchi, The exponential behaviour and stabilizability of stochastic 2D-Navier–Stokes equations, J. Differential Equations 179 (2) (2002) 714–737.
[18] T. Caraballo, A.M. Márquez-Durán, J. Real, The asymptotic behaviour of a stochastic 3D LANS-α model, Appl. Math. Optim. 53 (2) (2006) 141–161.
[19] T. Caraballo, J. Real, T. Taniguchi, On the existence and uniqueness of solutions to stochastic three-dimensional Lagrangian averaged Navier–Stokes equations, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 462 (2066) (2006) 459–479.
[20] P.-L. Chow, R.Z. Khasminskii, Stationary solutions of nonlinear stochastic evolution equations, Stoch. Anal. Appl. 15 (5) (1997) 671–699.
[21] I. Chueshov, A. Millet, Stochastic 2D hydrodynamical type systems: well posedness and large deviations, Appl. Math. Optim. 61 (3) (2010) 379–420.
[22] P. Constantin, C. Foias, R. Temam, Attractors representing turbulent flows, Mem. Amer. Math. Soc. 53 (314) (1985), vii+67 pp.
[23] P. Constantin, C. Foias, R. Temam, Phys. D Nonlinear Phenom. 30 (3) (1988) 284–296.
[24] G. Da Prato, A. Debussche, 2D stochastic Navier–Stokes equations with a time-periodic forcing term, J. Dynam. Differential Equations 20 (2) (2008) 301–335.
[25] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge University Press, 1992.
[26] G. Deugoue, M. Sango, On the strong solution for the 3D stochastic Leray-alpha model, Bound. Value Probl. 2010 (2010) 723018, http://dx.doi.org/10.1155/2010/723018, 31 pages.
[27] Z. Dong, Z. Jianliang, Martingale solutions and Markov selection of stochastic 3D Navier–Stokes equations with jump, J. Differential Equations 250 (2011) 2737–2778.
[28] M. Fabian, P. Habala, P. Hajek, V. Montesinos, V. Zizler, Banach Space Theory. The Basis for Linear and Nonlinear Analysis, CMS Books Math./Ouvrages Math. SMC, Springer, New York, 2011.
[29] F. Flandoli, M. Gubinelli, M. Hairer, M. Romito, Rigourous remarks about scaling laws in turbulent fluid, Comm. Math. Phys. 278 (2008) 1–29.
[30] J. Freshe, M. Ruzicka, Non-homogeneous generalized Newtonian fluids, Math. Z. 260 (2) (2008) 353–375.

[31] B. Guo, C. Guo, J. Zhang, Martingale and stationary solutions for stochastic non-Newtonian fluids, Differential Integral Equations 23 (3&4) (2010) 303–326.
[32] I. Gyöngy, N.V. Krylov, On stochastics equations with respect to semimartingales. II. Itô formula in Banach spaces, Stochastics 6 (3–4) (1981/1982) 153–173.
[33] B. Hasselblatt, A. Katok, A First Course in Dynamics. With a Panorama of Recent Developments, Cambridge University Press, New York, 2003.
[34] E. Hausenblas, P.A. Razafimandimby, M. Sango, Martingale solution to differential type fluids of grade two driven by random force of Lévy type, Potential Anal. 38 (4) (2013) 1291–1331.
[35] N. Ikeda, S. Watanabe, Stochastic Differential Equations and Diffusion Processes, second edition, N.-Holl. Math. Libr., vol. 24, North-Holland Publishing Co., Amsterdam, 1989.
[36] A. Kupiainen, Statistical theories of turbulence, in: Advances in Mathematical Sciences and Applications, Gakkotosho, Tokyo, 2003.
[37] O.A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, Gordon and Breach, New York, 1969.
[38] O.A. Ladyzhenskaya, New equations for the description of the viscous incompressible fluids and solvability in the large of the boundary value problems for them, in: Boundary Value Problems of Mathematical Physics V, American Mathematical Society, Providence, RI, 1970.
[39] J. Malek, J. Necas, A. Novotny, Measure-valued solutions and asymptotic behavior of a multipolar model of a boundary layer, Czechoslovak Math. J. 42 (3) (1992) 549–576.
[40] J. Malek, J. Necas, M. Rokyta, M. Ruzicka, Weak and Measure-Valued Solutions to Evolutionary PDEs, Appl. Math. Math. Comput., vol. 13, Chapman & Hall, London, 1996.
[41] R. Mikulevicius, B.L. Rozovskii, Stochastic Navier–Stokes equations and turbulent flows, SIAM J. Math. Anal. 35 (5) (2004) 1250–1310.
[42] J. Necas, A. Novotny, M. Silhavy, Global solution to the compressible isothermal multipolar fluids, J. Math. Anal. Appl. 162 (1991) 223–242.
[43] J. Necas, M. Silhavy, Multipolar viscous fluids, Quart. Appl. Math. XLIX (2) (1991) 247–266.
[44] E. Pardoux, Equations aux dérivées partielles stochastiques monotones, Thèse de Doctorat, Université Paris-Sud, 1975.
[45] S. Peszat, J. Zabczyk, Stochastic Partial Differential Equations with Levy Noise. An Evolution Equation Approach, Encyclopedia Math. Appl., vol. 113, Cambridge University Press, 2007.
[46] P.A. Razafimandimby, On stochastic models describing the motions of randomly forced linear viscoelastic fluids, J. Inequal. Appl. 2010 (2010) 932053, http://dx.doi.org/10.1155/2010/932053, 27 pages.
[47] P.A. Razafimandimby, M. Sango, Weak solutions of a stochastic model for two-dimensional second grade fluids, Bound. Value Probl. 2010 (2010) 636140, http://dx.doi.org/10.1155/2010/636140, 47 pages.
[48] P.A. Razafimandimby, M. Sango, Asymptotic behavior of solutions of stochastic evolution equations for second grade fluids, C. R. Math. Acad. Sci. Paris 348 (13–14) (2010) 787–790.
[49] P.A. Razafimandimby, M. Sango, Strong solution for a stochastic model of two-dimensional second grade fluids: existence, uniqueness and asymptotic behaviour, Nonlinear Anal. Theory Methods Appl. 75 (11) (2012) 4251–4270.
[50] P.A. Razafimandimby, M. Sango, On the exponential behaviour of stochastic evolution equations for non-Newtonian fluids, Appl. Anal. 91 (12) (2012) 2217–2233.
[51] J.C. Robinson, Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors, Cambridge Texts Appl. Math., Cambridge University Press, Cambridge, 2001.
[52] B. Rüdiger, Stochastic integration with respect to compensated Poisson random measure on separable Banach spaces, Stoch. Stoch. Rep. 76 (3) (2004) 213–242.
[53] M. Sango, Magnetohydrodynamic turbulent flows: existence results, Phys. D Nonlinear Phenom. 239 (12) (2010) 912–923.
[54] M. Sango, Density dependent stochastic Navier–Stokes equations with non Lipschitz random forcing, Rev. Math. Phys. 22 (6) (2010) 669–697.
[55] T. Taniguchi, The existence and asymptotic behaviour of energy solutions to stochastic 2D functional Navier–Stokes equations driven by Levy processes, J. Math. Anal. Appl. 385 (2) (2012) 634–654.
[56] R. Temam, Navier–Stokes Equations, North-Holland, 1979.
[57] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Appl. Math. Sci., vol. 68, Springer-Verlag, New York, 1988.