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Smoothing effects of the initial-boundary value problem for logarithmic type quasilinear parabolic equations

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A R T I C L E I N F O

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A B S T R A C T

We give existence theorems of global solutions in $L^\infty_{loc}((0,\infty);W_0^{1,\infty})$ to the initial boundary value problem for quasilinear degenerate parabolic equations of the form $u_t - \mathrm{div}\{\sigma(|\nabla u|^2)\nabla u\} = 0$, where the class of $\sigma(v^2)$ includes the logarithmic case $\sigma(|\nabla u|^2) = \log(1+|\nabla u|^2)$ for a typical example. We assume that the initial data belong to $W_0^{1,p_0}, p_0 \geq 2$, or $L^r, r \geq 1$, and we derive precise estimates for $\|\nabla u(t)\|_\infty$ near $t = 0$.

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1. Introduction

In this paper we consider the initial-boundary value problem of the quasilinear parabolic equation of the form:

$$u_t - \mathrm{div}\{\sigma(|\nabla u|^2)\nabla u\} = 0 \text{ in } \Omega \times (0,\infty) \tag{1.1}$$

with the initial-boundary conditions

$$u(x,0) = u_0(x) \text{ and } u(x,t)|_{\partial\Omega} = 0, \tag{1.2}$$

where Ω is a bounded domain in R^N with $C^{2,\alpha}, \alpha > 0$, class boundary $\partial\Omega$. Concerning $\sigma(v^2)$ we assume

Hyp.A. $\sigma(\cdot)$ is a nonnegative function in $C^{1,\alpha}((0,\infty)) \cap C([0,\infty)), 0 < \alpha \leq 1$, satisfying:

(1)

$$\sigma(v^2) + 2\sigma'(v^2)v^2 \geq k_0\sigma(v^2).$$

(2)

$$|\sigma'(v^2)|v^2 \leq k_1\sigma(v^2).$$

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(3)
$$k_0|v|^{\tilde{l}} \le \sigma(v^2) \le k_1|v|^L \text{ if } |v| \ge 1, \text{ with some } \tilde{l}, L \ge 0.$$

(4) There exists $\nu \ge 0$ and $m \ge 0$ such that for any $K \ge 1$,
$$\sigma(v^2) \ge k_0 K^{-\nu}|v|^m \text{ if } |v| \le K.$$

In the above k_0, k_1 are some positive constants and we assume $\tilde{l} \le m$.

The functions $\sigma(v^2) = \log(1 + v^2)$ and $\sigma(v^2) = |v|^m$ satisfy Hyp.A with $\nu = m = 2$, $\tilde{l} = 0$, any $L > 0$, and $\nu = 0$, $\tilde{l} = L = m$, respectively. These functions have a common property in the sense that they are growing up to infinity as $|v| \to \infty$ and degenerate at $v = 0$. When $\sigma(v^2) = |v|^m$, $m > 0$, the equation is called as m-Laplacian type or p-Laplacian type, and the problem (1.1)–(1.2) and related problems have been investigated by many authors from various points of view (cf. [1–4,6,5,7,17–20,8,9,13–16] etc.). However, the techniques treating the nonlinearity $\sigma = |v|^m$ do not seem to be directly applied to the logarithmic type nonlinearity because $\sigma = \log(1 + v^2)$ has not the property such that $k_0|v|^m \le \sigma(v) \le k_1|v|^m$, $k_0, k_1 > 0$ for any $m \ge 0$.

Recently we have proved in [11] the existence of global classical solutions of some general parabolic equations as in Hyp.A (except for (3), (4)) with the additional condition $\sigma(v^2) \ge k_0 > 0$, and as an application we have discussed the problem (1.1)–(1.2) with $\sigma(|\nabla u|^2) = \log(1 + |\nabla u|^2)$ and shown that if $u_0 \in W_0^{1,p_0}, p_0 > 2$, the problem admits a unique weak solution $u(t)$ in $L^\infty([0,\infty); W_0^{1,p_0}) \cap W^{1,2}([0,\infty); L^2)$, satisfying $\Gamma(t) \equiv \frac{1}{2}\int_\Omega \int_0^{\|\nabla u(t)\|^2} \log(1+\eta)\eta dx \le (\Gamma(0) + Ct)^{-2}$. Further, we have shown that if $u_0 \in W_0^{1,\infty}$, the solution belongs to $L^\infty([0,\infty); W_0^{1,\infty}) \cap W^{1,2}([0,\infty); L^2)$ and satisfies the decay estimate $\|\nabla u(t)\|_\infty \le C(\|\nabla u_0\|_\infty)(1 + t)^{-1/2}$.

Since our problem (1.1)–(1.2) is of parabolic type we can expect some smoothing effect near $t = 0$ and it is desirable for the above logarithmic case to show the global existence of solution in the class $L^\infty_{loc}((0,\infty); W_0^{1,\infty})$ for the initial data $u_0 \in W_0^{1,p_0}, p_0 > 2$, or more weakly $u_0 \in L_r, r \ge 1$. The object of this paper is to establish such results and derive precise estimates for $\|\nabla u(t)\|_\infty$ near $t = 0$ for a wider class of quasilinear parabolic equations satisfying Hyp.A. For the proof we employ Moser's technique (cf. [20,1,8,13,14] etc.) and a delicate 'loan' method (see section 5).

Our class of functions $\sigma(v^2)$ in Hyp.A includes $\{\log(1 + |v|^{m_1})\}^{m_2}, m_1, m_2 \ge 0$ (where $L > 0, \tilde{l} = 0, \nu = m = m_1 m_2$), $|v|^{m_1}\log(1 + |v|^{m_2})$ (where $L > m_1, \tilde{l} = m_1, \nu = m_2, m = m_1 + m_2$) and $|v|^{m_1}/\sqrt{1 + |v|^{m_2}}$ with $m_1 \ge m_2/2 \ge 0$ (where $L = \tilde{l} = m_1 - m_2/2$, $\nu = m_2/2$, $m = m_1$) etc. Since the most typical example is $\sigma(v^2) = \log(1 + v^2)$ and we are interested in the case $\tilde{l} < m$ we call, conveniently, our class of functions in Hyp.A as 'logarithmic type'. We note that if $\tilde{l} \ge m$, then we have $\sigma(v^2) \ge k_0|v|^{\tilde{l}}$ for all v with some $k_0 > 0$, and the problem becomes easier. Indeed, all of the results below hold with m replaced by $\tilde{m} = \max\{m, \tilde{l}\}$.

2. Statement of the results

We use only familiar function spaces and omit their definitions. But, we note that a function u belongs to $W_0^{1,\infty}$ iff $u \in W_0^{1,p}$ for any $p \ge 1$ and $|\nabla u| \in L^\infty$. We denote by $\|\cdot\|_p$ the L^p norm on Ω. We use $\|\cdot\|$ for $\|\cdot\|_2$ and the inner product in L^2 is denoted by $(,)$. We set

$$\Gamma(t) = \frac{1}{2}\int_\Omega \int_0^{|\nabla u(t)|^2} \sigma(\tau)d\tau dx \text{ and } \tilde{\Gamma}(t) = \int_\Omega \sigma(|\nabla u(t)|^2)|\nabla u(t)|^2 dx$$

for functions $u(x,t)$ if the right-hand sides are convergent. By Hyp.A,(1),(2) we see

$$\tilde{k}_0\tilde{\Gamma}(t) \le \Gamma(t) \le \tilde{k}_1\tilde{\Gamma}(t) \tag{2.1}$$

with some $\tilde{k}_0, \tilde{k}_1 > 0$. Indeed, by Hyp.A,(1) we have

$$k_0 \int_0^{v^2} \sigma(\eta) d\eta \leq \int_0^{v^2} (\sigma(\eta) + 2\sigma'(\eta)\eta) d\eta$$

$$= \int_0^{v^2} \left(2\frac{d}{d\eta}(\sigma(\eta)\eta) - \sigma(\eta) \right) d\eta = 2\sigma(v^2)v^2 - \int_0^{v^2} \sigma(\eta) d\eta.$$

Hence,

$$\int_0^{v^2} \sigma(\eta) d\eta \leq \frac{2}{k_0 + 1} \sigma(v^2) v^2.$$

On the other hand, by Hyp.A,(2) we have

$$k_1 \int_0^{v^2} \sigma(\eta) d\eta \geq \int_0^{v^2} \sigma'(\eta)\eta d\eta = \int_0^{v^2} (\sigma(\eta)\eta)' d\eta - \int_0^{v^2} \sigma(\eta) d\eta,$$

which implies

$$\int_0^{v^2} \sigma(\eta) d\eta \geq \frac{1}{k_1 + 1} \sigma(v^2) v^2.$$

Thus, (2.1) holds.

We employ the following definitions of solution of the problem (1.1)–(1.2).

Definition 2.1. Let $u_0 \in W_0^{1,p_0}$ for some $p_0 \geq L + 2$. A function $u(t)$ belonging to $L^{p_0}([0, \infty); W_0^{1,p_0}) \cap W^{1,2}([0, \infty); L^2)$ is called a solution of the problem (1.1)–(1.2) iff

$$\int_0^t (u_t(s), \phi(s)) ds + \int_0^t \int_\Omega \sigma(|\nabla u(s)|^2) \nabla u(s) \cdot \nabla \phi(s) dx ds = 0$$

for all $\phi(\cdot) \in L^{p_0}([0, \infty); W_0^{1,p_0})$, and $u(0) = u_0$.

Definition 2.2. Let $u_0 \in L^r$ for some $r \geq 1$. A function $u(t)$ belonging to $L^{p_0}_{loc}((0, \infty); W_0^{1,p_0}) \cap W^{1,2}_{loc}((0, \infty); L^2) \cap C([0, \infty); L^r)$ for some $p_0 \geq L + 2$ is called a solution of the problem (1.1)–(1.2) iff

$$\int_\delta^t (u_t(s), \phi(s)) ds + \int_\delta^t \int_\Omega \sigma(|\nabla u(s)|^2) \nabla u(s) \cdot \nabla \phi(s) dx ds = 0$$

for any $0 < \delta < t$ and for any $\phi(\cdot) \in L^{p_0}_{loc}((0, \infty); W_0^{1,p_0}) \cap W^{1,2}_{loc}((0, \infty); L^2) \cap C([0, \infty); L^r)$, and $u(0) = u_0$.

Remark 2.1. By the condition Hyp.A,(3) we see that if $p_0 \geq L + 2$ and $|\nabla u| \in L^{p_0}$, then $\sigma(|\nabla u|^2)|\nabla u| \in L^{p_0/(p_0-1)}$.

Our results read as follows.

Theorem 2.1. *Let* $u_0 \in W_0^{1,p_0}$ *for some* p_0 *such that* $p_0 \geq L+2$ *and* $p_0 > N(\nu - m)/2$. *Then there exists a unique solution* $u(t) \in L^\infty([0,\infty); W_0^{1,p_0}) \cap L^\infty_{loc}((0,\infty); W_0^{1,\infty}) \cap W^{1,2}([0,\infty); L^2)$ *of the problem (1.1)–(1.2) in the sense of Definition 2.1, satisfying the estimates*

$$\Gamma(t) \leq (\Gamma(0)^{-m/(m+2)} + mC_1^{-1}t)^{-(m+2)/m} \text{ and } \int_0^\infty \|u_t(s)\|^2 ds \leq \Gamma(0), \tag{2.2}$$

$$\|\nabla u(t)\|_{p_0}^2 \leq \|\nabla u_0\|_{p_0}^2 + Cp_0^\alpha \Gamma(0)^{2/(m+2)}, 0 \leq t < \infty, \tag{2.3}$$

with a certain $\alpha > 1$, *and*

$$\|\nabla u(t)\|_\infty \leq \begin{cases} C_1\left(\|\nabla u_0\|_{p_0} + \Gamma(0)^{1/(m+2)}\right)^{2p_0/(mN+2p_0)} t^{-N/2p_0}, 0 < t \leq 1, \\ C_1(1+t)^{-1/m}, t \geq 1, \end{cases} \tag{2.4}$$

where C_1 *denotes a constant continuously depending on* $\|\nabla u_0\|_{p_0}$ *and* $\Gamma(0)$, *which may be different from line to line.*

Theorem 2.2. *Let* $\nu < m + 4/N$ *in Hyp.A,(4). Let* $u_0 \in L^r$ *for some* $r \geq 1$, *where if* $1 \leq r < 2$ *we assume*

$$2(4+NL)r + (L+2)\tilde{l}(2r + N(2-r)) \geq 4LN. \tag{2.5}$$

Then the problem (1.1)–(1.2) admits a unique solution $u(t) \in L^\infty_{loc}((0,\infty); W_0^{1,\infty}) \cap W^{1,2}_{loc}((0,\infty); L^2) \cap C([0,\infty); L^r)$ *in the sense of Definition 2.2, satisfying*

$$\|u(t)\|_r \leq \|u_0\|_r, 0 \leq t < \infty, \tag{2.6}$$

$$\Gamma(t) \leq \begin{cases} C_0\|u_0\|_r^{2(1-\theta)(m+2+2\theta)/(m+2)} t^{-2\nu_0}, 0 < t \leq 1, \\ C_0\left(\|u_0\|_r^{-2m(1-\theta)(m+2+2\theta)/(m+2)^2} + m(t-1)\right)^{-(m+2)/m}, t \geq 1, \end{cases} \tag{2.7}$$

with

$$\theta = \frac{N(2-r)^+}{2r + (2-r)^+N} \text{ and } \nu_0 = \frac{2r + (2-r)^+N}{\tilde{l}(2r + (2-r)^+N) + 4r},$$

and

$$\|\nabla u(t)\|_\infty \leq \begin{cases} C_0\|u_0\|_r^{8(1-\theta)/(m+2)(mN+4)} t^{-(N+4\nu_0)/4}, 0 < t \leq 1, \\ C_0(1+t)^{-1/m}, 1 \leq t < \infty. \end{cases} \tag{2.8}$$

Further we have

$$\int_\delta^\infty \|u_t(t)\|^2 dt \leq \Gamma(\delta) \leq C_0\|u_0\|_r^{2(1-\theta)(m+2+2\theta)/(m+2)} \delta^{-2\nu_0}, 0 < \delta \leq 1. \tag{2.9}$$

In the above C_0 *denotes a constant continuously depending on* $\|u_0\|_r$ *which may be different from line to line.*

Remark 2.2. When $m = 0$ the first inequality of (2.2), the second inequalities of (2.4) and (2.8) should be replaced by $\Gamma(t) \le \Gamma(0)e^{-\lambda t}$, $\|\nabla u(t)\|_\infty \le C_1 e^{-\lambda t}$ and $\|\nabla u(t)\|_\infty \le C_0 e^{-\lambda t}$, respectively with some $\lambda > 0$.

Remark 2.3. For all of the examples stated in the introduction we can take $\nu \le m$, and the condition $\nu < m + 4$ in Theorem 2.2 does not seem to be restrictive.

Remark 2.4. The assumption $\sigma \in C^{1,\alpha}((0,\infty))$ in Hyp.A is made only for the construction of approximate functions $\sigma_\epsilon(v^2) \in C^{1,\alpha}([0,\infty))$, $\epsilon > 0$, such that $\sigma_\epsilon(v^2) \to \sigma$ in $C([0,\infty)$ as $\epsilon \to 0$ (see the section 3). Therefore Theorems 2.1, 2.2 can be applied also to an example like $\sigma(v^2) = \min\{|v|^{m_1}, |v|^{m_2}), m_1, m_2 \ge 0$ for which we can easily construct such approximate functions $\sigma_\epsilon(v^2)$.

3. Estimate for $\Gamma(t)$

Let $\epsilon > 0$ and we first take $u_0 \in C_0^3(\Omega)$. We consider the approximate problem

$$u_t - \operatorname{div}\{\sigma_\epsilon(|\nabla u|^2)\nabla u\} = 0 \text{ in } \Omega \times (0,\infty), \tag{3.1}$$

with the initial-boundary conditions

$$u(x,0) = u_0(x) \text{ and } u(x,t)|_{\partial\Omega} = 0, \tag{3.2}$$

where $\sigma_\epsilon(|\nabla u|^2) = \sigma(\epsilon + |\nabla u|^2)$. Then σ_ϵ belongs to $C^{1,\alpha}([0,\infty))$ and satisfies Hyp.A (with the same k_0, k_1), and hence, (2.1) holds with $\Gamma(t)$ and $\tilde{\Gamma}(t)$ replaced by $\Gamma_\epsilon(t)$ and $\tilde{\Gamma}_\epsilon(t)$, respectively, where we set

$$\Gamma_\epsilon(t) = \frac{1}{2}\int_\Omega \int_0^{|\nabla u(s)|^2} \sigma_\epsilon(\tau)d\tau dx$$

and

$$\tilde{\Gamma}_\epsilon(t) = \int_\Omega \sigma_\epsilon(|\nabla u(t)|^2)|\nabla u(t)|^2 dx.$$

Further we know $\sigma_\epsilon(v^2) \ge C_\epsilon > 0$. Therefore the problem (3.1)–(3.2) admits a unique classical solution $u_\epsilon(t) \in C^1([0,\infty); C(\bar{\Omega})) \cap C([0,\infty); C^2(\bar{\Omega})$. This fact is proved in [11] on the basis of a classical result in [5]. Our solution $u(t)$ of the original problem will be given as a limit of $u_\epsilon(t)$ as $\epsilon \to 0$. For the case $u_0 \in W_0^{1,p_0}$ or L^r we further take a sequence $\{u_{0,n}\} \subset C_0^3(\Omega)$ such that $u_{0,n} \to u_0$ in W_0^{1,p_0} or L^r and consider approximate solutions $u_n(t)$ with $u_n(0) = u_{0,n}$. We shall derive various estimates for $u_\epsilon(t)$ essentially independent of ϵ, which will be required for the proofs of Theorems 2.1 and 2.2. For simplicity of notation we write $u(t)$ for $u_\epsilon(t)$.

Proposition 3.1. *Let $r \ge 1$. Then, for the approximate solution $u(t)$ we have*

$$\|u(t)\|_r \le \|u_0\|_r, 0 \le t < \infty, \tag{3.3}$$

$$\|\nabla u(t)\| \le C_0\|u_0\|_r^{2(1-\theta)/(m+2)} \text{ and } \Gamma_\epsilon(t) \le C_0\|u_0\|_r^{2(1-\theta)(m+2+2\theta)/(m+2)} t^{-2\nu_0} \tag{3.4}$$

for $0 < t \le 1$ with

$$\theta = \frac{N(2-r)^+}{2r+(2-r)^+N} \text{ and } \nu_0 = \frac{2r+(2-r)^+N}{\tilde{l}(2r+(2-r)^+N)+4r},$$

$$\Gamma_\epsilon(t) \le C_0\left(C_0^{-1}\|u_0\|_r^{-m/(m+2)\nu_0} + m(t-1)\right)^{-(m+2)/m}, 1 \le t, \tag{3.5}$$

and

$$\Gamma_\epsilon(t) \le \left(\Gamma_\epsilon(0)^{-m/(m+2)} + mC_1^{-1}t\right)^{-(m+2)/m}, 0 \le t < \infty. \tag{3.6}$$

When $m = 0$ we replace (3.5) and (3.6) by $\Gamma_\epsilon(t) \le C_0\|u_0\|_r e^{-\lambda t}$ and $\Gamma_\epsilon(t) \le \Gamma_\epsilon(0)e^{-\lambda t}$, respectively, with some $\lambda > 0$.

Proof. If $r \ge 2$ we multiply the equation (3.1) by $|u|^{r-2}u$ and integrate it to obtain

$$\frac{1}{r}\|u(t)\|_r^r + r\int_0^t\int_\Omega \sigma_\epsilon(|\nabla u|^2)|\nabla u|^2|u|^{r-2}dxds = \frac{1}{r}\|u_0\|_r^r \tag{3.7}$$

which implies (3.3). When $1 \le r < 2$ we use a C^1 nondecreasing function $\rho_\delta(u), \delta > 0$, for $|u|^{r-2}u$ such that $\rho_\delta(u) = |u|^{r-2}u$ if $|u| \ge \delta$. Taking the limit as $\delta \to 0$ in the resulted inequality, we obtain (3.3).

To derive (3.4) a device is needed (a rather simple 'loan' method). Let $\tilde{K} > 0$ and $\tilde{\lambda} > 0$. Then we may assume

$$\|\nabla u(t)\| \le \tilde{K}t^{-\tilde{\lambda}}, 0 < t \le T_\epsilon, \tag{3.8}$$

with some $T_\epsilon \le 1$. Multiplying the equation by u_t and integrating it we have

$$\frac{d}{dt}\Gamma_\epsilon(t) + \|u_t(s)\|^2 = 0. \tag{3.9}$$

On the other hand, multiplying the equation by u and integrating it we have

$$\begin{aligned}\tilde{\Gamma}_\epsilon(t) = -(u_t, u) &\le \|u_t(t)\|\|u(t)\| \le C\|u_t(t)\|\|u(t)\|_r^{1-\theta}\|\nabla u(t)\|^\theta \\ &\le C\|u_t(t)\|(\tilde{K}t^{-\tilde{\lambda}})^\theta\|u_0\|_r^{1-\theta}, 0 < t \le T_\epsilon,\end{aligned} \tag{3.10}$$

with $\theta = N(2-r)^+/(2r+2N-rN)$. We denote by C a general positive constant which may be changed from line to line.

It follows from (3.9) and (3.10) that

$$\frac{d}{dt}\Gamma_\epsilon(t) + C^{-1}\tilde{K}^{-2\theta}t^{2\theta\tilde{\lambda}}\|u_0\|_r^{2(\theta-1)}\Gamma_\epsilon(t)^2 \le 0, 0 \le t \le T_\epsilon. \tag{3.11}$$

Solving (3.11) we have

$$\begin{aligned}\Gamma_\epsilon(t) &\le \left(\Gamma_\epsilon(0)^{-1} + C^{-1}\tilde{K}^{-2\theta}\|u_0\|_r^{-2(1-\theta)}t^{2\theta\tilde{\lambda}+1}\right)^{-1} \\ &\le C\tilde{K}^{2\theta}\|u_0\|_r^{2(1-\theta)}t^{-(2\theta\tilde{\lambda}+1)}, 0 < t \le T_\epsilon \le 1.\end{aligned} \tag{3.12}$$

Here, setting $\Omega_1 = \{x \in \Omega | |\nabla u(x,t)| \le 1\}$ and $\Omega_2 = \Omega \setminus \Omega_1$ we have from Hyp.A,(3), (4),

$$\|\nabla u(t)\|^2 = \int_{\Omega_1} |\nabla u(t)|^2 dx + \int_{\Omega_1} |\nabla u(t)|^2 dx$$
$$\leq C \left(\int_{\Omega_1} |\nabla u|^{m+2} \right)^{2/(m+2)} dx + C \left(\int_{\Omega_2} |\nabla u(t)|^{\tilde{l}+2} dx \right)^{2/(\tilde{l}+2)}$$
$$\leq C \left(\Gamma_\epsilon(t)^{2/(m+2)} + \Gamma_\epsilon(t)^{2/(\tilde{l}+2)} \right). \tag{3.13}$$

Then, by (3.12) and (3.13) we see

$$\|\nabla u(t)\| \leq C \left(\tilde{K}^{2\theta/(m+2)} \|u_0\|_r^{2(1-\theta)/(m+2)} + \tilde{K}^{2\theta/(\tilde{l}+2)} \|u_0\|^{2(1-\theta)/(\tilde{l}+2)} \right)$$
$$\times t^{-(2\theta\tilde{\lambda}+1)/(\tilde{l}+2)}, 0 < t \leq T_\epsilon. \tag{3.14}$$

Now we choose $\tilde{\lambda}$ as $(2\theta\tilde{\lambda}+1)/(\tilde{l}+2) = \tilde{\lambda}$, that is,

$$\tilde{\lambda} = \frac{2r + N(2-r)^+}{\tilde{l}(2r + N(2-r)^+) + 4r} \equiv \nu_0.$$

Then (3.14) implies

$$\|\nabla u(t)\| \leq C \left(\tilde{K}^{2\theta/(m+2)} + \tilde{K}^{2\theta/(\tilde{l}+2)} \|u_0\|^{2(1-\theta)(m-l)/(m+2)(\tilde{l}+2)} \right) \|u_0\|_r^{2(1-\theta)/(m+2)}$$
$$\times t^{-\tilde{\lambda}}, 0 < t \leq T_\epsilon. \tag{3.15}$$

Since $\theta < 1$ we can take a constant $\tilde{K} = \tilde{K}(\|u_0\|_r)(> 0)$ continuously depending on $\|u_0\|_r$ such that

$$C \left(\tilde{K}^{2\theta/(m+2)} + \tilde{K}^{2\theta/(\tilde{l}+2)} \|u_0\|^{2(1-\theta)(m-l)/(m+2)(\tilde{l}+2)} \right) \|u_0\|_r^{2(1-\theta)/(m+2)} < \tilde{K},$$

and we have from (3.15)

$$\|\nabla u(t)\| < \tilde{K} t^{-\tilde{\lambda}}, 0 < t \leq T_\epsilon \leq 1. \tag{3.16}$$

Due to (3.8) and (3.16) we can take $T_\epsilon = 1$ and the following estimate holds:

$$\|\nabla u(t)\| \leq C(\|u_0\|_r) \|u_0\|_r^{2(1-\theta)/(m+2)} t^{-\nu_0}, 0 < t \leq 1. \tag{3.17}$$

We also obtain from (3.12),

$$\Gamma_\epsilon(t) \leq C(\|u_0\|_r) \|u_0\|_r^{2(1-\theta)(m+2+2\theta)/(m+2)} t^{-2\nu_0}, 0 < t \leq 1. \tag{3.18}$$

We proceed to the decay estimate for $\Gamma_\epsilon(t)$, $t \geq 1$. We first see from (3.9) that $\Gamma_\epsilon(t) \leq \Gamma_\epsilon(1)$, $t \geq 1$, and hence, by (3.13),

$$\|u(t)\| \leq C\|\nabla u(t)\| \leq C(1 + \Gamma_\epsilon(1)^{(m-\tilde{l}))/(m+2)}) \Gamma_\epsilon(t)^{1/(m+2)}.$$

Thus, instead of (3.11), we have

$$\frac{d}{dt}\Gamma_\epsilon(t) + C^{-1}(\Gamma(1)+1)^{-(m-\tilde{l})/(\tilde{l}+2)(m+2)} \Gamma_\epsilon(t)^{2(m+1)/(m+2)} \leq 0, \tag{3.19}$$

which gives

$$\Gamma_\epsilon(t) \le \left(\Gamma_\epsilon(1)^{-m/(m+2)} + C^{-1}m(\Gamma_\epsilon(1)+1)^{-(m-\tilde{l})/(\tilde{l}+2)(m+2)}(t-1)\right)^{-(m+2)/m} \tag{3.20}$$

$$\le C_0\left(C_0^{-1}\|u_0\|_r^{-2m(1-\theta)(m+2+2\theta)/(m+2)^2} + m(t-1)\right)^{-(m+2)/m}, t \ge 1. \tag{3.21}$$

(When $m = 0$ we replace (3.21) by $\Gamma_\epsilon(t) \le C_0\|u_0\|_r^{2(1-\theta)(1+\theta)}e^{-\lambda t}$ with some $\lambda > 0$.)

Concerning the estimates depending on $\|\nabla u_0\|_{p_0}$ we first see $\Gamma_\epsilon(t) \le \Gamma_\epsilon(0)$ by (3.9), and $\|u(t)\| \le C\|\nabla u(t)\| \le C_1\Gamma_\epsilon(t)^{1/(m+2)}$ by (3.13). Thus, by the same argument as the one obtaining (3.20) we get (3.6).

Proposition 3.1 and (3.9) give immediately the following estimates for $\|u_t(t)\|$.

Proposition 3.2. *For the approximate solutions $u(t) = u_\epsilon(t)$ we have*

$$\int_0^\infty \|u_t(t)\|^2 dt \le \Gamma_\epsilon(0) \tag{3.22}$$

and

$$\int_\delta^\infty \|u_t(t)\|^2 dt \le \Gamma_\epsilon(\delta) \le C_0\|u_0\|_r^{2(1-\theta)(m+2+2\theta)/(m+2)}\delta^{-2\nu_0} \tag{3.23}$$

for any $\delta, 0 < \delta < 1$.

4. Estimates for $\|\nabla u(t)\|_p, 0 < t \le T_\epsilon$, with $2 \le p < \infty$

We estimate $\|\nabla u(t)\|_p, 0 < t \le T_\epsilon$, for $p, 2 \le p < \infty$.

First we recall the basic inequality for the solution $u(t) = u_\epsilon(t)$.

Proposition 4.1. *For any $p, 2 \le p < \infty$, we have*

$$\frac{1}{p}\frac{d}{dt}\|\nabla u(t)\|_p^p + \frac{\epsilon_0}{p^2}\left(\|\sqrt{\sigma_\epsilon}|\nabla u|^{p/2}\|_{H_1}^2 + \|\sqrt{\sigma_\epsilon}\nabla(|\nabla u|^{p/2})\|^2\right)$$
$$\le Cp^2\|\sqrt{\sigma_\epsilon(|\nabla u|^2)}|\nabla u|^{p/2}\|^2. \tag{4.1}$$

Proof. The proof follows by multiplying the equation by $-\nabla(|\nabla u|^{p-2}\nabla u)$, integrating it by parts and estimating carefully the boundary integral. For details see [11].

Here, by the Gagliardo–Nirenberg inequality we have

$$Cp^2\|\sqrt{\sigma}|\nabla u|^{p/2}\|^2$$
$$\le Cp^2\|\sqrt{\sigma}|\nabla u|^{p/2}\|_1^{2(1-\theta)}\|\sqrt{\sigma}|\nabla u|^{p/2}\|_{H_1}^{2\theta}, \theta = N/(N+2),$$
$$\le \frac{\epsilon_0}{2p^2}\|\sqrt{\sigma}|\nabla u|^{p/2}\|_{H_1}^2 + Cp^\alpha\|\sqrt{\sigma}|\nabla u|^{p/2}\|_1^2$$

with $\alpha = 2(N+4)/N$, and hence (4.1) implies,

$$\frac{1}{p}\frac{d}{dt}\|\nabla u(t)\|_p^p + \frac{\epsilon_0}{p^2}\left(\|\sqrt{\sigma_\epsilon}|\nabla u|^{p/2}\|_{H_1}^2 + \|\sqrt{\sigma_\epsilon}\nabla(|\nabla u|^{p/2})\|^2\right)$$
$$\leq Cp^\alpha\|\sqrt{\sigma_\epsilon(|\nabla u|^2)}|\nabla u|^{p/2}\|_1^2 \tag{4.2}$$

where we have changed $\epsilon_0/2$ by ϵ_0. Further we easily see that

$$\|\sqrt{\sigma_\epsilon(|\nabla u|^2)}|\nabla u|^{p/2}\|_1^2 \leq C\Gamma_\epsilon(t)\|\nabla u(t)\|_{p-2}^{p-2}$$

and the inequality (4.2) implies, in particular,

$$\frac{d}{dt}\|\nabla u(t)\|_{p_0}^2 \leq Cp_0^\alpha\Gamma_\epsilon(t), p_0 \geq 2. \tag{4.3}$$

Using the estimate (3.6) we obtain from (4.3),

$$\|\nabla u(t)\|_{p_0}^2 \leq \|\nabla u_0\|_{p_0}^2 + Cp_0^\alpha\int_0^t\left(\Gamma_\epsilon(0)^{-m/(m+2)} + mC_1^{-1}s\right)^{-(m+2)/m}ds$$
$$\leq \|\nabla u_0\|_{p_0}^2 + Cp_0^\alpha\Gamma_\epsilon(0)^{2/(m+2)}), 0 \leq t < \infty. \tag{4.4}$$

((4.4) will show (2.3).) Also, by Hyp.A,(4), we see

$$\|\sqrt{\sigma_\epsilon(|\nabla u|^2)}|\nabla u|^{p/2}\|_1^2 \leq C\int_{\Omega_1}|\nabla u|^p dx + \int_{\Omega_2}\sigma_\epsilon(|\nabla u|^2)dx\int_{\Omega_2}|\nabla u|^p dx.$$

Here, by Hyp.A,(3),

$$\int_{\Omega_2}\sigma_\epsilon(|\nabla u|^2)dx \leq C\int_{\Omega_2}(\sigma_\epsilon(|\nabla u|^2)|\nabla u|^2)^{L/(L+2)}dx$$
$$\leq C\Gamma_\epsilon(t)^{L/(L+2)} \leq \begin{cases} C_0 t^{-2L\nu_0/(L+2)} \\ C_1 \end{cases} \tag{4.5}$$

for $0 < t \leq 1$.

Therefore we obtain from (4.2),

$$\frac{1}{p}\frac{d}{dt}\|\nabla u(t)\|_p^p + \frac{\epsilon_0}{p^2}\left(\|\sqrt{\sigma_\epsilon}|\nabla u|^{p/2}\|_{H_1}^2 + \|\sqrt{\sigma_\epsilon}\nabla(|\nabla u|^{p/2})\|^2\right)$$
$$\leq \begin{cases} C_0p^\alpha t^{-2L\nu_0/(L+2)}\|\nabla u(t)\|_p^p \\ C_1p^\alpha\|\nabla u(t)\|_p^p \end{cases} \tag{4.6}$$

for $0 < t \leq 1$. These are the starting inequalities to estimate $\|\nabla u(t)\|_\infty, 0 < t \leq 1$.

Proposition 4.2. *If $1 \leq r < 2$ we make the condition (2.5) stated in Theorem 2.2. Let $K > 1$ and assume*

$$\|\nabla u(t)\|_\infty \leq Kt^{-\lambda}, 0 < t \leq T_\epsilon \leq 1, \tag{4.7}$$

with some $\lambda \geq 0$. Then, making the additional assumption

$$\|\nabla u(t)\|_q \leq \eta_q t^{-\lambda_q}, 0 < t \leq T_\epsilon, \tag{4.8}$$

for some $q \geq 2$, we obtain

$$\|\nabla u(t)\|_p \leq \left(C_0\epsilon_0^{-1}p^{\alpha+2}K^{\nu}\right)^{N(p-q)/p(mN+2q)}\eta_q^{1-mN(p-q)/(mN+2q)p} \times t^{-(m\lambda+1)N(p-q)/p(mN+2q)-\lambda_q(mN+2p)q/(mN+2q)p}, 0 < t \leq T_\epsilon, \tag{4.9}$$

for $p \geq q$, where we recall $C_0 = C(\|u_0\|_r)$.

The estimate (4.9) also holds even if we replace C_0 by $C_1 = C(\Gamma(0), \|\nabla u_0\|_{p_0})$ and in this case the assumption (2.5) is unnecessary.

Proof. By (4.7) and the Hyp. A,(4), we see from (4.6) that

$$\frac{1}{p}\frac{d}{dt}\|\nabla u(t)\|_p^p + \frac{\epsilon_0}{p^2}K^{-\nu}t^{m\lambda}\||\nabla u|^{(p+m)/2}\|_{H_1}^2 \leq \begin{cases} C_0p^{\alpha}t^{-2L\nu_0/(L+2)}\|\nabla u(t)\|_p^p \\ C_1p^{\alpha}\|\nabla u(t)\|_p^p \end{cases} \tag{4.10}$$

for $0 < t \leq T_\epsilon$, where ϵ_0 is changed. By the Gagliardo–Nirenberg inequality we see

$$\|\nabla u(t)\|_p \leq C^{1/p}\|\nabla u(t)\|_q^{1-\tilde{\theta}}\||\nabla u|^{(p+m)/2}\|_{H_1}^{2\tilde{\theta}/(p+m)} \tag{4.11}$$

with

$$\tilde{\theta} = \left(\frac{p+m}{2}(\frac{1}{q} - \frac{1}{p})\right)\left(\frac{1}{N} - \frac{1}{2} + \frac{p+m}{2}\cdot\frac{1}{q}\right)^{-1} = \frac{N(p+m)(p-q)}{p(mN+2q+N(p-q))}. \tag{4.12}$$

It follows from (4.10) and (4.11) that

$$\frac{d}{dt}\|\nabla u(t)\|_p + \frac{\epsilon_0}{p^2}K^{-\nu}t^{m\lambda}\|\nabla u(t)\|_p^{((1-\tilde{\theta})p+m)/\tilde{\theta}+1}\|\nabla u(t)\|_q^{-(1-\tilde{\theta})(p+m)/\tilde{\theta}} \leq \begin{cases} C_0p^{\alpha}t^{-2L\nu_0/(L+2)}\|\nabla u(t)\|_p \\ C_1p^{\alpha}\|\nabla u(t)\|_p \end{cases}, 0 < t \leq T_\epsilon,$$

and hence, by the assumption (4.8),

$$\frac{d}{dt}\|\nabla u(t)\|_p + \frac{\epsilon_0}{p^2}K^{-\nu}t^{m\lambda+\lambda_q(1-\tilde{\theta})(p+m)/\tilde{\theta}}\|\nabla u(t)\|_p^{((1-\tilde{\theta})p+m)/\tilde{\theta}+1} \leq \begin{cases} C_0p^{\alpha}t^{-2L\nu_0/(L+2)}\|\nabla u(t)\|_p \\ C_1p^{\alpha}\|\nabla u(t)\|_p \end{cases} \tag{4.13}$$

for $0 < t \leq T_\epsilon$.

Now, we know the following lemma concerning a singular differential inequality which is a special case of Lemma 2.2 in [16].

Lemma 4.1. *Let $y(t)$ be an absolutely continuous function on $(0, T], T > 0$, and satisfy the inequality*

$$\frac{d}{dt}y(t) + At^{\nu\alpha-1}y^{1+\alpha}(t) \leq Bt^{-\delta}y(t), 0 < t \leq T,$$

where we assume $A > 0, B \geq 0, \nu\alpha \geq 1$ and $0 \leq \delta \leq 1$. Then we have

$$y(t) \leq A^{-1/\alpha}(\nu + BT^{1-\delta})^{1/\alpha}t^{-\nu}, 0 < t \leq T.$$

Applying this with $T = T_\epsilon \leq 1$ to the first inequality of (4.13) we obtain (4.9) after some careful calculations. For this we have used the assumption (2.5) which is equivalent to $(\delta \equiv)2L\nu_0/(L+2) \leq 1$. In (4.9) we can replace C_0 by C_1 by using the second inequality of (4.12).

We prepare the following proposition which is easily deduced from Proposition 4.2.

Proposition 4.3. *Under the assumption (4.7), we have for $p > p_0 \geq 2$,*

$$\begin{aligned}\|\nabla u(t)\|_p \leq \left(C_1\epsilon_0^{-1}p^{\alpha+2}K^\nu\right)^{N(p-p_0)/p(mN+2p_0)} \\ \times \left((\|\nabla u_0\|_{p_0} + \Gamma_\epsilon(0)^{1/(m+2)}\right)^{1-mN(p-p_0)/(mN+2p_0)p} \\ \times t^{-(m\lambda+1)N(p-p_0)/(mN+2p_0)p}, 0 < t \leq T_\epsilon.\end{aligned} \tag{4.14}$$

We have also for $p > 2$,

$$\begin{aligned}\|\nabla u(t)\|_p \leq \left(C_0\epsilon_0^{-1}p^{\alpha+2}K^\nu\right)^{N(p-2)/(mN+4)p}\|u_0\|_r^{2(1-\theta)/(m+2)-2mN(p-2)(1-\theta)/p(mN+4)(m+2)} \\ \times t^{-(m\lambda+1)N(p-2)/p(mN+4)-2\nu_0(mN+2p)/(mN+4)p}, 0 < t \leq T_\epsilon \leq 1,\end{aligned} \tag{4.15}$$

where we recall

$$\theta = \frac{N(2-r)^+}{2r+N(2-r)^+} \text{ and } \nu_0 = \frac{2r+(2-r)^+N}{\tilde{l}(2r+N(2-r)^+)+4r}.$$

(The condition (2.5) is required for the estimate (4.14).)

Proof. We know by (4.4) and (3.17),

$$\|\nabla u(t)\|_{p_0}^2 \leq \|\nabla u_0\|_{p_0}^2 + Cp_0^\alpha\Gamma_\epsilon(0)^{2/(m+2)}$$

and

$$\|\nabla u(t)\| \leq C_0\|u_0\|_r^{2(1-\theta)/(m+2)}t^{-\nu_0}, 0 < t \leq T_\epsilon \leq 1.$$

Taking $q = p_0$ and $q = 2$ and applying (4.9) to these cases we get the estimates (4.13) and (4.15), respectively.

5. Estimates for $\|\nabla u(t)\|_\infty, 0 < t \leq 1$

On the assumption (4.6) we shall derive estimates for $\|\nabla u(t)\|_\infty$, $0 < t \leq T_\epsilon \leq 1$, for the approximate smooth solution $u(t)$ based on the results in previous sections. The aim is to derive an estimate like $\|\nabla u(t)\|_\infty \leq C(K)t^{-\lambda}$, $0 < t \leq T_\epsilon \leq 1$ with some $C(K)$ and a certain $\lambda > 0$. In this estimate, if we can take a large $K > 1$ such that $C(K) < K$ we can conclude, by a continuity principle, that $\|\nabla u(t)\|_\infty < Kt^{-\lambda}$ for $0 < t \leq 1$. Such an argument we call 'a loan method'. The argument is delicate. For other types of 'loan' method see [10,12,11] and the references cited therein. We use Moser's iteration method (cf. [1,8,13,14]). First we consider the estimate depending on $\|\nabla u_0\|_{p_0}$.

We take $p_1 > p_0$ and define $p_n, n \geq 2$, by $p_n + m = 2p_{n-1}$, that is, $p_n = 2^{n-1}(p_1 - m) + m$, $n = 1, 2, \cdots$. We shall derive, by induction, the estimate

$$\|\nabla u(t)\|_{p_n} \leq \eta_n t^{-\lambda_n}, 0 < t \leq T_\epsilon \leq 1, \tag{5.1}$$

where η_1 and λ_1 are determined through (4.13) as follows:

$$\eta_1 = \left(C_1\epsilon_0^{-1}p_1^{\alpha+2}K^\nu\right)^{N(p_1-p_0)/p_1(mN+2p_0)} \times \left(\|\nabla u_0\|_{p_0} + Cp_0^{\alpha/2}\Gamma_\epsilon(0)^{1/(m+2)}\right)^{1-mN(p_1-p_0)/(mN+2p_0)p_1} \tag{5.2}$$

and

$$\lambda_1 = (m\lambda+1)N(p_1-p_0)/(mN+2p_0)p_1. \tag{5.3}$$

Then we see from (4.8) with $p = p_n, q = p_{n-1}$ and from the assumption of induction that

$$\|\nabla u(t)\|_{p_n} \le \left(C_1\epsilon_0^{-1}p_n^{\alpha+2}K^\nu\right)^{N(p_n-p_{n-1})/p_n(mN+2p_{n-1})} \eta_{n-1}^{1-mN(p_n-p_{n-1})/(mN+2p_{n-1})p_n} \times t^{-(m\lambda+1)N(p_n-p_{n-1})/p_n(mN+2p_{n-1})-\lambda_{n-1}(mN+2p_n)p_{n-1}/(mN+2p_{n-1})p_n}. \tag{5.4}$$

This means that (5.1) is valid for $n \ge 2$ if we define

$$\eta_n = \left(C_1\epsilon_0^{-1}p_n^{\alpha+2}K^\nu\right)^{N(p_n-p_{n-1})/p_n(mN+2p_{n-1})} \eta_{n-1}^{1-mN(p_n-p_{n-1})/(mN+2p_{n-1})p_n} \tag{5.5}$$

and

$$\lambda_n = (m\lambda+1)N(p_n-p_{n-1})/p_n(mN+2p_{n-1}) + \lambda_{n-1}(mN+2p_n)p_{n-1}/(mN+2p_{n-1})p_n. \tag{5.6}$$

Setting

$$\beta_n = \frac{p_n(mN+2p_{n-1})}{N(p_n-p_{n-1})} \tag{5.7}$$

we see

$$\eta_n = \left(C_1\epsilon_0^{-1}p_n^{\alpha+2}K^\nu\right)^{1/\beta_n} \eta_{n-1}^{1-m/\beta_n} \tag{5.5'}$$

and

$$\lambda_n = \frac{m\lambda+1-(m-\beta_n)\lambda_{n-1}}{\beta_n}. \tag{5.6'}$$

From (5.6′) we have

$$\lambda_n - \frac{m\lambda+1}{m} = (1-\frac{m}{\beta_n})(\lambda_{n-1} - \frac{m\lambda+1}{m}) = \Pi_{k=2}^n(1-\frac{m}{\beta_k})(\lambda_1 - \frac{m\lambda+1}{m}).$$

Here, setting $w_n = 2p_n + mN$, we know

$$1 - \frac{m}{\beta_n} = \frac{w_n}{p_n}\cdot\frac{p_{n-1}}{w_{n-1}}$$

and hence,

$$\Pi_{k=2}^n(1-\frac{m}{\beta_k}) = \frac{w_n}{p_n}\cdot\frac{p_1}{w_1} \to \frac{2p_1}{2p_1+mN} \text{ as } n\to\infty. \tag{5.8}$$

Consequently,

$$\lambda_n \to \frac{2p_1\lambda_1}{2p_1 + mN} + \frac{N(m\lambda + 1)}{mN + 2p_1} \equiv \bar{\lambda}. \tag{5.9}$$

Further, by (5.3) we see

$$\bar{\lambda} = \frac{(m\lambda + 1)N}{mN + 2p_0}, \tag{5.10}$$

which is independent of p_1.

Next, we consider η_n. We may assume $C_1\epsilon_0^{-1}p_n^6 K^\nu > 1$ and by (5.5′),

$$\begin{aligned} \log\, \eta_n &\leq \frac{(\alpha+2)\log\, p_n + \nu\log\, K + C_1}{\beta_n} + (1 - \frac{m}{\beta_n})\log\, \eta_{n-1} \\ &\leq \sum_{k=1}^{n} \frac{(\alpha+2)\log\, p_k + \nu\log\, K + C_1}{\beta_k} + \Pi_{k=2}^n(1 - \frac{m}{\beta_k})\log\, \eta_1, \end{aligned} \tag{5.11}$$

where we have used the fact $\beta_n > m$.

Since $\beta_n \geq p_n(mN + 2p_n)/N(p_n - m) > 2p_n/N$ we easily see

$$\sum_{k=1}^{n} \frac{1}{\beta_k} \leq \frac{N}{2}\sum_{k=1}^{\infty}\frac{1}{p_k} < \frac{2N}{p_1} \tag{5.12}$$

if $p_1 \geq m + 2$. Further we know by (5.2) and (5.8),

$$\begin{aligned} &\lim_{n\to\infty} \Pi_{k=2}^n(1 - \frac{m}{\beta_k})\log\, \eta_1 \\ &\leq \log\, C_1 + \frac{2p_0}{(mN + 2p_0)}\log\, \left(\|\nabla u_0\|_{p_0} + Cp_0^{\alpha/2}\Gamma_\epsilon(0)^{1/(m+2)}\right) \\ &\qquad + \nu(p_1)\log\, K, \end{aligned} \tag{5.13}$$

where we set

$$\nu(p_1) = \frac{2p_1 N\nu(p_1 - p_0}{(2p_1 + mN)(mN + 2p_0)}.$$

Thus we have from (5.11) and (5.13)

$$\begin{aligned} \overline{\lim}_{n\to\infty}\log\, \eta_n &\leq \log\, C_1 + \frac{2p_0}{mN + 2p_0}\, \log\, (\|\nabla u_0\|_{p_0} + Cp_0^{\alpha/2}\Gamma_\epsilon(0)^{1/(m+2)}) \\ &\quad + (\mu(p_1) + \nu(p_1))\log\, K \end{aligned} \tag{5.14}$$

with $\mu(p_1) = \nu\sum_{k=1}^\infty 1/\beta_k$. Note that

$$\lim_{p_1\to\infty} \nu(p_1) = \frac{\nu N}{mN + 2p_0} \text{ and } \lim_{p_1\to\infty} \mu(p_1) = 0.$$

We assume here $p_0 > N(\nu - m)/2$. Then $\nu N/(mN + 2p_0) < 1$. Therefore, taking a sufficiently large p_1 and fixing it, we obtain

$$\overline{\lim}_{n\to\infty}\eta_n \leq C_1(\|\nabla u_0\|_{p_0} + \Gamma_\epsilon(0)^{1/(m+2)})^{2p_0/(mN+2p_0)}K^\mu \tag{5.15}$$

with some $\mu < 1$. It follows from (5.1), (5.9) and (5.15) that

$$\|\nabla u(t)\|_\infty \le C_1(\|\nabla u_0\|_{p_0} + \Gamma_\epsilon(0)^{1/(m+2)})^{2p_0/(mN+2p_0)} K^\mu t^{-\bar\lambda}, 0 < t \le T_\epsilon \le 1, \tag{5.16}$$

where we recall $\bar\lambda = (m\lambda + 1)N/(mN + 2p_0)$. Now we take λ satisfying $\lambda = \bar\lambda$, that is,

$$\lambda = \frac{N}{2p_0}. \tag{5.17}$$

Then we have

$$\|\nabla u(t)\|_\infty \le C_1(\|\nabla u_0\|_{p_0} + \Gamma_\epsilon(0)^{1/(m+2)})^{2p_0/(mN+2p_0)} K^\mu t^{-\lambda}, 0 < t \le T_\epsilon \le 1. \tag{5.18}$$

Now, letting $p_0 \ge L + 2$ we can take a large $K = K(\|\nabla u_0\|_{p_0})$ which is independent of $\epsilon, 0 < \epsilon \ll 1$, and depends continuously on $\|\nabla u_0\|_{p_0}$ such that

$$C_1(\|\nabla u_0\|_{p_0} + \Gamma_\epsilon(0)^{1/(m+2)})^{2p_0/(mN+2p_0)} K^\mu < K. \tag{5.19}$$

Then we arrive at the desired estimate

$$\|\nabla u(t)\|_\infty < K t^{-\lambda}, 0 < t \le T_\epsilon \le 1. \tag{5.20}$$

Thus, starting from the assumption (4.6) which is certainly valid for a small $T_\epsilon > 0$, we have derived a sharper estimate (5.20). This means that the following estimate holds:

$$\|\nabla u(t)\|_\infty < K t^{-\lambda}, 0 < t \le 1. \tag{5.21}$$

Further, all of the estimates established so far for $0 < t \le T_\epsilon$ are in fact valid for $0 < t \le 1$. We summarize the above result.

Proposition 5.1. *Let $p_0 > N(\nu - m)/2$ and $p_0 \ge L + 2$. Then there exists $C_1 = C(\|\nabla u_0\|_{p_0}) > 0$ such that the approximate solution $u(t) = u_\epsilon(t)$ satisfies*

$$\|\nabla u(t)\|_\infty \le C_1 \left(\|\nabla u_0\|_{p_0} + \Gamma_\epsilon(0)^{1/(m+2)}\right)^{2p_0/(mN+2p_0)} t^{-N/2p_0}, 0 < t \le 1. \tag{5.22}$$

Next, we shall derive a similar estimate to (5.22) which depends only on $\|u_0\|_r$. For this we refine the above argument. We shall derive again the estimate (5.1) where in the present case we determine η_1 and λ_1 through (4.14) as follows:

$$\eta_1 = C_0 \left(\epsilon_0^{-1} p_1^{\alpha+2} K^\nu\right)^{N(p_1-2)/p_1(mN+4)} \|u_0\|_r^{4(2p_1+mN)(1-\theta)/p_1(m+2)(mN+4)} \tag{5.23}$$

and

$$\lambda_1 = (m\lambda + 1)N(p_1 - 2)/p_1(mN + 4) + 2\nu_0(mN + 2p_1)/(mN + 4)p_1. \tag{5.24}$$

We knew already that (5.1) is valid for n if we define η_n, λ_n by (5.5) and (5.6), respectively. Thus, by the same argument as above we have (see (5.8) and (5.9))

$$\begin{aligned} \lambda_n \to & \frac{p_1}{p_1 + N}(\lambda_1 - \frac{m\lambda + 1}{2}) + \frac{m\lambda + 1}{m} \\ = & \frac{2p_1 + \lambda_1 + N(m\lambda + 1)}{2p_1 + mN} \equiv \bar\lambda \end{aligned}$$

as $n \to \infty$, and

$$\overline{\lim}_{n\to\infty}\log\, \eta_n \le \sum_{k=1}^{\infty} \frac{(\alpha+2)\log\, p_k + \nu\log\, K + C_0}{\beta_k} + \Pi_{k=2}^{\infty}(1-\frac{m}{\beta_k})\log\, \eta_1$$
$$\le \log\, C_0 + \nu \sum_{k=1}^{\infty} \frac{1}{\beta_k}\, \log\, K + \frac{2p_1}{2p_1+mN}\log\, \eta_1.$$

Substituting (5.24) we see

$$\bar{\lambda} = \frac{N(m\lambda+1)+4\nu_0}{mN+4} \tag{5.25}$$

which is independent of p_1. We take λ satisfying $\lambda = \bar{\lambda}$, that is,

$$\lambda = \frac{N+4\nu_0}{4}. \tag{5.26}$$

Next, substituting (5.23) into the inequality for η_n we have

$$\overline{\lim}_{n\to\infty}\log\, \eta_n \le \log\, C_0 + \log\, p_1 + (\mu(p_1)+\nu(p_1))\log\, K$$
$$+\frac{8p_1(1-\theta)}{(m+2)(mN+4)}\log\, \|u_0\|_r, \tag{5.27}$$

where we set $\nu(p_1) = 2\nu N(p_1-2)/(2p_1+mN)(mN+4)$. Thus we obtain

$$\|\nabla u(t)\|_\infty \le C_0\|u_0\|_r^{8(1-\theta)/(m+2)(4+mN)}K^{\mu(p_1)+\nu(p_1)}t^{-\lambda}, 0<t\le T_\epsilon \le 1. \tag{5.28}$$

We see that $\mu(p_1) \to 0$ and $\nu(p_1) \to \nu N/(mN+4)$ as $p_1 \to \infty$. Let us assume here $\nu < m+4/N$. Then we can fix a large p_1 and take a large $K = C(\|u_0\|_r)$ such that

$$\|\nabla u(t)\|_\infty \le C_0\|u_0\|_r^{8(1-\theta)/(m+2)(4+mN)}K^{\mu}t^{-\lambda} < Kt^{-\lambda}, 0<t\le T_\epsilon \le 1, \tag{5.29}$$

with some $\mu < 1$. Thus we conclude that the estimate (5.29) holds in fact for $0 < t \le 1$ and all of the estimates depending on $\|u_0\|_r$ derived so far under the assumption $\|\nabla u(t)\|_\infty \le Kt^{-\lambda}$, $0 < t \le T_\epsilon$, hold in fact for $0 < t \le 1$.

We summarize the result.

Proposition 5.2. *Assume that $\nu < m+4/N$. Then there exists a large $K = K(\|u\|_0)$ continuously depending on $\|u_0\|_r$ such that the estimate (5.29) holds for $T_\epsilon = 1$ for the approximate solution $u(t) = u_\epsilon(t)$.*

6. Decay estimate for $\|\nabla u(t)\|_\infty, t \ge 1$

Finally we derive the boundedness and also the decay estimate for $\|\nabla u(t)\|_\infty$ as $t \to \infty$. When $\sigma(|\nabla u|^2) = \log\, (1+|\nabla u|^2)$ we know the estimate $\|\nabla u(t)\|_\infty \le C(\|\nabla u_0\|_\infty)(1+t)^{-1/2}$, $0 \le t < \infty$ (see [11]). The general case is treated similarly. So we give an outline of the proof. It suffices to consider an assumed smooth solution $u(t)$ for the original problem.

Since $\|\nabla u(1)\|_\infty \le C_0$ we see from an argument similar to the one deriving (4.3) that

$$\|\nabla u(t)\|_{p_1} \le C_0 p_1^{\alpha} < \infty, 1 \le t < \infty, \tag{6.1}$$

for any $p_1 \geq m+2$. We return to the inequality (4.1). Using a similar argument to the one deriving (4.12) we have

$$\frac{1}{p}\frac{d}{dt}\|\nabla u(t)\|_p^p + \frac{\epsilon_0}{p^2}\left(\|\sqrt{\sigma_\epsilon}|\nabla u|^{p/2}\|_{H_1}^2 + \|\sqrt{\sigma_\epsilon}\nabla(|\nabla u|^{p/2})\|^2\right)$$
$$\leq C_0 p^\alpha \|\nabla u\|_{p-2}^{p-2} \leq C_0 p^\alpha (1+\|\nabla u(t)\|_p^p). \tag{6.2}$$

Let $K > \|\nabla u(1)\|_\infty$. Then we may assume

$$\|\nabla u(t)\|_\infty \leq K, 1 \leq t \leq T, \tag{6.3}$$

for some $T > 1$. Setting $p_n = 2p_{n-1} - m$ with $p_1 \geq m+2$ we can derive, by induction, the estimate

$$\|\nabla u(t)\|_{p_n} \leq \eta_n, 1 \leq t \leq T, \tag{6.4}$$

with $\eta_1 = \max\{1, C\|\nabla u(1)\|_\infty, \sup_{1\leq t<\infty}, \|\nabla u(t)\|_{p_1}\}$. Indeed, by the inequality

$$\|\nabla u(t)\|_{p_n} \leq C^{1/p_n}\|\nabla u(t)\|_{p_{n-1}}^{1-\theta_n}\||\nabla u(t)|^{(p_n+m)/2}\|_{H_1}^{2\theta_n/(p_n+m)}$$

with $\theta_n = 2N(1-p_{n-1}/p_n)/(N+2)$ we have from (6.2),

$$\frac{1}{p_n}\frac{d}{dt}\|\nabla u(t)\|_{p_n}^{p_n} + \frac{\epsilon_0}{p_n^2 K^\nu}\eta_{n-1}^{(p_n+m)(1-\theta_n)/\theta_n}\|\nabla u(t)\|_{p_n}^{(p_n+m)/\theta_n}$$
$$\leq C_0 p_n^\alpha (1+\|\nabla u(t)\|_{p_n}^{p_n}), 1 \leq t \leq T. \tag{6.5}$$

If $\|\nabla u(t)\|_{p_n} \geq 1$ for some t we see

$$\frac{d}{dt}\|\nabla u(t)\|_{p_n} + \left(\frac{\epsilon_0}{p_n^2 K^\nu}\eta_{n-1}^{(p_n+m)(1-\theta_n)/\theta_n}\|\nabla u(t)\|_{p_n}^{(p_n(1-\theta_n)+m)/\theta_n} - 2C_0 p_n^\alpha\right)\|\nabla u(t)\|_{p_n} \leq 0 \tag{6.6}$$

at the time t, which implies for all $t, 1 \leq t \leq T$,

$$\|\nabla u(t)\|_{p_n} \leq \max\{1, C\|\nabla u(1)\|_\infty, \tilde{\eta}_n\}, \tag{6.7}$$

where we set

$$\tilde{\eta}_n \equiv \left(C_0 p_n^{\alpha+2}K^\nu \eta_{n-1}^{(p_n+m)(1-\theta_n)/\theta_n}\right)^{\theta_n/((1-\theta_n)p_n+m)} = \left(C_0 p_n^{\alpha+2}K^\nu\right)^{m/\beta_n}\eta_{n-1}^{1-m/\beta_n}$$

with $\beta_n = m((1-\theta_n)p_n + m)/\theta_n$.

Since $\eta_{n-1} \geq 1$ and we may assume $C_0 p_n^2 K^\nu \geq 1$, the right-hand side of (6.7) is dominated by

$$\max\{C\|\nabla u(1)\|_\infty, \left(C_0 p_n^{\alpha+2}K^\nu\right)^{m/\beta_n}\eta_{n-1}\} = \left(C_0 p_n^{\mu+2}K^\nu\right)^{m/\beta_n}\eta_{n-1}.$$

Thus we have

$$\|\nabla u(t)\|_{p_n} \leq \left(C_0 p_n^{\alpha+2}K^\nu\right)^{m/\beta_n}\eta_{n-1} \equiv \eta_n. \tag{6.8}$$

We see as in the argument deriving (5.14) that

$$\eta_n \leq C_0 p_1^\alpha \|\nabla u(1)\|_{p_1} K^{\mu(p_1)} \tag{6.9}$$

with some $\mu(p_1)$ which tends to 0 as $p_1 \to \infty$. Thus, we can take a large p_1 to obtain

$$\|\nabla u(t)\|_\infty \leq C_0 K^\mu, 1 \leq t \leq T \tag{6.10}$$

with some $\mu < 1$, where we fix p_1. Therefore, by taking a large $K = K(\|u_0\|_r)$, we see $\|\nabla u(t)\|_\infty < K, 1 \leq t \leq T$. We conclude that

$$\|\nabla u(t)\|_\infty \leq C_0 < \infty, 1 \leq t < \infty. \tag{6.11}$$

It is clear that C_0 can be replaced by C_1.

We proceed to the decay estimate for $\|\nabla u(t)\|_\infty$. Once the boundedness of $\|\nabla u(t)\|_\infty$ has been established we see

$$\sigma(|\nabla u(t)|^2) \geq C_0^{-1} |\nabla u(t)|^m, t \geq 1,$$

with some positive constant C_0. (We can replace C_0 by C_1.) Therefore we have from (4.1) and (3.5) (or (3.6))

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|\nabla u(t)\|_p^p + \frac{1}{C_0 p^2}\||\nabla u|^{(p+m)/2}\|_{H_1}^2 &\leq C p^\mu \Gamma(t) \|\nabla u(t)\|_{p-2}^{p-2} \\ &\leq C_0 p^\alpha (1+t)^{-(m+2)/m} \|\nabla u(t)\|_p^{p-2}, 1 \leq t < \infty.\end{aligned} \tag{6.12}$$

Setting $\mathbf{w}(\tau) = (1+t)^{1/m}\nabla u(t)$ and $\tau = \log(1+t)$, (6.12) is rewritten as

$$\begin{aligned}&\frac{1}{p}\frac{d}{d\tau}\|\mathbf{w}(\tau)\|_p^p + \frac{1}{C_1 p^2}\|\mathbf{w}(\tau)|^{(p+m)/2}\|_{H_1}^2 \\ &\leq C_0 p^\alpha (\|\mathbf{w}(\tau)\|_p^p + \|\mathbf{w}(\tau)\|_p^{p-2}) \leq C_0 p^\alpha (\|\mathbf{w}(\tau)\|_p^p + 1).\end{aligned} \tag{6.13}$$

This is essentially the same form as (6.2) with K fixed. Further we have, instead of (3.13),

$$\begin{aligned}\|\nabla u(t)\|_{m+2}^{m+2} &\leq C \int_{\Omega_1} \sigma_\epsilon(|\nabla u(t)|^2)|\nabla u(t)|^2 dx + C_0 \int_{\Omega_2} |\nabla u(t)|^2 dx \\ &\leq C_0 \Gamma(t) \leq C_0 (1+t)^{-(m+2)/m}, t \geq 1.\end{aligned}$$

Therefore we have

$$\|\mathbf{w}(t)\|_{m+2} \leq (1+t)^{-1/m}\|\nabla u(t)\|_{m+2} \leq C_0, \ t \geq 1. \tag{6.14}$$

Thus, repeating an argument similar to the one deriving (6.11) with $p_1 = m+2$ we can derive the estimate

$$\|\mathbf{w}(\tau)\|_\infty \leq C_0 < \infty, \ \log 2 \leq \tau < \infty$$

and consequently,

$$\|\nabla u(t)\|_\infty \leq C_0 (1+t)^{-1/m}, 1 \leq t < \infty. \tag{6.15}$$

Proposition 6.1. *The approximate solution $u(t)$ satisfies the estimate (6.15). We can replace also C_0 by C_1.*

7. Proofs of Theorems 2.1 and 2.2

We have proved that the set of approximate solutions $u_\epsilon(t), 0 < \epsilon < 1$, is bounded in $L^\infty([0,\infty); W^{1,p_0}) \cap L^\infty_{loc}((0,\infty); W^{1,\infty}_0) \cap W^{1,2}([0,\infty); L^2)$ and the boundedness depends on $\|\nabla u_0\|_{p_0}, p_0 \geq L+2$, and further, the set is also bounded in $L^\infty([0,\infty); L^r) \cap L^\infty_{loc}((0,\infty); W^{1,\infty}_0) \cap W^{1,2}_{loc}((0,\infty); L^2)$ and the boundedness depends on $\|u_0\|_r$. We begin with the proof of Theorem 2.1.

We first assume that $u_0 \in C^3_0(\Omega)$. By the first boundedness of $u_\epsilon(t)$ we can extract a subsequence as $\epsilon \to 0$, which we denote again by $u_\epsilon(t)$ for simplicity, such that

$$\begin{gathered} u_\epsilon(t) \to u(t) \text{ weakly* in } L^\infty_{loc}([0,\infty); L^{p_0}), \\ \nabla u_\epsilon \to \nabla u(t) \text{ weakly* in } L^\infty_{loc}((0,\infty); L^\infty)), \\ u_\epsilon(t) \to u(t) \text{ strongly in } L^2_{loc}([0,\infty); L^2), \\ u_{\epsilon,t} \to u_t(t) \text{ weakly in } L^2_{loc}([0,\infty); L^2) \end{gathered}$$

and

$$\begin{gathered} A_\epsilon(\nabla u_\epsilon) \equiv -\operatorname{div}\{\sigma_\epsilon(|\nabla u_\epsilon(t)|^2))\nabla u_\epsilon(t)\} \to \chi(t) \\ \text{weakly in } L^{p_0/(p_0-1)}([0,\infty); W^{-1,p_0/(p_0-1)}) \end{gathered}$$

in the sense that

$$\begin{aligned} < A_\epsilon(\nabla u_\epsilon), \phi(t) >_T \equiv & \int_0^T \int_\Omega \sigma_\epsilon(|\nabla u_\epsilon(t)|^2))\nabla u_\epsilon(t) \cdot \nabla\phi(t) dx dt \\ & \to < \chi(t), \phi(t) >_T \end{aligned}$$

for any $T > 0$ and any $\phi(t) \in L^{p_0}([0,T]; W^{1,p_0}_0)$, where $<,>_T$ denotes the pairing of

$$L^{p_0/(p_0-1)}([0,T]; W^{-1,p_0/(p_0-1)}_0) \text{ and } L^{p_0}([0,T]; W^{1,p_0}_0).$$

The limit function $u(t)$ satisfies

$$\int_0^T (u_t(t), \phi(t)) + < \chi(t), \phi(t) >_T = 0 \tag{7.1}$$

for any $T > 0$ and for any $\phi(t) \in L^p_0([0,T]; W^{1,p_0}_0)$, and also we have

$$u(t) = \int_0^t u_t(s) ds + u_0 \text{ in } L^2, 0 \leq t < \infty. \tag{7.2}$$

All of the estimates established for $u_\epsilon(t)$ are still valid for $u(t)$ (with $\epsilon = 0$). To complete the proof it suffices to show that $\chi(t) = -\operatorname{div}\{\sigma(|\nabla u(t)|^2)\nabla u(t)\}$. For this we note that if $p_0 \geq L+2$,

$$\begin{aligned} & \lim_{\epsilon \to 0} \sigma_\epsilon(|\nabla u(t)|^2))\nabla u(t) \\ & = \sigma(|\nabla u(t)|^2))\nabla u(t) \text{ in } L^{p_0/(p_0-1)}_{loc}([0,\infty; L^{p_0/(p_0-1)}) \end{aligned} \tag{7.3}$$

for any $u \in L^{p_0}_{loc}([0,\infty); W^{1,p_0}_0)$, which follows from Hyp.A,(3). Further, we see by Hyp.A,(1),

$$(\sigma_\epsilon(|\nabla u|^2)\nabla u - \sigma_\epsilon(|\nabla v|^2)\nabla v, \nabla u - \nabla v)$$
$$\geq (\sigma_\epsilon(|\nabla u|^2)|\nabla u| - \sigma_\epsilon(|\nabla v|^2)|\nabla v|, |\nabla u| - |\nabla v|) \geq 0.$$

Then the identity $\chi(t) = -\mathrm{div}\{\sigma(|\nabla u(t)|^2)\nabla u(t)\}$ follows from the standard monotonicity argument on the operator $A_\epsilon(\nabla u) \equiv -\mathrm{div}\{\sigma_\epsilon(|\nabla u(t)|^2)\nabla u(t)\}$ in $L^{p_0/(p_0-1)}([0,T]; W^{-1,p_0/(p_0-1)})$, $T > 0$.

The uniqueness follows easily also from the monotonicity of $A(|\nabla u(t)|) = -\mathrm{div}\{\sigma(|\nabla u(t)|^2)\nabla u(t)\}$.

Next, we assume that $u_0 \in W^{1,p_0}_0$. Then we can take a sequence $\{u_{0,n}\} \subset C^3_0(\Omega)$ such that $u_{0,n} \to u_0$ in W^{1,p_0}_0 as $n \to \infty$. The solutions $u_n(t)$ with $u_n(0) = u_{0,n}$ satisfy essentially the same estimates for $u_\epsilon(t)$ with u_0 replaced by $u_{0,n}$, and repeating the above argument with $u_\epsilon(t)$ replaced by $u_n(t)$ we get the desired weak solution $u(t)$ in the sense of Definition 2.1. It is clear that all of the estimates (3.4), (3.5), (3.6), (3.18) and (5.29) with $T_\epsilon = 1$ (and $\epsilon = 0$) hold for this $u(t)$.

Finally, when $u_0 \in L^r, r \geq 1$, we take a sequence $\{u_{0,n}\} \subset C^3_0(\Omega)$ such that $u_{0,n} \to u_0$ in L^r as $n \to \infty$. The corresponding solutions $u_n(t)$ satisfy all of the estimates for $u_\epsilon(t)$ with u_0 replaced by $u_{0,n}$ (and with $\epsilon = 0$), in particular, the estimates depending on $\|u_{0,n}\|_r$. To check the convergency of $u_n(t)$ we first note that

$$\|u_m(t) - u_n(t)\|_r \leq \|u_{0,m} - u_{0,n}\|_r \tag{7.4}$$

which follows by multiplying the difference of two equations for $u_n(t)$ and $u_m(t)$ by $|u_n(t)-u_m(t)|^{r-2}(u_n(t)-u_m(t))$ and integrating it (when $1 \leq r < 2$ we make a device as in the one deriving the estimate (3.3)). Thus $\{u_n(t)\}$ converges uniformly to a function $u(t) \in C([0,\infty); L^r)$. Of course we see $u(0) = \lim_{n\to\infty} u_n(0) = u_0$. Along a subsequence, $\{u_n(t)\}$ converges to $u(t) \in L^\infty_{loc}((0,\infty); W^{1,\infty}_0) \cap W^{1,2}_{loc}((0,\infty); L^2) \cap C([0,\infty); L^r)$ in the following way:

$$u_n(t) \to u(t) \text{ in } C([0,\infty); L^r) \text{ and weakly* in } L^\infty_{loc}((0,\infty); W^{1,\infty}_0),$$
$$u_{n,t} \to u_t(t) \text{ weakly in } L^2_{loc}((0,\infty); L^2)$$

and

$$< A(\nabla u_n(t), \phi(t) >_{\delta,T} = \int_\delta^T \int_\Omega \sigma(\nabla u_n(t)|^2))\nabla u_n(t) \cdot \phi(t)\,dxdt$$
$$\to < \chi(t), \phi(t) >_{\delta,T}$$

for any $0 < \delta < T$ and any $\phi(t) \in L^{p_0}([0,T]; W^{1,p_0}_0)$. Note that

$$\int_\delta^T (u_t(t), \phi(t)) + < \chi(t), \phi(t) >_{\delta,T} = 0$$

for any $\phi(t) \in L^\infty_{loc}((0,\infty); W^{1,p_0}_0) \cap W^{1,2}_{loc}((0,\infty); L^2), p_0 \geq L + 2$. Then, the monotonicity argument shows that $\chi(t) = A(\nabla u(t)) = -\mathrm{div}\{\sigma(|\nabla u(t)|^2)\nabla u(t)\}$ in $L^{p_0/(p_0-1)}_{loc}((0,\infty); W^{-1,p_0/(p_0-1)})$. Therefore $u(t)$ is a solution in the sense of Definition 2.2.

Let $u_1(t), u_2(t)$ be two possible solutions with $u_1(0) = u_2(0) = u_0$ in the same class as above. Then we have easily

$$\|u_1(t) - u_2(t)\|_r \leq \|u_1(\delta) - u_2(\delta)\|_r \to 0 \text{ as } \delta \to 0, \tag{7.5}$$

which implies $u_1(t) = u_2(t)$. The uniqueness for he case $u_0 \in L^r$ is also proved.

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