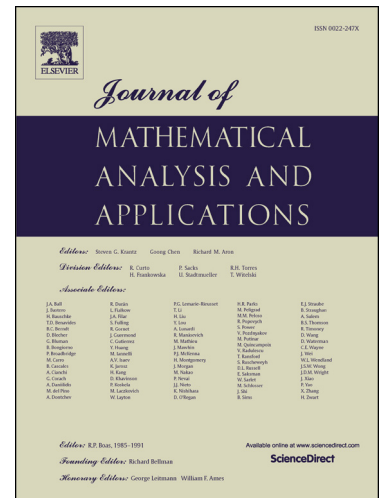




Quantum Markov States on Cayley trees

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Abstract

It is known that any locally faithful quantum Markov state (QMS) on one dimensional setting can be considered as a Gibbs state associated with Hamiltonian with commuting nearest-neighbor interactions. In our previous results, we have investigated quantum Markov states (QMS) associated with Ising type models with competing interactions, which are expected to be QMS, but up to now, there is no any characterization of QMS over trees. We notice that these QMS do not have one-dimensional analogues, hence results of related to one dimensional QMS are not applicable. Therefore, the main aim of the present paper is to describe of QMS over Cayley trees. Namely, we prove that any QMS (associated with localized conditional expectations) can be realized as integral of product states w.t.r. a Gibbs measure. Moreover, it is established that any locally faithful QMS associated with localized conditional expectations can be considered as a Gibbs state corresponding to Hamiltonians (on the Cayley tree) with commuting competing interactions.

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Key words: Quantum Markov state; localized; Cayley tree; disintegration; Ising type model; chain.

1 Introduction

It is known that [18], in quantum statistical mechanics, concrete systems are identified with states on corresponding algebras. In many cases, the algebra can be chosen to be a quasi-local algebra of observables. The states on these algebras satisfying Kubo–Martin–Schwinger (KMS) boundary condition, as is known, describe equilibrium states of the quantum system under consideration. On the other hand, for classical systems with the finite radius of interaction, limiting Gibbs measures are known to be Markov

random fields, see e.g. [20, 25, 35]. In connection with this, there is a problem of constructing analogues of non commutative Markov chains, which arises from quantum statistical mechanics and quantum field theory in a natural way [21]. This problem was firstly explored in [1] by introducing non commutative Markov chains on the algebra of quasi-local observables. The reader is referred to [4]–[8],[22, 27, 28, 29] and the references cited therein, for recent development of the theory of quantum stochastic processes and their applications.

The investigation of a particular class of quantum Markov chains, called *quantum Markov states (QMS)*, was pursued in [4, 8, 14], where connections with properties of the modular operator of the states under consideration were established [7, 26]. This provides natural applications to temperature states arising from suitable quantum spin models, that is natural connections with the KMS boundary condition.¹

In [4], the most general one dimensional quantum Markov states have been considered. Among the other results concerning the structure of such states, a connection with classes of local Hamiltonians satisfying certain commutation relations and quantum Markov states has been obtained. The situation arising from quantum Markov states on the chain, describes one dimensional models of statistical mechanics with mutually commuting nearest neighbor interactions. Namely, one dimensional quantum Markov states are very near to be (diagonal liftings of) “Ising type” models, apart from noncommuting boundary terms.

One of the basic open problems in quantum probability is the construction of a theory of quantum Markov fields, that are quantum processes with multi-dimensional index set [5]. This program concerns the generalization of the theory of Markov fields (see [20],[25]) to a non-commutative setting, naturally arising in quantum statistical mechanics and quantum field theory.

First attempts to construct quantum analogues of classical Markov fields have been done in [3]–[5],[8, 30]. In these papers the notion of *quantum Markov state*, introduced in [7], extended to fields as a sub-class of the quantum Markov chains. In [6] a more general definition of quantum Markov states and chains, including all the presently known examples, have been extended. In the mentioned papers quantum Markov fields were considered over multidimensional integer lattice $\mathbb{Z}$ which, due to the existence of loops, did not allow to construct explicit examples of such kind of fields. It is known [17, 38] that explicit Gibbs measures can be obtained on regular trees, therefore, in [13, 32], quantum Markov fields (or quantum Markov chains (QMC)) has been constructed over such trees. Moreover, certain concrete examples were provided. This direction opened a new direction in the study and construction of QMC via investigation of lattice models on trees [9]–[12],[31]. Mostly, the existing works based on certain models over the Cayley trees (or Bathe lattices) [34]. In fact, even if several definitions of quantum Markov fields on trees (and more generally on graphs) have been proposed, a really satisfactory, general theory is still missing and physically interesting examples of such fields in dimension $d \geq 2$ are very few.

On the other hand, taking into account results of [4] any QMS (one dimensional

¹Most of the states arising from Markov processes considered in [15, 22, 23] describe ground states (i.e. states at zero temperature) of suitable models of quantum spin chains.

setting) can be considered as Gibbs state associated with Hamiltonian with commuting nearest-neighbor interactions. The models considered in [12, 31, 32, 33] satisfy this type of condition, and hence, roughly speaking, QMC considered there, are expected to be QMS, but up to now, there is no any characterization of QMS over trees. On the other hand, those QMC do not have one-dimensional analogues, hence results of [4] are not applicable. Therefore, main aim of the present paper, is to fill this gap, i.e. we are going to describe of QMS over Cayley trees. This will allow us, in our further investigations, to distinguish which QMC may satisfy KMS conditions (see [8, 26]).

We emphasize that any notion of Markovianity strongly depends on an underlying notion of localization and it is known that, both in the classical and the quantum case, if the localization is sufficiently rough, then any state can be considered as Markov chain. Therefore, if one considers the the localization given by the levels of the tree, a Markov field simply becomes a non-homogeneous Markov chain and in this case the structure of the subclass of Markov states is known [4] .

In this paper, we investigate Markov property not only w.r.t. levels of the Cayley tree but also w.r.t. its finer localization structure of the considered tree through considering suitable quasi-conditional expectation called *localized*, which keeps into account this finer localization and to prove the structure theorem corresponding to this localization. An interesting consequence of this structure theorem is that the notion of competing interactions, previously introduced by hands [31, 32], now emerges as a consequence of the intrinsic denition combined with the structure theorem. Therefore, the present paper's main result differs from the non-homogeneous one dimensional case studied in [4].

Notice in our previous work [16] we investigated Markov property on the finer structure of general graphs for backward quantum Markov fields through a specific tessellation on the set of vertices.

Let us outline the organization of the paper. After preliminary information (see Section 2), in Section 3 we recall definition of quantum Markov chains and states on Cayley trees. In Section 4, localized conditional expectations (connected to the tree) are considered and described. In section 5, a Gibbs measure (see [25, 36] for Gibbs states on Cayley trees) is constructed by means of QMS associated with localized conditional expectations. In section 6, using the results of sections 4 and 5, we prove that any QMS (associated with localized conditional expectations) can be realized as integral of product states w.t.r. the Gibbs measure. In section 7, we prove a reconstruction result. Finally, in section 8, we will establish that any locally faithful QMS associated with localized conditional expectations can be considered as Gibbs state corresponding to Hamiltonians (on the Cayley tree) with commuting competing interactions which implies that all QMC considered in [31, 32] are indeed QMS.

2 Cayley tree

Let $\Gamma_+^k = (L, E)$ be a semi-infinite Cayley tree of order $k \geq 1$ with the root x^0 (i.e. each vertex of Γ_+^k has exactly $k + 1$ edges, except for the root x^0 , which has k edges). Here L is the set of vertices and E is the set of edges. The vertices x and y are called *nearest*

neighbors and they are denoted by $l = \langle x, y \rangle$ if there exists an edge connecting them. A collection of the pairs $\langle x, x_1 \rangle, \dots, \langle x_{d-1}, y \rangle$ is called a *path* from the point x to the point y . The distance $d(x, y)$, $x, y \in V$, on the Cayley tree, is the length of the shortest path from x to y .

Let us set

$$W_n = \{x \in L : d(x, x_0) = n\}, \quad \Lambda_n = \bigcup_{k=0}^n W_k, \quad \Lambda_{[n,m]} = \bigcup_{k=n}^m W_k, \quad (n < m)$$

$$E_n = \{ \langle x, y \rangle \in E : x, y \in \Lambda_n \}, \quad \Lambda_n^c = \bigcup_{k=n}^{\infty} W_k$$

Recall a coordinate structure in Γ_+^k : every vertex x (except for x^0) of Γ_+^k has coordinates $(i_1, \dots, i_n)$, here $i_m \in \{1, \dots, k\}$, $1 \leq m \leq n$ and for the vertex x^0 we put (0) . Namely, the symbol (0) constitutes level 0, and the sites $(i_1, \dots, i_n)$ form level n (i.e. $d(x^0, x) = n$) of the lattice.

For $x \in \Gamma_+^k$, $x = (i_1, \dots, i_n)$ denote

$$S(x) = \{(x, i) : 1 \leq i \leq k\}.$$

Here (x, i) means that $(i_1, \dots, i_n, i)$. This set is called a set of *direct successors* of x .

Two vertices $x, y \in V$ is called *one level next-nearest-neighbor vertices* if there is a vertex $z \in V$ such that $x, y \in S(z)$, and they are denoted by $\succ x, y \prec$. In this case the vertices x, z, y is called *ternary* and denoted by $\langle x, z, y \rangle$.

Let us rewrite the elements of W_n in the following lexicographic order (w.r.t. the coordinate system)

$$\overrightarrow{W_n} := (x_{W_n}^{(1)}, x_{W_n}^{(2)}, \dots, x_{W_n}^{(|W_n|)}).$$

Note that $|W_n| = k^n$. In this lexicographic order, the vertices $x_{W_n}^{(1)}, x_{W_n}^{(2)}, \dots, x_{W_n}^{(|W_n|)}$ of W_n can be represented in terms of the coordinate system as follows

$$\begin{aligned} x_{W_n}^{(1)} &= (1, 1, \dots, 1, 1), & x_{W_n}^{(2)} &= (1, 1, \dots, 1, 2), & \dots & & x_{W_n}^{(k)} &= (1, 1, \dots, 1, k), & (1) \\ x_{W_n}^{(k+1)} &= (1, 1, \dots, 2, 1), & x_{W_n}^{(2)} &= (1, 1, \dots, 2, 2), & \dots & & x_{W_n}^{(2k)} &= (1, 1, \dots, 2, k), \end{aligned}$$

$\vdots$

$$x_{W_n}^{(|W_n|-k+1)} = (k, k, \dots, k, 1), \quad x_{W_n}^{(|W_n|-k+2)} = (k, k, \dots, k, 2), \quad \dots \quad x_{W_n}^{(|W_n|)} = (k, k, \dots, k, k).$$

Analogously, for a given vertex x , we shall use the following notation for the set of direct successors of x :

$$\overrightarrow{S(x)} := ((x, 1), (x, 2), \dots, (x, k)), \quad \overleftarrow{S(x)} := ((x, k), (x, k-1), \dots, (x, 1)).$$

3 Quantum Markov chains and states

The algebra of observables $\mathcal{B}_x$ for any single site $x \in L$ will be taken as the algebra M_d of the complex $d \times d$ matrices. The algebra of observables localized in the finite volume $\Lambda \subset L$ is then given by $\mathcal{B}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{B}_x$. As usual if $\Lambda^1 \subset \Lambda^2 \subset L$, then $\mathcal{B}_{\Lambda^1}$ is identified as a subalgebra of $\mathcal{B}_{\Lambda^2}$ by tensoring with unit matrices on the sites $x \in \Lambda^2 \setminus \Lambda^1$. Note that, in the sequel, by $\mathcal{B}_{\Lambda,+}$ we denote the set of all positive elements of $\mathcal{B}_\Lambda$ (note that an element is positive if its spectrum is located in $\mathbb{R}_+$). The full algebra $\mathcal{B}_L$ of the tree is obtained in the usual manner by an inductive limit

$$\mathcal{B}_L = \overline{\bigcup_{\Lambda_n} \mathcal{B}_{\Lambda_n}}.$$

In what follows, by $\mathcal{S}(\mathcal{B}_\Lambda)$ we will denote the set of all states defined on the algebra $\mathcal{B}_\Lambda$.

Consider a triplet $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$ of unital C^* -algebras. Recall [2] that a *quasi-conditional expectation* with respect to the given triplet is a completely positive (CP), identity-preserving linear map $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{B}$ such that $\mathcal{E}(ca) = c\mathcal{E}(a)$, for all $a \in \mathcal{A}$, $c \in \mathcal{C}$.

In what follows, by (Umegaki) *conditional expectation* $E : \mathcal{A} \rightarrow \mathcal{B}$ we mean a norm-one projection of the C^* -algebra $\mathcal{A}$ onto a C^* -subalgebra (with the same identity $\mathbf{1}$) $\mathcal{B}$. The map E is automatically a completely positive, identity-preserving $\mathcal{B}$ -module map [37]. If $\mathcal{A}$ is a matrix algebra, then the structure of a conditional expectation is well-known [8]. Let us recall some facts. Assume that $\mathcal{A}$ is a full matrix algebra, and consider the (finite) set of $\{P_i\}$ of minimal central projections of the range $\mathcal{B}$ of E , we have

$$E(x) = \sum_i E(P_i x P_i) P_i.$$

Then E is uniquely determined by its values on the reduced algebras

$$\mathcal{A}_{P_i} := P_i \mathcal{A} P_i = N_i \otimes \bar{N}_i,$$

where $N_i \sim \mathcal{B}_{P_i} := \mathcal{B} P_i$ and $\bar{N}_i := \mathcal{B}' P_i$ (here the commutant $\mathcal{B}'$ is considered relative to $\mathcal{A}$). Moreover, there exist states ϕ_i on $\bar{N}_i$ such that

$$E(P_i(a \otimes \bar{a})P_i) = \phi_i(\bar{a})P_i(a \otimes \mathbf{1})P_i. \quad (2)$$

For the general theory of operator algebras we refer to [18, 37].

Definition 3.1. [6, 13] Let φ be a state on $\mathcal{B}_L$. Then φ is called a (backward) quantum Markov chain, associated to $\{\Lambda_n\}$, if there exists a quasi-conditional expectation $\mathcal{E}_{\Lambda_n}$ with respect to the triple $\mathcal{B}_{\Lambda_{n-1}} \subseteq \mathcal{B}_{\Lambda_n} \subseteq \mathcal{B}_{\Lambda_{n+1}}$ for each $n \in \mathbb{N}$ and an initial state $\rho_0 \in \mathcal{S}(\mathcal{B}_{\Lambda_0})$ such that

$$\varphi = \lim_{n \rightarrow \infty} \rho_0 \circ \mathcal{E}_{\Lambda_0} \circ \mathcal{E}_{\Lambda_1} \circ \cdots \circ \mathcal{E}_{\Lambda_n}$$

in the weak- $*$ topology.

Definition 3.2. [6] A quantum Markov chain φ is said to be quantum Markov state with respect to the sequence $\{\mathcal{E}_{\Lambda_j}\}$ of quasi-conditional expectations if one has

$$\varphi|_{\mathcal{B}_{\Lambda_j}} \circ \mathcal{E}_{\Lambda_j} = \varphi|_{\mathcal{B}_{\Lambda_{j+1}}}, \quad j \in \mathbb{N}.$$

In what follows, we always assume that the states are *locally faithful* (i.e. states on $\mathcal{B}$ with faithful restrictions to local subalgebras). By the standard way (see [5, 6]) one can show that the Markov property defined above can be stated by a sequence of global quasi-conditional expectations, or equally well by sequences of local or global conditional expectations. By putting $e_n := \mathcal{E}_{\Lambda_n}|_{\mathcal{B}_{\Lambda_{[n, n+1]}}}$, it will be enough to consider the ergodic limits

$$\mathcal{E}^{(n)} := \lim_m \frac{1}{m} \sum_{h=0}^{m-1} (e_n)^h,$$

which give rise to a sequence of two-step conditional expectations, called *transition expectations* in the sequel.

For $j > 0$, we define the conditional expectation E_j from $\mathcal{B}_{\Lambda_{j+1}}$ into $\mathcal{B}_{\Lambda_j}$ by:

$$E_j(a_{W_0} \otimes \cdots \otimes a_{W_j} \otimes a_{W_{j+1}}) = a_{W_0} \otimes \cdots \otimes a_{W_{j-1}} \otimes \mathcal{E}^{(j)}(a_{W_j} \otimes a_{W_{j+1}})$$

One can prove the following

Proposition 3.3. Let φ be a state on the $\mathcal{B}$. The following assertions are equivalent.

- (i) φ is a quantum Markov state;
- (ii) the properties listed in Definitions 3.1 and 3.2 are satisfied if one replaces the quasi-conditional expectations $\mathcal{E}_{\Lambda_n}$ with Umegaki conditional expectations E_n .

The next result describes the quantum Markov states. Note that if the tree Γ_+^k is one dimensional (i.e: $k=1$), then the similar result has been proven in [8, 4].

Theorem 3.4. Let $\varphi \in \mathcal{S}(\mathcal{B}_L)$. Then φ is a quantum Markov state w.r.t the sequence of transition expectations $\{\mathcal{E}^{(j)}\}_{j \geq 0}$ if and only if

$$\varphi(a) = \varphi(\mathcal{E}^{(0)}(a_{W_0} \otimes \cdots \otimes \mathcal{E}^{(n-1)}(a_{W_{n-1}} \otimes \mathcal{E}^{(n)}(a_{W_n} \otimes a_{W_{n+1}}) \cdots))) \quad (3)$$

for every $n \in \mathbb{N}$, and $a = a_{W_0} \otimes \cdots \otimes a_{W_n} \otimes a_{W_{n+1}}$ any linear generator of $\mathcal{B}_{\Lambda_{n+1}}$, with $a_{W_j} \in \mathcal{B}_{W_j}$ for $j = 1 \cdots n+1$.

Proof. Suppose that φ is a quantum Markov state w.r.t the sequence $\{\mathcal{E}^{(j)}\}_{j \geq 0}$. Then for any $a = a_{W_0} \otimes a_{W_1} \otimes \cdots \otimes a_{W_n} \otimes a_{W_{n+1}} \in \mathcal{B}_{\Lambda_{n+1}}$, by means of the Markov property one gets

$$\varphi(a) = \varphi(a_{W_0} \cdots \otimes a_{W_{n-1}} \otimes \mathcal{E}^{(n)}(a_{W_n} \otimes a_{W_{n+1}}))$$

Then by repeating the application of the Markov property n more times we obtain (3).

Conversely, assume that φ satisfies the chain of conditions (3) Then for a fixed

$$a = a_{W_0} \otimes \cdots \otimes a_{W_n} \otimes a_{W_{n+1}} \quad (4)$$

from $\mathcal{B}_{\Lambda_{n+1}}$ by (3) one finds

$$\varphi(a) = \varphi \left(\mathcal{E}^{(0)} \left(a_{W_0} \otimes \cdots \otimes \mathcal{E}^{(n-1)} \left(a_{W_{n-1}} \otimes \mathcal{E}^{(n)} \left(a_{W_n} \otimes a_{W_{n+1}} \right) \cdots \right) \right) \right)$$

And again by application of (3) we get

$$\varphi \left(\mathcal{E}^{(0)} \left(a_{W_0} \otimes \cdots \otimes \mathcal{E}^{(n-1)} \left(a_{W_{n-1}} \otimes \mathcal{E}^{(n)} \left(a_{W_n} \otimes a_{W_{n+1}} \right) \cdots \right) \right) \right) = \varphi \left(a_{W_0} \otimes \cdots \otimes a_{W_{n-1}} \otimes \widehat{a}_{W_n} \right)$$

where $\widehat{a}_{W_n} = \mathcal{E}^{(n)} \left(a_{W_n} \otimes a_{W_{n+1}} \right)$. Then keeping mind the equality

$$a_{W_0} \otimes \cdots \otimes a_{W_{n-1}} \otimes \widehat{a}_{W_n} = E_n(a)$$

we obtain the Markov property for φ

$$\varphi|_{\mathcal{B}_{\Lambda_{n+1}}} (a) = \varphi|_{\mathcal{B}_{\Lambda_n}} (E_n(a)),$$

since the elements of the form (4) generate $\mathcal{B}_{\Lambda_{n+1}}$. □

4 Localized conditional expectations

In this section, we consider and describe localized conditional expectations. Namely, let $\mathcal{E}^{(j)}$ be a transition expectation from $\mathcal{B}_{\Lambda_{[j,j+1]}}$ to $\mathcal{B}_{W_j}$. The transition expectation $\mathcal{E}^{(j)}$ is called *localized* if one has

$$\mathcal{E}^{(j)} = \bigotimes_{x \in W_j} \mathcal{E}_x^{(j)}, \quad (5)$$

where $\mathcal{E}_x^{(j)} : \mathcal{B}_x \otimes \mathcal{B}_{(x,1)} \otimes \cdots \otimes \mathcal{B}_{(x,k)} \rightarrow \mathcal{B}_x$ is a conditional expectation.

Remark 4.1. We notice that if one considers conditional expectations without localization property, then the results of [4] can be applied to the considered QMS and one can get the disintegration of QMS, which would be not enough for its finer representation. Roughly speaking, in that case, the Hamiltonians (see Section 8) would be defined on the levels W_j and their structure would not be described. Therefore, for our need, we have to impose the localization, which would yield a desired representation of QMS.

In what follows, we will use techniques of [8, 24] related to conditional expectations.

For each $j \geq 0$ and $x \in W_j$ by $\mathcal{R}(\mathcal{E}_x^{(j)})$ we denote the range of the transition expectation $\mathcal{E}_x^{(j)}$. Consider its central Z_x^j (i.e. the center of $\mathcal{R}(\mathcal{E}_x^j)$), together with its spectrum Ω_x^j , and the set of minimal central projections $\{P_{\omega_x}^j\}_{\omega_x \in \Omega_x}$ (which is finite whenever $\mathcal{R}(\mathcal{E}_x^{(j)})$ is finite dimensional). For the tail of simplicity, central projections $P_{\omega_x}^j$ will be denoted P_{ω_x} henceforth. We have

$$\sum_{\Omega_x} P_{\omega_x} = 1$$

We define for $j \geq 0$ and $x \in W_j$.

$$R_x^j := \bigoplus_{\Omega_x} P_{\omega_x} \mathcal{B}_x P_{\omega_x}$$

And define the C^* -subalgebra $\mathcal{R}$ of $\mathcal{B}_L$ by:

$$\mathcal{R} := \bigotimes_{j \geq 0} \left(\bigotimes_{x \in W_j} R_x^j \right) \quad (6)$$

Then we obtain, in a canonical way, a conditional expectation

$$E : \mathcal{B}_L \mapsto \mathcal{R}$$

defined to be the (infinite) tensor product of the following conditional expectations:

$$a \in \mathcal{B}_x \mapsto \sum_{\omega_x \in \Omega_x} P_{\omega_x} a P_{\omega_x} \quad (7)$$

Note that for the algebra R_x^j one has that $P_{\omega_x} R_x^j P_{\omega_x}$ is a factor of $B(H_{\omega_x})$ with $H_{\omega_x} = P_{\omega_x} H$.

Then $P_{\omega_x} R_x^j P_{\omega_x} = P_{\omega_x} B(H_{\omega_{x,0}}) \otimes \mathbf{1}_{H_{\omega_{x,1}}} P_{\omega_x}$, where $H_{\omega_x} = H_{\omega_{x,0}} \otimes H_{\omega_{x,1}}$. Using this fact and the fact that the family of central projections $\{P_{j_x}\}$ is orthogonal, we obtain

$$\begin{aligned} R_x^j &= \sum_{\omega_x \in \Omega_x} P_{\omega_x} R_x^j P_{\omega_x} \\ &= \bigoplus_{\omega_x \in \Omega_x} P_{\omega_x} B(H_{\omega_{x,0}}) \otimes \mathbf{1}_{H_{\omega_{x,1}}} P_{\omega_x} \end{aligned} \quad (8)$$

and

$$R_x^{j'} = \bigoplus_{\Omega_x} P_{\omega_x} \mathbf{1}_{H_{\omega_{x,0}}} \otimes B(H_{\omega_{x,1}}) P_{\omega_x}$$

Then the transition expectation $\mathcal{E}_x^{(j)}$ can be written in a canonical form: for $a \in \mathcal{B}_x^j \otimes \mathcal{B}_{S(x)}^{j+1}$ one has

$$\mathcal{E}_x^{(j)}(a) = \sum_{\Omega_x} P_{\omega_x} \Phi_{\omega_x}(P_{\omega_x} a P_{\omega_x}) P_{\omega_x} \quad (9)$$

where $\Phi_{\omega_x} : B(H_{\omega_{x,0}}) \otimes B(H_{\omega_{x,1}}) \otimes \mathcal{B}_{S(x)}^{(j+1)} \longrightarrow B(H_{\omega_{x,0}}) \otimes \mathbf{1}_{\omega_{x,1}}$ is the Umegaki conditional expectation defined by a state ϕ_{ω_x} on the algebra $B(H_{\omega_{x,1}}) \otimes \mathcal{B}_{S(x)}^{(j+1)}$ in the following way:

$$\Phi_{\omega_x}(a_{\omega_{x,0}} \otimes a_{\omega_{x,1}} \otimes b) = \phi_{\omega_x}(a_{\omega_{x,1}} \otimes b) a_{\omega_{x,0}} \otimes \mathbf{1}_{\omega_{x,1}}$$

Denote $\Omega^{(j)} := \prod_{x \in \overrightarrow{W_j}} \Omega_x$, and for $\omega^{(j)} = (\omega_x)_{x \in \overrightarrow{W_j}} \in \Omega^{(j)}$, we define a projection $P_{\omega^{(j)}}$ defined by:

$$P_{\omega^{(j)}} := \bigotimes_{x \in \overrightarrow{W_j}} P_{\omega_x}.$$

Theorem 4.2. *Let $j \in \mathbb{N}$ and let for $\omega^{(j)} = (\omega_x)_{x \in \overrightarrow{W_j}}$,*

$$H_{\omega^{(j)},0} := \bigotimes_{x \in \overrightarrow{W_j}} H_{\omega_{x,0}}, \quad H_{\omega^{(j)},1} := \bigotimes_{x \in \overrightarrow{W_j}} H_{\omega_{x,1}}.$$

Then

(i) The range $R^{(j)}$ of the transition expectation $\mathcal{E}^{(j)}$ satisfies

$$R^{(j)} \cong \bigoplus_{\omega^{(j)} \in \Omega^{(j)}} (P_{\omega^{(j)}} B(H_{\omega^{(j)},0})) \otimes (\mathbf{1}_{H_{\omega^{(j)},1}} P_{\omega^{(j)}}) \quad (10)$$

(ii) Furthermore, for $a \in \mathcal{B}_{W_j} \otimes \mathcal{B}_{W_{j+1}}$ one has

$$\mathcal{E}^{(j)}(a) = \sum_{\omega^j \in \Omega^{(j)}} P_{\omega^{(j)}} \Phi_{\omega^{(j)}}(P_{\omega^{(j)}} a P_{\omega^{(j)}}) P_{\omega^{(j)}} \quad (11)$$

where $\Phi_{\omega^{(j)}} = \bigotimes_{x \in W_j} \Phi_{\omega_x}$.

Proof. (i) Due to $\mathcal{E}^{(j)} = \bigotimes_{x \in \overrightarrow{W_j}} \mathcal{E}_x^{(j)}$ the rang of $\mathcal{E}^{(j)}$ is given by:

$$\begin{aligned} R^{(j)} &= \bigotimes_{x \in \overrightarrow{W_j}} R_x^{(j)} \\ &= \bigotimes_{x \in \overrightarrow{W_j}} \left[\bigoplus_{\omega_x \in \Omega_x} P_{\omega_x} B(H_{\omega_x,0}) \otimes \mathbf{1}_{H_{\omega_x,1}} P_{\omega_x} \right] \\ &= \bigoplus_{\{\omega_{x_{W_j}(k)} \in \Omega_{x_{W_j}(k)}; k=1, \dots, |W_j|\}} \left(P_{\omega_{x_{W_j}(1)}} B(H_{\omega_{x_{W_j}(1)},0}) \otimes \mathbf{1}_{H_{\omega_{x_{W_j}(1)},1}} P_{\omega_{x_{W_j}(1)}} \right) \otimes \\ &\quad \dots \otimes \left(P_{\omega_{x_{W_j}(|W_j|)}} B(H_{\omega_{x_{W_j}(|W_j|),0}) \otimes \mathbf{1}_{H_{\omega_{x_{W_j}(|W_j|),1}} P_{\omega_{x_{W_j}(|W_j|)}} \right) \\ &\cong \bigoplus_{\{\omega_{x_{W_j}(k)} \in \Omega_{x_{W_j}(k)}; k=1, \dots, |W_j|\}} \left(\bigotimes_{k=1}^{|W_j|} P_{\omega_{x_{W_j}(k)}} B(H_{\omega_{x_{W_j}(k)},0}) \right) \otimes \left(\bigotimes_{k=1}^{|W_j|} \mathbf{1}_{H_{\omega_{x_{W_j}(k)},1}} P_{\omega_{x_{W_j}(k)}} \right) \\ &\cong \bigoplus_{\omega^{(j)} \in \Omega^{(j)}} (P_{\omega^{(j)}} B(H_{\omega^{(j)},0})) \otimes (\mathbf{1}_{H_{\omega^{(j)},1}} P_{\omega^{(j)}}) \end{aligned} \quad (12)$$

(ii) Let $a = a_0^{(j)} \otimes a_1^{(j)} \otimes b_{j+1} \in \mathcal{B}_{W_n} \otimes \mathcal{B}_{W_{n+1}}$ be a localized element with

$$a_0^{(j)} = \bigotimes_{x \in W_j} a_{x,0} \in B(H_0^{(j)}), \quad a_1^{(j)} = \bigotimes_{x \in W_j} a_{x,1} \in B(H_1^{(j)}), \quad b_{j+1} = \bigotimes_{y \in W_{j+1}} b_y \in \mathcal{B}_{W_{j+1}}.$$

Then we have

$$\begin{aligned} \mathcal{E}^{(j)}(a) &= \bigotimes_{x \in W_j} \mathcal{E}_x^{(j)}(a_{x,0} \otimes a_{x,1} \otimes b_{\overrightarrow{S(x)}}) \\ &= \bigotimes_{x \in W_j} \left[\sum_{\omega_x} P_{\omega_x} \Phi_{\omega_x}(P_{\omega_x} a P_{\omega_x}) P_{\omega_x} \right] \\ &= \sum_{\{\omega_x \in \Omega_{(x)} | x \in W_j\}} \bigotimes_{x \in W_j} P_{\omega_x} \Phi_{\omega_x}(P_{\omega_x} a P_{\omega_x}) P_{\omega_x} \\ &= \sum_{\omega^{(j)} \in \Omega^{(j)}} P_{\omega^{(j)}} \Phi_{\omega^{(j)}}(P_{\omega^{(j)}} a P_{\omega^{(j)}}) P_{\omega^{(j)}} \end{aligned}$$

The last one can be extended to all elements of $\mathcal{B}_{W_j} \otimes \mathcal{B}_{W_{j+1}}$. $\square$

Remark 4.3. For $\omega^{(j)} = (\omega_x)_{x \in W_j} \in \Omega^{(j)}$ a product state $\phi_{\omega^{(j)}}$ on $B(H_{\omega^{(j)},1}) \otimes \mathcal{B}_{W_{j+1}}$ is defined on the localized elements by:

$$\phi_{\omega^{(j)}} \left(\bigotimes_{x \in W_j} (a_{\omega_x,1} \otimes a_{\overrightarrow{S(x)}}) \right) = \prod_{x \in W_j} \phi_{\omega_x}(a_{\omega_x,1} \otimes a_{\overrightarrow{S(x)}}) \quad (13)$$

for every $a_{\omega_x,1} \in B(H_{\omega_x,1})$ and every $a_{\overrightarrow{S(x)}} \in \mathcal{B}_{\overrightarrow{S(x)}}$.

5 A classical Gibbs measure

In this section, we considering a quantum Markov state φ on the quasi-local algebra $\mathcal{B}_L$ w.r.t. the sequence of localized conditional expectations $\{\mathcal{E}^{(j)}\}$. Assume, as before, that for every $x \in \Gamma^k(x_0)$, $\mathcal{B}_x \cong \mathcal{A}$ with $\mathcal{A}$ is a fixed finite dimensional C*-algebra. Then the range $\mathcal{R}(\mathcal{E}_x^{(j)})$ of $\mathcal{E}_x^{(j)}$ is finite dimensional. The center $Z_x^{(j)}$ is finite dimensional and the index set $\mathcal{I}_x$ is in one to one bijection with the spectrum $\Omega_x^{(j)} := \text{spec}(Z_x^{(j)})$, then we denote the set of minimal central projections by $\{P_{\omega_x} \mid \omega_x \in \Omega_x\}$ and the states $\{\phi_{\omega_x} \mid j_x \in \Omega_x\}$ by $\{\phi_{\omega_x} \mid \omega_x \in \Omega_x\}$ (see Section 4).

Remark 5.1. 1. Let $\Lambda \subset_{fin} L$, if ψ_x is a given state on $\mathcal{B}_x$ for all $x \in \Lambda$, we denote the product state ψ_Λ of $\{\psi_i \mid i \in \Lambda\}$ defined on $\mathcal{B}_\Lambda$ by $\psi_\Lambda = \bigotimes_{x \in \Lambda} \psi_x$.

2. Let $x \in L$, we denote simply $\psi_{\overrightarrow{S(x)}} := \bigotimes_{y \in \overrightarrow{S(x)}} \psi_y$ the product state of $\{\psi_y \in \mathcal{S}(\mathcal{B}_y) \mid y \in \overrightarrow{S(x)}\}$ and $P_\psi = \bigotimes_{y \in \overrightarrow{S(x)}} P_{\psi_y}$ its support projection for P_{ψ_y} the support projection of ψ_y .

Lemma 5.2. Let Z^j be the center of $\mathcal{R}(\mathcal{E}^{(j)})$ and $\Omega_j := \text{spec}(Z^j)$. Then

1. $Z^j = \bigotimes_{x \in W_j} Z_x^j$.
2. $\Omega^{(j)} = \prod_{x \in W_j} \Omega_x^{(j)} \left(i.e.: \forall \omega_j \in \Omega_j, \exists \{\omega_x\}_{x \in \overrightarrow{W_j}} \mid \omega^{(j)} = \bigotimes_{x \in \overrightarrow{W_j}} \omega_x \right)$

For the sake of simplicity, let us denote the spectrum of $Z := Z(\mathcal{A})$ as follow $\text{spec}(Z) := \{\omega_1, \omega_2, \dots, \omega_q\}$ and the set of associate minimal central projection by $\{P_1, P_2, \dots, P_q\}$. Without loss of generality we may suppose that:

$$\forall x \in L, \forall \omega_x \in \Omega_x, \quad P_{\omega_x} = i_{\Lambda_{x,x}}(P_{\omega_r}) \quad \text{for some } r \in \{1, 2, \dots, q\}$$

where $i_{\Lambda_{x,x}}$ the canonical embedding of $\mathcal{A}$ at the x^{th} position on $\mathcal{B}_L$.

Let $\Phi := \{1, 2, \dots, q\}$ we can also take $\Omega = \Phi^L$ equipped with the σ -algebra $\mathcal{F}$ generated by its cylinder subsets. For $A \subset L$ a spin configuration σ_A on A is defined as a function $x \in A \mapsto \sigma_A(x) \in \Phi$. The set of all configurations coincides with $\Omega_A = \Phi^A$.

Remark 5.3. 1. For $A, B \subset L$, with $A \cap B = \emptyset$, and $\sigma \in \Omega_A, \sigma' \in \Omega_B$, we denote by $\sigma \vee \sigma'$ the configuration defined on $A \cup B$ such that $\sigma \vee \sigma'|_A = \sigma$ and $\sigma \vee \sigma'|_B = \sigma'$.

2. Let $j \geq 0$ the set $\Omega^{(j)}$ is equal Ω_{W_j} through the following identification:

$$\begin{aligned}\sigma_{\omega^{(j)}} : W_j &\rightarrow \Phi \\ x &\mapsto \omega_x\end{aligned}$$

If $\sigma = \sigma_{\omega^{(j)}}$ we denote the projection $P_{\omega^{(j)}}$ by P_σ and the state $\phi_{\omega^{(j)}}$ by ϕ_σ , and for $i \in \{0, 1\}$ the Hilbert spaces $H_{\omega_x, i}$ and $H_{\omega^{(j)}, i}$ will be denoted respectively by $H_{\sigma(x), i}$ and $H_{\overrightarrow{S(x)}, i}$

Now we define by induction the sequence $\{\mu_{\Lambda_n}\}_{n \geq 0}$ as follows:

$$\begin{cases} \mu_{\Lambda_0}(\sigma_0) = \varphi\left(i_{\Lambda_0}(P_{\omega_{\sigma_0(x_0)}})\right), & \forall \sigma_0 \in \Omega_{\Lambda_0}, \\ \mu_{\Lambda_n}(\sigma) = \mu_{\Lambda_0}(\sigma|_{\Lambda_0}) \prod_{k=1}^{n-1} \prod_{x \in W_k} \pi_{\omega_{\sigma(x)}, \omega_{\overrightarrow{S(x)}}}, & \forall \sigma \in \Omega_{\Lambda_n}, n \geq 1, \end{cases}$$

where $\pi_{\omega_x, \omega_{\overrightarrow{S(x)}}} = \phi_{\omega_x}(\mathbf{1}_{H_{\omega_x, 1}} \otimes P_{\omega_{\overrightarrow{S(x)}}})$

Definition 5.4. A sequence $\{P_{\Lambda_n}\}_{n \geq 0}$ of probability measures on Ω is said to be compatible if for all $n \geq 0$ and $\sigma_n \in \Omega_{\Lambda_n}$

$$\sum_{\sigma' \in \Omega_{W_{n+1}}} P_{\Lambda_{n+1}}(\sigma_n \vee \sigma') = P_{\Lambda_n}(\sigma_n) \quad (14)$$

Proposition 5.5. The sequence $\{\mu_{\Lambda_n}\}_{n \geq 0}$ is compatible.

Proof. Let $n \geq 0$ and $\sigma \in \Omega_{\Lambda_n}$. For all $\sigma' \in \Omega_{W_{n+1}}$ one has:

$$\begin{aligned} \mu_{\Lambda_{n+1}}(\sigma \vee \sigma') &= \mu_{\Lambda_0}(\sigma \vee \sigma'|_{\Lambda_0}) \prod_{k=1}^n \prod_{x \in W_k} \pi_{\omega_{\sigma \vee \sigma'(x)}, \omega_{\overrightarrow{S(x)}}} \\ &= \left(\mu_{\Lambda_0}(\sigma|_{\Lambda_0}) \prod_{k=1}^{n-1} \prod_{x \in W_k} \pi_{\omega_{\sigma(x)}, \omega_{\overrightarrow{S(x)}}} \right) \prod_{x \in W_n} \pi_{\omega_{\sigma(x)}, \omega_{\overrightarrow{S(x)}}} \\ &= \mu_{\Lambda_n}(\sigma) \prod_{x \in W_n} \pi_{\omega_{\sigma(x)}, \omega_{\overrightarrow{S(x)}}} \\ &= \mu_{\Lambda_n}(\sigma) \phi_{\sigma|_{W_n}}(\mathbf{1}_{H_{\sigma|_{W_n}, 1}} \otimes P_{\sigma'}) \end{aligned}$$

According to $\sum_{\sigma' \in W_{n+1}} P_{\sigma'} = 1$ with $P_{\sigma'} = \otimes_{x \in W_{n+1}} P_{\sigma'(x)}$ we find

$$\begin{aligned} \sum_{\sigma' \in \Omega_{W_{n+1}}} \mu_{\Lambda_{n+1}}(\sigma \vee \sigma') &= \mu_{\Lambda_n}(\sigma) \sum_{\sigma' \in \Omega_{W_{n+1}}} \phi_{\sigma|_{W_n}}(\mathbf{1}_{H_{\sigma|_{W_n}, 1}} \otimes P_{\sigma'}) \\ &= \mu_{\Lambda_n}(\sigma) \cdot \phi_{\sigma|_{W_n}}(\mathbf{1}_{H_{\sigma|_{W_n}, 1}} \otimes (\sum_{\sigma' \in W_{n+1}} P_{\sigma'})) \\ &= \mu_{\Lambda_n}(\sigma) \cdot \phi_{\sigma|_{W_j}}(\mathbf{1}_{H_{\sigma|_{W_j}, 1}} \otimes \mathbf{1}_{W_{n+1}}) \\ &= \mu_{\Lambda_n}(\sigma). \end{aligned}$$

This completes the proof. $\square$

Due to the previous proposition and to the fact that $\mu_{\Lambda_{n+1}|\Omega_{\Lambda_n}} = \mu_{\Lambda_n}$, the Kolmogorov consistency theorem ensures the existence of a probability measure μ on Ω such that $\mu|_{\Omega_{\Lambda_n}} = \mu_{\Lambda_n}$.

Let us consider the Hamiltonian on Ω_{Λ_n} defined by:

$$\mathcal{H}_{\Lambda_n}(\sigma) := \sum_{k=1}^{n-1} \sum_{x \in W_k} \rho\left(\sigma(x), \sigma(\overrightarrow{S(x)})\right) + \rho_0(\sigma|_{\Lambda_0}) \quad (15)$$

where $\rho_0(\sigma_0) = \ln(\mu(\sigma_0))$ for all $\sigma_0 \in \Omega_{\Lambda_0}$ and $\rho\left(\sigma(x), \sigma(\overrightarrow{S(x)})\right) = \ln\left(\pi_{\omega_{\sigma(x)}, \omega_{\overrightarrow{S(x)}}}\right)$ for $x \in \Gamma^k(x_0)$.

Then the measure μ can be viewed as a Gibbs measure for the Hamiltonian in the following sense:

$$\mu_{\Lambda_n}(\sigma) = \frac{e^{\mathcal{H}_{\Lambda_n}(\sigma)}}{Z_n}, \quad \forall \sigma \in \Omega_{\Lambda_n}, \forall n \geq 0, \quad (16)$$

where $Z_n = \sum_{\sigma' \in \Omega_{\Lambda_n}} e^{\mathcal{H}_{\Lambda_n}(\sigma')}$

Remark 5.6. 1. The Hamiltonian $\mathcal{H}$ is well defined because the state φ is locally faithful.

2. In our case $Z_n = \mu_{\Lambda_n}(\Omega_{\Lambda_n}) = 1$.

6 Disintegration of the state φ

We will use the same notations as the previous section. Let $\sigma \in \Omega$, for $x \in L$ we denote $\mathcal{B}_{\sigma(x)} := P_{\sigma(x)} \mathcal{B}_x P_{\sigma(x)}$. One can define a quasi-local algebra $\mathcal{B}_\sigma$ in the following way:

$$\begin{aligned} \mathcal{B}_\sigma &:= \bigotimes_{j \geq 0} \left(\bigotimes_{x \in \overrightarrow{W_j}} \mathcal{B}_{P_{\sigma(x)}} \right) \\ &\equiv B(H_{\sigma(x_0), 0}) \otimes \left(\bigotimes_{j \geq 0} \left(B(H_{\sigma(x), 1}) \otimes B(H_{\sigma(\overrightarrow{S(x)}), 0}) \right) \right) \end{aligned} \quad (17)$$

A completely positive identity preserving map $E_\sigma : \mathcal{B}_L \mapsto \mathcal{B}_\sigma$ is defined as the (infinite) tensor product of the mappings

$$a \in \mathcal{B}_x \mapsto P_{\sigma(x)} a P_{\sigma(x)} \quad (18)$$

The map E_σ satisfies $E_\sigma = E_\sigma \circ E$, where E is defined by (7).

Define a state ψ_σ on $\mathcal{B}_\sigma$ as follows:

$$\psi_{\sigma|_{\Lambda_n}} := \eta_{\sigma(x_0)}^{(0)} \otimes \left(\bigotimes_{j \geq 0} \left(\bigotimes_{x \in W_j} \eta_{\sigma(x), \sigma(\overrightarrow{S(x)})}^{(j)} \right) \right) \otimes \left(\bigotimes_{y \in W_n} \hat{\eta}_{\sigma(y)}^{(n)} \right) \quad (19)$$

where the state $\eta_{\sigma(x_0)}^{(0)}$ is defined on $B(H_{\sigma(x_0),0})$ by

$$\eta_{\sigma(x_0)}^{(0)}(a) := \frac{\varphi\left(i_{\Lambda_0}\left(P_{\sigma(x_0)}(a \otimes \mathbf{1}_{H_{\sigma(x),1}})P_{\sigma(x_0)}\right)\right)}{\varphi(P_{\sigma(x_0)})} \quad (20)$$

For $x \in W_j$, the state $\eta_{\sigma(x)}^{(j)}$ is defined on $B(H_{\sigma(x),1}) \otimes B(H_{\sigma(\overrightarrow{S(x)}),0})$ as follows:

$$\eta_{\sigma(x),\sigma(\overrightarrow{S(x)})}^{(j)}(a \otimes b) := \frac{\phi_{\sigma(x)}\left(a \otimes i_{\Lambda_{\overrightarrow{S(x)}}}\left(P_{\sigma(\overrightarrow{S(x)})}(b \otimes \mathbf{1}_{H_{\sigma(\overrightarrow{S(x)}),1}})P_{\sigma(\overrightarrow{S(x)})}\right)\right)}{\pi_{\sigma(x),\sigma(\overrightarrow{S(x)})}} \quad (21)$$

And for $y \in W_n$, the state $\hat{\eta}_{\sigma(y)}^{(n)}$ is defined on $B(H_{\sigma(y),1})$ as follows:

$$\hat{\eta}_{\sigma(y)}^{(n)}(a) := \sum_{\sigma' \in \Omega_{\overrightarrow{S(x)}}} \eta_{\sigma(x),\sigma'(\overrightarrow{S(x)})}^{(n)} \quad (22)$$

Let $\{\mathcal{E}_{\sigma(x)}^j\}_{j \geq 0, x \in W_j}$ be a family of transition expectations:

$$\mathcal{E}_{\sigma(x)}^j : \mathcal{B}_{P_{\sigma(x)}} \otimes \mathcal{B}_{P_{\sigma(\overrightarrow{S(x)})}} \mapsto \mathcal{B}_{P_{\sigma(x)}}$$

defined by

$$\begin{aligned} & \mathcal{E}_{\sigma(x)}^j \left((a_0 \otimes a_1) \otimes (b_0 \otimes (\otimes_{y \in \overrightarrow{S(x)}} b_{y,1})) \right) \\ &= \left(\eta_{\sigma(x),\sigma(\overrightarrow{S(x)})}^{(j)}(a_1 \otimes b_0) \left(\prod_{y \in \overrightarrow{S(x)}} \eta_{\sigma(y),\sigma(\overrightarrow{S(y)})}^{(j+1)}(b_{y,1} \otimes \mathbf{1}_{H_{\sigma(y),0}}) \right) \right) a_0 \otimes \mathbf{1} \end{aligned} \quad (23)$$

And let $\mathcal{E}_{\sigma[W_j]}^j : \mathcal{B}_{\sigma(W_j)}^{j+1} \rightarrow \mathcal{B}_{\sigma(W_j)}^j$ be a transition expectation defined by:

$$\mathcal{E}_{\sigma[W_j]}^j := \bigotimes_{x \in W_j} \mathcal{E}_{\sigma(x)}^j. \quad (24)$$

One can prove the following fact.

Proposition 6.1. *The state ψ_σ satisfies the Markov property w.r.t the family of transition expectations $\{\mathcal{E}_\sigma^j\}_{j \geq 0}$ given by (24).*

Theorem 6.2. *Let φ be a Markov state w.r.t the family of localized transitions expectations $\{\mathcal{E}^j\}_{j \geq 0}$. Assume that Ω and μ are as in Section 5 and the quasi-local algebra $\mathcal{B}_\sigma$ is given by (17), the map E_σ by tensor of the maps (24); the state ψ_σ on $\mathcal{B}_\sigma$ is defined by (19). Then the state φ admits a disintegration*

$$\varphi = \int_{\Omega} \varphi_\sigma \mu(d\sigma),$$

where $\sigma \in \Omega \mapsto \varphi_\sigma \in \mathcal{S}(\mathcal{B}_L)$ is a $\sigma(\mathcal{B}_L^*, \mathcal{B}_L)$ -measurable map satisfying, for μ -almost all $\sigma \in \Omega$,

$$\varphi_\sigma = \psi_\sigma \circ E_\sigma$$

Proof. Let φ be a Markov state w.r.t the sequence $\{\mathcal{E}_j\}_{j \geq 0}$. We define a commutative C^* -subalgebra $\mathcal{C}$ defined by:

$$\mathcal{C} := \bigotimes_{j \geq 0} \left(\bigotimes_{x \in W_j} Z_x^j \right)$$

Following [4], let (H, π) be the GNS representation of $\mathcal{R}$ associate with $\varphi|_{\mathcal{R}}$. Then $\pi(\mathcal{C})'' \subset \pi(\mathcal{R})' \cap \pi(\mathcal{R})''$ (see [39, section III.2, Theorem 7.2]) hence, the representation π can be disintegrated as follows:

$$\pi = \int_{\Omega}^{\oplus} \pi_{\sigma} \mu(d\sigma)$$

where $\sigma \mapsto \pi_{\sigma}$ is a weakly measurable field of representation of $\mathcal{R}$ (see [39, Theorem IV.8.25],[4]). Furthermore one can find a measurable field $\sigma \mapsto \zeta_{\sigma}$ of unit vectors such that, for each $a \in \mathcal{R}$, we get

$$\varphi(a) = \int_{\Omega} \langle \pi_{\sigma}(a) \zeta_{\sigma}, \zeta_{\sigma} \rangle \mu(d\sigma)$$

As φ is a Markov state, it's invariant under E and the previous expression can be extended on $\mathcal{B}_L$ by

$$\varphi = \int_{\Omega} \varphi_{\sigma} \mu(d\sigma)$$

for the $\sigma(\mathcal{B}_L^*, \mathcal{B}_L)$ -measurable field φ_{σ} defined as

$$\varphi_{\sigma} := \langle \pi_{\sigma}(E(\cdot)) \zeta_{\sigma}, \zeta_{\sigma} \rangle .$$

Let us now prove the second part of the theorem about the expression of φ_{σ} . For $\bar{\sigma} \in \Omega$ take an element $A \in \mathcal{B}_{\Lambda_n}$ given by

$$A = \bigotimes_{j \geq 0} \left(\bigotimes_{x \in W_j} P_{\bar{\sigma}(x)}(a_{\bar{\sigma}(x),0} \otimes a_{\bar{\sigma}(x),1}) P_{\bar{\sigma}(x)} \right)$$

and consider

$$Z = \bigotimes_{j \geq 0} \left(\bigotimes_{x \in W_j} P_{\bar{\sigma}(x)} \right)$$

Then we obtain

$$\varphi(i_{\Lambda_n}(AZ)) = \int_{\Omega} z(\sigma) \varphi_{\sigma}(i_{\Lambda_n}(AZ)) \mu(d\sigma)$$

where

$$z(\sigma) = \delta_{\sigma|_{W_n}, \bar{\sigma}|_{W_n}}$$

Now for $j \in \{0, \dots, n\}$, define

$$A_j = \bigotimes_{x \in W_j} P_{\sigma(x)}(a_{\sigma(x),1} \otimes a_{\sigma(x),1}) P_{\sigma(x)} .$$

Using the Markov property, $\varphi(i_{\Lambda_n}(AZ))$ can be computed as follows:

$$\begin{aligned}\varphi(i_{\Lambda_n}(AZ)) &= z(\sigma)\varphi(i_{\Lambda_n}(A)) = z(\sigma) \sum_{\sigma_{n+1} \in \Omega_{W_{n+1}}} \varphi(i_{\Lambda_{n+1}}(A \otimes P_{\sigma_{n+1}})) \\ &= z(\sigma) \sum_{\sigma_{n+1} \in \Omega_{n+1}} \varphi(\mathcal{E}^{(0)}(A_0 \otimes \mathcal{E}^{(1)}(A_1 \otimes \dots \\ &\quad \otimes \mathcal{E}^{(n-1)}(A_{n-1} \otimes \mathcal{E}^{(n)}(A_n \otimes P_{\sigma_{n+1}}))))\end{aligned}$$

According to the localized property of $\mathcal{E}^{(j)}$ one has

$$\begin{aligned}\mathcal{E}^{(n)}(A_n \otimes P_{\sigma_{n+1}}) &= \bigotimes_{x \in W_n} \mathcal{E}_x^{(n)}(P_{\sigma(x)}(a_{\sigma(x),0} \otimes a_{\sigma(x),1})P_{\sigma(x)} \otimes P_{\sigma_{n+1}(\overrightarrow{S(x)})}) \\ &= \bigotimes_{x \in W_n} \phi_{\sigma(x)} \left(P_{\sigma(x)} a_{\sigma(x)} P_{\sigma(x)} \otimes P_{\sigma_{n+1}(\overrightarrow{S(x)})} \right) P_{\sigma(x)}(a_{\sigma(x),0} \otimes \mathbf{1}_{\sigma(x),1}) \\ &= \prod_{x \in W_n} \phi_{\sigma(x)} \left(P_{\sigma(x)} a_{\sigma(x)} P_{\sigma(x)} \otimes P_{\sigma_{n+1}(\overrightarrow{S(x)})} \right) \bigotimes_{x \in W_n} P_{\sigma(x)}(a_{\sigma(x),0} \otimes \mathbf{1}_{\sigma(x),1}) \\ &= \prod_{x \in W_n} \eta_{\sigma(x), \sigma_{n+1}(\overrightarrow{S(x)})}^{(n)} (a_{\sigma(x),1} \otimes \mathbf{1}_{\sigma_{n+1}(\overrightarrow{S(x)}),0}) \pi_{\sigma(x), \sigma_{n+1}(\overrightarrow{S(x)})}\end{aligned}$$

Let $j \in \{0, \dots, n-1\}$ and denote

$$A_{j+1,0} = \bigotimes_{x \in W_{j+1}} P_{\sigma(x)}(a_{\sigma(x),0} \otimes \mathbf{1}_{s(x),1})P_{\sigma(x)}$$

and $a_{\overrightarrow{S(x)},0} = \bigotimes_{y \in \overrightarrow{S(x)}} a_{\sigma(y),0}$. Then we get

$$\begin{aligned}\mathcal{E}^{(j)}(A_j \otimes A_{j+1,0}) &= \bigotimes_{x \in W_n} \mathcal{E}_x^{(j)} \left(P_{\sigma(x)}(a_{\sigma(x),0} \otimes a_{\sigma(x),1})P_{\sigma(x)} \otimes P_{\sigma(\overrightarrow{S(x)})}(a_{\sigma(\overrightarrow{S(x)},0)} \otimes \mathbf{1}_{\overrightarrow{S(x)},1})P_{\sigma(\overrightarrow{S(x)})} \right) \\ &= \bigotimes_{x \in W_j} \phi_{\sigma(x)} \left(P_{\sigma(x)} a_{\sigma(x),1} P_{\sigma(x)} \otimes P_{\sigma(\overrightarrow{S(x)})} \right) P_{\sigma(x)}(a_{\sigma(x),0} \otimes \mathbf{1}_{\sigma(x),1}) \\ &= \prod_{x \in W_j} \phi_{\sigma(x)} \left(P_{\sigma(x)} a_{\sigma(x),1} P_{\sigma(x)} \otimes P_{\sigma(\overrightarrow{S(x)})}(a_{\sigma(\overrightarrow{S(x)},0)} \otimes a_{\sigma(\overrightarrow{S(x)},1)})P_{\sigma(\overrightarrow{S(x)})} \right) \\ &\quad \times \bigotimes_{x \in W_n} P_{\sigma(x)}(a_{\sigma(x),0} \otimes \mathbf{1}_{\sigma(x),1}) \\ &= \prod_{x \in W_j} \eta_{\sigma(x), \sigma(\overrightarrow{S(x)})}^{(j)} (a_{\sigma(x),1} \otimes \mathbf{1}_{\sigma(\overrightarrow{S(x)},0)}) \pi_{\sigma(x), \sigma(\overrightarrow{S(x)})} \bigotimes_{x \in W_n} P_{\sigma(x)}(a_{\sigma(x),0} \otimes \mathbf{1}_{\sigma(x),1})\end{aligned}$$

Hence,

$$\begin{aligned}
 \varphi(i_{\Lambda_n}(AZ)) &= z(\sigma) \varphi \left(\sum_{\sigma_{n+1} \in \Omega_{n+1}} \prod_{j=1}^{n-1} \prod_{x \in W_j} \eta_x^{(j)}(a_{\sigma(x),0} \otimes a_{\sigma(\overrightarrow{S(x)}),0}) \cdot \pi_{\sigma(x),\sigma(\overrightarrow{S(x)})} \times a_{\sigma(x_0),0} \otimes \mathbf{1}_{\sigma(x_0),1} \right) \\
 &= z(\sigma) \sum_{\sigma_{n+1} \in \Omega_{W_{n+1}}} \psi_{\sigma \vee \sigma_{n+1}}(A \otimes \mathbf{1}_{W_{n+1}}) \\
 &= \int_{\Omega} z(\sigma) \psi_{\sigma} \circ E_{\sigma}(A) \mu(d\sigma)
 \end{aligned}$$

Consequently, one has

$$\int_{\Omega} z(\sigma) \varphi_{\sigma} \mu(d\sigma) = \int_{\Omega} z(\sigma) \psi_{\sigma}(E_{\sigma}(a)) \mu(d\sigma)$$

for each fixed localized operator $A \in \mathcal{B}_L$ and each function $z \in C(\Omega)$ depending only on finitely many variables. As such functions are dense on $C(\Omega)$ and due to the uniqueness of the Radon-Nikodym derivative, for each localized $A \in \mathcal{B}_L$, there exists a measurable set $\Omega_A \subset \Omega$ of full μ -measure such that for $\sigma \in \Omega_0$ one has:

$$\varphi_{\sigma}(A) = \psi_{\sigma}(E_{\sigma}(A)) \quad (25)$$

By considering linear combination with rational coefficients, one can find dense subset $\mathcal{B}_{L,0}$ of localized operators such that (25) still satisfied for each element of $\mathcal{B}_{L,0}$.

Let $a \in \mathcal{B}_L$ and $(a_n)_n$ be a sequence of $\mathcal{B}_{L,0}$ that converges to a . Then for $\sigma \in \Omega$ one has:

$$\varphi_{\sigma}(a) = \lim_n \varphi_{\sigma}(a_n) = \lim_n \psi_{\sigma}(E_{\sigma}(a_n)) = \psi_{\sigma}(E_{\sigma}(a))$$

Thus the equation (25) holds for all $\alpha \in \mathcal{B}_L$. $\square$

Corollary 6.3. *Let*

$$\varphi(a) = \int_{\Omega} \varphi_{\sigma} \mu(d\sigma)$$

be the disintegration of the Markov state φ as in theorem (6.2). Then ϕ_{σ} is a factor state for μ -almost all $\sigma \in \Omega$.

The proof is similar to [4, Corollary 3.3].

7 A reconstruction theorem

In this section we study the converse direction of the disintegration result in the previous section.

Let us consider for $j \geq 0$ and $x \in W_j$, a commutative subalgebra $\mathcal{Z}_x^j$ of $\mathcal{B}_x$ with spectrum its Ω_x^j together with its family of projections $\{P_{\sigma(x)}^j\}_{\sigma(x) \in \Omega_x^j}$. For $\sigma \in \Omega$ we assume that the following distributions are given:

- $\pi_{\sigma(x_0)}^0 > 0$; (initial distribution)

- For $j \geq 0$ and $x \in W_j$ $\pi_{\sigma(x), \sigma(\overrightarrow{S(x)})}^j > 0$; (transition probabilities)

such that for $n \geq 0$:

$$\sum_{\sigma' \in \Omega_{W_{n+1}}} \prod_{x \in W_j} \pi_{\sigma(x), \sigma'(\overrightarrow{S(x)})}^j = 1$$

Then a Markov measure μ on $\Omega := \prod_{j \geq 0} \prod_{x \in W_j} \Omega_x^j$ is defined as follows: For $n \geq 1$ and $\sigma \in \Omega_{\Lambda_n}$:

$$\mu(\sigma) = \pi_{\sigma(x_0)}^0 \prod_{j=0}^{n-1} \prod_{x \in W_j} \pi_{\sigma(x), \sigma(\overrightarrow{S(x)})}^j$$

Now for $j \geq 0$ and $x \in \mathcal{B}_x^j$ and $P_{\sigma(x)}$ a given central projection we set

$$\mathcal{B}_{P_{\sigma(x)}}^j := P_{\sigma(x)} \mathcal{B}_x^j P_{\sigma(x)} = N_{\sigma(x)}^j \otimes \overline{N}_{\sigma(x)}^j,$$

where $N_{\sigma(x)}^j$ and $\overline{N}_{\sigma(x)}^j$ are finite dimensional factors.

For $\sigma \in \Omega$, let $\eta_{\sigma(x_0)}^0$ be a state on $N_{\sigma(x_0)}$ and $\eta_{\sigma(x), \sigma(\overrightarrow{S(x)})}^j$ be a state on $\overline{N}_{\sigma(x)}^j \otimes N_{\sigma(S(x))}^{j+1}$ with $N_{\sigma(S(x))}^{j+1} = \bigotimes_{y \in W_{n+1}} N_{\sigma(y)}$. For each $\sigma \in \Omega$ we define the state ψ_σ by (19) on the quasi-local algebra $\mathcal{B}_\sigma$ defined by (17). Let $E_\sigma : \mathcal{B}_L \rightarrow \mathcal{B}_\sigma$ be given in (18), together with the $\sigma(\mathcal{B}_L^*, \mathcal{B}_L)$ -measurable map

$$\sigma \in \Omega \mapsto \varphi_\sigma := \psi_\sigma \circ E_\sigma$$

Theorem 7.1. *For the same notations as above, the state φ on $\mathcal{B}_L$ given by*

$$\varphi := \int_{\Omega} \varphi_\sigma \mu(d\sigma)$$

is a Markov state w.r.t the sequence of transition expectations $\{\mathcal{E}^j\}_{j \geq 0}$, determined by the states $\phi_{\sigma(x)}^j$ satisfying, for each $j \geq 0, x \in W_j$ and $\sigma \in \Omega$ the following equality

$$\begin{aligned} & \phi_{\sigma(x)}^j(\overline{a} \otimes P_{\sigma(\overrightarrow{S(x)})}^{j+1}(b \otimes \overline{b})P_{\sigma(\overrightarrow{S(x)})}^{j+1}) \\ &= \sum_{\sigma_{j+2} \in \Omega_{W_{j+2}}} \pi_{\sigma(x), \sigma(\overrightarrow{S(x)})}^j \eta_{\sigma(x), \sigma(\overrightarrow{S(x)})}^j(\overline{a} \otimes b) \prod_{y \in \overrightarrow{S(x)}} \pi_{\sigma(y), \sigma(\overrightarrow{S(y)})}^{j+1} \eta_{\sigma(y), \sigma(\overrightarrow{S(y)})}^{j+1}(\overline{b}_y \otimes \mathbf{1}) \end{aligned} \quad (26)$$

with $a \in N_{\sigma(x)}^j$, $\overline{a} \in \overline{N}_{\sigma(x)}^j$, $b = \bigotimes_{y \in S(x)} b_{\sigma(y)} \in N_{\overrightarrow{S(x)}}^{j+1}$ and $\overline{b} = \bigotimes_{y \in \overrightarrow{S(x)}} \overline{b}_{\sigma(y)} \in \overline{N}_{\overrightarrow{S(x)}}^j$.

Proof. For $\sigma \in \Omega$ the state ψ_σ is well-defined, in addition it's a quantum Markov state w.r.t the sequence $\{\mathcal{E}_{\sigma|W_j}^j\}_{(j \geq 0)}$ of the transition expectations given by (24).

Let $n \geq 0$ and take an element

$$A = \bigotimes_{j=0}^n \bigotimes_{x \in W_j} P_{\overline{\sigma}(x)}^j(a_{\overline{\sigma}(x)} \otimes \overline{a}_{\overline{\sigma}(x)})P_{\overline{\sigma}(x)}^j$$

in $\mathcal{B}_{\Lambda_n}$. Using (3.4) it is enough to proof that for such an element A the equality

$$\varphi(A) = \varphi(B)$$

with

$$B = \mathcal{E}^{(0)}(A_{W_0} \otimes \cdots \otimes A_{W_{n-1}} \otimes \mathcal{E}^{(n)}(A_{W_{n-1}} \otimes A_{W_n}) \cdots)$$

$$A_{W_j} = \bigotimes_{x \in W_j} P_{\bar{\sigma}(x)}^j(a_{\bar{\sigma}(x)} \otimes \bar{a}_{\bar{\sigma}}) P_{\bar{\sigma}(x)}^j.$$

For $\sigma \in \Omega_{\Lambda_n}$ and $E_{\sigma}(A) = \delta_{\sigma \upharpoonright \Lambda_n, \bar{\sigma} \upharpoonright \Lambda_n} A$

$$\varphi_{\sigma}(A) = \varphi_{\sigma}(\mathcal{E}_{\sigma}^{(0)}(A_{\sigma(W_0)} \otimes \cdots \otimes A_{\sigma(W_{n-1})} \otimes \mathcal{E}_{\sigma}^{(n-1)}(A_{\sigma(W_{n-1})} \otimes A_{\sigma(W_n)}) \cdots),$$

where $A_{\sigma(x)} = a_{\sigma(x)} \otimes \bar{a}_{\sigma(x)}$.

One has

$$\begin{aligned} \varphi_{\sigma}(i_{\Lambda_n}(A)) &= \delta_{\sigma \upharpoonright \Lambda_n, \bar{\sigma} \upharpoonright \Lambda_n} \sum_{\sigma' \in \Omega_{W_{n+1}}} \psi_{\sigma} \left(\bigotimes_{j=1}^n \bigotimes_{x \in W_j} (P_{\sigma(x)}^j(a_{\sigma(x)} \otimes \bar{a}_{\sigma(x)}) P_{\sigma(x)}^j) \right) \\ &\quad \times \prod_{y \in W_n} \eta_{\sigma(y), \sigma'(\overrightarrow{S(y)})}(\bar{a}_{\sigma(y)} \otimes \mathbf{1}_{\sigma'(\overrightarrow{S(y)})}) \\ &= \delta_{\sigma \upharpoonright \Lambda_n, \bar{\sigma} \upharpoonright \Lambda_n} \sum_{\sigma' \in \Omega_{W_{n+1}}} \eta_{\sigma(x_0), \sigma'(x_0)}^{(0)}(a_{\sigma(x_0)}) \left(\prod_{j=0}^{n-1} \prod_{x \in W_j} \eta_{\sigma(x), \sigma(\overrightarrow{S(x)})}^{(j)}(\bar{a}_{\sigma(x)} \otimes a_{\sigma(\overrightarrow{S(x)})}) \right) \\ &\quad \times \prod_{y \in W_n} \eta_{\sigma(y), \sigma'(\overrightarrow{S(y)})}(\bar{a}_{\sigma(y)} \otimes \mathbf{1}_{\sigma'(\overrightarrow{S(y)})}) \end{aligned}$$

Then

$$\begin{aligned} \varphi(i_{\Lambda_n}(A)) &= \int_{\Omega} \varphi_{\sigma}(i_{\Lambda_n}(A)) \mu(d\sigma) \\ &= \sum_{\sigma' \in \Omega_{W_{n+1}}} \pi_{(x_0)}^0 \eta_{(x_0)}^{(0)}(a_{(x_0)}) \prod_{j=0}^{n-1} \prod_{x \in W_j} \pi_{(x), (\overrightarrow{S(x)})}^j \eta_{(x), (\overrightarrow{S(x)})}^{(j)}(\bar{a}_{(x)} \otimes a_{(\overrightarrow{S(x)})}) \\ &\quad \times \prod_{y \in W_n} \pi_{(y), (\overrightarrow{S(y)})}^n \eta_{(y), \sigma'(\overrightarrow{S(y)})}(\bar{a}_{(y)} \otimes \mathbf{1}_{\sigma'(\overrightarrow{S(y)})}) \end{aligned}$$

Conversely we have

$$\begin{aligned}
 \mathcal{E}_\sigma^{(n-1)} \left(A_{(W_{n-1})} \otimes A_{(W_n)} \right) &= \delta_{\sigma \upharpoonright \Lambda_n, \bar{\sigma} \upharpoonright \Lambda_n} \bigotimes_{x \in W_{n-1}} \mathcal{E}_{\sigma(x)} \left(a_{\sigma(x)} \otimes \bar{a}_{\sigma(x)} \otimes (a_{\sigma(\overrightarrow{S(x)})} \otimes \bar{a}_{\sigma(\overrightarrow{S(x)})}) \right) \\
 &= \delta_{\sigma \upharpoonright \Lambda_n, \bar{\sigma} \upharpoonright \Lambda_n} \bigotimes_{x \in W_{n-1}} \sum_{\sigma' \in \Omega_{\overrightarrow{S(x)}}} \phi_{\sigma(x)}^{(n-1)} (\bar{a}_{\sigma(x)} \otimes (a_{\sigma(\overrightarrow{S(x)})} \otimes \mathbf{1})) \\
 &\quad \times \prod_{y \in \Omega_{W_n}} \phi_{\sigma(y)}^{(n)} (\bar{a}_{\sigma(y)} \otimes \mathbf{1}_{\sigma'(\overrightarrow{S(y)})}) (P_{\sigma(x)}^n a_{\sigma(x)} \otimes \mathbf{1} P_{\sigma(x)}^n) \\
 &= \delta_{\sigma \upharpoonright \Lambda_n, \bar{\sigma} \upharpoonright \Lambda_n} \sum_{\sigma' \in \Omega_{W_{n+1}}} \prod_{x \in W_{n-1}} \phi_{\sigma(x)}^{(n-1)} (\bar{a}_{\sigma(x)} \otimes (a_{\sigma(\overrightarrow{S(x)})} \otimes \mathbf{1})) \\
 &\quad \times \prod_{y \in \overrightarrow{S(x)}} \phi_{\sigma(y)}^{(n)} (\bar{a}_{\sigma(y)} \otimes \mathbf{1}_{\sigma'(\overrightarrow{S(y)})}) \bigotimes_{x \in W_n} P_{\sigma(x)}^n (a_{\sigma(x)} \otimes \mathbf{1}) P_{\sigma(x)}^n
 \end{aligned}$$

After small computation, we get

$$\begin{aligned}
 \varphi_\sigma(B) &= \delta_{\sigma \upharpoonright \Lambda_n, \bar{\sigma} \upharpoonright \Lambda_n} \sum_{\sigma' \in \Omega_{W_{n+1}}} \phi_{\sigma(x_0)}^{(0)} (a_{\sigma(x_0)}) \prod_{j=0}^{n-1} \prod_{x \in W_j} \phi_{\sigma(x)}^{(j)} (\bar{a}_{\sigma(x)} \otimes (a_{\sigma(\overrightarrow{S(x)})} \otimes \mathbf{1})) \\
 &\quad \times \prod_{y \in W_n} \phi_{\sigma(y)}^{(n)} (\bar{a}_{\sigma(y)} \otimes \mathbf{1}_{\sigma'(\overrightarrow{S(y)})})
 \end{aligned}$$

Then by taking into account that $\phi_{\sigma(x_0)}^{(0)}(\dots) = \pi_{\sigma(x_0)}^0 \eta_{\sigma(x_0)}^{(0)}(\cdot)$ in addition to the expression of $\phi_{\sigma(x), \sigma(\overrightarrow{S(x)})}^j$ given in the theorem, we obtain

$$\begin{aligned}
 \varphi(i_{\Lambda_n}(B)) &= \int_{\Omega} \varphi_\sigma(B) \mu(d\sigma) \\
 &= \sum_{\sigma' \in \Omega_{W_{n+1}}} \phi_{\sigma(x_0)}^{(0)} (a_{\sigma(x_0)}) \prod_{j=0}^{n-1} \prod_{x \in W_j} \left(\phi_{\sigma(x)}^{(j)} (\bar{a}_{\sigma(x)} \otimes (a_{\sigma(\overrightarrow{S(x)})} \otimes \mathbf{1})) \right) \prod_{y \in W_n} \phi_{\sigma(y)}^{(n)} (\bar{a}_{\sigma(y)} \otimes \mathbf{1}_{\sigma'(\overrightarrow{S(y)})}) \\
 &= \sum_{\sigma' \in \Omega_{W_{n+1}}} \pi_{\sigma(x_0)}^0 \eta_{\sigma(x_0)}^{(0)} (a_{\sigma(x_0)}) \prod_{j=0}^{n-1} \prod_{x \in W_j} \left(\pi_{\sigma(x), \sigma(\overrightarrow{S(x)})}^{(j)} \eta_{\sigma(x), \sigma(\overrightarrow{S(x)})}^{(j)} (\bar{a}_{\sigma(x)} \otimes a_{\sigma(\overrightarrow{S(x)})}) \right) \\
 &\quad \times \prod_{y \in W_n} \pi_{\sigma(y), \sigma'(\overrightarrow{S(y)})}^{(n)} \eta_{\sigma(y), \sigma'(\overrightarrow{S(y)})}^{(n)} (\bar{a}_{\sigma(y)} \otimes \mathbf{1}_{\sigma'(\overrightarrow{S(y)})}) \\
 &= \varphi(i_{\Lambda_n}(A)).
 \end{aligned}$$

Hence, the proof is complete. $\square$

8 Connection with statistical mechanics

In this section we study the link between Markov states on the Cayley tree and the Ising potentials through the Markov property.

Let us assume that we have a locally faithful Markov state φ on the quasi-local algebra $\mathcal{B}_L$, then a potential h_Λ is canonically defined for each finite subset $\Lambda \subset L$ as follows:

$$\varphi|_{\mathcal{B}_\Lambda} = \text{Tr}_{\mathcal{B}_\Lambda}(e^{-h_\Lambda \cdot}) \quad (27)$$

The set of potentials $\{h_\Lambda\}_{\Lambda \subset_{fin} L}$ satisfy normalization conditions

$$\text{Tr}_{\mathcal{B}_\Lambda}(e^{-h_\Lambda}) = 1$$

together with compatibility conditions

$$(\text{Tr}_{\mathcal{B}_{\Lambda' \setminus \Lambda}} \otimes \mathbf{1}_{\mathcal{B}_\Lambda})(e^{-h_{\Lambda'}}) = e^{-h_\Lambda}$$

for finite subsets $\Lambda \subset \Lambda' \subset_{fin} L$. In particular for each $n \geq 0$, one has

$$(\text{Tr}_{\mathcal{B}_{W_{n+1}}} \otimes \mathbf{1}_{\mathcal{B}_{\Lambda_n}})(e^{-h_{\Lambda_{n+1}}}) = e^{-h_{\Lambda_n}}.$$

Theorem 8.1. *Let φ be a locally faithful state on $\mathcal{B}_L$. Then the following assertions are equivalent:*

- (i) φ is a Markov state w.r.t. the localized sequence $\{\mathcal{E}^{(j)}\}_{j \geq 0}$ of transition expectations;
- (ii) The sequence of potentials $\{h_{\Lambda_n}\}$ associated to φ by (27), can be recovered by

$$h_{\Lambda_n} = H_{W_0} + \sum_{j=0}^{n-1} H_{W_j, W_{j+1}} + \hat{H}_{W_n} \quad (28)$$

where the sequences $\{H_{W_j}\}_{j \geq 0}$, $\{\hat{H}_{W_j}\}_{j \geq 0}$ and $\{H_{W_j, W_{j+1}}\}_{j \geq 0}$ of self-adjoint operators localized in $\mathcal{B}_{W_j}$ and $\mathcal{B}_{\Lambda_{j,j+1}}$, respectively, and satisfying commutation relations

$$\begin{aligned} [H_{W_n}, H_{W_n, W_{n+1}}] &= 0, & [H_{W_n, W_{n+1}}, \hat{H}_{W_{n+1}}] &= 0, \\ [H_{W_n}, \hat{H}_{W_n}] &= 0, & [H_{W_n, W_{n+1}}, H_{W_{n+1}, W_{n+2}}] &= 0. \end{aligned} \quad (29)$$

Proof. (i) $\Rightarrow$ (ii). Let φ be a locally faithful Markov state w.r.t. the sequence $\{\mathcal{E}^{(j)}\}_{j \geq 0}$ of transition expectations. For every $j \geq 0$, $x \in W_j$ and $\sigma \in \Omega$ we define the following set of potentials $\{h_{\sigma(x_0)}^0\}$, $\{h_{\sigma(x), \sigma(\overrightarrow{S(x)})}^j\}$ and $\{\hat{h}_{\sigma(x)}^{(j)}\}$ related to the following positive functionals. Namely, the potential $h_{\sigma(x_0)}^0$ is related to $\pi_{\sigma(x_0)}^0 \eta_{\sigma(x_0)}^0$ on N^0 , the potential $h_{\sigma(x), \sigma(\overrightarrow{S(x)})}^j$ is related to $\pi_{\sigma(x), \sigma(\overrightarrow{S(x)})}^j \eta_{\sigma(x), \sigma(\overrightarrow{S(x)})}^j$ on $\overline{N}_{\sigma(x)}^j \otimes N_{\sigma(\overrightarrow{S(x)})}^{j+1}$ and $\{\hat{h}_{\sigma(x)}^{(j)}\}$ is related to

$$\sum_{\sigma' \in \Omega_{\overrightarrow{S(x)}}} \pi_{\sigma(x), \sigma'(\overrightarrow{S(x)})}^j \eta_{\sigma(x), \sigma'(\overrightarrow{S(x)})}^j (\cdot \otimes \mathbf{1})$$

on $\overline{N}_{\sigma(x)}^j$. The potential $\{\hat{h}_{\sigma(x)}^{(j)}\}$ is related to

$$\sum_{\sigma' \in \Omega_{\overrightarrow{S(x)}}} \pi_{\sigma(x), \sigma'(\overrightarrow{S(x)})}^j \eta_{\sigma(x), \sigma'(\overrightarrow{S(x)})}^j$$

Take any localized element

$$a = \bigotimes_{j=0}^n \bigotimes_{x \in W_j} a_{\sigma(x)} \otimes \bar{a}_{\sigma(x)}$$

from $\mathcal{B}_{\sigma(\Lambda_n)}$. Then one has

$$\begin{aligned} \varphi_{\sigma}(a) &= \psi_{\sigma}(a) \\ &= \eta_{x_0}^0(a_{\sigma(x_0)}) \prod_{j=0}^{n-1} \prod_{x \in W_j} \eta_{\sigma(x), \sigma(\overrightarrow{S(x)})}^j(\bar{a}_{\sigma(x)} \otimes a_{\overrightarrow{S(x)}}) \prod_{y \in W_n} \eta_{\sigma(y), \sigma(\overrightarrow{S(y)})}^j(\bar{a}_{\sigma(y)} \otimes \mathbf{1}_{\overrightarrow{S(y)}}) \\ &= \text{Tr}(e^{-h_{\sigma(x_0)}^0} a_{\sigma(x_0)}) \prod_{j=0}^{n-1} \prod_{x \in W_j} \text{Tr}(e^{-h_{\sigma(x), \sigma(\overrightarrow{S(x)})}^j} \bar{a}_{\sigma(x)} \otimes a_{\overrightarrow{S(x)}}) \prod_{y \in W_n} \text{Tr}(e^{-\hat{h}_{\sigma(y)}^n} \bar{a}_{\sigma(y)}) \end{aligned}$$

While in the last expression the traces are taken on disjoint tensors, we then get

$$\varphi_{\sigma}(a) = \text{Tr}(e^{-h_{\sigma(\Lambda_n)}} a)$$

with

$$h_{\sigma(\Lambda_n)} = h_{\sigma(x_0)}^0 + \sum_{j=0}^{n-1} \sum_{x \in W_j} h_{\sigma(x), \sigma(\overrightarrow{S(x)})}^j + \sum_{y \in W_n} \hat{h}_{\sigma(y)}^n$$

Then we define

$$\begin{aligned} H_0 &:= \sum_{\sigma \in \Omega_{W_j}} P_{\sigma(x)}^j(h_{\sigma(x)}^j \otimes \mathbf{1}) P_{\sigma(x)}^j \\ H_{x, \overrightarrow{S(x)}} &:= \sum_{\sigma \in \Omega_{\{x, \overrightarrow{S(x)}\}}} (P_{\sigma(x)}^j \otimes P_{\sigma(\overrightarrow{S(x)})}^{j+1})(\mathbf{1} \otimes h_{\sigma(x), \sigma(\overrightarrow{S(x)})}^j \otimes \mathbf{1})(P_{\sigma(x)}^j \otimes P_{\sigma(\overrightarrow{S(x)})}^{j+1}) \\ \hat{H}_x &:= \sum_{\sigma \in \Omega_x} P_{\sigma(x)}^j(\mathbf{1} \otimes \hat{h}_{\sigma(x)}^j) P_{\sigma(x)}^j \end{aligned}$$

and

$$\begin{aligned} H_{j,j+1} &= \sum_{x \in W_j} H_{x, \overrightarrow{S(x)}} \\ \hat{H}_j &:= \sum_{x \in W_j} \hat{H}_x \end{aligned}$$

Finally we set

$$h_{\Lambda_n} = H_0 + \sum_{j=0}^{n-1} H_{j,j+1} + \hat{H}_n.$$

Then h_{Λ_n} is the potential related to the state φ on $\mathcal{B}_{\Lambda_n}$. (ii) $\Rightarrow$ (i). For $n \in \mathbb{N}, x \in W_n$ we consider the map $\mathcal{E}_x^{(n)} : \mathcal{B}_x \otimes \mathcal{B}_{\overrightarrow{S(x)}} \rightarrow \mathcal{B}_x$ defined by

$$\mathcal{E}_x^{(n)}(a) = (\text{Tr}_{\overrightarrow{S(x)}} \otimes \mathbf{1}_x)(A_{x, \overrightarrow{S(x)}}^* a A_{x, \overrightarrow{S(x)}})$$

with

$$A_{x, \overrightarrow{S(x)}} = e^{-\frac{1}{2}H_{x, \overrightarrow{S(x)}}} e^{-\frac{1}{2}\widehat{H}_{\overrightarrow{S(x)}}} e^{\frac{1}{2}\widehat{H}_x}$$

considering $K_{n,n+1} = \bigotimes_{x \in W_n} A_{x, \overrightarrow{S(x)}}$ (or also by taking $K_{n,n+1} = e^{-\frac{1}{2}H_{n,n+1}} e^{-\frac{1}{2}\widehat{H}_{n+1}} e^{\frac{1}{2}\widehat{H}_n}$). We get a transition expectation

$$\mathcal{E}^{(n)}(a) = (\text{Tr}_{W_{n+1}} \otimes \mathbf{1}_{W_n})(K_{n,n+1}^* a K_{n,n+1})$$

and its corresponding quasi-conditional expectation is defined by

$$E^n = \mathbf{1}_{\mathcal{B}_{\Lambda_{n-1}}} \otimes \mathcal{E}^{(n)}.$$

One can easily check that φ is a Markov state w.r.t the sequence $\{E^n\}$ of conditional expectations. $\square$

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