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Distortion of locally biholomorphic Bloch mappings on bounded symmetric domains

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We generalize Bonk's distortion theorem on the unit disc in the complex plane to locally biholomorphic mappings on finite dimensional bounded symmetric domains. As an application, we obtain a lower bound for the Bloch constant for various classes of locally biholomorphic Bloch mappings.

1. Introduction

Let $\mathbb{U} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ be the unit disc in $\mathbb{C}$ and let $f : \mathbb{U} \to \mathbb{C}$ be a holomorphic function with $f'(0) = 1$. The celebrated Bloch's theorem states that f maps a domain in $\mathbb{U}$ biholomorphically onto a disc with radius $r(f)$ greater than some positive absolute constant. The 'best possible' constant $\mathbf{B}$ for all such functions, that is,

$$\mathbf{B} = \inf\{r(f) : f \text{ is holomorphic on } \mathbb{U} \text{ and } f'(0) = 1\},$$

is called the Bloch constant. Bonk proved in [2] the following distortion theorem.

Theorem 1.1. *If $f : \mathbb{U} \to \mathbb{C}$ is a holomorphic function such that $f'(0) = 1$ and $\sup_{\zeta\in\mathbb{U}}(1 - |\zeta|^2)|f'(\zeta)| \le 1$, then the real part $\Re f'(\zeta)$ satisfies*

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$$\Re f'(\zeta) \geq \frac{1-\sqrt{3}|\zeta|}{\left(1-\frac{|\zeta|}{\sqrt{3}}\right)^3}, \quad |\zeta| \leq \frac{1}{\sqrt{3}}.$$

The above distortion theorem implies readily a result of Ahlfors [1] that the Bloch constant $\mathbf{B}$ is greater than $\sqrt{3}/4$ (see [2]). This lower bound was further improved in [2] to $\mathbf{B} > \frac{\sqrt{3}}{4} + 10^{-14}$, and in [5] to $\mathbf{B} \geq \frac{\sqrt{3}}{4} + 2 \times 10^{-4}$.

Bonk's distortion theorem has been extended by Liu in [20, Theorem 7] to the family $H_{\text{loc}}(\mathbb{B}^n, \mathbb{C}^n)$ of $\mathbb{C}^n$-valued locally biholomorphic mappings on the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, as follows.

Theorem 1.2. *If $f \in H_{\text{loc}}(\mathbb{B}^n, \mathbb{C}^n)$, $\|f\|_0 = 1$ and $\det Df(0) = 1$, then*

$$|\det Df(z)| \geq \Re \det Df(z) \geq \frac{\exp\left(\frac{-(n+1)\|z\|}{1-\|z\|}\right)}{(1-\|z\|)^{n+1}}, \quad z \in \mathbb{B}^n.$$

This inequality is sharp.

We refer to Definition 3.2 for the definition of the above prenorm $\|f\|_0$. Bloch's theorem fails in dimension 2. Nevertheless, one can define the Bloch constant for various families of Bloch mappings in higher dimensions. Using the above distortion theorem, lower and upper bounds for such a Bloch constant for $\mathbb{B}^n$ were obtained in [20]. For the class $H_{\text{loc}}(\mathbb{U}^n, \mathbb{C}^n)$ of locally biholomorphic mappings on the unit polydisc $\mathbb{U}^n$ in $\mathbb{C}^n$, the following distortion theorem has been shown by Wang and Liu [27, Theorem 3.2].

Theorem 1.3. *If $f \in H_{\text{loc}}(\mathbb{U}^n, \mathbb{C}^n)$, $\|f\|_0 = 1$ and $\det Df(0) = 1$, then*

$$|\det Df(z)| \geq \Re \det Df(z) \geq \frac{\exp\left(\frac{-2n\|z\|}{1-\|z\|}\right)}{(1-\|z\|)^{2n}}, \quad z \in \mathbb{U}^n.$$

This inequality is sharp.

This theorem was also used in [27] to derive a lower bound of the Bloch constant for classes of locally biholomorphic Bloch mappings on $\mathbb{U}^n$.

Both the Euclidean unit ball and the unit polydisc in $\mathbb{C}^n$ are examples of bounded symmetric domains in $\mathbb{C}^n$. The following natural questions arise.

Question 1.4. *Can we explain the difference of the exponents in the distortion bounds in Theorems 1.2 and 1.3?*

Question 1.5. *Can we extend Bonk's distortion theorem to other bounded symmetric domains in $\mathbb{C}^n$?*

We give an affirmative answer to both questions in this paper and as an application, we derive a lower bound of the Bloch constant for various classes of locally biholomorphic Bloch mappings on a finite dimensional bounded symmetric domain.

A finite dimensional bounded symmetric domain can be realized as the open unit ball B_X of a finite dimensional JB*-triple X, which carries a Jordan algebraic structure and is formed by the complex space $\mathbb{C}^n$ equipped with the Carathéodory norm (cf. [6, p. 153]), where $n = \dim X$. Such a realization can be viewed as a generalization of the Riemann Mapping Theorem, and enables us to use Jordan theory to derive analytic results for bounded symmetric domains.

The novelty of our approach is the use of Jordan theory. Indeed, the exponent of the distortion bound depends on the '*diameter*' $2c(B_X)$ of the ball B_X with respect to the Bergman metric at 0, defined by Hamada, Honda and Kohr in [13]. The constant $c(B_X)$ depends on the Jordan structure of the underlying JB*-triple X. For the Euclidean unit ball $\mathbb{B}^n$ and the unit polydisc $\mathbb{U}^n$, we have $c(\mathbb{B}^n) = (n+1)/2$ and $c(\mathbb{U}^n) = n$, respectively.

The lower bound of the Bloch constant obtained in Theorem 5.6 for classes of locally biholomorphic Bloch mappings on B_X is also given in terms of $2c(B_X)$.

We prove a distortion theorem in Theorem 4.1 for the class $H_{\mathrm{loc}}(B_X, \mathbb{C}^n)$ of $\mathbb{C}^n$-valued locally biholomorphic mappings on such a unit ball B_X and a special case of the theorem asserts that

$$|\det Df(z)| \geq \frac{1}{(1-\|z\|)^{2c(B_X)}} \exp\left\{ \frac{-2c(B_X)\|z\|}{1-\|z\|} \right\}$$

for $f \in H_{\mathrm{loc}}(B_X, \mathbb{C}^n)$, $\|f\|_0 = 1$ and $\det Df(0) = 1$. This generalizes Theorems 1.2 and 1.3, and also explains the difference of the exponents in the first question.

Our results also generalize simultaneously other results on Bonk's distortion theorem for locally univalent Bloch functions in one complex variable in [3,21], and those for locally biholomorphic Bloch mappings in several complex variables in [25]. We refer to [7,12–14] for other distortion theorems for normalized locally biholomorphic mappings on unit balls of finite dimensional JB*-triples.

2. Bounded symmetric domains and JB*-triples

Let B_X be the unit ball of a complex Banach space X. We will denote by $H(B_X, Y)$ the space of holomorphic mappings from B_X to a complex Banach space Y. A holomorphic mapping $f : B_X \to Y$ is said to be *locally biholomorphic* if the Fréchet derivative $Df(x)$ has a bounded inverse for each $x \in B_X$. A holomorphic mapping $f : B_X \to Y$ is said to be *biholomorphic* if $f(B_X)$ is a domain in Y, f^{-1} exists and is holomorphic on $f(B_X)$.

Let $L(X, Y)$ denote the Banach space of continuous linear operators from X to Y.

Finite dimensional bounded symmetric domains have been classified by Cartan [4]. The irreducible ones come in four classical series of Cartan domains (cf. [16,19]) and two exceptional domains. They can be described as the open unit balls of some finite dimensional JB*-triples (cf. [15] and [6, Theorem 2.5.9]). In this context, Cartan's classification has been extended by Kaup in [18, Theorem 5.4], which asserts that every bounded symmetric domain, including the infinite dimensional ones, is biholomorphic to the open unit ball of a JB*-triple, and conversely, the open unit ball of a JB*-triple is a bounded symmetric domain.

A JB*-*triple* is a complex Banach space X equipped with a continuous Jordan triple product

$$(x, y, z) \in X \times X \times X \mapsto \{x, y, z\} \in X$$

satisfying

(i) $\{x, y, z\}$ is symmetric bilinear in the outer variables, but conjugate linear in the middle variable,
(ii) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$,
(iii) $x \,\square\, x \in L(X, X)$ is a hermitian operator with spectrum $\geqq 0$,
(iv) $\|\{x, x, x\}\| = \|x\|^3$

for $a, b, x, y, z \in X$, where the *box operator* $x \,\square\, y : X \to X$ is defined by $x \,\square\, y(\cdot) = \{x, y, \cdot\}$.

Example 2.1. (i) A complex Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ is a JB*-triple with the triple product

$$\{x,y,z\} = \frac{1}{2}(\langle x,y\rangle z + \langle z,y\rangle x).$$

(ii) A C*-algebra $\mathcal{A}$ is a JB*-triple in the triple product

$$\{a,b,c\} = \frac{1}{2}(ab^*c + cb^*a).$$

(iii) The complex space $\mathbb{C}^n$ is also a JB*-triple when it is equipped with the ℓ_∞ norm $\|\cdot\|_\infty$ and the triple product

$$\{x,y,z\} = (x_i\overline{y_i}z_i)_{1\le i\le n},\ x=(x_i)_{1\le i\le n},\ y=(y_i)_{1\le i\le n},\ z=(z_i)_{1\le i\le n}\in\mathbb{C}^n.$$

The unit polydisc $\mathbb{U}^n$ is the unit ball of $(\mathbb{C}^n, \|\cdot\|_\infty)$.

We refer to [6,22,23] for relevant details of JB*-triples and references. We recall some of them which will be needed later.

An element u in a JB*-triple X is called a tripotent if $\{u,u,u\}=u$. Two tripotents u and v are said to be orthogonal to each other if $u\,\Box\, v=0$, which is equivalent to $v\,\Box\, u=0$ (cf. [6, Corollary 1.2.46]). A tripotent u is said to be *maximal* if the only tripotent which is orthogonal to u is 0. A tripotent u is said to be *minimal* if it cannot be written as a sum of two non-zero orthogonal tripotents. A *frame* is a maximal family of pairwise orthogonal minimal tripotents. In a finite dimensional JB*-triple X, the cardinality of all frames is the same, and is called the *rank* of X. As usual, we denote by $\mathrm{Aut}(B_X)$ the automorphism group of the open unit ball B_X of a JB*-triple X, consisting of biholomorphic self-maps of B_X.

Now let X be a finite dimensional JB*-triple. Then its open unit ball B_X is (biholomorphic to) a bounded symmetric domain in some $\mathbb{C}^n$.

Given a tripotent $v\in X$, the possible eigenvalues of the box operator $v\,\Box\, v$ are 0, 1/2 or 1, which induces the following eigenspace decomposition of X:

$$X = V_0(v)\oplus V_1(v)\oplus V_2(v),$$

called the *Peirce decomposition* of X, where $V_j(v)=\{x\in X : 2(v\,\Box\, v)x = jx\}$ for $j=0,1,2$. Let u be a maximal tripotent in X. Then, there exist orthogonal tripotents $u_1,\dots,u_r$ such that $u=u_1+\cdots+u_r$, where r is the rank of X [23, Proposition VI.3.2]. Since $u_1,\dots,u_r$ are linearly independent in $V_2(u)$, we have

$$\dim V_2(u)\ge r. \tag{2.1}$$

We recall that the constant $c(B_X)$, introduced in [13], is defined by

$$c(B_X) = \frac{1}{2}\sup_{x,y\in B_X}|h_0(x,y)|,$$

where h_0 is the Bergman metric at 0. One can view $2c(B_X)$ as the 'diameter' of B_X measured by the metric h_0. In [13], it is proved that

$$c(B_X) = \frac{1}{2}(\dim V_1(u) + 2\dim V_2(u)), \tag{2.2}$$

where u is an arbitrary maximal tripotent in X. From (2.1), (2.2) and the fact that $V_0(u)=0$, we deduce that

$$\frac{1}{2}(\dim X + r) \le c(B_X) \le \dim X, \tag{2.3}$$

where r is the rank of X.

Let $(X, \|\cdot\|)$ be a JB*-triple and let $H(B_X, \mathbb{C}^n)$ denote the space of holomorphic mappings from B_X to $\mathbb{C}^n$, where $\mathbb{C}^n$ is equipped with the Euclidean norm $\|\cdot\|_e$. The norm of a bounded operator $A \in L(X, \mathbb{C}^n)$ will be denoted by

$$\|A\|_{X,e} = \sup\left\{\|Az\|_e : \|z\| = 1\right\}.$$

The norm of $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ will be abbreviated to

$$\|A\|_e = \sup\{\|Az\|_e : \|z\|_e = 1\}.$$

3. Bloch mappings

The notion of a $\mathbb{C}^n$-valued Bloch mapping on a finite dimensional bounded symmetric domain, under the name of *normal mapping of finite order*, was first introduced by Hahn [9]. Several equivalent definitions for complex-valued Bloch functions on a finite dimensional bounded homogeneous domain have been given by Timoney in [24]. $\mathbb{C}^n$-valued Bloch mappings on the Euclidean ball of $\mathbb{C}^n$ have also been studied in [20]. The following definition for a Bloch mapping from a finite dimensional bounded symmetric domain to $\mathbb{C}^n$ given by Hamada [10] is a direct extension of the one in [24, Theorem 3.4 (4)] and [20].

Definition 3.1. Let B_X be the unit ball of a finite dimensional JB*-triple X. A mapping $f \in H(B_X, \mathbb{C}^n)$ is called a *Bloch mapping* if the family

$$F_f = \{f \circ \varphi - f(\varphi(0)) : \varphi \in \mathrm{Aut}(B_X)\}$$

is *normal*, that is, every sequence in F_f contains a subsequence converging uniformly on compact subsets of B_X.

Equivalently, $f \in H(B_X, \mathbb{C}^n)$ is a Bloch mapping if

$$\|f\|_{\mathcal{B}} = \sup\left\{\|D(f \circ \varphi)(0)\|_{X,e} : \varphi \in \mathrm{Aut}(B_X)\right\} < \infty$$

(cf. [20,24]), where $\|f\|_{\mathcal{B}}$ is called the *Bloch semi-norm* of f.

For $1 \le K \le +\infty$, we will denote by $\beta(B_X, \mathbb{C}^n, K)$ the set of Bloch mappings $f \in H(B_X, \mathbb{C}^n)$ with $\|f\|_{\mathcal{B}} \le K$.

We note that, in the above definition of a $\mathbb{C}^n$-valued Bloch mapping, we do not require that the domain B_X has the same dimension n, although this is the case in the following results. We recall that an n-dimensional JB*-triple X is the complex space $(\mathbb{C}^n, \|\cdot\|_X)$ equipped with the Carathéodory norm $\|\cdot\|_X$.

Definition 3.2. Let B_X be the unit ball of an n-dimensional JB*-triple X. We define the *prenorm* $\|f\|_0$ of $f \in H(B_X, \mathbb{C}^n)$ by

$$\|f\|_0 = \sup\left\{(1 - \|z\|^2)^{c(B_X)/n} |\det Df(z)|^{1/n} : z \in B_X\right\}.$$

We will make use of the following two lemmas which have been proved in [10].

Lemma 3.3. *Let B_X be the unit ball of an n-dimensional JB^*-triple X and $f \in H(B_X, \mathbb{C}^n)$.*

(i) *If f is a Bloch mapping on B_X, then we have*

$$\|Df(z)\|_{X,e} \leq \frac{\|f\|_{\mathcal{B}}}{1 - \|z\|^2}, \quad z \in B_X$$

and

$$\|f\|_0 \leq \sup\left\{|\det Dg(0)|^{1/n} : g \in F_f\right\} < +\infty.$$

(ii) *If $\|f\|_0 < +\infty$, then*

$$|\det Df(z)| \leq \frac{\|f\|_0^n}{(1 - \|z\|^2)^{c(B_X)}}, \quad z \in B_X.$$

(iii) *If $\|f\|_0 = 1$ and $\det Df(0) = 1$, then $|\det Df(z)| = 1 + o(\|z\|)$.*

For $x \in X \setminus \{0\}$, the set

$$T(x) = \{l_x \in X^* : \ l_x(x) = \|x\|, \ \|l_x\| = 1\}$$

of support functionals of x is nonempty in view of the Hahn–Banach theorem. Let $H(\mathbb{U}) = H(\mathbb{U}, \mathbb{C})$ denote the set of holomorphic functions on the unit disc $\mathbb{U}$ in $\mathbb{C}$.

Lemma 3.4. *Fix a point u in the topological boundary ∂B_X and let*

$$f(z) = \left(\int_0^{l_u(z)} \psi(t)dt\right) u + z - l_u(z)u, \quad z \in B_X,$$

where $l_u \in T(u)$ and $\psi \in H(\mathbb{U})$. Then $f \in H(B_X, \mathbb{C}^n)$, $f(0) = 0$ and $\det Df(z) = \psi(l_u(z))$ for $z \in B_X$.

Next, we recall some basic facts concerning subdomains in the unit disc $\mathbb{U}$.

Definition 3.5. Let $\Omega \subset \mathbb{C}$ be a domain containing the origin and let f and g be holomorphic functions on Ω. We say that f is *subordinate* to g if there exists a holomorphic function $v : \Omega \to \Omega$ such that $v(0) = 0$ and $f = g \circ v$. We write $f \prec g$ to denote this subordination relation.

For $a \in \mathbb{C}$ and $r > 0$, we let

$$\mathbb{U}(a, r) = \{\zeta \in \mathbb{C} : |\zeta - a| < r\}$$

and let $\Delta(1, r)$ be a horodisc in $\mathbb{U}$, that is,

$$\Delta(1, r) = \left\{\zeta \in \mathbb{U} : \frac{|1 - \zeta|^2}{1 - |\zeta|^2} < r\right\} = \mathbb{U}\left(\frac{1}{1 + r}, \frac{r}{1 + r}\right).$$

The boundary $\partial\Delta(1, r)$ is a circle internally tangent to the unit circle at 1. Given $r > 1$, Wang [25, Lemma 1] has obtained the following lemma.

Lemma 3.6. *Let $r > 1$. Assume that $h \in H(\mathbb{U})$, $h(0) = a \in \mathbb{R}$ and that there exists a positive number $s > 0$ such that $h(\Delta(1,r)) \subset \{w : \Re w < s\}$. Then*

(i) $h(\zeta) \prec G_0(\zeta) = b\frac{\zeta+1}{\zeta-1} + b + a$ *on* $\Delta(1,r)$, *where* $b = \frac{r(s-a)}{r-1} > 0$.
(ii) $\Re h(x) \geq G_0(x) = \frac{2bx}{x-1} + a$ *for* $0 < x < 1$, *and equality holds for some x if and only if* $h = G_0$.
(iii) $\Re h(-x) \leq G_0(-x) = \frac{2bx}{x+1} + a$ *for* $0 < x \leq \frac{r-1}{r+1}$, *and equality holds for some x if and only if* $h = G_0$.

The following lemma can be derived directly from the classical Julia's lemma (see [27, Lemma 2.2] and [21, p. 327]).

Lemma 3.7. *Let g be a holomorphic function on $\mathbb{U} \cup \{1\}$. Assume that $g(\mathbb{U}) \subset \mathbb{U} \setminus \{0\}$ and $g(1) = 1$. Then $g'(1) = \alpha > 0$ and*

$$|g(x)| \geq \exp\left\{-2\alpha\frac{1-x}{1+x}\right\}, \quad \textit{for all } x \in (-1,1).$$

4. Distortion theorems

In this section, we prove a distortion theorem for locally biholomorphic mappings on the unit ball B_X of a finite dimensional JB*-triple X, which is a generalization of [20, Theorem 7], [25, Theorem 1], [26, Corollary 1.1] and [27, Theorem 3.2].

Theorem 4.1. *Let B_X be the unit ball of an n-dimensional JB*-triple X. Let $\alpha \in (0,1]$ and let $m(\alpha)$ be the unique root of the equation*

$$e^{-c(B_X)x}(1+x)^{c(B_X)} = \alpha \tag{4.1}$$

in the interval $[0,+\infty)$. If $f \in H_{\text{loc}}(B_X,\mathbb{C}^n)$, $\|f\|_0 = 1$ and $\det Df(0) = \alpha$, then we have

(i)

$$|\det Df(z)| \geq \frac{\alpha}{(1-\|z\|)^{2c(B_X)}} \exp\left\{(1+m(\alpha))\frac{-2c(B_X)\|z\|}{1-\|z\|}\right\} \tag{4.2}$$

for $z \in B_X$;

(ii)

$$|\det Df(z)| \leq \frac{\alpha}{(1+\|z\|)^{2c(B_X)}} \exp\left\{(1+m(\alpha))\frac{2c(B_X)\|z\|}{1+\|z\|}\right\} \tag{4.3}$$

for $\|z\| \leq \dfrac{m(\alpha)}{2+m(\alpha)}$.

The estimates in (4.2) and (4.3) are sharp.

Proof. We shall use arguments similar to those in the proof of [25, Theorem 1]. Let $c = c(B_X)$ and let

$$r(t) = e^{-ct}(1+t)^c, \quad t \in [0,+\infty).$$

Then $r(t)$ is decreasing on $[0,+\infty)$, $r(0) = 1$ and $r(+\infty) = 0$. Therefore, there exists a unique $m(\alpha) \in [0,+\infty)$ such that

$$e^{-cm(\alpha)}(1+m(\alpha))^c = \alpha.$$

Let $z \in B_X \setminus \{0\}$ be fixed and let $u = z/\|z\|$.

(i) First, we consider the case $\alpha \in (0,1)$. Then $m(\alpha) > 0$. Let

$$g(\zeta) = (1-\zeta)^{2c} \det Df(\zeta u), \quad \zeta \in \mathbb{U}.$$

Then $g \in H(\mathbb{U})$, $g(\zeta) \neq 0$ on $\mathbb{U}$ and $g(0) = \alpha$. Since $\|f\|_0 = 1$, Lemma 3.3 (ii) yields

$$|g(\zeta)| \leq \left(\frac{|1-\zeta|^2}{1-|\zeta|^2} \right)^c.$$

Let $h(\zeta) = \log g(\zeta)$, where the branch of the logarithm is chosen such that $h(0) = \log g(0) = \log \alpha$ is real. Then we have

$$\Re h(\zeta) = \log |g(\zeta)| \leq c \log \frac{|1-\zeta|^2}{1-|\zeta|^2}, \quad \zeta \in \mathbb{U}.$$

Therefore we have

$$h(\Delta(1, 1+m(\alpha))) \subset \{w : \Re w < c \log(1+m(\alpha))\}.$$

In view of Lemma 3.6 (i), we obtain $h \prec G_0$ on $\Delta(1, 1+m(\alpha))$, where

$$G_0(\zeta) = b \frac{\zeta+1}{\zeta-1} + b + \log \alpha \text{ and}$$

$$b = \frac{1+m(\alpha)}{m(\alpha)} (c \log(1+m(\alpha)) - \log \alpha) = c(1+m(\alpha)).$$

For the last equality, we use the identity

$$e^{-cm(\alpha)}(1+m(\alpha))^c = \alpha.$$

For any $x \in (0,1)$, we deduce from Lemma 3.6 (ii) that

$$\log |g(x)| = \Re h(x) \geq c(1+m(\alpha)) \frac{2x}{x-1} + \log \alpha.$$

This implies that

$$|g(x)| \geq \alpha \exp \left\{ c(1+m(\alpha)) \frac{-2x}{1-x} \right\}.$$

Putting $x = \|z\|$ in the above inequality, we obtain the inequality (4.2) for $\alpha \in (0,1)$.

Next, we consider the case $\alpha = 1$ for which $m(\alpha) = 0$. Let

$$g(\zeta) = \left(\frac{1+\zeta}{2} \right)^{2c} \det Df \left(\frac{1-\zeta}{2} u \right), \quad \zeta \in \mathbb{U}.$$

Then g is holomorphic on $\mathbb{U} \cup \{1\}$ and $g(1) = 1$. Since $\|f\|_0 = 1$ and $\det Df(0) = 1$, we have from Lemma 3.3 (ii) and (iii) that $g'(1) = c$ and

$$
\begin{aligned}
|g(\zeta)| &= \left|\frac{1+\zeta}{2}\right|^{2c} \left|\det Df\left(\frac{1-\zeta}{2}u_1\right)\right| \\
&\le \left(\left|1-\frac{1-\zeta}{2}\right|^2 \frac{1}{1-\left|\frac{1-\zeta}{2}\right|^2}\right)^c \\
&< 1
\end{aligned}
$$

for $\zeta \in \mathbb{U}$, where

$$
\frac{1-\zeta}{2} \in \mathbb{U}\left(\frac{1}{2},\frac{1}{2}\right) = \left\{\xi \in \mathbb{U} : \frac{|1-\xi|^2}{1-|\xi|^2} < 1\right\}.
$$

This implies $g(\mathbb{U}) \subset \mathbb{U} \setminus \{0\}$. By Lemma 3.7, we obtain

$$
|g(x)| \ge \exp\left\{-2c\frac{1-x}{1+x}\right\}
$$

for all $x \in (-1, 1)$. Putting $x = 1 - 2\|z\|$ in the above inequality, we obtain the inequality (4.2) for $\alpha = 1$.

(ii) If $\alpha = 1$, then $m(\alpha) = 0$ and the inequality (4.3) holds trivially.

Now let $\alpha \in (0, 1)$ and let

$$
g(\zeta) = (1-\zeta)^{2c} \det Df(-\zeta u) \qquad (\zeta \in \mathbb{U}).
$$

We define the mappings h and G_0 as in the proof of (i).

By the arguments in (i) and Lemma 3.6 (iii), we derive

$$
\Re h(-x) \le G_0(-x) = 2c(1+m(\alpha))\frac{x}{x+1} + \log\alpha
$$

for $0 < x \le \dfrac{m(\alpha)}{2+m(\alpha)}$. Putting $x = \|z\|$ in the above inequality, one obtains the inequality (4.3).

Finally, we will show that the estimates (4.2) and (4.3) are sharp. Indeed, fix any $u \in \partial B_X$ and let

$$
F(z) = \left(\int_0^{l_u(z)} \psi(t)dt\right) u + z - l_u(z)u,
$$

where $l_u \in T(u)$ and

$$
\psi(\zeta) = \frac{\alpha}{(1-\zeta)^{2c}} \exp\left\{(1+m(\alpha))\frac{-2c\zeta}{1-\zeta}\right\} \in H(\mathbb{U}).
$$

Then $F \in H(B_X, \mathbb{C}^n)$, $F(0) = 0$ and $\det DF(z) = \psi(l_u(z))$ by Lemma 3.4. Therefore $\det DF(0) = \psi(0) = \alpha$. For any $z \in B_X$, let $\zeta = l_u(z)$. Since $e^{-cm(\alpha)}(1+m(\alpha))^c = \alpha$, we have

$$
\begin{aligned}
(1-\|z\|^2)^c |\det DF(z)| &\le (1-|l_u(z)|^2)^c |\psi(l_u(z))| \\
&= \left(\frac{1-|\zeta|^2}{|1-\zeta|^2}\right)^c \alpha \left|\exp\left((1+m(\alpha))\frac{-2c\zeta}{1-\zeta}\right)\right| \\
&= \left(\frac{1-|\zeta|^2}{|1-\zeta|^2} b \exp\left(1 - b\Re\left(1+\frac{2\zeta}{1-\zeta}\right)\right)\right)^c
\end{aligned}
$$

$$= (bt\exp(1-bt))^c$$
$$\leq 1,$$

where $b = 1 + m(\alpha)$ and

$$t = \frac{1-|\zeta|^2}{|1-\zeta|^2} > 0.$$

Note that in the last inequality, we have used the inequality

$$xe^{1-x} \leq 1 \quad \text{for } x > 0.$$

Therefore $\|F\|_0 \leq 1$. Let $z = \zeta u$. Then $\|z\| = |\zeta|$, $l_u(z) = \zeta$ and the equality $(1-\|z\|^2)^c|\det DF(z)| = 1$ holds when $t = 1/b$. This implies that $\|F\|_0 = 1$. Since $\det DF(\pm\|z\|u) = \psi(\pm\|z\|)$ for all $z \in B_X$, F attains the equalities in (4.2) and (4.3). This completes the proof. □

Remark 4.2. (i) If $B_X = \mathbb{B}^n$, where $\mathbb{B}^n$ is the Euclidean unit ball in $\mathbb{C}^n$, then $c(\mathbb{B}^n) = (n+1)/2$ by [13], and hence Theorem 4.1 reduces to [20, Theorem 7] and [25, Theorem 1]. In particular, for the unit disc $B_X = \mathbb{U}$ in $\mathbb{C}$, Theorem 4.1 reduces to [3, Theorem 3].

(ii) Let $\mathbb{U}^n$ be the unit polydisc in $\mathbb{C}^n$. The Bergman metric at 0 is given by

$$h_0(u,v) = 2\sum_{j=1}^{n} u_j \overline{v}_j.$$

Hence $c(\mathbb{U}^n) = n$ and if $\alpha = 1$, Theorem 4.1 reduces to [27, Theorem 3.2].

5. Bloch constant

Given the open unit ball B_X of an n-dimensional JB*-triple $X = (\mathbb{C}^n, \|\cdot\|_X)$, we will assume throughout this section that

$$B_X \supset \mathbb{B}^n. \tag{5.1}$$

This assumption is not too restrictive since the unit polydisc satisfies this condition and for any B_X, there exists a constant $r > 0$ such that the ball $B = rB_X$ satisfies (5.1). Under the above assumption, we give a lower estimate for the radius of the largest univalent ball in the image of f centered at $f(0)$.

Let $\mathbb{B}^n(b, r)$ denote the Euclidean ball with center b and radius r. For $f \in H(B_X, \mathbb{C}^n)$, a *schlicht ball* of f centered at $f(a)$ is a Euclidean ball $\mathbb{B}^n(f(a), r)$ such that f maps an open set $G \subset B_X$ with $a \in G$ biholomorphically onto this ball.

For a point $a \in B_X$, let $r(a, f)$ be the radius of the largest schlicht ball of f centered at $f(a)$, that is,

$$r(a,f) = \sup\{r > 0 : \mathbb{B}^n(f(a), r) \subset f(B_X),\ f^{-1} \text{ is biholomorphic on } \mathbb{B}^n(f(a), r)\}.$$

Let $r(f) = \sup\{r(a, f) : a \in B_X\}$. For the class $\beta(B_X, \mathbb{C}^n, K) \cap H_{\text{loc}}(B_X, \mathbb{C}^n)$, one can define the Bloch constant to be

$$\mathcal{B}_{\text{loc}}(K) = \inf\{r(f) : f \in \beta(B_X, \mathbb{C}^n, K) \cap H_{\text{loc}}(B_X, \mathbb{C}^n), \det Df(0) = 1\}.$$

As in [8, Theorem 1.3], we obtain the following result.

Proposition 5.1. *Let B_X be the unit ball of an n-dimensional JB*-triple X. For any $K \geq 1$, there exists $f \in \beta(B_X, \mathbb{C}^n, K) \cap H_{\text{loc}}(B_X, \mathbb{C}^n)$ such that $\mathcal{B}_{\text{loc}}(K) = r(f)$, $\det Df(0) = 1$ and $\|f\|_0 = 1$.*

Definition 5.2. A point $z_0 \in B_X$ is called a *critical point* of $f \in H(B_X, \mathbb{C}^n)$ if $\det Df(z_0) = 0$. In this case $f(z_0)$ is called a *critical value* of f.

The following lemma is a generalization of [20, Lemma 2] to the unit ball of a finite dimensional JB*-triple. Since the proof of [20, Lemma 2] can be applied directly to our case, we omit it.

Lemma 5.3. *Let B_X be the unit ball of an n-dimensional JB*-triple X. Let $f \in H(B_X, \mathbb{C}^n)$ and G be an open subset of B_X with $a \in G$. If f maps G biholomorphically onto the schlicht ball $\mathbb{B}^n(f(a), r(a, f))$, then either G and B_X have a common boundary point or there exists a critical value $f(z_0)$ on the boundary of the ball $\mathbb{B}^n(f(a), r(a, f))$ with the critical point z_0 on the boundary of G.*

The following lemma was proved by Hamada and Kohr in [11].

Lemma 5.4. *Assume that the condition (5.1) holds. Let $A \in L(X, \mathbb{C}^n)$. Then the following inequality holds:*

$$\|Aw\|_e \geq \frac{|\det A|}{\|A\|_{X,e}^{n-1}} \quad (w \in \partial B_X) \quad \text{if } \|A\|_{X,e} > 0. \tag{5.2}$$

Remark 5.5. The inequality (5.2) need not hold if we use the JB*-norm $\|\cdot\|_X$ for the codomain $\mathbb{C}^n$. Indeed, let $X = (\mathbb{C}^2, \|\cdot\|_\infty)$ and let $A \in L(X, \mathbb{C}^2)$ be given by

$$A = \begin{bmatrix} 1 & a \\ 0 & 2 \end{bmatrix},$$

where $a \in (0, 1)$. Then $\det A = 2$ and $\|A\|_{\infty,\infty} = 2$. Then we have

$$\left\| A \begin{bmatrix} 1 \\ -\varepsilon \end{bmatrix} \right\|_\infty < 1 = \frac{\det A}{\|A\|_{\infty,\infty}}$$

for small $\varepsilon > 0$.

For a locally biholomorphic Bloch mapping f, we obtain the following lower estimate for the radius of the largest schlicht ball of f centered at $f(0)$. The following result is a generalization of [20, Theorem 8], [25, Theorem 2] and [27, Theorem 3.4] to the unit ball of a finite dimensional JB*-triple.

Theorem 5.6. *Let B_X be the unit ball of an n-dimensional JB*-triple X. Also, assume that the condition (5.1) is satisfied. If $f \in \beta(B_X, \mathbb{C}^n, K) \cap H_{\text{loc}}(B_X, \mathbb{C}^n)$, $\|f\|_0 = 1$ and $\det Df(0) = \alpha \in (0, 1]$, then we have*

$$\begin{aligned} r(0, f) &\geq K^{1-n}\alpha \int_0^1 \frac{(1-t^2)^{n-1}}{(1-t)^{2c(B_X)}} \exp\left\{(1 + m(\alpha))\frac{-2c(B_X)t}{1-t}\right\} dt \\ &\geq \frac{\alpha K^{1-n}}{2c(B_X)(1 + m(\alpha))} \end{aligned}$$

where $m(\alpha)$ is the unique root of the equation

$$e^{-c(B_X)x}(1+x)^{c(B_X)} = \alpha$$

in the interval $[0, +\infty)$.

Proof. We shall use arguments similar to those in the proof of [25, Theorem 2]. Write $c = c(B_X)$. By Lemma 5.3, $r(0, f)$ is equal to the Euclidean distance from $f(0)$ to a boundary point of $f(B_X)$ since f is locally biholomorphic on B_X. Hence there exists a line segment Γ of Euclidean length $r(0, f)$ from $f(0)$ to a point in $\partial f(B_X)$. Note that $r(0, f)$ is the largest nonnegative number r such that there exists a domain $V \subset B_X$ which is mapped biholomorphically onto $\mathbb{B}^n(f(0), r)$ by f. Let $\gamma = (f|_V)^{-1}(\Gamma)$. Then γ is a smooth curve which is not relatively compact in B_X. By Lemma 5.4, we have

$$\begin{aligned} r(0, f) &= \int_\Gamma \|dw\|_e = \int_\gamma \|Df(z)dz\|_e = \int_\gamma \left\| Df(z)\frac{dz}{\|dz\|} \right\|_e \|dz\| \\ &\geq \int_\gamma \frac{|\det Df(z)|}{\|Df(z)\|_{X,e}^{n-1}} \|dz\|. \end{aligned}$$

From Theorem 4.1 (i) and Lemma 3.3 (i), we deduce

$$\begin{aligned} &\int_\gamma \frac{|\det Df(z)|}{\|Df(z)\|_{X,e}^{n-1}} \|dz\| \\ &\geq K^{1-n}\alpha \int_\gamma \frac{(1-\|z\|^2)^{n-1}}{(1-\|z\|)^{2c(B_X)}} \exp\left\{(1+m(\alpha))\frac{-2c(B_X)\|z\|}{1-\|z\|}\right\} \|dz\| \\ &\geq K^{1-n}\alpha \int_\gamma \frac{(1-\|z\|^2)^{n-1}}{(1-\|z\|)^{2c(B_X)}} \exp\left\{(1+m(\alpha))\frac{-2c(B_X)\|z\|}{1-\|z\|}\right\} d\|z\|, \end{aligned}$$

where $d\|z\| \leq \|dz\|$ a.e. on γ by [17, Lemma 1.3]. Therefore, we have

$$r(0, f) \geq K^{1-n}\alpha \int_0^1 \frac{(1-t^2)^{n-1}}{(1-t)^{2c(B_X)}} \exp\left\{(1+m(\alpha))\frac{-2c(B_X)t}{1-t}\right\} dt.$$

Since $c(B_X) \geq (n+1)/2$ by (2.3), we also have

$$\begin{aligned} r(0, f) &\geq K^{1-n}\alpha \int_0^1 \frac{1}{(1-t)^2} \exp\left\{(1+m(\alpha))\frac{-2c(B_X)t}{1-t}\right\} dt \\ &\geq \frac{\alpha K^{1-n}}{2c(B_X)(1+m(\alpha))}. \end{aligned}$$

This completes the proof. □

Remark 5.7. (i) If B_X is the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, then $c(\mathbb{B}^n) = (n+1)/2$ and Theorem 5.6 reduces to [20, Theorem 8] and [25, Theorem 2]. For $n = 1$, Theorem 5.6 reduces to [3, Corollary 3].

(ii) If B_X is the unit polydisc $\mathbb{U}^n$ in $\mathbb{C}^n$, then $c(\mathbb{U}^n) = n$ and for $\alpha = 1$, Theorem 5.6 reduces to [27, Theorem 3.4].

When $\alpha = 1$, we obtain the following result which is a generalization of [20, Corollary, p. 362] and [27, Theorem 3.4] to the unit ball of a finite dimensional JB*-triple.

Corollary 5.8. *Let B_X be the unit ball of an n-dimensional JB^*-triple X.*

(i) *Assume that the condition (5.1) is satisfied. Then we have*

$$\mathcal{B}_{\mathrm{loc}}(K) \geq C_0(K, B_X, n),$$

where

$$C_0(K, B_X, n) = K^{1-n} \int_0^1 \frac{(1-t^2)^{n-1}}{(1-t)^{2c(B_X)}} \exp\left\{\frac{-2c(B_X)t}{1-t}\right\} dt.$$

(ii) *If $n \geq 2$ and $\inf\{\|z\|_e : z \in \partial B_X\} = 1$ holds, then we have*

$$\tilde{K}^{1-n} \geq \mathcal{B}_{\mathrm{loc}}(K),$$

where $\tilde{K} = K / \sup\{\|z\|_e : z \in \partial B_X\}$.

Proof. (i) From Proposition 5.1 and Theorem 5.6, we obtain the inequality $\mathcal{B}_{\mathrm{loc}}(K) \geq C_0(K, B_X, n)$.

(ii) Since $\inf\{\|z\|_e : z \in \partial B_X\} = 1$ holds, there exists $e_1 \in \partial B_X$ with $\|e_1\|_e = 1$. Let $e_1, e_2, \dots, e_n$ be an orthonormal basis of $\mathbb{C}^n$ with respect to the Euclidean inner product and let $f(z) = \tilde{K}^{1-n} z_1 e_1 + \tilde{K} z_2 e_2 + \cdots + \tilde{K} z_n e_n$. Since

$$\|Df(\varphi(0))D\varphi(0)\|_{X.e} \leq \tilde{K}\|D\varphi(0)\|_{X.e} \leq K$$

for all $\varphi \in \mathrm{Aut}(B_X)$, we have $f \in \beta(B_X, \mathbb{C}^n, K) \cap H_{\mathrm{loc}}(B_X, \mathbb{C}^n)$. Therefore $\mathcal{B}_{\mathrm{loc}}(K) \leq r(f) = \tilde{K}^{1-n}$. $\square$

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