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Spectral analysis of certain groups of isometries on Hardy and Bergman spaces ☆

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ABSTRACT

Using the similarity theory of semigroups as well as spectral theory, we obtain the resolvents of the generators of strongly continuous groups of isometries on the Hardy and Bergman spaces. These groups are obtained as weighted composition operators associated with specific automorphisms of the upper half-plane. The resulting resolvents are given as integral operators for which we determine the norms and spectra.

1. Introduction

Let $\mathbb{C}$ be the complex plane. The set $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, is called the (open) unit disc. Let dA denote the area measure on $\mathbb{D}$, and for $\alpha \in \mathbb{R}$, $\alpha > -1$, we define a positive Borel measure dm_α on $\mathbb{D}$ by $dm_\alpha(z) = (1 - |z|^2)^\alpha dA(z)$. On the other hand, the set $\mathbb{U} = \{\omega \in \mathbb{C} : \Im(\omega) > 0\}$ denotes the upper half of the complex plane $\mathbb{C}$, and where $\Im(\omega)$ stands for the imaginary part of ω. For $\alpha > -1$, we define a weighted measure on $\mathbb{U}$ by $d\mu_\alpha(\omega) = (\Im(\omega))^\alpha\, dA(\omega)$. The Cayley transform $\psi(z) := \frac{i(1+z)}{1-z}$ maps the unit disc $\mathbb{D}$ conformally onto the upper half-plane $\mathbb{U}$ with inverse $\psi^{-1}(\omega) = \frac{\omega-i}{\omega+i}$.

For an open subset Ω of $\mathbb{C}$, let $\mathcal{H}(\Omega)$ denote the Fréchet space of analytic functions $f : \Omega \to \mathbb{C}$ endowed with the topology of uniform convergence on compact subsets of Ω. Let $\mathrm{Aut}(\Omega) \subset \mathcal{H}(\Omega)$ denote the group of biholomorphic maps $f : \Omega \to \Omega$. For $1 \le p < \infty$, the Hardy spaces of the upper half plane, $H^p(\mathbb{U})$, are defined as

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$$H^P(\mathbb{U}) := \left\{ f \in \mathcal{H}(\mathbb{U}) : \|f\|_{H^p(\mathbb{U})} := \sup_{y>0} \left(\int_{-\infty}^{\infty} |f(x+iy)|^p dx \right)^{1/p} < \infty \right\},$$

while the Hardy spaces of the unit disc, $H^p(\mathbb{D})$, by

$$H^P(\mathbb{D}) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|^p_{H^p(\mathbb{D})} := \sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta < \infty \right\}.$$

We note that every function $f \in H^p(\mathbb{U})$ (or $H^p(\mathbb{D})$) has non-tangential boundary values almost everywhere on $\partial\mathbb{U}$ (or $\partial\mathbb{D}$), see for example [8]. In particular, H^p-functions may be identified with their boundary values and with this convention,

$$\|f\|_{H^p(\mathbb{U})} = \left(\int_{-\infty}^{\infty} |f(x)|^p \, dx \right)^{\frac{1}{p}} \text{ and } \|f\|_{H^p(\mathbb{D})} = \left(\int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \right)^{\frac{1}{p}}.$$

On the other hand, for $1 \le p < \infty$, $\alpha > -1$, the weighted Bergman spaces on the upper half plane, $L^p_a(\mathbb{U}, \mu_\alpha)$, are defined by

$$L^p_a(\mathbb{U}, \mu_\alpha) := \left\{ f \in \mathcal{H}(\mathbb{U}) : \|f\|_{L^p_a(\mathbb{U},\mu_\alpha)} = \left(\int_{\mathbb{U}} |f(z)|^p \, d\mu_\alpha(z) \right)^{\frac{1}{p}} < \infty \right\},$$

while the corresponding spaces on the disc, $L^p_a(\mathbb{D}, m_\alpha)$, by

$$L^p_a(\mathbb{D}, m_\alpha) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{L^p_a(\mathbb{D},m_\alpha)} = \left(\int_{\mathbb{D}} |f(z)|^p \, dm_\alpha(z) \right)^{\frac{1}{p}} < \infty \right\}.$$

In particular, $L^p_a(\cdot) = L^P(\cdot) \cap \mathcal{H}(\cdot)$ where $L^p(\cdot)$ denotes the classical Lebesgue spaces. For a comprehensive theory of Hardy and Bergman spaces, we refer to [8,9,12,15,16]. As noted in [1] and [3], the Hardy space $H^p(\cdot)$ behaves in many ways as the limiting case of $L^p_a(\cdot)$ as $\alpha \to -1^+$. Therefore, we shall let X denote either the Hardy space $H^p(\mathbb{U})$ or the weighted Bergman space $L^p_a(\mathbb{U}, \mu_\alpha)$, and we associate with each X, a parameter $\gamma = \frac{\alpha+2}{p}$, where $\alpha = -1$ in the case that $X = H^p(\mathbb{U})$. Also, we shall let $X(\mathbb{D})$ denote the corresponding space of analytic functions on the unit disc $\mathbb{D}$.

If X is an arbitrary Banach space, let $\mathcal{L}(X)$ denote the algebra of bounded linear operators on X. For a linear operator T with domain $\mathcal{D}(T) \subset X$, denote the spectrum and point spectrum of T by $\sigma(T, X)$ and $\sigma_p(T, X)$ respectively. The resolvent set of T is $\rho(T, X) = \mathbb{C} \setminus \sigma(T, X)$ while $r(T)$ denotes its spectral radius. For a good account of the theory of spectra, see [6,7,13]. If X and Y are arbitrary Banach spaces and $U \in \mathcal{L}(X, Y)$ is an invertible operator, then clearly $(A_t)_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ is a strongly continuous group if and only if $B_t := UA_tU^{-1}$, $t \in \mathbb{R}$, is a strongly continuous group in $\mathcal{L}(Y)$. In this case, if $(A_t)_{t\in\mathbb{R}}$ has generator Γ, then $(B_t)_{t\in\mathbb{R}}$ has generator $\Delta = U\Gamma U^{-1}$ with domain $\mathcal{D}(\Delta) = U\mathcal{D}(\Gamma) := \{y \in Y : Uy \in \mathcal{D}(\Gamma)\}$. Moreover, $\sigma_p(\Delta, Y) = \sigma_p(\Gamma, X)$, and $\sigma(\Delta, Y) = \sigma(\Gamma, X)$, since if λ is in the resolvent set $\rho(\Gamma, X) := \mathbb{C} \setminus \sigma(\Gamma, X)$, we have that $R(\lambda, \Delta) = UR(\lambda, \Gamma)U^{-1}$. See for example [10, Chapter II] and [13, Chapter 3].

2. Groups of automorphisms of the upper half plane

Motivated by the work of Arvanitidis and Siskakis in [2] where a specific automorphism of the upper half plane $\mathbb{U}$ was considered and used to study the Cesàro operator $\mathcal{C}f(z) = \frac{1}{z}\int_0^z f(\xi)\,d\xi$ on $\mathbb{U}$, the current author together with three others in [3] identified and classified all the one parameter groups of automorphisms of $\mathbb{U}$ into three distinct classes (scaling, translation and rotation) according to the location of the fixed points. More precisely, we state the following result;

Theorem 2.1 *([3, Theorem 2.2]). Let $\varphi : \mathbb{R} \to \mathrm{Aut}(\mathbb{U})$ be a nontrivial continuous group homomorphism. Then exactly one of the following cases holds:*

1. *There exists $k > 0$, $k \neq 1$, and $g \in \mathrm{Aut}(\mathbb{U})$ so that $\varphi_t(z) = g^{-1}(k^t g(z))$ for all $z \in \mathbb{U}$ and $t \in \mathbb{R}$.*
2. *There exists $k \in \mathbb{R}$, $k \neq 0$, and $g \in \mathrm{Aut}(\mathbb{U})$ so that $\varphi_t(z) = g^{-1}(g(z) + kt)$ for all $z \in \mathbb{U}$ and $t \in \mathbb{R}$.*
3. *There exists $k \in \mathbb{R}$, $k \neq 0$, and a conformal mapping g of $\mathbb{U}$ onto $\mathbb{D}$ such that $\varphi_t(z) = g^{-1}(e^{ikt}g(z))$ for all $z \in \mathbb{U}$ and $t \in \mathbb{R}$. Equivalently, there exist $\theta \in \mathbb{R} \setminus \{0\}$ and $h \in \mathrm{Aut}(\mathbb{U})$ so that*

$$\varphi_t(z) = h^{-1}\left(\frac{h(z)\cos(\theta t) - \sin(\theta t)}{h(z)\sin(\theta t) + \cos(\theta t)}\right).$$

Since every continuous one-parameter semigroup $\varphi : \mathbb{R}^+ \to \mathrm{Aut}(\mathbb{U})$ extends uniquely to a continuous one-parameter group via $\varphi(t) = \varphi^{-1}(-t)$ for $t < 0$, continuous one-parameter semigroups of automorphisms are also of three types. We shall obtain two specific automorphisms of the upper half plane which interestingly correspond to the assertions 2 and 3 of Theorem 2.1 above. An example of an automorphism corresponding to the first assertion is $\varphi_t(z) = e^{-t}z$, $z \in \mathbb{U}$, $t \in \mathbb{R}$ which was considered in detail in [2] and [3, Section 3].

Let us first consider assertion 2 of Theorem 2.1. Let $g(z) = \frac{z}{1-z}$, then a straightforward calculation shows that $\Im(g(z)) = \frac{\Im(z)}{|1-z|^2} > 0$ and therefore $g(\mathbb{U}) \subseteq \mathbb{U}$. Clearly, $g \in \mathrm{Aut}(\mathbb{U})$ with the inverse $g^{-1}(z) = \frac{z}{1+z}$. Now, taking $k = 1$, we have

$$\begin{aligned}\varphi_t(z) = g^{-1}(g(z) + t) &= \frac{\frac{z}{1-z} + t}{1 + \frac{z}{1-z} + t} \\ &= \frac{(1-t)z + t}{-tz + 1 + t}. \qquad (2.1)\end{aligned}$$

Now consider assertion 3 of Theorem 2.1. Here, we consider $g : \mathbb{U} \to \mathbb{D}$ conformal and the natural candidate is the Cayley transform ψ with inverse ψ^{-1} already defined. Therefore, if we take $g(z) = \psi^{-1}(z) = \frac{z-i}{z+i}$ with $g^{-1}(z) = \psi(z) = \frac{i(1+z)}{1-z}$, then for $k = -2$, a direct computation yields; $1 + e^{i\theta} = 2\cos\left(\frac{\theta}{2}\right)e^{\frac{i\theta}{2}}$ and $1 - e^{i\theta} = -2i\sin\left(\frac{\theta}{2}\right)e^{\frac{i\theta}{2}}$, and thus

$$\begin{aligned}\varphi_t(z) = g^{-1}(e^{-2it}g(z)) &= \frac{i\left(1 + e^{-2it}\frac{z-i}{z+i}\right)}{1 - e^{-2it}\frac{z-i}{z+i}} \\ &= \frac{i\left((1+e^{-2it})z + i(1-e^{-2it})\right)}{(1-e^{-2it})z + i(1+e^{-2it})} \\ &= \frac{i\left((2\cos(-t)e^{-it})z + i(-2i\sin(-t)e^{-it})\right)}{(-2i\sin(-t)e^{-it})z + i(2\cos(-t)e^{-it})} \\ &= \frac{z\cos t - \sin t}{z\sin t + \cos t}. \qquad (2.2)\end{aligned}$$

Apparently, (2.1) and (2.2) are one-parameter groups of automorphisms of the upper half-plane $\mathbb{U}$.

Let $\{V_1, V_2\} = \{\mathbb{D}, \mathbb{U}\}$, and let $LF(V_i, V_j)$ denote the collection of conformal mappings from V_i onto V_j. Then $LF(V_i, V_i) = \mathrm{Aut}(V_i)$, and if $h \in LF(V_i, V_j)$, then $g \in \mathrm{Aut}(V_j) \mapsto h^{-1} \circ g \circ h \in \mathrm{Aut}(V_i)$ is an isomorphism from $\mathrm{Aut}(V_i)$ onto $\mathrm{Aut}(V_j)$. For each $g \in LF(V_i, V_j)$, we define a weighted composition operator $S_g : \mathcal{H}(V_j) \to \mathcal{H}(V_i)$, by

$$S_g f(z) = (g'(z))^\gamma f(g(z)), \quad \text{for all } z \in V_i. \tag{2.3}$$

We note that if $g \in LF(V_i, V_j)$ and $h \in LF(V_j, V_i)$, then it is clear by chain rule that $S_h S_g = S_{gh}$ and $S_g^{-1} = S_{g^{-1}}$. Indeed,

$$\begin{aligned} S_h S_g f(z) &= S_h\left((g')^\gamma f(g(z))\right) = (h')^\gamma \left(g'(h)\right)^\gamma f\left(g(h(z))\right) \\ &= \left((g \circ h)'(z)\right)^\gamma f\left(g \circ h\right)(z) = S_{g \circ h} f(z). \quad \text{In particular } S_{g^{-1}} = S_g^{-1}. \end{aligned}$$

Following [3, Proposition 2.1 and Theorem 2.3], the group $(S_{\varphi_t})_{t \in \mathbb{R}}$ is a strongly continuous surjective isometry in $\mathcal{L}(X)$ where X is either $H^p(\mathbb{U})$ or $L^p_a(\mathbb{U}, \mu_\alpha)$. Semigroups and groups of isometries on the Hardy spaces of the unit disc have been well studied in literature in the past few decades. See for instance [4,5,11] and references therein. The corresponding study on the upper half plane $\mathbb{U}$ is much less complete. For a complete account of the theory of strongly continuous semigroups of Banach space operators, we refer to [7, Chapter VIII], [10] or [14].

In this paper, we shall carry out a complete spectral analysis of the groups of weighted composition operators associated with the automorphism groups given by the equations (2.1) and (2.2). These groups of composition operators turns out to be strongly continuous groups of surjective isometries. Specifically, we employ the theory of similar semigroups detailed in section 1 as well as the results obtained in [3] to determine the infinitesimal generators of these groups of isometries. We further determine their spectra, their point spectra, as well as their resolvents. Finally, we then obtain the norms and the spectra of the resolvent operators on both Hardy and Bergman spaces.

3. The translation group

For every $z \in \mathbb{U}$, let $u_t = z + t$. Then by equation (2.1),

$$\begin{aligned} \varphi_t(z) := \frac{(1-t)z + t}{-tz + 1 + t} &= g^{-1}(g(z) + t) \\ &= g^{-1} \circ u_t \circ g(z). \end{aligned}$$

Therefore it can be easily verified that $S_{\varphi_t} = S_g S_{u_t} S_g^{-1}$. From [3, Section 4], we see that, if Γ is the infinitesimal generator of the group S_{u_t} on X, then the following theorem holds,

Theorem 3.1.

1. $\Gamma f(z) = f'(z)$ *with domain* $\mathcal{D}(\Gamma) = \{f \in X : f' \in X\}$.
2. $\sigma_p(\Gamma, X) = \emptyset$ *and* $\sigma(\Gamma, X) = \{is : s \geq 0\}$.
3. *If* $\lambda \in \rho(\Gamma)$, *then* $R(\lambda, \Gamma)h(z) = e^{\lambda z} \int_z^\infty e^{-\lambda \omega} h(\omega)\, d\omega := R_\lambda h(z)$.
4. *For* $1 < p < \infty$, *if* $\lambda \in \rho(\Gamma)$, $R_\lambda^* = -R_{-\bar{\lambda}}$.

Proof. For assertions 1–3 see [3, Section 4]. To prove (4), recall from [15] that for $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, $(H^p(\mathbb{U}))^* \approx H^q(\mathbb{U})$ and $(L^p_a(\mathbb{U}, \mu_\alpha))^* \approx L^q_a(\mathbb{U}, \mu_\alpha)$ under the sesquilinear pairings given by respectively,

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$$\langle f, g\rangle = \int\limits_{\mathbb{R}} f(x)\overline{g(x)}\,dx \quad (f \in H^p(\mathbb{U}),\ g \in H^q(\mathbb{U})), \tag{3.1}$$

and

$$\langle f, g\rangle = \int\limits_{\mathbb{U}} f(\omega)\overline{g(\omega)}\,d\mu_\alpha \quad (f \in L^p_a(\mu_\alpha),\ g \in L^q_a(\mu_\alpha)). \tag{3.2}$$

We take note that under these pairings, the adjoint operator is conjugate linear. Let $T_t f(z) = S_{\varphi_t} f(z) = f(z+t)$ for every $f \in X$, and define $T_{-t}g(z) = g(z-t)$ for all $g \in X^*$. Then $(T_t)_{t\in\mathbb{R}}$ and $(T_{-t})_{t\in\mathbb{R}}$ are adjoints of each other, that is, $T_t^* = T_{-t}$. Indeed, if $X = H^p(\mathbb{U})$, then $X^* = H^q(\mathbb{U})$ and for all $f \in X$, $g \in X^*$, $z = x + yi \in \mathbb{U}$, we have

$$\langle T_t f, g\rangle = \int\limits_{\mathbb{R}} f(x+t)\overline{g(x)}\,dx = \int\limits_{\mathbb{R}} f(u)\overline{g(u-t)}\,du = \langle f, T_{-t}g\rangle.$$

Similarly, if $X = L^p_a(\mathbb{U}, \mu_\alpha)$, we have $X^* = L^q_a(\mathbb{U}, \mu_\alpha)$ and for every $f \in X$, $z \in \mathbb{U}$,

$$\begin{aligned}\langle T_t f, g\rangle &= \int\limits_{\mathbb{U}} f(z+t)\overline{g(z)}(\Im(z))^\alpha\,dA(z)\\ &= \int\limits_{\mathbb{U}} f(\omega)\overline{g(\omega-t)}(\Im(\omega))^\alpha\,dA(\omega) = \langle f, T_{-t}g\rangle, \qquad \text{as desired.}\end{aligned}$$

Now, since X is reflexive, it follows from [14, Corollaries 10.2, 10.6] that $\Gamma_p^* = -\Gamma_q$ and that for $\lambda \in \rho(\Gamma, X)$, we have

$$\begin{aligned}R_\lambda^* &= R(\lambda, \Gamma_p)^* = R(\bar{\lambda}, \Gamma_p^*) = R(\bar{\lambda}, -\Gamma_q)\\ &= -R(-\bar{\lambda}, \Gamma_q) = -R_{-\bar{\lambda}}, \qquad \text{as claimed.} \quad \square\end{aligned}$$

The next theorem which is our main result in this section details the analysis of the group of isometries obtained from the automorphism group given by (2.1).

Theorem 3.2. *Let $X = H^p(\mathbb{U})$ or $L^p_a(\mathbb{U}, \mu_\alpha)$, $1 \le p < \infty$. Let $\varphi_t \in \mathrm{Aut}(\mathbb{U})$ be given by $\varphi_t(z) = \frac{(1-t)z+t}{-tz+1+t}$ for $t \in \mathbb{R}$, $z \in \mathbb{U}$, and S_{φ_t} be the corresponding group of isometries. Then,*

1. *The infinitesimal generator Δ of $S_{\varphi_t} \subset \mathcal{L}(X)$ is given by*

$$\Delta(h(z)) = -2\gamma(1-z)h(z) + (1-z)^2 h'(z)$$

   with domain $\mathcal{D}(\Delta) = \left\{h \in X : -2\gamma(1-z)h(z) + (1-z)^2 h'(z) \in X\right\}$.
2. *$\sigma_p(\Delta, X) = \emptyset$ and $\sigma(\Delta, X) = \{is : s \ge 0\}$.*
3. *If $\lambda \in \rho(\Delta)$, then*

$$R(\lambda, \Delta)h(z) = \frac{1}{(1-z)^{2\gamma}} e^{\lambda \frac{z}{1-z}} \int\limits_z^\infty e^{\lambda\frac{\omega}{1-\omega}}(1-\omega)^{2\gamma-2} h(\omega)\,d\omega.$$

Proof. Since Γ is the generator of the group S_{u_t}, and $S_{\varphi_t} = S_g S_{u_t} S_g^{-1}$ and as remarked in the introduction, it follows that the generator Δ of the group S_{φ_t} is given by

$$\Delta = S_g \Gamma S_g^{-1} \text{ with domain } \mathcal{D}(\Delta) = S_g \mathcal{D}(\Gamma).$$

Now, let $f' \in X$, then $f \in \mathcal{D}(\Gamma)$ and $h := S_g f$ belongs to $\mathcal{D}(\Delta)$ with $f = S_g^{-1} h$. Then

$$\begin{aligned} \Delta(h(z)) &= S_g \Gamma S_g^{-1} h(z) = S_g \Gamma f(z) = S_g f'(z) \\ &= (g'(z))^\gamma f'(g(z)) = \frac{1}{(1-z)^{2\gamma}} f'(g(z)). \end{aligned} \tag{3.3}$$

But $f(z) = S_g^{-1} h(z) = S_{g^{-1}} h(z) = \frac{1}{(1+z)^{2\gamma}} h\left(g^{-1}(z)\right)$, implying that

$$\begin{aligned} f'(z) &= -2\gamma(1+z)^{-2\gamma-1} h(g^{-1}(z)) + \frac{1}{(1+z)^{2\gamma+2}} h'(g^{-1}(z)) \\ &= (1+z)^{-2\gamma-2}\left(-2\gamma(1+z)h(g^{-1}(z)) + h'(g^{-1}(z))\right), \quad \text{so that} \end{aligned}$$

$$\begin{aligned} f'(g(z)) &= \left(1 + \frac{z}{1-z}\right)^{-2\gamma-2} \left(-2\gamma\left(1+\frac{z}{1-z}\right) h(z) + h'(z)\right) \\ &= (1-z)^{2\gamma}\left(-2\gamma(1-z)h(z) + (1-z)^2 h'(z)\right). \end{aligned}$$

Therefore, equation (3.3) becomes $\Delta(h(z)) = -2\gamma(1-z)h(z) + (1-z)^2 h'(z)$, as desired, with the domain

$$\begin{aligned} \mathcal{D}(\Delta) &= S_g \mathcal{D}(\Gamma) = \{S_g f : f \in \mathcal{D}(\Gamma)\} = \{h = S_g f : S_g f' \in X\} \\ &= \left\{h \in X : -2\gamma(1-z)h(z) + (1-z)^2 h'(z) \in X\right\}. \end{aligned}$$

Since $\Delta = S_g \Gamma S_g^{-1}$ and S_g is invertible, it's clear again from the theory of similar semigroups that $\sigma_p(\Delta, X) = \sigma_p(\Gamma, X) = \emptyset$ and $\sigma(\Delta, X) = \sigma(\Gamma, X) = \{is : s \geq 0\}$.

For the resolvents, we have: If $\lambda \in \rho(\Delta, X) = \rho(\Gamma, X)$, then

$$R(\lambda, \Delta) = S_g R(\lambda, \Gamma) S_g^{-1},$$

and thus we have,

$$\begin{aligned} R(\lambda, \Delta) h(z) &= S_g R(\lambda, \Gamma) S_g^{-1} h(z) = S_g \left(e^{\lambda z} \int_z^\infty e^{-\lambda\omega} S_g^{-1} h(\omega)\, d\omega \right) \\ &= S_g \left(e^{\lambda z} \int_z^\infty e^{-\lambda\omega} \frac{1}{(1+\omega)^{2\gamma}} h(g^{-1}(\omega))\, d\omega \right) \\ &= \frac{1}{(1-z)^{2\gamma}} e^{\lambda g(z)} \int_z^\infty e^{-\lambda g(\omega)} \frac{1}{(1+g(\omega))^{2\gamma}} h(\omega)\, dg(\omega) \\ &= \frac{1}{(1-z)^{2\gamma}} e^{\lambda \frac{z}{1-z}} \int_z^\infty e^{-\lambda \frac{\omega}{1-\omega}} (1-\omega)^{2\gamma-2} h(\omega)\, d\omega, \end{aligned}$$

which completes the proof. □

We end this section by determining the norm and spectra of the resolvent operator.

Theorem 3.3. *If $\Re(\lambda) \neq 0$, denote the circle $\left|z - \frac{1}{2\Re(\lambda)}\right| = \frac{1}{|2\Re(\lambda)|}$ by C_λ, and if $\lambda = ib$ for some $b < 0$, take C_λ to be the imaginary axis.*

1. *If $\Re(\lambda) \neq 0$ and $\Im(\lambda) \geq 0$, then $\sigma(R(\lambda, \Delta))$ is the arc of the circle C_λ from $\frac{1}{\lambda}$ to 0 that contains the upper half of C_λ. Moreover, $\|R(\lambda, \Delta)\| = r(R(\lambda, \Delta)) = \frac{1}{|\Re(\lambda)|}$.*
2. *If $\Im(\lambda) < 0$, then $\sigma(R(\lambda, \Delta))$ is the arc of the circle C_λ from $1/\lambda$ to 0 contained in the upper half of C_λ. In this case, $\|R(\lambda, \Delta)\| = r(R(\lambda, \Delta)) = \frac{1}{|\lambda|}$.*

Proof. Take note that $R(\lambda, \Delta) = S_g R(\lambda, \Gamma) S_g^{-1}$ where S_g is invertible. Using the well known fact that similar operators have the same spectrum, the result follows immediately from [3, Proposition 4.3]. $\square$

4. The rotation group

In this case for all $z \in \mathbb{U}$, let $u_t(z) = e^{-2it}z$. Then as argued similarly in the previous section,

$$\begin{aligned} \varphi_t(z) &:= \frac{z\cos t - \sin t}{z \sin t + \cos t} = g^{-1}(e^{-2it}g(z)) \\ &= g^{-1} \circ u_t \circ g(z). \end{aligned}$$

Now by definition, $S_{u_t}f(z) = (u_t')^\gamma f(u_t(z)) = e^{-2it}f(e^{-2it}z)$. Comparing with the group of isometries under consideration for the rotation group (see [3, Section 5]), $T_t f(z) = e^{ict}f(e^{ikt})$ whose infinitesimal generator we denote by $\Gamma_{c,k}$, we see that $c = -2\gamma$ and $k = -2$. Therefore the infinitesimal generator of the group S_{u_t} will be denoted by $\Gamma_{-2\gamma,-2}$ and whose properties can be summarized in the next theorem. Before stating the theorem, we take note that the operator $M_z f(z) := zf(z)$ is bounded and bounded below on $X(\mathbb{D})$ with range $\mathcal{R}(M_z) = \{f \in X(\mathbb{D}) : f(0) = 0\}$.

Theorem 4.1. *For $X(\mathbb{D}) = H^p(\mathbb{D})$ or $L^p_a(\mathbb{D}, m_\alpha)$, let S_{u_t} be the group of isometries on $X(\mathbb{D})$ defined above, and $\Gamma_{-2\gamma,-2}$ be its generator. Then*

1. *$\Gamma_{-2\gamma,-2}f(z) = i\,(-2\gamma f(z) - 2zf'(z))$ for every $f \in X(\mathbb{D})$, with domain $\mathcal{D}(\Gamma_{-2\gamma,-2}, X(\mathbb{D})) = \{f \in X(\mathbb{D}) : f' \in X(\mathbb{D})\}$.*
2. *$\sigma(\Gamma_{-2\gamma,-2}, X(\mathbb{D})) = \sigma_p(\Gamma_{-2\gamma,-2}, X(\mathbb{D})) = \{-2(\gamma + n)i : n \in \mathbb{Z}_+\}$, and for each $n \geq 0$, $\ker\,(-2(\gamma + n)i - \Gamma_{-2\gamma,-2}) = span(z^n)$.*
3. *If $\mu \in \rho(\Gamma_{-2\gamma,-2})$, then $\mathcal{R}(M_z^m)$ is $R(\mu, \Gamma_{-2\gamma,-2})$-invariant for every $m \in \mathbb{Z}_+$, $m > \Im\,(-(\mu + 2\gamma i)/2)$. Moreover, if $h \in \mathcal{R}(M_z^m)$, then*

$$R(\mu, \Gamma_{-2\gamma,-2})h(z) = -\frac{i}{2}z^{(\frac{\mu+2i\gamma}{2})i}\int_0^z \omega^{-(\frac{\mu-2i\gamma}{2})i-1}h(\omega)\,d\omega := R_\mu h(z).$$

4. *For $1 < p < \infty$, if $\mu \in \rho(\Gamma_{-2\gamma,-2})$, then $R_\mu^* = -R_{-\bar{\mu}}$.*

Before we prove this theorem, we first give the following two Lemmas;

Lemma 4.2. *Let $X(\mathbb{D})$ denote one of the spaces $H^p(\mathbb{D})$ or $L^p_a(\mathbb{D}, m_\alpha)$, $1 \leq p < \infty$. Then the infinitesimal generator $\Gamma_{c,k}$ of the group $(T_t)_{t\in\mathbb{R}} \subset \mathcal{L}(X(\mathbb{D}))$ is $\Gamma_{c,k}f(z) = i\,(cf(z) + kzf'(z))$ with domain $\mathcal{D}(\Gamma_{c,k}) = \{f \in X(\mathbb{D}) : f' \in X(\mathbb{D})\}$.*

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Proof. By the definition,

$$\begin{aligned}\Gamma_{c,k} f(z) &= \frac{\partial}{\partial_t}\left(e^{ict} f(e^{ikt} z)\right)\Big|_{t=0}\\ &= \left(ice^{ict} f(e^{ikt} z) + e^{ict} ike^{ikt} z f'(e^{ikt} z)\right)\big|_{t=0}\\ &= i(cf(z) + kzf'(z)).\end{aligned}$$

Therefore the domain $\mathcal{D}(\Gamma_{c,k}) \subset \{f \in X(\mathbb{D}) : zf' \in X(\mathbb{D})\}$. But $zf' \in X(\mathbb{D})$ implies that $zf' \in \mathcal{R}(M_z)$ and therefore $f' \in X(\mathbb{D})$. Thus $\{f \in X(\mathbb{D}) : zf' \in X(\mathbb{D})\} = \{f \in X(\mathbb{D}) : f' \in X(\mathbb{D})\}$.

Conversely if $f \in X(\mathbb{D})$ is such that $zf' \in X$, then $F(z) = i(cf(z) + kzf'(z)) \in X(\mathbb{D})$, and for all $t > 0$

$$\begin{aligned}\frac{T_t f(z) - f(z)}{t} &= \frac{1}{t}\int_0^t \partial_s(T_s f(z))\, ds\\ &= \frac{1}{t}\int_0^t e^{ics}\left[i(cf(e^{iks} z) + k(e^{iks} z) f'(e^{iks} z))\right]\, ds = \frac{1}{t}\int_0^t T_s F(z)\, ds.\end{aligned}$$

Now, strong continuity of $(T_s)_{s\geq 0}$ implies that

$$\left\| \frac{1}{t}\int_0^t T_s F\, ds - F \right\| \leq \frac{1}{t}\int_0^t \|T_s F - F\|\, ds \to 0 \text{ as } t \to 0^+.$$

Thus, $\mathcal{D}(\Gamma_{c,k}) = \{f \in X(\mathbb{D}) : f' \in X(\mathbb{D})\}$. □

Lemma 4.3. *Let $X(\mathbb{D})$ denote one of the spaces $H^p(\mathbb{D})$ or $L^p_a(\mathbb{D}, m_\alpha)$, $1 \leq p < \infty$. Then*

1. $\Gamma_{c,k} = ic + k\Gamma_{0,1}$ *with domain* $\mathcal{D}(\Gamma_{c,k}) = \mathcal{D}(\Gamma_{0,1}) = \{f : f' \in X(\mathbb{D})\}$.
2. $\sigma(\Gamma_{c,k}) = \{ic + k\sigma(\Gamma_{0,1})\}$, *and* $\sigma_p(\Gamma_{c,k}) = \{ic + k\sigma_p(\Gamma_{0,1})\}$.

In fact, $\lambda \in \rho(\Gamma_{0,1})$ if and only if $ic + k\lambda \in \rho(\Gamma_{c,k})$, and

$$R(ic + k\lambda, \Gamma_{c,k}) = \frac{1}{k} R(\lambda, \Gamma_{0,1}). \tag{4.1}$$

Proof. From Lemma 4.2, $\Gamma_{0,1} f(z) = izf'(z)$ for all $f \in X(\mathbb{D})$. Therefore,

$$\Gamma_{c,k} f(z) = i\left(cf(z) + kzf'(z)\right) = icf(z) + k\Gamma_{0,1} f(z),$$

with same domain as claimed.

Now, let $\lambda \in \rho(\Gamma_{0,1})$, then

$$\begin{aligned}(ic + k\lambda - \Gamma_{c,k})\frac{1}{k} R(\lambda, \Gamma_{0,1}) &= \left(ic + k\lambda - (ic + k\Gamma_{0,1})\right)\frac{1}{k} R(\lambda, \Gamma_{0,1})\\ &= \frac{k}{k}(\lambda - \Gamma_{0,1}) R(\lambda, \Gamma_{0,1}) = I,\end{aligned}$$

and if $f \in \mathcal{D}(\Gamma_{c,k})$, then

$$\begin{aligned}\frac{1}{k}R(\lambda,\Gamma_{0,1})(ic+k\lambda-\Gamma_{c,k})f &= \frac{1}{k}R(\lambda,\Gamma_{0,1})\left(ic+k\lambda-(ic+k\Gamma_{0,1})\right)f\\ &= \frac{k}{k}R(\lambda,\Gamma_{0,1})(\lambda-\Gamma_{0,1})f = f.\end{aligned}$$

Conversely, if $\mu \in \rho(\Gamma_{c,k})$, let $\mu = ic + k\lambda$ so that $\lambda = \frac{\mu-ic}{k}$. Then

$$\begin{aligned}(\lambda-\Gamma_{0,1})kR(\mu,\Gamma_{c,k}) &= k\left(\frac{\mu-ic}{k}-\Gamma_{0,1}\right)R(\mu,\Gamma_{c,k}) = (\mu-ic-k\Gamma_{0,1})R(\mu,\Gamma_{c,k})\\ &= \left(\mu-(ic+k\Gamma_{0,1})\right)R(\mu,\Gamma_{c,k}) = (\lambda-\Gamma_{c,k})R(\mu,\Gamma_{c,k}) = I,\end{aligned}$$

and if $f \in \mathcal{D}(\Gamma_{0,1})$, then

$$\begin{aligned}kR(\mu,\Gamma_{c,k})(\lambda-\Gamma_{0,1})f &= R(\mu,\Gamma_{c,k})(\mu-ic-k\Gamma_{0,1})f\\ &= R(\mu,\Gamma_{c,k})(\mu-\Gamma_{c,k})f = f.\end{aligned}$$

Thus, $\sigma(\Gamma_{c,k}) = \{ic+k\lambda : \lambda \in \sigma(\Gamma_{0,1})\}$, $\sigma_p(\Gamma_{c,k}) = \{ic+k\lambda : \lambda \in \sigma_p(\Gamma_{0,1})\}$, and for all $\lambda \in \rho(\Gamma_{0,1})$, $R(ic+k\lambda,\Gamma_{c,k}) = \frac{1}{k}R(\lambda,\Gamma_{0,1})$, as desired. $\square$

Proof of Theorem 4.1. Assertions 1–3 can easily be obtained from the above Lemmas 4.2 and 4.3 together with the results contained in [3, Section 5]. We omit the details.

To prove assertion (4), recall from [15,16] that for $1 < p, q < \infty$, $\frac{1}{p}+\frac{1}{q} = 1$ and $\alpha > -1$, $(L^p_a(\mathbb{D}, m_\alpha))^* \approx L^q_a(\mathbb{D}, m_\alpha)$ and $(H^p(\mathbb{D}))^* \approx H^q(\mathbb{D})$ under the sesquilinear pairings given respectively by

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}\,dm_\alpha(z) \quad (f \in L^p_a(\mathbb{D}, m_\alpha),\ g \in L^q_a(\mathbb{D}, m_\alpha)),$$

and

$$\langle f, g\rangle = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\,d\theta \quad (f \in H^p(\mathbb{D}),\, g \in H^q(\mathbb{D})).$$

Again, under the above pairings, the adjoint operator is conjugate linear. Let $\gamma = \frac{\alpha+2}{p}$ and $T_t f(z) = e^{-2i\gamma t} f(e^{-2it}z)$ for every $f \in X(\mathbb{D})$, and define $T_{-t}g(z) = e^{2i\gamma t} f(e^{2it}z)$ for all $g \in X(\mathbb{D})^*$. Then $(T_t)_{t\in\mathbb{R}}$ and $(T_{-t})_{t\in\mathbb{R}}$ are adjoints of each other; that is, $T^*_t = T_{-t}$. To see this, we proceed as follows: If $X(\mathbb{D}) = L^p_a(\mathbb{D}, m_\alpha)$, then $X^* = L^q_a(\mathbb{D}, m_\alpha)$ and for all $f \in X(\mathbb{D})$, $g \in X(\mathbb{D})^*$, we have,

$$\begin{aligned}\langle T_t f, g\rangle &= \int_{\mathbb{D}} e^{-2i\gamma t} f(e^{-2it}z)\overline{g(z)}\,dm_\alpha(z)\\ &= \int_{\mathbb{D}} e^{-2i\gamma t} f(e^{-2it}z)\overline{g(z)}(1-|z|^2)^\alpha\,dA(z)\\ &= \int_{\mathbb{D}} f(\omega)\overline{e^{2i\gamma t}g(e^{2it}\omega)}\,dm_\alpha(\omega) = \langle f, T_{-t}g\rangle;\end{aligned}$$

If $X(\mathbb{D}) = H^p(\mathbb{D})$, then $X^* = H^q(\mathbb{D})$ and for all $f \in X(\mathbb{D})$, $g \in X(\mathbb{D})^*$, we have

$$\langle T_t f, g\rangle = \int_0^{2\pi} e^{-2i\gamma t} f(e^{-2it}e^{i\theta})\overline{g(e^{i\theta})}\, d\theta = \int_0^{2\pi} f(e^{i(-2t+\theta)})\overline{e^{2i\gamma t} g(e^{i\theta})}\, d\theta$$
$$= \int_0^{2\pi} f(e^{i\omega})\overline{e^{2i\gamma t} g(e^{i(\omega+2t)})}\, d\omega = \langle f, T_{-t}g\rangle, \qquad \text{as desired.}$$

Since $X(\mathbb{D}) = H^p(\mathbb{D})$ or $L^p_a(\mathbb{D}, m_\alpha)$, $1 < p < \infty$, is reflexive, it follows from semigroup theory that $\Gamma^*_{-2\gamma,-2} = -\Gamma_{-2\gamma,-2}$ and that if $\mu \in \rho(\Gamma_{-2\gamma,-2}, X(\mathbb{D}))$, then

$$R^*_\mu = (R(\mu, \Gamma_{-2\gamma,-2}))^* = R(\bar\mu, \Gamma^*_{-2\gamma,-2}) = R(\bar\mu, -\Gamma_{-2\gamma,-2})$$
$$= -R(-\bar\mu, \Gamma_{-2\gamma,-2}) = -R_{-\bar\mu}, \qquad \text{as claimed.} \quad \square$$

Our main result in this section details the analysis of the group of isometries obtained from the automorphism group given by (2.2) as we give in the following theorem;

Theorem 4.4. *Let $X = H^p(\mathbb{U})$ or $L^p_a(\mathbb{U}, \mu_\alpha)$, $1 \le p < \infty$, $\alpha > -1$. Let $\varphi_t \in \mathrm{Aut}(\mathbb{U})$ be given by $\varphi_t(z) = \frac{z\cos t - \sin t}{z \sin t + \cos t}$, for all $t \in \mathbb{R}$, $z \in \mathbb{U}$, and the corresponding group of isometries on X by $S_{\varphi_t} f(z) := (\varphi_t')^\gamma f(\varphi_t(z))$. Then*

1. *The infinitesimal generator Δ of the group $S_{\varphi_t} \subset \mathcal{L}(X)$ is given by*
$$\Delta(h(z)) = -2\gamma z h(z) - (1+z^2)h'(z),$$
with domain $\mathcal{D}(\Delta) = \{h \in X(\mathbb{D}) : 2\gamma(\omega+i)h + (\omega+i)^2 h' \in X\}$.
2. *$\sigma_p(\Delta) = \sigma(\Delta) = \{-2(\gamma+n)i : n \in \mathbb{Z}_+\}$, and for each $n \ge 0$, $\ker(-2(\gamma+n)i - \Delta) = span(S_g z^n)$.*
3. *If $\mu \in \rho(\Delta)$ and if $m \in \mathbb{Z}_+$ is such that $m > \Im(-(\mu + 2i\gamma)/2)$. Then, if $h \in \mathcal{R}(M^m_z)$, we have*
$$R(\mu,\Delta)h(z) = (z-i)^{\frac{\mu+2i\gamma}{2}i}(z+i)^{-(\frac{\mu+2i\gamma}{2}i+2\gamma)}\int_0^z (\omega-i)^{-(\frac{\mu+2i\gamma}{2})i-1}(\omega+i)^{\frac{\mu+2i\gamma}{2}i+2\gamma-1}h(\omega)\, d\omega. \tag{4.2}$$
4. *$R(\mu,\Delta)$ is compact on $X(\mathbb{D})$.*
5. *$\sigma(R(\mu,\Delta)) = \sigma_p(R(\mu,\Delta)) = \left\{w \in \mathbb{C} : \left|w - \frac{1}{2\Re(\mu)}\right| = \frac{1}{2\Re(\mu)}\right\}$. Moreover,*
$$r(R(\mu,\Delta)) = \|R(\mu,\Delta)\| = \tfrac{1}{2\Re(\mu)}.$$

Proof. Since $\varphi_t = g^{-1}\circ u_t \circ g$, it follows that $S_{\varphi_t} = S_g S_{u_t} S_{g^{-1}} = S_g S_{u_t} S_g^{-1}$. Let Δ be the generator of S_{φ_t} and $\Gamma := \Gamma_{-2\gamma,-2}$ be the generator of S_{u_t}, then as noted before,

$$\Delta = S_g \Gamma S_g^{-1} \text{ with domain } \mathcal{D}(\Delta) = S_g\mathcal{D}(\Gamma).$$

As in the proof of Theorem 3.2, let $f' \in X(\mathbb{D})$. Then $f \in \mathcal{D}(\Gamma)$ and $h := S_g f$ belongs to $\mathcal{D}(\Delta)$ with $f = S_g^{-1}h$. Then

$$\Delta(h(z)) = S_g\Gamma S_g^{-1}h(z) = S_g \Gamma f(z) = S_g\left(-2\gamma i f(z) - 2izf'(z)\right)$$
$$= -\frac{(2i)^\gamma}{(z+i)^{2\gamma}}\left(2\gamma i f(g(z)) + 2ig(z)f'(g(z))\right).$$

But $f(z) = S_g^{-1}h(z) = S_{g^{-1}}h(z) = \frac{(2i)^\gamma}{(1-z)^{2\gamma}}h(g^{-1}(z))$, implying that $f(g(z)) = \frac{1}{(2i)^\gamma}(z+i)^{2\gamma}h(z)$. Moreover, $f'(z) = \frac{(2i)^\gamma}{(1-z)^{2\gamma+2}}\left(2\gamma(1-z)h(g^{-1}(z)) + 2ih'(g^{-1}(z))\right)$, implying that

$$f'(g(z)) = \frac{1}{(2i)^{\gamma+1}}(z+i)^{2\gamma+1}\left(2\gamma h(z) + (z+i)h'(z)\right).$$

Therefore,

$$\begin{aligned}\Delta(h(z)) &= -\left(2i\gamma h(z) + 2\gamma(z-i)h(z) + (z-i)(z+i)h'(z)\right)\\ &= -2\gamma z h(z) - (1+z^2)h'(z), \text{ as desired.}\end{aligned}$$

As noted before, the domain of Δ, $\mathcal{D}(\Delta)$ is given by $\mathcal{D}(\Delta) = S_g\mathcal{D}(\Gamma) = \{S_g f : f \in \mathcal{D}(\Gamma)\}$. Now $h \in \mathcal{D}(\Delta) \Leftrightarrow S_g^{-1}h \in \mathcal{D}(\Gamma) \Leftrightarrow (S_{g^{-1}}h)' \in X(\mathbb{D})$. But

$$\begin{aligned}(S_{g^{-1}}h)' &= ((\psi')^\gamma h \circ \psi)'\\ &= \frac{(2i)^\gamma}{(1-z)^{2\gamma}}\left(\frac{2\gamma}{1-\psi^{-1}\circ\psi(z)}h(\psi(z)) + \frac{2i}{1-\psi^{-1}\circ\psi(z)}h'(\psi(z))\right)\\ &= S_{g^{-1}}\left(\frac{2\gamma}{1-\psi^{-1}(\omega)}h(\omega) + \frac{2i}{1-\psi^{-1}(\omega)}h'(\omega)\right).\end{aligned}$$

Therefore,

$$\begin{aligned}h \in \mathcal{D}(\Delta) &\Leftrightarrow S_{g^{-1}}\left(\frac{2\gamma}{1-\psi^{-1}(\omega)}h(\omega) + \frac{2i}{1-\psi^{-1}(\omega)}h'(\omega)\right) \in X(\mathbb{D})\\ &\Leftrightarrow \left(\frac{2\gamma}{1-\psi^{-1}(\omega)}h(\omega) + \frac{2i}{1-\psi^{-1}(\omega)}h'(\omega)\right) \in X\\ &\Leftrightarrow \frac{\omega+i}{2i}\left[2\gamma h(\omega) + (\omega+i)h'(\omega)\right] \in X,\end{aligned}$$

which implies that $\mathcal{D}(\Delta) = \{h \in X(\mathbb{D}) : 2\gamma h(\omega) + (\omega+i)h'(\omega) \in X\}$.

Again, it's clear from Section 1 that,

$$\sigma_p(\Delta) = \sigma_p(\Gamma) = \sigma(\Gamma) = \sigma(\Delta) = \{-2(\gamma+n)i : n \in \mathbb{Z}_+\},$$

but with $\ker(-2(\gamma+n)i - \Delta) = \mathrm{span}(S_g z^n)$ for each $n \geq 0$.

For the resolvents, if $\mu \in \rho(\Delta) = \rho(\Gamma)$, then for $m \in \mathbb{Z}_+$, $m > \Im\left(-(\mu+2\gamma i)/2\right)$, and if $h \in \mathcal{R}(M_z^m)$, we have $R(\mu,\Delta) = S_g R(\mu,\Gamma)S_g^{-1}$ and so

$$\begin{aligned}R(\mu,\Delta)h(z) &= S_g\left(-\frac{i}{2}z^{\frac{\mu+2\gamma i}{2}i}\int_0^z \omega^{-(\frac{\mu+2\gamma i}{2})i-1}S_{g^{-1}}h(\omega)\,d\omega\right)\\ &= S_g\left(-\frac{i}{2}z^{\frac{\mu+2\gamma i}{2}i}\int_0^z \omega^{-(\frac{\mu+2\gamma i}{2})i-1}\frac{(2i)^\gamma}{(1-\omega)^{2\gamma}}h(g^{-1}(\omega))\,d\omega\right)\\ &= -\frac{i}{2}\cdot\frac{(2i)^\gamma}{(z+i)^{2\gamma}}(g(z))^{\frac{\mu+2\gamma i}{2}i}\int_0^z (g(\omega))^{-(\frac{\mu+2\gamma i}{2})i-1}\frac{(2i)^\gamma}{(1-g(\omega))^{2\gamma}}h(\omega)\,\frac{dg}{d\omega}\,d\omega\\ &= \left(\frac{z-i}{(z+i)^{2\gamma}}\right)^{\frac{\mu+2i\gamma}{2}i}\int_0^z (\omega-i)^{-(\frac{\mu+2i\gamma}{2})i-1}(\omega+i)^{\frac{\mu+2i\gamma}{2}i+2\gamma-1}h(\omega)\,d\omega.\end{aligned}$$

The compactness of the resolvent operator $R(\mu, \Delta)$ follows from the compactness of $R(\mu, \Gamma)$ by a similarity argument.

The spectral mapping theorem and the assertion 2 of this theorem imply that for all $\mu \in \rho(\Delta)$,

$$\begin{aligned}\sigma(R(\mu,\Delta)) &= \left\{\frac{1}{\mu - z} : z \in \sigma(\Delta)\right\} \cup \{0\} \\ &= \left\{\frac{1}{\mu + 2(\gamma + n)i} : n \in \mathbb{Z}_+\right\} \cup \{0\} \\ &= \left\{w \in \mathbb{C} : \left|w - \frac{1}{2\Re(\mu)}\right| = \frac{1}{2\Re(\mu)}\right\}.\end{aligned}$$

The equality $\sigma(R(\mu, \Delta)) = \sigma_p(R(\mu, \Delta))$ follows from the compactness of $R(\mu, \Delta)$ as given by [10, Corollary V.1.15]. From the spectrum, it is clear that the spectral radius of the resolvent is $r(R(\mu, \Delta)) = \frac{1}{\Re(\mu)}$. Moreover, the Hille–Yosida theorem yields $r(R(\mu, \Delta)) = \frac{1}{\Re(\mu)} \leq \|R(\mu, \Delta)\| \leq \frac{1}{\Re(\mu)}$. $\square$

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References

[1] E. Albrecht, T.L. Miller, M.M. Neumann, Spectral properties of generalized Cesàro operators on Hardy and weighted Bergman spaces, Arch. Math. (Basel) 85 (2005) 446–459.
[2] A.G. Arvanitidis, A.G. Siskakis, Cesàro operators on the Hardy spaces of the half plane, Canad. Math. Bull. 56 (2013) 229–240.
[3] S. Ballamoole, J.O. Bonyo, T.L. Miller, V.G. Miller, Cesàro operators on the Hardy and Bergman spaces of the half plane, Complex Anal. Oper. Theory 10 (2016) 187–203.
[4] E. Berkson, One-parameter semigroups of isometries into H^p, Pacific J. Math. 86 (1980) 403–413.
[5] E. Berkson, H. Porta, Hermitian operators and one-parameter groups of isometries in Hardy spaces, Trans. Amer. Math. Soc. 185 (1973) 331–344.
[6] J.B. Conway, A Course in Functional Analysis, Springer-Verlag, New York, 1985.
[7] N. Dunford, J.T. Schwartz, Linear Operators Part I, Interscience Publishers, New York, 1958.
[8] P. Duren, Theory of H^p Spaces, Academic Press, New York, 1970.
[9] P. Duren, A. Schuster, Bergman Spaces, Mathematical Surveys and Monographs, vol. 100, Amer. Math. Soc., Providence, RI, 2004.
[10] K.-J. Engel, R. Nagel, A Short Course on Operator Semigroups, Universitext, Springer, New York, 2006.
[11] F. Forelli, The isometries of H^p, Canad. J. Math. 16 (1964) 721–728.
[12] J.B. Garnett, Bounded Analytic Functions, revised first edition, Graduate Texts in Mathematics, Springer, Berlin, 2010.
[13] K.B. Laursen, M.M. Neumann, An Introduction to Local Spectral Theory, Clarendon Press, Oxford, 2000.
[14] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences, vol. 40, Springer, New York, 1983.
[15] M.M. Peloso, Classical Spaces of Holomorphic Functions, Technical report, Università di Milano, 2014.
[16] K. Zhu, Operator Theory in Function Spaces, Mathematical Surveys and Monographs, vol. 138, Amer. Math. Soc., Providence, 2007.

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