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C^*-algebra of nonlocal convolution type operators

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ABSTRACT

The C^*-subalgebra $\mathfrak{B}$ of all bounded linear operators on the space $L^2(\mathbb{R})$, which is generated by all multiplication operators by piecewise slowly oscillating functions, by all convolution operators with piecewise slowly oscillating symbols and by the range of a unitary representation of the group of all affine mappings on $\mathbb{R}$, is studied. A faithful representation of the quotient C^*-algebra $\mathfrak{B}^\pi = \mathfrak{B}/\mathcal{K}$ in a Hilbert space, where $\mathcal{K}$ is the ideal of compact operators on the space $L^2(\mathbb{R})$, is constructed by applying an appropriate spectral measure decompositions, a local-trajectory method and the Fredholm symbol calculus for the C^*-algebra of convolution type operators without shifts. This gives a Fredholm symbol calculus for the C^*-algebra $\mathfrak{B}$ and a Fredholm criterion for the operators $B \in \mathfrak{B}$.

1. Introduction

Let $\mathcal{B} := \mathcal{B}(L^2(\mathbb{R}))$ be the C^*-algebra of all bounded linear operators acting on the Lebesgue space $L^2(\mathbb{R})$ and let $\mathcal{K} := \mathcal{K}(L^2(\mathbb{R}))$ be the ideal of all compact operators in $\mathcal{B}$. An operator $B \in \mathcal{B}$ is called *Fredholm* if its image is closed and the spaces $\ker B$ and $\ker B^*$ are finite-dimensional, or equivalently, the coset $B^\pi := B + \mathcal{K}$ is invertible in the Calkin algebra $\mathcal{B}^\pi := \mathcal{B}/\mathcal{K}$ (see, e.g., [20]). Put $A \simeq B$ if $A - B \in \mathcal{K}$.

Consider the unital C^*-algebras of convolution type operators

$$\mathfrak{A} := \operatorname{alg}\left\{aI, W^0(b) :\ a, b \in PSO^\diamond\right\} \subset \mathcal{B}, \quad \mathcal{Z} := \operatorname{alg}\left\{aI, W^0(b) :\ a, b \in SO^\diamond\right\} \subset \mathfrak{A} \tag{1.1}$$

generated by the multiplication operators aI and the convolution operators $W^0(b) := \mathcal{F}^{-1}b\mathcal{F}$, where $a, b \in PSO^\diamond$ and $a, b \in SO^\diamond$, respectively, and $\mathcal{F}$ is the Fourier transform: $(\mathcal{F}\varphi)(x) = \int_{\mathbb{R}} e^{ixy}\varphi(y)dy$ for $x \in \mathbb{R}$. Here $SO^\diamond$ is the C^*-algebra of functions admitting slowly oscillating discontinuities at every point $\lambda \in \mathbb{R} \cup \{\infty\}$

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and $PSO^\diamond$ is the C^*-algebra of piecewise slowly oscillating functions (see their definitions in Section 2). Since the C^*-algebras $\mathfrak{A}$ and $\mathcal{Z}$ have the same classes of discontinuous data for multiplication and convolution operators, these algebras are invariant under the transform $A \to \mathcal{F}^{-1}A\mathcal{F}$.

Let G be the solvable group of all orientation-preserving affine mappings $g_{k,h} : x \mapsto kx+h$ $(k > 0,\ h \in \mathbb{R})$ on $\mathbb{R}$ with product $g_{k_1,h_1}g_{k_2,h_2} = g_{k_2k_1,k_2h_1+h_2}$. The shifts $g \in G$ possess the common fixed point ∞ for all $g \in G$ and distinct fixed points $h/(1-k)$ for $g_{k,h} \in G$ if $k \neq 1$. Consider the unitary shift operators

$$U_g :\ L^2(\mathbb{R}) \to L^2(\mathbb{R}),\quad U_g f := |g'|^{1/2}(f \circ g),\quad g \in G. \tag{1.2}$$

The aim of this paper is to elaborate a Fredholm symbol calculus for the C^*-algebra of nonlocal convolution type operators

$$\mathfrak{B} := \mathrm{alg}(\mathfrak{A}, U_G) := \mathrm{alg}\{aI, W^0(b), U_g :\ a, b \in PSO^\diamond,\ g \in G\} \subset \mathcal{B} \tag{1.3}$$

generated by all operators $A \in \mathfrak{A}$ and by all unitary shift operators U_g $(g \in G)$, or equivalently, to construct a faithful representation of the quotient C^*-algebra $\mathfrak{B}^\pi := \mathfrak{B}/\mathcal{K}$ in an appropriate Hilbert space, where the C^*-algebra $\mathfrak{A}$ is given by (1.1) and $\mathcal{K} \subset \mathcal{Z} \subset \mathfrak{A}$ (see [29, Lemma 6.1]). To this end we apply the local-trajectory method and spectral measures (see [22], [24] and [5]), suitable spectral measure decompositions and the Fredholm symbol calculus for the C^*-algebra of convolution type operators with piecewise slowly oscillating data elaborated in [27–29], with its improvement obtained in [25] in the setting of weighted Lebesgue spaces with Muckenhoupt weights that involve the two idempotents theorem (see [11], [35]), as well as results of [30] on convolution type operators with translations. Making use of the Fredholm symbol calculus for the C^*-algebra $\mathfrak{B}$, we establish a Fredholm criterion for the operators $B \in \mathfrak{B}$ in terms of their Fredholm symbols.

The study of the spectral properties of operators from algebras generated by multiplication operators by functions and by convolution operators that reflect the Fourier duality of multiplication and differentiation, and extended by shift operators is an interesting and complicated mathematical problem. The difficulty here is that all three types of operators do not act separately on the Fourier and non-Fourier side. This makes the algebras generated by these operators highly noncommutative and extremely hard to study. Therefore, in view of complicated nature of the algebra, we decompose this algebra into several subalgebras studied by different non-trivial methods. As a result, a complete description of a Fredholm symbol calculus in terms of operators of multiplication by infinite matrix functions acting on suitable Hilbert spaces is obtained.

The C^*-algebra $\mathfrak{C} \subset \mathcal{B}(L^2(\mathbb{T}))$ of nonlocal singular integral operators generated by the Cauchy singular integral operator $S_\mathbb{T}$, by the operators of multiplications by piecewise quasicontinuous (PQC) functions [36], and by the unitary shift operators U_g $(g \in G)$, where G is a discrete amenable [21] group of shifts acting freely on $\mathbb{T}$, was studied in [13]. Recall that the group of shifts G *acts freely* on $\mathbb{T}$ if the points $g(t)$ $(t \in \mathbb{T},\ g \in G)$ are pairwise distinct. The C^*-algebra $\mathfrak{E} \subset \mathcal{B}(L^2(\mathbb{T}))$ generated by all rotation operators on $\mathbb{T}$, by all multiplication operators by piecewise slowly oscillating functions on $\mathbb{T}$ and by the operators $e_{h,\lambda}S_\mathbb{T}e_{h,\lambda}^{-1}I$ $(h \in \mathbb{R},\ \lambda \in \mathbb{T})$, where $e_{h,\lambda}(t) = \exp(h(t+\lambda)/(t-\lambda))$ for $t \in \mathbb{T} \setminus \{\lambda\}$, was studied in [4]. The C^*-algebra $\mathfrak{D} \subset \mathcal{B}(L^2(\mathbb{T}))$ generated by the Cauchy singular integral operator $S_\mathbb{T}$, by the operators of multiplications by piecewise slowly oscillating functions on $\mathbb{T}$, and by the unitary shift operators U_g $(g \in G)$, where G is a discrete amenable group of shifts acting topologically freely on $\mathbb{T}$ and having the same finite set of fixed points, was studied in [5] (for more general actions of G see also [6–9]).

On the other hand, more complicated C^*-algebras $\mathfrak{B} = \mathrm{alg}(\mathfrak{A}, U_G)$ of nonlocal convolution type operators were studied only in the case of piecewise continuous data (see [22], [23]). Algebras of convolution type operators $\mathfrak{A}$ with piecewise continuous data were studied by R.V. Duduchava, R. Schneider, S. Roch and B. Silbermann, A. Böttcher and I.M. Spitkovsky (see [11], [14], [16], [19], [35] and the references therein). In the present paper, applying results of [27–29], [25] for the C^*-algebra $\mathfrak{A}$ of convolution type operators with $PSO^\diamond$ data, we study the C^*-algebra $\mathfrak{B}$ of nonlocal convolution type operators with such data. Since $\mathfrak{B}^\pi$ is an example of C^*-algebras associated with C^*-dynamical systems and the action of the group G on the maximal ideal space of the central subalgebra $\mathcal{Z}^\pi := \mathcal{Z}/\mathcal{K}$ of the quotient C^*-algebra $\mathfrak{A}^\pi := \mathfrak{A}/\mathcal{K}$ is not topologically free, for studying the invertibility in $\mathfrak{B}^\pi$ we apply a version of the local-trajectory method combined with using spectral measures (see [22], [24], [5]). For other versions of the local-trajectory method and their applications see [1–3].

The paper is organized as follows. In Section 2 we define the C^*-algebras $SO^\diamond$ and $PSO^\diamond$ and describe their maximal ideal spaces, describe the Gelfand transform for the central subalgebra $\mathcal{Z}^\pi$ of $\mathfrak{A}^\pi$ and construct a faithful representation of the quotient C^*-algebra $\mathfrak{A}^\pi$ in a Hilbert space. In Section 3 we present main results of the paper: a Fredholm symbol calculus for the C^*-algebra $\mathfrak{B}$ given by (1.3), a Fredholm criterion for the operators $B \in \mathfrak{B}$ and the faithful representation of the C^*-algebra $\mathfrak{B}^\pi = \mathfrak{B}/\mathcal{K}$ in a Hilbert space.

The local-trajectory method elaborated in [22], [24] to study the invertibility in the abstract C^*-algebra $\mathfrak{B} = \mathrm{alg}(\mathfrak{A}, U_G)$ generated by a unital C^*-subalgebra $\mathfrak{A}$ and a unitary representation U of an amenable group G is stated in Section 4. In contrast to the local-trajectory methods developed in [1–3], the method used here is related to the Allan-Douglas local principle (see, e.g., [18], [16]) and supply us with a convenient machinery for studying C^*-algebras of nonlocal type operators with discontinuous data in case $\mathfrak{A}$ has a non-trivial central subalgebra $\mathcal{Z}$.

In Section 5 we introduce another central C^*-algebra $\widetilde{\mathcal{Z}}^\pi$ of $\mathfrak{A}^\pi$ that properly contains $\mathcal{Z}^\pi$ and leads to simpler local representatives of the cosets $A^\pi \in \mathfrak{A}^\pi$. Since the action of the group G on the maximal ideal space of $\widetilde{\mathcal{Z}}^\pi$ is not topologically free, applying spectral measures, we construct here a spectral decomposition of the C^*-algebra $\mathfrak{B}^\pi$ and give an abstract Fredholm criterion for the operators $B \in \mathfrak{B}$ in terms of invertibility of their images in the C^*-algebras $\mathfrak{B}_{\mathbb{R},\infty}$, $\mathfrak{B}_{\infty,\mathbb{R}\setminus\{0\}}$, $\mathfrak{B}^\circ_{\infty,0}$, $\mathfrak{B}_{\infty,0}$ and $\mathfrak{B}_{\infty,\infty}$ related to the spectral decomposition mentioned above.

In Section 6 we study the invertibility of the operators in the C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty}$, making use of two representations Φ_1 and Φ_2 in Hilbert spaces, where Φ_1 is defined by analogy with [9] and Φ_2 is based on applying the local-trajectory method and the lifting theorem. Sections 7–9 are devoted to studying the invertibility in the C^*-algebras $\mathfrak{B}_{\infty,\mathbb{R}\setminus\{0\}}$ and $\mathfrak{B}^\circ_{\infty,0}$ with applications of spectral measures, the local-trajectory method and results from [30]. In Section 10 we show that the invertibility in the C^*-algebras $\mathfrak{B}_{\infty,0}$ and $\mathfrak{B}_{\infty,\infty}$ follows from that in the C^*-algebras $\mathfrak{B}_{\infty,\mathbb{R}\setminus\{0\}}$ and $\mathfrak{B}_{\mathbb{R},\infty}$, respectively. Section 11 contains the proofs of the main results of the paper on the basis of previous sections.

2. The C^*-algebra $\mathfrak{A}$ of convolution type operators

2.1. The C^*-algebras $SO^\diamond$ and $PSO^\diamond$

Let $\dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ and $\overline{\mathbb{R}} := [-\infty, +\infty]$. For a bounded measurable function $f : \dot{\mathbb{R}} \to \mathbb{C}$ and a set $I \subset \dot{\mathbb{R}}$, let $\operatorname{osc}(f, I) := \operatorname{ess\,sup}\big\{|f(t) - f(s)| : t, s \in I\big\}$. Similarly to [4], we say that a function $f \in L^\infty(\mathbb{R})$ is called *slowly oscillating at a point* $\lambda \in \dot{\mathbb{R}}$ if for every (equivalently, for some) $r \in (0, 1)$,

$$\lim_{x\to+0} \operatorname{osc}\big(f, \lambda + ([-x, -rx] \cup [rx, x])\big) = 0 \ \text{ if } \ \lambda \in \mathbb{R},$$
$$\lim_{x\to+\infty} \operatorname{osc}\big(f, [-x, -rx] \cup [rx, x]\big) = 0 \ \text{ if } \ \lambda = \infty.$$

For every $\lambda \in \dot{\mathbb{R}}$, let SO_λ denote the C^*-subalgebra of $L^\infty(\mathbb{R})$ defined by

$$SO_\lambda := \big\{f \in C(\dot{\mathbb{R}} \setminus \{\lambda\}) \cap L^\infty(\mathbb{R}) : \ f \ \text{ slowly oscillates at } \ \lambda\big\}.$$

Let $SO^\diamond$ be the minimal C^*-subalgebra of $L^\infty(\mathbb{R})$ that contains all the C^*-algebras SO_λ with $\lambda \in \dot{\mathbb{R}}$, let PC denote the C^*-algebra of all piecewise continuous functions, that is, functions in $L^\infty(\mathbb{R})$ that have one-sided limits at each point $t \in \dot{\mathbb{R}}$, and let $PSO^\diamond$ be the C^*-subalgebra of $L^\infty(\mathbb{R})$ generated by the C^*-algebras PC and $SO^\diamond$. All these algebras contain $C(\dot{\mathbb{R}})$. Elements of the algebras $SO^\diamond$ and $PSO^\diamond$ are called, respectively, *slowly oscillating* and *piecewise slowly oscillating* functions.

Identifying the points $\lambda \in \dot{\mathbb{R}}$ with the evaluation functionals δ_λ on $\dot{\mathbb{R}}$ given by $\delta_\lambda(f) = f(\lambda)$ for $f \in C(\dot{\mathbb{R}})$, we infer that the maximal ideal space $M(SO^\diamond)$ of $SO^\diamond$ is of the form $M(SO^\diamond) = \bigcup_{\lambda\in\dot{\mathbb{R}}} M_\lambda(SO^\diamond)$, where $M_\lambda(SO^\diamond) := \{\xi \in M(SO^\diamond) : \xi|_{C(\dot{\mathbb{R}})} = \delta_\lambda\}$ are fibers of $M(SO^\diamond)$ over points $\lambda \in \dot{\mathbb{R}}$. Similarly, $M(PSO^\diamond) = \bigcup_{\lambda\in\dot{\mathbb{R}}} M_\lambda(PSO^\diamond)$. Applying [29, Corollary 2.2] and [10, Proposition 5], we infer that for every $\lambda \in \dot{\mathbb{R}}$,

$$M_\lambda(SO^\diamond) = M_\lambda(SO_\lambda) = M_\infty(SO_\infty) = (\mathrm{clos}_{SO^*_\infty}\mathbb{R}) \setminus \mathbb{R}, \tag{2.1}$$

where $\mathrm{clos}_{SO^*_\infty}\mathbb{R}$ is the weak-star closure of $\mathbb{R}$ in SO^*_∞, the dual space of SO_∞ (cf. [12, Proposition 4.1]).

The maximal ideal space $M(PC)$ of the algebra PC can be identified with $\dot{\mathbb{R}} \times \{0,1\}$: for each $a \in PC$,

$$a(\lambda,0) = a(\lambda-0),\quad a(\lambda,1) = a(\lambda+0)\ \text{ if }\ \lambda\in\mathbb{R};\quad a(\lambda,0) = a(+\infty),\quad a(\lambda,1) = a(-\infty)\ \text{ if }\ \lambda=\infty.$$

The maximal ideal space $M(PSO^\diamond)$ of the algebra $PSO^\diamond$ has a similar form: $M(PSO^\diamond) = M(SO^\diamond)\times\{0,1\}$. Identifying characters $\zeta \in M_\lambda(PSO^\diamond)$ for $\lambda\in\dot{\mathbb{R}}$ with pairs $(\xi,\mu) \in M_\lambda(SO^\diamond)\times M_\lambda(PC)$ by [27, Lemma 3.4], where $M_\lambda(PC) = \{0,1\}$, we get the following characterization of the fiber $M_\lambda(PSO^\diamond)$ (cf. [4, Theorem 4.6]).

Theorem 2.1 *([27], Theorem 3.5). If $(\xi,\mu) \in M_\lambda(SO^\diamond)\times\{0,1\}$ and $\lambda\in\dot{\mathbb{R}}$, then $(\xi,\mu)|_{SO^\diamond} = \xi$, $(\xi,\mu)|_{C(\dot{\mathbb{R}})} = \lambda$, $(\xi,\mu)|_{PC} = (\lambda,\mu)$.*

As usual, we write $a(\xi) := \xi(a)$ for $a \in SO^\diamond$ and $\xi \in M(SO^\diamond)$. For $c \in PSO^\diamond$ and $\xi \in M(SO^\diamond)$, we put

$$c(\xi^-) := c(\xi,0)\ \text{ and }\ c(\xi^+) := c(\xi,1), \tag{2.2}$$

where $c(\xi,\mu) = (\xi,\mu)c$ for $(\xi,\mu) \in M(SO^\diamond)\times\{0,1\}$. The Gelfand topology on $M(PSO^\diamond)$ can be described as follows. A base of neighborhoods for $(\xi,\mu) \in M(PSO^\diamond)$ consists of all open sets of the form

$$U_{(\xi,\mu)} = \begin{cases}(U_{\xi,\lambda}\times\{0\})\cup(U^-_{\xi,\lambda}\times\{0,1\}) & \text{if } \mu=0,\\ (U_{\xi,\lambda}\times\{1\})\cup(U^+_{\xi,\lambda}\times\{0,1\}) & \text{if } \mu=1,\end{cases} \tag{2.3}$$

where $U_{\xi,\lambda} = U_\xi \cap M_\lambda(SO^\diamond)$ if $\xi\in M_\lambda(SO^\diamond)$ for some $\lambda\in\dot{\mathbb{R}}$, U_ξ is an open neighborhood of ξ in $M(SO^\diamond)$, and $U^-_{\xi,\lambda}$, $U^+_{\xi,\lambda}$ consist of all $\zeta\in U_\xi$ whose restrictions $\tau = \zeta|_{C(\dot{\mathbb{R}})}$ belong, respectively, to the sets $(\lambda-\varepsilon,\lambda)$ and $(\lambda,\lambda+\varepsilon)$ with $\varepsilon>0$ if $\lambda\in\mathbb{R}$, and to the sets $(\varepsilon,+\infty)$ and $(-\infty,-\varepsilon)$ with $\varepsilon\in\mathbb{R}$ if $\lambda=\infty$.

2.2. Faithful representation of the quotient C^*-algebra $\mathfrak{A}^\pi$

Consider the C^*-algebras $\mathfrak{A}$ and $\mathcal{Z}$ given by (1.1). As $\mathcal{K}\subset\mathcal{Z}\subset\mathfrak{A}$, it follows from [29, Theorem 4.4] that $\mathcal{Z}^\pi = \mathcal{Z}/\mathcal{K}$ is a central C^*-subalgebra of the quotient C^*-algebra $\mathfrak{A}^\pi = \mathfrak{A}/\mathcal{K}$. Put

$$\begin{gathered}\Omega_{\mathbb{R},\infty} := \bigcup_{t\in\mathbb{R}} M_t(SO^\diamond)\times M_\infty(SO^\diamond),\quad \Omega_{\infty,\mathbb{R}} := M_\infty(SO^\diamond)\times\bigcup_{t\in\mathbb{R}} M_t(SO^\diamond),\\ \Omega_{\infty,\infty} := M_\infty(SO^\diamond)\times M_\infty(SO^\diamond).\end{gathered} \tag{2.4}$$

Theorem 2.2 *([29], Theorem 6.2). The maximal ideal space $M(\mathcal{Z}^\pi)$ of the commutative C^*-algebra $\mathcal{Z}^\pi$ is homeomorphic to the set $\Omega := \Omega_{\mathbb{R},\infty}\cup\Omega_{\infty,\mathbb{R}}\cup\Omega_{\infty,\infty}$ equipped with topology induced by the product topology of $M(SO^\diamond)\times M(SO^\diamond)$, and the Gelfand transform $\Gamma : \mathcal{Z}^\pi \to C(\Omega)$, $A^\pi \mapsto \mathcal{A}(\cdot,\cdot)$ is defined on the generators $A^\pi = (aW^0(b))^\pi$ $(a,b\in SO^\diamond)$ of the algebra $\mathcal{Z}^\pi$ by $\mathcal{A}(\xi,\eta) = a(\xi)b(\eta)$ for all $(\xi,\eta)\in\Omega$.*

Following [28, Subsection 3.2] and [25, Section 3] and using (2.4), we consider the set

$$\widetilde{\Omega} = \big(\Omega_{\mathbb{R},\infty} \times \overline{\mathbb{R}}\big) \cup \big(\Omega_{\infty,\mathbb{R}} \times \overline{\mathbb{R}}\big) \cup \big(\Omega_{\infty,\infty} \times \{\pm\infty\}\big), \tag{2.5}$$

where, by Theorem 2.2, the sets $\Omega_{\mathbb{R},\infty}$ and $\Omega_{\infty,\mathbb{R}}$ given by (2.4) are open in Ω, while the set $\Omega_{\infty,\infty}$ is closed in Ω. According to [28, Section 4.4], for each $(\xi,\eta,x) \in \widetilde{\Omega}$, we define the mapping

$$\begin{aligned} &\Psi_{\xi,\eta,x} : \ \big\{aI : a \in PSO^\diamond\big\} \cup \big\{W^0(b) : b \in PSO^\diamond\big\} \to \mathbb{C}^{2\times 2}, \\ &\Psi_{\xi,\eta,x}(aI) = \operatorname{diag}\big\{a(\xi^+), a(\xi^-)\big\}, \\ &\Psi_{\xi,\eta,x}(W^0(b)) = \begin{bmatrix} b(\eta^+)\mu(x) + b(\eta^-)(1-\mu(x)) & [b(\eta^+) - b(\eta^-)]\varrho(x) \\ [b(\eta^+) - b(\eta^-)]\varrho(x) & b(\eta^+)(1-\mu(x)) + b(\eta^-)\mu(x) \end{bmatrix}, \end{aligned} \tag{2.6}$$

where $a(\xi^\pm)$ and $b(\eta^\pm)$ are defined by (2.2), and

$$\mu(x) := (1 + \tanh(\pi x))/2, \quad \varrho(x) := 1/\cosh(\pi x) \quad \text{for all} \quad x \in \overline{\mathbb{R}}. \tag{2.7}$$

Theorem 2.3 *([30], Theorem 3.2). The mappings $\Psi_{\xi,\eta,x}$ $\big((\xi,\eta,x) \in \widetilde{\Omega}\big)$ given on the generators of the C^*-algebra $\mathfrak{A}$ by formulas (2.6)–(2.7) extend to C^*-algebra homomorphisms $\Psi_{\xi,\eta,x} : \mathfrak{A} \to \mathbb{C}^{2\times 2}$. An operator $A \in \mathfrak{A}$ is Fredholm on the space $L^2(\mathbb{R})$ if and only if*

$$\det \Psi_{\xi,\eta,x}(A) \neq 0 \ \ \textit{for all} \ \ (\xi,\eta,x) \in \widetilde{\Omega}. \tag{2.8}$$

To any operator $A \in \mathfrak{A}$ we assign its *Fredholm symbol*, that is, the bounded matrix function

$$\mathcal{A} : \ \widetilde{\Omega} \to \mathbb{C}^{2\times 2}, \quad (\xi,\eta,x) \mapsto \mathcal{A}(\xi,\eta,x) := \Psi_{\xi,\eta,x}(A).$$

Let $B(\widetilde{\Omega}, \mathbb{C}^{2\times 2})$ denote the C^*-algebra of all bounded $\mathbb{C}^{2\times 2}$-valued functions on $\widetilde{\Omega}$.

Theorem 2.4 *([30], Theorem 3.3). The Fredholm symbol mapping $\Psi : \mathfrak{A} \to B(\widetilde{\Omega}, \mathbb{C}^{2\times 2})$, $A \mapsto \mathcal{A}(\cdot,\cdot,\cdot)$, is a C^*-algebra homomorphism whose kernel $\ker \Psi$ coincides with the ideal $\mathcal{K}$ of all compact operators on the space $L^2(\mathbb{R})$ and the image $\Psi(\mathfrak{A})$ is a C^*-subalgebra of $B(\widetilde{\Omega}, \mathbb{C}^{2\times 2})$.*

Corollary 2.5 *([30], Corollary 3.4). The mapping $\Psi_0 : A^\pi \mapsto \bigoplus_{(\xi,\eta,x)\in\widetilde{\Omega}} \mathcal{A}(\xi,\eta,x)I$ is a faithful representation of the quotient C^*-algebra $\mathfrak{A}^\pi$ in the Hilbert space $\bigoplus_{(\xi,\eta,x)\in\widetilde{\Omega}} \mathbb{C}^2$.*

3. Faithful representation of the C^*-algebra $\mathfrak{B}^\pi$: main results

Consider the C^*-algebra $\mathfrak{B} \subset \mathcal{B}(L^2(\mathbb{R}))$ generated by the operators $A \in \mathfrak{A}$ and the unitary shift operators $U_{g_{k,h}}$ $(k > 0,\ h \in \mathbb{R})$ given by (1.2), where $g_{k,h} : x \mapsto kx + h$ for $x \in \mathbb{R}$. Put

$$k_g := k \ \ \text{and} \ \ h_g := h \ \ \text{for all} \ \ g = g_{k,h}. \tag{3.1}$$

The group G consisting of the shifts $g_{k,h}$ $(k > 0,\ h \in \mathbb{R})$ is the semidirect product $\widetilde{G} \rtimes \widehat{G}$ of its subgroups

$$\widetilde{G} := \{g_{k,0} : k > 0\} \quad \text{and} \quad \widehat{G} := \{g_{1,h} : h \in \mathbb{R}\}. \tag{3.2}$$

Hence $g_{k_1,h_1} g_{k_2,h_2} = g_{k_2k_1, k_2h_1+h_2}$ and $g_{k,h}(x) = g_{1,h}[g_{k,0}(x)]$ for $x \in \mathbb{R}$.

Given $t, \tau \in \dot{\mathbb{R}}$ and $g \in G$, we define the set $Y_{t,\tau}$ and the function $\delta_g : \dot{\mathbb{R}} \times \dot{\mathbb{R}} \to \{0, 1\}$ by

$$Y_{t,\tau} := \{g \in G : \ g(t) = \tau\}, \quad \delta_g(t,\tau) = 1 \ \text{ if } \ g(t) = \tau, \quad \delta_g(t,\tau) = 0 \ \text{ if } \ g(t) \neq \tau. \tag{3.3}$$

Fix $t_0 \in \mathbb{R}$. Then its G-orbit $G(t_0) := \{g(t_0) : g \in G\}$ coincides with $\mathbb{R}$. For each $\tau \in \mathbb{R}$, we fix the shift $g_\tau := g_{1,\tau-t_0} \in Y_{t_0,\tau}$. Hence $g_{t_0} = e$, the unit of G. Observe that, for every $g \in Y_{t,\tau}$ with $t, \tau \in \mathbb{R}$, we have

$$\widetilde{g}_{t,\tau} := g_t g g_\tau^{-1} = g_\tau^{-1} \circ g \circ g_t \in Y_{t_0,t_0}. \tag{3.4}$$

Let $\mathbb{R}_+ := (0, +\infty)$ and $\mathbb{R}_- := (-\infty, 0)$. Given $t \in \mathbb{R}$, we consider the sets

$$\begin{gathered} \Omega_{t,\infty} := M_t(SO^\diamond) \times M_\infty(SO^\diamond), \quad \Omega_{\infty,t} := M_\infty(SO^\diamond) \times M_t(SO^\diamond), \\ \Delta^\circ_{t,\infty} := \Omega_{t,\infty} \times \mathbb{R}, \quad \dot{\Delta}_{t,\infty} := \Omega_{t,\infty} \times \dot{\mathbb{R}}, \quad \widetilde{\Omega}_{t,\infty} := \Omega_{t,\infty} \times \overline{\mathbb{R}}, \quad \widehat{\Omega}_{t,\infty} := \Omega_{t,\infty} \times \{\pm\infty\}, \\ \Delta^\circ_{\infty,t} := \Omega_{\infty,t} \times \mathbb{R}, \quad \dot{\Delta}_{\infty,t} := \Omega_{\infty,t} \times \dot{\mathbb{R}}, \quad \widetilde{\Omega}_{\infty,t} := \Omega_{\infty,t} \times \overline{\mathbb{R}}, \quad \widehat{\Omega}_{\infty,t} := \Omega_{\infty,t} \times \{\pm\infty\}. \end{gathered} \tag{3.5}$$

Fix $t_0 \in \mathbb{R}$ and $t_\pm \in \mathbb{R}_\pm$. With the C^*-algebra $\mathfrak{B}$ we associate the Hilbert space

$$\mathcal{H} := \mathcal{H}_{\mathbb{R},\infty,1} \oplus \mathcal{H}_{\mathbb{R},\infty,2} \oplus \mathcal{H}_{\infty,\mathbb{R}_-} \oplus \mathcal{H}_{\infty,0} \oplus \mathcal{H}_{\infty,\mathbb{R}_+}, \tag{3.6}$$

where the non-separable Hilbert spaces

$$\begin{aligned} \mathcal{H}_{\mathbb{R},\infty,1} &:= l^2(\Delta^\circ_{t_0,\infty},\ l^2(\mathbb{R}, \mathbb{C}^2)), \quad & \mathcal{H}_{\mathbb{R},\infty,2} &:= l^2(\widehat{\Omega}_{t_0,\infty},\ l^2(G, \mathbb{C}^2)), \\ \mathcal{H}_{\infty,0} &:= l^2(\Delta^\circ_{\infty,0},\ \mathbb{C}^2), & \mathcal{H}_{\infty,\mathbb{R}_\pm} &:= l^2(\widetilde{\Omega}_{\infty,t_\pm},\ l^2(\mathbb{R}_+, \mathbb{C}^2)) \end{aligned} \tag{3.7}$$

consist, respectively, of $l^2(\mathbb{R}, \mathbb{C}^2)$)-valued functions defined on the set $\Delta^\circ_{t_0,\infty}$, of $l^2(G, \mathbb{C}^2)$-valued functions defined on the set $\widehat{\Omega}_{t_0,\infty}$, of $\mathbb{C}^2$-valued functions defined on the set $\Delta^\circ_{\infty,0}$ and of $l^2(\mathbb{R}_+, \mathbb{C}^2)$)-valued functions defined on the sets $\widetilde{\Omega}_{\infty,t_\pm}$, and these functions have at most countable sets of non-zero values. In its turn, for $X \in \{\mathbb{R}, G, \mathbb{R}_+\}$, $l^2(X, \mathbb{C}^2)$ is the non-separable Hilbert space consisting of all vectors $f = (f_\tau)_{\tau \in X}$ with at most countable sets of non-zero entries $f_\tau \in \mathbb{C}^2$ and the norm $\|f\| = (\sum_\tau \|f_\tau\|^2_{\mathbb{C}^2})^{1/2} < \infty$.

For the Hilbert space $\mathcal{H}$ given by (3.6), we construct the representation

$$\Phi : \mathfrak{B} \to \mathcal{B}(\mathcal{H}), \quad B \mapsto \Phi_1(B) \oplus \Phi_2(B) \oplus \Phi_-(B) \oplus \Phi_0(B) \oplus \Phi_+(B), \tag{3.8}$$

which is the direct sum of the following five C^*-algebra homomorphisms:

$$\begin{aligned} &\Phi_1 : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_{\mathbb{R},\infty,1}), \quad B \mapsto \mathrm{Sym}_1(B)I, \quad && \Phi_2 : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_{\mathbb{R},\infty,2}), \quad B \mapsto \mathrm{Sym}_2(B)I, \\ &\Phi_0 : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_{\infty,0}), \quad B \mapsto \mathrm{Sym}_0(B)I, && \Phi_\pm : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_{\infty,\mathbb{R}_\pm}), \quad B \mapsto \mathrm{Sym}_\pm(B)I, \end{aligned} \tag{3.9}$$

defined initially on the generators of the C^*-algebra $\mathfrak{B}$.

Here $\Phi_1(B)$ are operators of multiplication by infinite matrix $\mathrm{Sym}_1(B)$ given on the set $\Delta^\circ_{t_0,\infty}$, where the values of these matrix functions at the points $(\xi, \eta, x) \in \Delta^\circ_{t_0,\infty}$ define bounded linear operators on the Hilbert space $l^2(\mathbb{R}, \mathbb{C}^2)$ and are given on the generators of the C^*-algebra $\mathfrak{B}$ by

$$\begin{aligned} [\mathrm{Sym}_1(aI)](\xi, \eta, x) &:= \mathrm{diag}\left\{\mathrm{diag}\left\{(a \circ g_t)(\xi^+),\ (a \circ g_t)(\xi^-)\right\}\right\}_{t \in \mathbb{R}}, \\ [\mathrm{Sym}_1(W^0(b))](\xi, \eta, x) &:= \mathrm{diag}\left\{\begin{bmatrix} b(\eta^+)\mu(x) + b(\eta^-)(1-\mu(x)) & (b(\eta^+) - b(\eta^-))\varrho(x) \\ (b(\eta^+) - b(\eta^-))\varrho(x) & b(\eta^+)(1-\mu(x)) + b(\eta^-)\mu(x) \end{bmatrix}\right\}_{t \in \mathbb{R}}, \\ [\mathrm{Sym}_1(U_g)](\xi, \eta, x) &:= \left[\mathrm{diag}\left\{\delta_g(t,\tau) e^{ix \ln k_g},\ \delta_g(t,\tau) e^{ix \ln k_g}\right\}\right]_{t,\tau \in \mathbb{R}}, \end{aligned} \tag{3.10}$$

where $a, b \in PSO^\diamond$, $g \in G$, the functions $x \mapsto \mu(x)$ and $x \mapsto \varrho(x)$ here and below are given by (2.7), and k_g and $\delta_g(t, \tau)$ are given by (3.1) and (3.3), respectively.

Further, $\Phi_2(B)$ are operators of multiplication by infinite matrix functions $\mathrm{Sym}_2(B)$ given on the set $\widehat{\Omega}_{t_0,\infty}$, where the values of these matrix functions at the points $(\xi,\eta,x)\in\widehat{\Omega}_{t_0,\infty}$ define bounded linear operators on the space $l^2(G,\mathbb{C}^2)$ and are given on the generators of the C^*-algebra $\mathfrak{B}$ as follows:

$$\begin{aligned}
&[\mathrm{Sym}_2(aI)](\xi,\eta,x) := \mathrm{diag}\big\{\mathrm{diag}\{(a\circ g)(\xi^+),\ (a\circ g)(\xi^-)\}\big\}_{g\in G},\\
&[\mathrm{Sym}_2(W^0(b))](\xi,\eta,x) := \mathrm{diag}\big\{\mathrm{diag}\{b(\eta^+)\mu(x)+b(\eta^-)(1-\mu(x)),\ b(\eta^+)(1-\mu(x))+b(\eta^-)\mu(x)\}\big\}_{g\in G},\\
&[\mathrm{Sym}_2(U_g)](\xi,\eta,x) := [\delta_{hg,s}I_2]_{h,s\in G},
\end{aligned}\tag{3.11}$$

where $a,b\in PSO^\diamond$, $g\in G$ and $\delta_{h,s}$ is the Kronecker symbol on G.

In its turn, $\Phi_0(B)$ are operators of multiplication by 2×2 matrix functions $\mathrm{Sym}_0(B):\Delta^\circ_{\infty,0}\to\mathbb{C}^{2\times 2}$ whose values at the points $(\xi,\eta,x)\in\Delta^\circ_{\infty,0}$ are defined on the generators of the C^*-algebra $\mathfrak{B}$ by

$$\begin{aligned}
&[\mathrm{Sym}_0(aI)](\xi,\eta,x) := \mathrm{diag}\{a(\xi^+),a(\xi^-)\},\\
&[\mathrm{Sym}_0(W^0(b))](\xi,\eta,x) := \begin{bmatrix} b(\eta^+)\mu(x)+b(\eta^-)(1-\mu(x)) & [b(\eta^+)-b(\eta^-)]\varrho(x)\\ [b(\eta^+)-b(\eta^-)]\varrho(x) & b(\eta^+)(1-\mu(x))+b(\eta^-)\mu(x)\end{bmatrix},\\
&[\mathrm{Sym}_0(U_g)](\xi,\eta,x) := e^{-ix\ln k_g}I_2,
\end{aligned}\tag{3.12}$$

where $a,b\in PSO^\diamond$, $g\in G$, k_g is given by (3.1), and $I_2 := \mathrm{diag}\{1,1\}$.

Finally, $\Phi_\pm(B)$ are operators of multiplication by infinite matrix functions $\mathrm{Sym}_\pm(B)$ given on the sets $\widetilde{\Omega}_{\infty,t_\pm}$, and the values of these matrix functions at the points $(\xi,\eta,x)\in\widetilde{\Omega}_{\infty,t_\pm}$ define bounded linear operators on the Hilbert space $l^2(\mathbb{R}_+,\mathbb{C}^2)$, which are given on the generators of the C^*-algebra $\mathfrak{B}$ by

$$\begin{aligned}
&[\mathrm{Sym}_\pm(aI)](\xi,\eta,x) := \mathrm{diag}\big\{\mathrm{diag}\{a(\xi^+),a(\xi^-)\}\big\}_{t\in\mathbb{R}_+},\\
&[\mathrm{Sym}_\pm(W^0(b)](\xi,\eta,x) := \mathrm{diag}\left\{\begin{bmatrix} b_t(\eta^+)\mu(x)+b_t(\eta^-)(1-\mu(x)) & [b_t(\eta^+)-b_t(\eta^-)]\varrho(x)\\ [b_t(\eta^+)-b_t(\eta^-)]\varrho(x) & b_t(\eta^+)(1-\mu(x))+b_t(\eta^-)\mu(x)\end{bmatrix}\right\}_{t\in\mathbb{R}_+},\\
&[\mathrm{Sym}_\pm(U_{g_{k,h}})](\xi,\eta,x) := \big[\delta_{g_{k,0}}(t,\tau)e^{-iht_\pm/\tau}I_2\big]_{t,\tau\in\mathbb{R}_+},
\end{aligned}\tag{3.13}$$

where $a,b\in PSO^\diamond$, $g_{k,h}\in G$, $b_t := b\circ g_{t^{-1},0}$ for $t\in\mathbb{R}_+$, and $\delta_{g_{k,0}}$ is given by (3.3) for $g=g_{k,0}$.

We will prove below the following main results of the paper.

Theorem 3.1. *The map Φ defined on the generators of the C^*-algebra $\mathfrak{B}$ by formulas (3.8)–(3.13) extends to a C^*-algebra homomorphism of $\mathfrak{B}$ into the C^*-algebra $\mathcal{B}(\mathcal{H})$, and*

$$\|\Phi(B)\|_{\mathcal{B}(\mathcal{H})}\le\|B^\pi\| := \inf_{K\in\mathcal{K}}\|B+K\|_{\mathcal{B}(L^2(\mathbb{R}))}.$$

Theorem 3.2. *An operator $B\in\mathfrak{B}$ is Fredholm on the space $L^2(\mathbb{R})$ if and only if the operator $\Phi(B)$ is invertible on the Hilbert space $\mathcal{H}$, that is, if the following four conditions hold:*

(i) *the operator $[\mathrm{Sym}_1(B)](\xi,\eta,x)I$ is invertible on the space $l^2(\mathbb{R},\mathbb{C}^2)$ for every $(\xi,\eta,x)\in\Delta^\circ_{t_0,\infty}$ and*

$$\sup_{(\xi,\eta,x)\in\Delta^\circ_{t_0,\infty}}\big\|\big([\mathrm{Sym}_1(B)](\xi,\eta,x)I\big)^{-1}\big\|_{\mathcal{B}(l^2(\mathbb{R},\mathbb{C}^2))}<\infty;$$

(ii) *the operator $[\mathrm{Sym}_2(B)](\xi,\eta,x)I$ is invertible on the space $l^2(G,\mathbb{C}^2)$ for every $(\xi,\eta,x)\in\widehat{\Omega}_{t_0,\infty}$ and*

$$\sup_{(\xi,\eta,x)\in\widehat{\Omega}_{t_0,\infty}}\big\|\big([\mathrm{Sym}_2(B)](\xi,\eta,x)I\big)^{-1}\big\|_{\mathcal{B}(l^2(G,\mathbb{C}^2))}<\infty;$$

(iii) *for every* $(\xi,\eta,x) \in \Delta^\circ_{\infty,0}$ *the* 2×2 *matrix* $[\mathrm{Sym}_0(B)](\xi,\eta,x)$ *is invertible and*

$$\inf_{(\xi,\eta,x)\in\Delta^\circ_{\infty,0}} \big|\det\big([\mathrm{Sym}_0(B)](\xi,\eta,x)\big)\big| > 0;$$

(iv) *for every* $(\xi,\eta,x) \in \widetilde{\Omega}_{\infty,t_\pm}$ *the operators* $[\mathrm{Sym}_\pm(B)](\xi,\eta,x)I$ *are invertible on the space* $l^2(\mathbb{R}_+,\mathbb{C}^2)$ *and*

$$\sup_{(\xi,\eta,x)\in\widetilde{\Omega}_{\infty,t_\pm}} \big\|\big([\mathrm{Sym}_\pm(B)](\xi,\eta,x)I\big)^{-1}\big\|_{\mathcal{B}(l^2(\mathbb{R}_+,\mathbb{C}^2))} < \infty;$$

where the sets $\Delta^\circ_{t_0,\infty}$, $\Delta^\circ_{\infty,0}$, $\widehat{\Omega}_{t_0,\infty}$ *and* $\widetilde{\Omega}_{\infty,t_\pm}$ *are defined in* (3.5).

Theorem 3.2 immediately implies the following corollary.

Corollary 3.3. *The map* $\Phi^\pi := \Phi^\pi_1 \oplus \Phi^\pi_2 \oplus \Phi^\pi_- \oplus \Phi^\pi_0 \oplus \Phi^\pi_+ : \mathfrak{B}^\pi \to \mathcal{B}(\mathcal{H})$ *defined for every* $B \in \mathfrak{B}$ *by* $\Phi^\pi_1(B^\pi) := \Phi_1(B)$, $\Phi^\pi_2(B^\pi) := \Phi_2(B)$, $\Phi^\pi_0(B^\pi) := \Phi_0(B)$, *and* $\Phi^\pi_\pm(B^\pi) := \Phi_\pm(B)$ *is a faithful representation of the* C^**-algebra* $\mathfrak{B}^\pi$ *in the Hilbert space* $\mathcal{H}$ *given by* (3.6)–(3.7).

4. The local-trajectory method

To study the nonlocal C^*-algebra $\mathfrak{B}$ of the form (1.3), we apply the local-trajectory method. Let us recall its statements (see [22], [24]). In what follows we write $\mathcal{C} \cong \mathcal{D}$ if the unital C^*-algebras $\mathcal{C}$ and $\mathcal{D}$ are *-isomorphic and therefore isometrically *-isomorphic (see, e.g., [32, Theorem 2.1.7]).

Let Q be a unital C^*-algebra, $\mathfrak{A}$ a C^*-subalgebra of Q with unit I of Q, and let $\mathcal{Z}$ be a central C^*-subalgebra of $\mathfrak{A}$ with the same unit I. For a discrete group G with unit e, let $U : g \mapsto U_g$ be a homomorphism of the group G onto a group $U_G = \{U_g : g \in G\}$ of unitary elements of Q, where $U_{g_1g_2} = U_{g_1}U_{g_2}$. We denote by $\mathfrak{B} := \mathrm{alg}(\mathfrak{A}, U_G)$ the minimal C^*-subalgebra of Q containing $\mathfrak{A}$ and U_G. Assume that

(A1) *for all* $g \in G$ *the mappings* $\alpha_g : a \mapsto U_g a U_g^*$ *are* **-automorphisms of the* C^**-algebras* $\mathfrak{A}$ *and* $\mathcal{Z}$.

According to (A1), $\mathfrak{B}$ is the closure of the set $\mathfrak{B}^0$ consisting of all elements of the form $b = \sum a_g U_g$ where $a_g \in \mathfrak{A}$ and g runs through finite subsets of G.

Since the unital C^*-algebra $\mathcal{Z}$ is commutative, the Gelfand-Naimark theorem (see, e.g., [33, § 16]) implies that $\mathcal{Z} \cong C(M(\mathcal{Z}))$ where $C(M(\mathcal{Z}))$ is the C^*-algebra of all continuous complex-valued functions on the maximal ideal space $M(\mathcal{Z})$ of $\mathcal{Z}$. By (A1), each *-automorphism $\alpha_g : \mathcal{Z} \to \mathcal{Z}$ induces a homeomorphism $\beta_g : M(\mathcal{Z}) \to M(\mathcal{Z})$ given by the rule $z[\beta_g(m)] = [\alpha_g(z)](m)$ for all $z \in \mathcal{Z}$, $m \in M(\mathcal{Z})$ and $g \in G$, where $z(\cdot) \in C(M(\mathcal{Z}))$ is the Gelfand transform of $z \in \mathcal{Z}$. The set $G(m) := \{\beta_g(m) : g \in G\}$ is called the G*-orbit* of a point $m \in M(\mathcal{Z})$. In what follows we assume that

(A2) G *is an amenable discrete group.*

By [21], a discrete group G is called *amenable* if the C^*-algebra $l^\infty(G)$ of all bounded complex-valued functions on G with sup-norm has an invariant mean, that is, a positive linear functional ρ of norm 1 satisfying the condition $\rho(f) = \rho(_sf) = \rho(f_s)$ for all $s \in G$ and all $f \in l^\infty(G)$, where $(_sf)(g) = f(s^{-1}g)$ and $(f_s)(g) = f(gs)$ for all $g \in G$. Finite groups, commutative groups, subexponential groups and solvable groups are examples of amenable groups (see, e.g., [1], [21], [24]).

Let J_m be the closed two-sided ideal of $\mathfrak{A}$ generated by a maximal ideal $m \in M(\mathcal{Z})$ of the central C^*-algebra $\mathcal{Z} \subset \mathfrak{A}$. Then the Allan-Douglas local principle (see, e.g., [16, Theorem 1.35]) gives the following.

Theorem 4.1. *An element $a \in \mathfrak{A}$ is invertible in $\mathfrak{A}$ if and only if for every $m \in M(\mathcal{Z})$ the coset $a + J_m$ is invertible in the quotient C^*-algebra $\mathfrak{A}/J_m$.*

Let $\mathcal{P}_{\mathfrak{A}}$ be the set of all pure states (see, e.g., [17], [32]) of the C^*-algebra $\mathfrak{A}$ equipped with induced weak* topology. By [15, Lemma 4.1], if $\mu \in \mathcal{P}_{\mathfrak{A}}$, then $\ker \mu \supset J_m$, where $m := \mathcal{Z} \cap \ker \mu \in M(\mathcal{Z})$. We assume that

(A3) *there is a set $M_0 \subset M(\mathcal{Z})$ such that for every finite set $G_0 \subset G$ and every nonempty open set $W \subset \mathcal{P}_{\mathfrak{A}}$ there exists a state $\nu \in W$ such that $\beta_g(m_\nu) \neq m_\nu$ for all $g \in G_0 \setminus \{e\}$, where the point $m_\nu = \mathcal{Z} \cap \ker \nu$ belongs to the G-orbit $G(M_0) := \{\beta_g(m) : g \in G,\ m \in M_0\}$ of the set M_0.*

If the C^*-algebra $\mathfrak{A}$ is commutative itself, then $\mathcal{P}_{\mathfrak{A}}$ consists of all characters of $\mathfrak{A}$ (see, e.g., [32, Theorem 5.1.6]), which simplifies (A3).

For every $m \in M(\mathcal{Z})$, let $\widetilde{\pi}_m : \mathfrak{A}/J_m \to \mathcal{B}(\mathcal{H}_m)$ be an isometric (equivalently, faithful) representation of the quotient algebra $\mathfrak{A}/J_m$ in a Hilbert space $\mathcal{H}_m$, which exists by [32, Theorem 3.4.1]. Moreover, in view of (A1), the spaces $\mathcal{H}_m$ can be chosen equal for all m in the same G-orbit. Consider the representation

$$\pi'_m : \mathfrak{A} \to \mathcal{B}(\mathcal{H}_m), \quad A \mapsto (\widetilde{\pi}_m \circ \varrho_m)(A), \tag{4.1}$$

where $\varrho_m : \mathfrak{A} \to \mathfrak{A}/J_m$ is the canonical *-homomorphism. Let Ω be the set of G-orbits of all points $m \in M_0$ with $M_0 \subset M(\mathcal{Z})$ taken from (A3), let $\mathcal{H}_\omega = \mathcal{H}_m$ where $m = m_\omega$ is an arbitrary fixed point of an orbit $\omega \in \Omega$, and let $l^2(G, \mathcal{H}_\omega)$ be the Hilbert space of all functions $f : G \mapsto \mathcal{H}_\omega$ such that $f(g) \neq 0$ for at most countable set of points $g \in G$ and $\sum \|f(g)\|^2_{\mathcal{H}_\omega} < \infty$. For every $\omega \in \Omega$, we consider the representation $\pi_\omega : \mathfrak{B} \to \mathcal{B}(l^2(G, \mathcal{H}_\omega))$ defined for all $a \in \mathfrak{A}$, all $g, s \in G$ and all $f \in l^2(G, \mathcal{H}_\omega)$ by

$$[\pi_\omega(a)f](g) = \pi'_{m_\omega}(\alpha_g(a))f(g), \quad [\pi_\omega(U_s)f](g) = f(gs). \tag{4.2}$$

A slight modification of [24, Theorem 4.12], where the superfluous condition of the closedness of the set $M_0 \subset M(\mathcal{Z})$ was imposed, gives the following nonlocal version of Theorem 4.1 (see [5, Theorem 3.1]).

Theorem 4.2. *If assumptions* (A1)–(A3) *are satisfied, then an element $b \in \mathfrak{B}$ is invertible in $\mathfrak{B}$ if and only if for every orbit $\omega \in \Omega$ the operator $\pi_\omega(b)$ is invertible on the space $l^2(G, \mathcal{H}_\omega)$ and, for infinite set Ω, $\sup_{\omega \in \Omega} \|(\pi_\omega(b))^{-1}\| < \infty$.*

Corollary 4.3. *Under the conditions of Theorem 4.2, the mapping $\pi : b \mapsto \bigoplus_{\omega \in \Omega} \pi_\omega(b)$ is a faithful representation of the C^*-algebra $\mathfrak{B}$ in the Hilbert space $\bigoplus_{\omega \in \Omega} l^2(G, \mathcal{H}_\omega)$.*

5. A spectral measure decomposition of the C^*-algebra $\mathfrak{B}^\pi$

5.1. A central subalgebra $\widetilde{\mathcal{Z}}^\pi$ of $\mathfrak{A}^\pi$

Along with the C^*-algebra $\mathfrak{A} \subset \mathcal{B}(L^2(\mathbb{R}))$, we consider its C^*-subalgebra

$$\mathfrak{S} := \operatorname{alg}\big\{aI, S_{\mathbb{R}} :\ a \in PC\big\} \subset \mathcal{B}(L^2(\mathbb{R})) \tag{5.1}$$

generated by all aI $(a \in PC)$ and by the Cauchy singular integral operator $S_{\mathbb{R}}$ given by

$$(S_{\mathbb{R}} f)(x) = \lim_{\epsilon \to 0} \frac{1}{\pi i} \int\limits_{\mathbb{R} \setminus (x-\epsilon, x+\epsilon)} \frac{f(t)}{t - x}\, dt, \quad x \in \mathbb{R}. \tag{5.2}$$

As is well known, the ideal $\mathcal{K}$ of all compact operators in $\mathcal{B}(L^2(\mathbb{R}))$ is contained in the C^*-algebra $\mathfrak{S}$.

Let $\chi_\pm$ be the characteristic functions of $\mathbb{R}_\pm$, respectively. The operator V given for $\varphi \in L^2(\mathbb{R})$ by

$$(V\varphi)(z) = \frac{\chi_+(z)}{\pi i}\int_{\mathbb{R}} \frac{\varphi(y)\chi_+(y)}{y+z}\,dy - \frac{\chi_-(z)}{\pi i}\int_{\mathbb{R}} \frac{\varphi(y)\chi_-(y)}{y+z}\,dy, \quad z \in \mathbb{R}, \tag{5.3}$$

belongs to the C^*-algebra $\mathfrak{S}$ given by (5.1) because $V = (\chi_+ S_{\mathbb{R}}\chi_+ S_{\mathbb{R}}\chi_+ I + \chi_- S_{\mathbb{R}}\chi_- S_{\mathbb{R}}\chi_- I - I)^{1/2}$ (see also [26, Sections 2.3–2.4] and [5, Lemma 5.3]). The operator V has two fixed singularities: at 0 and ∞.

To each $t \in \mathbb{R}$ we assign the operator $V_t \in \mathcal{B}(L^2(\mathbb{R}))$ with only fixed singularity at t, which is given by

$$(V_t\varphi)(z) := \frac{\chi_t^+(z)}{\pi i}\int_{\mathbb{R}} \frac{\varphi(y)\chi_t^+(y)}{y+z-2t}\,dy - \frac{\chi_t^-(z)}{\pi i}\int_{\mathbb{R}} \frac{\varphi(y)\chi_t^-(y)}{y+z-2t}\,dy, \quad t, z \in \mathbb{R}, \tag{5.4}$$

where χ_t^- and χ_t^+ are, respectively, the characteristic functions of the intervals $(t-1, t)$ and $(t, t+1)$. The operators V_t for all $t \in \mathbb{R}$ belong to the C^*-algebra $\mathfrak{S}$ because $V_0 = \chi_0^+ V \chi_0^+ I + \chi_0^- V \chi_0^- I \in \mathfrak{S}$ along with (5.3) and $V_t = U_{g_{1,t}}^{-1} V_0 U_{g_{1,t}}$, where the map $A \mapsto U_{g_{1,t}}^{-1} A U_{g_{1,t}}$ is a *-automorphism of the C^*-algebra $\mathfrak{S}$.

Let $\mathcal{P}$ consist of all polynomials $\sum_{k=0}^n a_k u^k$ ($a_k \in \mathbb{C}$, $n = 0, 1, \dots$). Then

$$H_{P,t} := P(\chi_t^+ S_{\mathbb{R}} \chi_t^+ I - \chi_t^- S_{\mathbb{R}} \chi_t^- I) V_t \in \mathfrak{S} \quad \text{for all} \quad P \in \mathcal{P} \quad \text{and all} \quad t \in \mathbb{R}. \tag{5.5}$$

As $\mathfrak{S} \subset \mathfrak{A}$ and the map $A \mapsto W^0(A) := \mathcal{F}^{-1} A \mathcal{F}$ is a *-automorphism of the C^*-algebra $\mathfrak{A}$, the operators

$$\widetilde{V}_\tau := W^0(V_\tau) \quad \text{and} \quad \widetilde{H}_{P,\tau} := W^0(H_{P,\tau}) \quad \text{for all} \quad \tau \in \mathbb{R} \quad \text{and all} \quad P \in \mathcal{P} \tag{5.6}$$

belong to the C^*-algebra $\mathfrak{A}$ along with V_τ and $H_{P,\tau}$. We now introduce the C^*-algebra

$$\widetilde{\mathcal{Z}} := \operatorname{alg}\left\{ aI, W^0(b), H_{P,t}, \widetilde{H}_{P,\tau} :\ a, b \in SO^\diamond,\ P \in \mathcal{P},\ t, \tau \in \mathbb{R} \right\} \subset \mathcal{B}(L^2(\mathbb{R})) \tag{5.7}$$

generated by the operators aI, $W^0(b)$, $H_{P,t}$, $\widetilde{H}_{P,\tau}$ with given data. By [29, Lemma 6.1], $\mathcal{K} \subset \mathcal{Z} \subset \widetilde{\mathcal{Z}} \subset \mathfrak{A}$.

Lemma 5.1. *The quotient C^*-algebra $\widetilde{\mathcal{Z}}^\pi := \widetilde{\mathcal{Z}}/\mathcal{K}$ is a central subalgebra of the C^*-algebra $\mathfrak{A}^\pi = \mathfrak{A}/\mathcal{K}$.*

Proof. Applying formula (4.10) in [5], (2.6), Theorem 2.4, the map $A \mapsto \mathcal{F}^{-1} A \mathcal{F}$ and (5.8), we obtain

$$aH_{P,t} \simeq H_{P,t} aI, \quad W^0(b) H_{P,t} \simeq H_{P,t} W^0(b), \quad a\widetilde{H}_{P,t} \simeq \widetilde{H}_{P,t} aI, \quad W^0(b)\widetilde{H}_{P,t} \simeq \widetilde{H}_{P,t} W^0(b) \tag{5.8}$$

for all $a, b \in PSO^\diamond$, all $P \in \mathcal{P}$ and all $t \in \mathbb{R}$. Moreover, $H_{P,t}\widetilde{H}_{P,\tau} \simeq 0$ for all $t, \tau \in \mathbb{R}$, and

$$aW^0(b) \simeq W^0(b) aI \quad \text{for all} \quad (a, b), (b, a) \in SO^\diamond \times PSO^\diamond \quad \text{and all} \quad t \in \mathbb{R} \tag{5.9}$$

by [29, Theorem 4.6]. Finally, it follows from (5.8)–(5.9) that $\widetilde{\mathcal{Z}}^\pi$ is a central subalgebra of $\mathfrak{A}^\pi$. $\square$

Given $t \in \mathbb{R}$, along with (2.4), we define the sets

$$\begin{aligned}
&\widetilde{\Omega}_{\mathbb{R},\infty} := \Omega_{\mathbb{R},\infty} \times \overline{\mathbb{R}}, \quad \widetilde{\Omega}_{\infty,\mathbb{R}} := \Omega_{\infty,\mathbb{R}} \times \overline{\mathbb{R}}, \quad \widehat{\Omega}_{\infty,\infty} = \Omega_{\infty,\infty} \times \{\pm\infty\},\\
&\Delta^\circ_{\mathbb{R},\infty} := \Omega_{\mathbb{R},\infty} \times \mathbb{R}, \quad \dot{\Delta}_{\mathbb{R},\infty} := \Omega_{\mathbb{R},\infty} \times \dot{\mathbb{R}}, \quad \Delta_{\mathbb{R},\infty} := \Omega_{\mathbb{R},\infty} \times \{\infty\},\\
&\Delta^\circ_{\infty,\mathbb{R}} := \Omega_{\infty,\mathbb{R}} \times \mathbb{R}, \quad \dot{\Delta}_{\infty,\mathbb{R}} := \Omega_{\infty,\mathbb{R}} \times \dot{\mathbb{R}}, \quad \Delta_{\infty,\mathbb{R}} := \Omega_{\infty,\mathbb{R}} \times \{\infty\},\\
&\Delta_{t,\infty} := \Omega_{t,\infty} \times \{\infty\}, \quad \Delta_{\infty,t} := \Omega_{\infty,t} \times \{\infty\}, \quad \Delta_{\infty,\infty} := \Omega_{\infty,\infty} \times \{\infty\},
\end{aligned} \tag{5.10}$$

where $\Omega_{t,\infty}$ and $\Omega_{\infty,t}$ are given by (3.5). We also introduce the set $\dot{\mathfrak{N}}_{\xi,\eta}$ given by

$$\dot{\mathfrak{N}}_{\xi,\eta} := \{(\xi,\eta)\} \times \dot{\mathbb{R}} \quad \text{if} \quad (\xi,\eta) \in \Omega_{\mathbb{R},\infty} \cup \Omega_{\infty,\mathbb{R}}, \quad \dot{\mathfrak{N}}_{\xi,\eta} := \{(\xi,\eta)\} \times \{\infty\} \quad \text{if} \quad (\xi,\eta) \in \Omega_{\infty,\infty}. \tag{5.11}$$

Applying Theorems 2.3 and 2.4 to the C^*-algebra $\widetilde{\mathcal{Z}} \subset \mathfrak{A}$, we immediately infer that $\widetilde{\mathcal{Z}}^\pi \cong C(\widetilde{\Delta})$, where

$$\widetilde{\Delta} := \dot{\Delta}_{\mathbb{R},\infty} \cup \dot{\Delta}_{\infty,\mathbb{R}} \cup \Delta_{\infty,\infty} \tag{5.12}$$

is a compact Hausdorff space equipped with topology whose neighborhood base of the points $(\xi,\eta,x) \in \widetilde{\Delta}$ consists of open sets of the form

$$W_{(\xi,\eta,x)} = \begin{cases} \{(\xi,\eta)\} \times (x-\varepsilon, x+\varepsilon) & \text{if} \quad (\xi,\eta,x) \in \Delta^\circ_{\mathbb{R},\infty} \cup \Delta^\circ_{\infty,\mathbb{R}}, \\ \Big(\bigcup\limits_{(\zeta,\theta)\in U_\xi\times U_\eta} \dot{\mathfrak{N}}_{\zeta,\theta}\Big) \setminus \big(U_{\xi,t} \times U_{\eta,\tau} \times [-\varepsilon,\varepsilon]\big) & \text{if} \quad (\xi,\eta,x) \in \Delta_{\mathbb{R},\infty} \cup \Delta_{\infty,\mathbb{R}}, \\ \bigcup\limits_{(\zeta,\theta)\in U_\xi\times U_\eta} \dot{\mathfrak{N}}_{\zeta,\theta} & \text{if} \quad (\xi,\eta,x) \in \Delta_{\infty,\infty}, \end{cases} \tag{5.13}$$

where $\varepsilon > 0$, $\dot{\mathfrak{N}}_{\zeta,\theta}$ are given by (5.11), U_ξ and U_η are open neighborhoods of points $\xi,\eta \in M(SO^\diamond)$, $U_{\xi,t} = U_\xi \cap M_t(SO^\diamond)$ for $t = \xi|_{C(\dot{\mathbb{R}})} \in \dot{\mathbb{R}}$, $U_{\eta,\tau} = U_\eta \cap M_\tau(SO^\diamond)$ for $\tau = \eta|_{C(\dot{\mathbb{R}})} \in \dot{\mathbb{R}}$. This gives the following.

Lemma 5.2. *The maximal ideal space $M(\widetilde{\mathcal{Z}}^\pi)$ of the C^*-algebra $\widetilde{\mathcal{Z}}^\pi$ is homeomorphic to the compact Hausdorff space (5.12) whose topology is given by (5.13).*

By analogy with [5, Lemma 5.4], we obtain from (2.6) the following result.

Lemma 5.3. *Let g be an orientation-preserving diffeomorphism of $\dot{\mathbb{R}}$ onto itself, $t_0 \in \mathbb{R}$ and $v(x) = -i/\cosh(\pi x)$ for $x \in \overline{\mathbb{R}}$. If $g(t_0) = t_0$, then $U_g V_{t_0} \in \mathfrak{A}$ and*

$$\Psi_{\xi,\eta,x}(U_g V_{t_0}) = e^{ix\ln g'(t_0)} v(x) I_2 \quad \textit{if} \quad (\xi,\eta,x) \in \widetilde{\Omega}_{t_0,\infty}, \quad \Psi_{\xi,\eta,x}(U_g V_{t_0}) = 0_{2\times 2} \quad \textit{if} \quad (\xi,\eta,x) \in \widetilde{\Omega} \setminus \widetilde{\Omega}_{t_0,\infty},$$

where $\widetilde{\Omega}$ and $\widetilde{\Omega}_{t_0,\infty}$ are defined, respectively, by (2.5) and (3.5), and $I_2 = \operatorname{diag}\{1,1\}$. Similarly,

$$\Psi_{\xi,\eta,x}(\widetilde{V}_{t_0}) = v(x) I_2 \quad \textit{if} \quad (\xi,\eta,x) \in \widetilde{\Omega}_{\infty,t_0}, \quad \Psi_{\xi,\eta,x}(\widetilde{V}_{t_0}) = 0_{2\times 2} \quad \textit{if} \quad (\xi,\eta,x) \in \widetilde{\Omega} \setminus \widetilde{\Omega}_{\infty,t_0}. \tag{5.14}$$

5.2. Spectral measures and representations of the C^*-algebra $\mathfrak{B}^\pi$

Every orientation-preserving affine mapping $g_{k,h} : \mathbb{R} \to \mathbb{R}$ ($k > 0$, $h \in \mathbb{R}$) extends to the homeomorphism $\widetilde{g}_{k,h}$ of $M(SO^\diamond)$ onto itself by the rule: $a(\widetilde{g}_{k,h}(\xi)) = (a \circ g_{k,h})(\xi)$ for all $a \in SO^\diamond$ and all $\xi \in M(SO^\diamond)$. Since

$$U_{g_{k,h}}(aI)U^{-1}_{g_{k,h}} = (a \circ g_{k,h})I, \quad U_{g_{k,h}} W^0(b) U^{-1}_{g_{k,h}} = W^0(b \circ g_{k^{-1},0}) \quad \text{for all} \quad a, b \in SO^\diamond,$$

we conclude that every shift $g_{k,h} \in G$ induces on $\widetilde{\Delta} = M(\widetilde{\mathcal{Z}}^\pi)$ given by (5.12) the homeomorphism

$$\gamma_{k,h} : \widetilde{\Delta} \to \widetilde{\Delta}, \quad (\xi,\eta,x) \mapsto \big(\widetilde{g}_{k,h}(\xi),\, \widetilde{g}_{k^{-1},0}(\eta),\, x\big), \tag{5.15}$$

where $x \in \dot{\mathbb{R}}$ if $(\xi,\eta) \in \Omega_{\mathbb{R},\infty} \cup \Omega_{\infty,\mathbb{R}}$ and $x \in \{\pm\infty\}$ if $(\xi,\eta) \in \Omega_{\infty,\infty}$.

Let $\mathfrak{R}(\widetilde{\Delta})$ be the σ-algebra of all Borel subsets of $\widetilde{\Delta}$ and let

$$\varphi : \ \mathfrak{B}^\pi \to \mathcal{B}(\mathcal{H}_\varphi), \quad B^\pi \mapsto \varphi(B^\pi) \quad \text{and} \quad P_\varphi : \mathfrak{R}(\widetilde{\Delta}) \to \mathcal{B}(\mathcal{H}_\varphi) \tag{5.16}$$

be, respectively, an isometric representation of the C^*-algebra $\mathfrak{B}^\pi$ in an abstract Hilbert space $\mathcal{H}_\varphi$ and the spectral measure associated with representation φ and the commutative C^*-algebra $\widetilde{\mathcal{Z}}^\pi \subset \mathfrak{B}^\pi$. Put

$$\mathfrak{R}_G(\widetilde{\Delta}) := \big\{\Theta \in \mathfrak{R}(\widetilde{\Delta}) : \ \gamma_{k,h}(\Theta) = \Theta \ \text{ for all } \ g_{k,h} \in G\big\}. \tag{5.17}$$

Applying (5.12) and setting $\dot{\Delta}_{\infty,\mathbb{R}\setminus\{0\}} := (\Omega_{\infty,\mathbb{R}} \setminus \Omega_{\infty,0}) \times \dot{\mathbb{R}}$, we obtain the partition

$$\widetilde{\Delta} := \dot{\Delta}_{\mathbb{R},\infty} \cup \dot{\Delta}_{\infty,\mathbb{R}\setminus\{0\}} \cup \Delta^{\circ}_{\infty,0} \cup \Delta_{\infty,0} \cup \Delta_{\infty,\infty}, \tag{5.18}$$

where $\dot{\Delta}_{\infty,\mathbb{R}\setminus\{0\}}$ and the sets $\dot{\Delta}_{\mathbb{R},\infty}$ and $\Delta^{\circ}_{\infty,0}$ given by (5.10) and (3.5) are open in $\widetilde{\Delta}$, while the sets $\Delta_{\infty,0}$ and $\Delta_{\infty,\infty}$ given by (5.10) are closed in $\widetilde{\Delta}$, and all these sets are in $\mathfrak{R}_G(\widetilde{\Delta})$. Consider the C^*-subalgebras

$$\mathfrak{B}_{\mathbb{R},\infty}, \ \mathfrak{B}_{\infty,\mathbb{R}\setminus\{0\}}, \ \mathfrak{B}^{\circ}_{\infty,0}, \ \mathfrak{B}_{\infty,0}, \ \mathfrak{B}_{\infty,\infty} \tag{5.19}$$

of $\varphi(\mathfrak{B}^\pi)$ associated to decomposition (5.18). These algebras of the form

$$\mathfrak{B}(\Delta) := \text{alg}\,\big\{P_\varphi(\Delta)\varphi(A^\pi), \ P_\varphi(\Delta)\varphi(U^\pi_g) : \ A \in \mathfrak{A}, \ g \in G\big\}$$

are generated by the operators $P_\varphi(\Delta)\varphi(A^\pi)$ $(A \in \mathfrak{A})$ and $P_\varphi(\Delta)\varphi(U^\pi_g)$ $(g \in G)$, where Δ is one of the sets on the right of (5.18) and $P_\varphi(\Delta) \neq 0$ for these Δ by [24, Subsection 5.1] and [30, Lemma 6.1]. This gives the following abstract Fredholm criterion in terms of invertibility of operators in the C^*-algebras (5.19).

Theorem 5.4. *An operator B in the C^*-algebra $\mathfrak{B}$ given by (1.3) is Fredholm on the space $L^2(\mathbb{R})$ if and only if the following five assertions are fulfilled:*

(i) *the operator $B_{\mathbb{R},\infty} := P_\varphi(\dot{\Delta}_{\mathbb{R},\infty})\varphi(B^\pi)$ is invertible on the Hilbert space $\mathcal{H}_{\varphi,\mathbb{R},\infty} := P_\varphi(\dot{\Delta}_{\mathbb{R},\infty})\mathcal{H}_\varphi$;*
(ii) *the operator $B_{\infty,\mathbb{R}\setminus\{0\}} := P_\varphi(\dot{\Delta}_{\infty,\mathbb{R}\setminus\{0\}})\varphi(B^\pi)$ is invertible on the Hilbert space $\mathcal{H}_{\varphi,\infty,\mathbb{R}\setminus\{0\}} := P_\varphi(\dot{\Delta}_{\infty,\mathbb{R}\setminus\{0\}})\mathcal{H}_\varphi$;*
(iii) *the operator $B^{\circ}_{\infty,0} := P_\varphi(\Delta^{\circ}_{\infty,0})\varphi(B^\pi)$ is invertible on the Hilbert space $\mathcal{H}^{\circ}_{\varphi,\infty,0} := P_\varphi(\Delta^{\circ}_{\infty,0})\mathcal{H}_\varphi$;*
(iv) *the operator $B_{\infty,0} := P_\varphi(\Delta_{\infty,0})\varphi(B^\pi)$ is invertible on the Hilbert space $\mathcal{H}_{\varphi,\infty,0} := P_\varphi(\Delta_{\infty,0})\mathcal{H}_\varphi$;*
(v) *the operator $B_{\infty,\infty} := P_\varphi(\Delta_{\infty,\infty})\varphi(B^\pi)$ is invertible on the Hilbert space $\mathcal{H}_{\varphi,\infty,\infty} := P_\varphi(\Delta_{\infty,\infty})\mathcal{H}_\varphi$.*

Along with the abstract Hilbert space $\mathcal{H}_\varphi$, we consider the Hilbert space $\mathcal{H}_\phi := \bigoplus_{(\xi,\eta,x)\in\widetilde{\Omega}} \mathbb{C}^2$, where $\widetilde{\Omega}$ is given by (2.5), and introduce the representation ϕ and the spectral measure P_ϕ by

$$\phi : \ \mathfrak{A}^\pi \to \mathcal{B}(\mathcal{H}_\phi), \quad A^\pi \mapsto \bigoplus_{(\xi,\eta,x)\in\widetilde{\Omega}} \Psi_{\xi,\eta,x}(A)I, \quad P_\phi : \mathfrak{R}(\widetilde{\Delta}) \to \mathcal{B}(\mathcal{H}_\phi), \tag{5.20}$$

where ϕ is an isometric representation of $\mathfrak{A}^\pi$ in the Hilbert space $\mathcal{H}_\phi$ by Corollary 2.5, and the spectral measure P_ϕ is associated with representation ϕ and the central algebra $\widetilde{\mathcal{Z}}^\pi$. Below we need the subspaces

$$\mathcal{H}_{\phi,\mathbb{R},\infty} := P_\phi(\dot{\Delta}_{\mathbb{R},\infty})\mathcal{H}_\phi, \quad \mathcal{H}_{\phi,\infty,\mathbb{R}} := P_\phi(\dot{\Delta}_{\infty,\mathbb{R}})\mathcal{H}_\phi \quad \text{of} \quad \mathcal{H}_\phi, \tag{5.21}$$

which are isometrically isomorphic to the Hilbert spaces $\bigoplus_{(\xi,\eta,x)\in\widetilde{\Omega}_{\mathbb{R},\infty}} \mathbb{C}^2$ and $\bigoplus_{(\xi,\eta,x)\in\widetilde{\Omega}_{\infty,\mathbb{R}}} \mathbb{C}^2$, where the sets $\widetilde{\Omega}_{\mathbb{R},\infty}$ and $\widetilde{\Omega}_{\infty,\mathbb{R}}$ are given by (5.10).

6. The C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty}$

6.1. *The C^*-algebra $\mathfrak{A}_{\mathbb{R},\infty}$*

Along with the C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty} = \mathfrak{B}(\dot{\Delta}_{\mathbb{R},\infty})$, we consider its C^*-subalgebras

$$\mathfrak{A}_{\mathbb{R},\infty} := \big\{P_\varphi(\dot{\Delta}_{\mathbb{R},\infty})\varphi(A^\pi) : A \in \mathfrak{A}\big\}, \quad \widetilde{\mathcal{Z}}_{\mathbb{R},\infty} := \big\{P_\varphi(\dot{\Delta}_{\mathbb{R},\infty})\varphi(A^\pi) : A \in \widetilde{\mathcal{Z}}\big\}. \tag{6.1}$$

Lemma 5.1 implies that $\widetilde{\mathcal{Z}}_{\mathbb{R},\infty}$ is a central subalgebra of $\mathfrak{A}_{\mathbb{R},\infty}$. By [22, Lemmas 5.1, 5.2 and Corollary 5.3],

$$\widetilde{\mathcal{Z}}_{\mathbb{R},\infty} \cong C(\widetilde{\Delta}_{\mathbb{R},\infty}), \quad \widetilde{\Delta}_{\mathbb{R},\infty} := \dot{\Delta}_{\mathbb{R},\infty} \cup \Delta_{\infty,\infty}, \tag{6.2}$$

where the compact Hausdorff space $\widetilde{\Delta}_{\mathbb{R},\infty}$ is equipped with topology induced by the Gelfand topology of $\widetilde{\Delta}$ (see (5.13)). Applying (5.5) and [31, (5.24)], we infer that

$$U_g a U_g^{-1} = (a \circ g)I, \quad U_g H_{P,t} \simeq H_{P,g^{-1}(t)} U_g, \quad U_g S_{\mathbb{R}} U_g^{-1} = S_{\mathbb{R}} \tag{6.3}$$

for all $a \in PSO^\diamond$, all $g \in G$, all $t \in \mathbb{R}$ and all $P \in \mathcal{P}$, where $a \circ g \in PSO^\diamond$ (cf. [4, Lemma 4.2]). Further,

$$U_{g_{k,h}} W^0(b) U_{g_{k,h}}^{-1} = W^0(b \circ g_{k^{-1},0}) \ \text{ for all } \ b \in PSO^\diamond, \ k \in \mathbb{R}_+, \ h \in \mathbb{R}. \tag{6.4}$$

Similarly to [5, Theorem 6.4], we conclude that

$$(b \circ g)(\eta^\pm) = b(\eta^\pm) \ \text{ for all } \ b \in PSO^\diamond, \ g \in G, \ \eta \in M_\infty(SO^\diamond). \tag{6.5}$$

Taking any function $b \in PSO^\diamond$, we infer from [19, Lemma 7.1] that $[W^0(b - b_\infty^- \chi_- - b_\infty^+ \chi_+)]_{\mathbb{R},\infty} = 0$, where functions $b_\infty^\pm \in SO_\infty$ are such that $b_\infty^\pm(\eta) = b(\eta^\mp)$ for every $\eta \in M_\infty(SO^\diamond)$, and $\chi_\pm$ are the characteristic functions of $\mathbb{R}_\pm$. Hence, it follows from (6.4) and (6.5) that

$$[U_g W^0(b) U_g^{-1}]_{\mathbb{R},\infty} = [W^0(b)]_{\mathbb{R},\infty} \quad \text{for all } \ b \in PSO^\diamond \ \text{ and all } \ g \in G. \tag{6.6}$$

Since (5.14) implies that $(\widetilde{H}_{P,\tau})_{\mathbb{R},\infty} = 0$ for all $P \in \mathcal{P}$ and all $\tau \in \mathbb{R}$, we infer from (6.3) and (6.6) that for each $g \in G$ the mapping $\alpha_g : \ A_{\mathbb{R},\infty} \mapsto (U_g)_{\mathbb{R},\infty} A_{\mathbb{R},\infty} (U_g)^{-1}_{\mathbb{R},\infty}$ is a *-automorphism of the C^*-algebra $\mathfrak{A}_{\mathbb{R},\infty}$ and its central C^*-subalgebra $\widetilde{\mathcal{Z}}_{\mathbb{R},\infty}$. These *-automorphisms induce on the maximal ideal space $\widetilde{\Delta}_{\mathbb{R},\infty}$ of $\widetilde{\mathcal{Z}}_{\mathbb{R},\infty}$ given by (6.2) the group of homeomorphisms $\beta_g : \widetilde{\Delta}_{\mathbb{R},\infty} \to \widetilde{\Delta}_{\mathbb{R},\infty}$, $(\xi, \eta, x) \mapsto (\widetilde{g}(\xi), \eta, x)$ for all $g \in G$, where $\xi \mapsto \widetilde{g}(\xi)$ is the homeomorphism on $M(SO^\diamond)$ given by

$$a(\widetilde{g}(\xi)) = (a \circ g)(\xi) \quad \text{for all } \ a \in SO^\diamond \ \text{ and all } \ \xi \in M(SO^\diamond). \tag{6.7}$$

Letting $\mathfrak{R}_G(\widetilde{\Delta}_{\mathbb{R},\infty}) := \mathfrak{R}_G(\widetilde{\Delta}) \cap \widetilde{\Delta}_{\mathbb{R},\infty}$, we infer from [24] that $P_\varphi(\Theta) B_{\mathbb{R},\infty} = B_{\mathbb{R},\infty} P_\varphi(\Theta)$ for all $B_{\mathbb{R},\infty} \in \mathfrak{B}_{\mathbb{R},\infty}$ and all $\Theta \in \mathfrak{R}_G(\widetilde{\Delta}_{\mathbb{R},\infty})$. For each $g \in G$, the homeomorphism $(\xi, \eta) \mapsto (\widetilde{g}(\xi), \eta)$ sends the set $\Omega_{t,\infty}$ onto the set $\Omega_{g(t),\infty}$. Then, similarly to [7, Lemma 4.2], we get the following.

Lemma 6.1. *For every $t \in \mathbb{R}$ and every $g \in G$, $P_\varphi(\Delta^\circ_{t,\infty})(U_g)_{\mathbb{R},\infty} = (U_g)_{\mathbb{R},\infty} P_\varphi(\Delta^\circ_{g(t),\infty})$.*

Along with the C^*-algebra $\mathfrak{A}_{\mathbb{R},\infty}$ given by (6.1), we consider the C^*-algebra

$$\widehat{\mathfrak{A}}_{\mathbb{R},\infty} := P_\phi(\dot{\Delta}_{\mathbb{R},\infty}) \phi(\mathfrak{A}^\pi) = \big\{ \widehat{A}_{\mathbb{R},\infty} := P_\phi(\dot{\Delta}_{\mathbb{R},\infty}) \phi(A^\pi) : \ A \in \mathfrak{A} \big\}.$$

Theorem 6.2. *For each $t \in \mathbb{R}$, the mapping $P_\varphi(\Delta^\circ_{t,\infty}) A_{\mathbb{R},\infty} \mapsto P_\phi(\Delta^\circ_{t,\infty}) \widehat{A}_{\mathbb{R},\infty}$ is a C^*-algebra isomorphism of the C^*-algebra $\mathfrak{A}_{t,\infty} := P_\varphi(\Delta^\circ_{t,\infty}) \mathfrak{A}_{\mathbb{R},\infty}$ onto the C^*-algebra $\widehat{\mathfrak{A}}_{t,\infty} := P_\phi(\Delta^\circ_{t,\infty}) \widehat{\mathfrak{A}}_{\mathbb{R},\infty}$.*

Proof. By [5, Lemma 3.5], for the open Borel set $\Delta^\circ_{t,\infty}$ and each $A \in \mathfrak{A}$, we obtain

$$\| P_\varphi(\Delta^\circ_{t,\infty}) A_{\mathbb{R},\infty} \|_{\mathcal{B}(\mathcal{H}_{\varphi,\mathbb{R},\infty})} = \sup_{Z \in \widetilde{\mathcal{Z}}(\Delta^\circ_{t,\infty})} \| \varphi(Z^\pi A^\pi) \|_{\mathcal{B}(\mathcal{H}_{\varphi,\mathbb{R},\infty})}, \tag{6.8}$$

$$\| P_\phi(\Delta^\circ_{t,\infty}) \widehat{A}_{\mathbb{R},\infty} \|_{\mathcal{B}(\mathcal{H}_{\phi,\mathbb{R},\infty})} = \sup_{Z \in \widetilde{\mathcal{Z}}(\Delta^\circ_{t,\infty})} \| \phi(Z^\pi A^\pi) \|_{\mathcal{B}(\mathcal{H}_{\phi,\mathbb{R},\infty})}, \tag{6.9}$$

where $\widetilde{\mathcal{Z}}(\Delta^\circ_{t,\infty})$ consists of the operators $Z \in \widetilde{\mathcal{Z}}$ for which the Gelfand transform of Z^π is a real-valued function in $C(\widetilde{\Delta})$ with values in $[0,1]$ and support in $\dot{\Delta}_{t,\infty}$. Since φ and ϕ are isometric representations of the C^*-algebra $\mathfrak{A}^\pi$, we conclude that the right-hand sides of (6.8) and (6.9) are equal, and therefore

$$\|P_\varphi(\Delta^\circ_{t,\infty})A_{\mathbb{R},\infty}\|_{\mathcal{B}(\mathcal{H}_{\varphi,\mathbb{R},\infty})} = \|P_\phi(\Delta^\circ_{t,\infty})\widehat{A}_{\mathbb{R},\infty}\|_{\mathcal{B}(\mathcal{H}_{\phi,\mathbb{R},\infty})}$$

for all $A \in \mathfrak{A}$, which implies the assertion of the theorem. □

Theorem 6.3. *For every $t \in \mathbb{R}$, the map*

$$\mathrm{Sym}^\circ_{t,\infty}: \ \mathfrak{A}_{t,\infty} \to \widehat{\mathfrak{A}}_{t,\infty} \to \mathcal{B}(l^2(\widetilde{\Omega}_{t,\infty}, \mathbb{C}^2)), \quad P_\varphi(\Delta^\circ_{t,\infty})A_{\mathbb{R},\infty} \mapsto P_\phi(\Delta^\circ_{t,\infty})\widehat{A}_{\mathbb{R},\infty} \mapsto \Psi(A)|_{\widetilde{\Omega}_{t,\infty}} I \tag{6.10}$$

is an isometric C^*-algebra homomorphism. For every $t \in \mathbb{R}$ and each $A \in \mathfrak{A}$, the operator $P_\varphi(\Delta^\circ_{t,\infty})A_{\mathbb{R},\infty}$ is invertible on the Hilbert space $P_\varphi(\Delta^\circ_{t,\infty})\mathcal{H}_{\varphi,\mathbb{R},\infty}$ if and only if $\det[\Psi_{\xi,\eta,x}(A)] \neq 0$ for all $(\xi,\eta,x) \in \widetilde{\Omega}_{t,\infty}$.

Proof. The C^*-algebra $\widehat{\mathfrak{A}}_{t,\infty}$ is *-isomorphic to the C^*-subalgebra $\mathcal{B}_1 \subset \mathcal{B}(l^2(\Delta^\circ_{t,\infty}, \mathbb{C}^2))$ of multiplication operators by bounded matrix functions $F_A : \Delta^\circ_{t,\infty} \to \mathbb{C}^{2\times 2}$, $(\xi,\eta,x) \mapsto \Psi_{\xi,\eta,x}(A)$ for all $A \in \mathfrak{A}$, which, in its turn, is *-isomorphic to the C^*-subalgebra $\mathcal{B}_2 \subset \mathcal{B}(l^2(\widetilde{\Omega}_{t,\infty}, \mathbb{C}^2))$ of multiplication operators by bounded matrix functions $\widetilde{F}_A : \widetilde{\Omega}_{t,\infty} \to \mathbb{C}^{2\times 2}$, $(\xi,\eta,x) \mapsto \Psi_{\xi,\eta,x}(A)$ because the matrix functions $x \mapsto \Psi_{\xi,\eta,x}(A)$ are continuous on $\overline{\mathbb{R}}$ for every $(\xi,\eta) \in \Omega_{t,\infty}$ in view of (2.6). Hence, $\|F_A I\|_{\mathcal{B}(l^2(\Delta^\circ_{t,\infty},\mathbb{C}^2))} = \|\widetilde{F}_A I\|_{\mathcal{B}(l^2(\widetilde{\Omega}_{t,\infty},\mathbb{C}^2))}$ for all $A \in \mathfrak{A}$. Thus, the map $P_\phi(\Delta^\circ_{t,\infty})\widehat{A}_{\mathbb{R},\infty} \mapsto F_A I \mapsto \widetilde{F}_A I$ is an isometric *-homomorphism of $\widehat{\mathfrak{A}}_{t,\infty}$ into $\mathcal{B}(l^2(\widetilde{\Omega}_{t,\infty}, \mathbb{C}^2))$. Involving Theorem 6.2, we see that the map (6.10) is an isometric C^*-algebra homomorphism. It remains to apply (2.8) for all $(\xi,\eta,x) \in \widetilde{\Omega}_{t,\infty}$. □

6.2. The homomorphism Φ_1

Fix $t_0 \in \mathbb{R}$. The set of all G-orbits of points $t \in \dot{\mathbb{R}}$ consists of only two G-orbits: the one-point orbit $G(\infty) = \{\infty\}$ and the non-countable orbit $\omega := G(t_0)$. Let $\mathfrak{H} = \mathfrak{H}_{t_0}$ be the closed two-sided ideal of the C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty}$ generated by the operator $(V_{t_0})_{\mathbb{R},\infty}$, where the operator V_{t_0} is given by (5.4).

Consider the dense subalgebra $\mathfrak{B}^0$ of $\mathfrak{B}$ consisting of all operators of the form

$$\sum_{i=1}^{n} T_{i,1}\,T_{i,2}\dots T_{i,j_i} + K \quad (T_{i,k} \in \{aI, W^0(b), U_g:\ a,b \in PSO^0,\ g \in G\},\ n, j_i \in \mathbb{N},\ K \in \mathcal{K}),$$

where PSO^0 is the non-closed algebra consisting of all functions in $PSO^\diamond$ with finite sets of discontinuities. Analogously we define the non-closed subalgebra $\mathfrak{A}^0$ of $\mathfrak{A}$ generated by aI and $W^0(b)$, where $a,b \in PSO^0$.

Given $B \in \mathfrak{B}^0$, the operator $B_{\mathbb{R},\infty}$ can be written in the form

$$B_{\mathbb{R},\infty} = \sum_{g\in F}(A_g)_{\mathbb{R},\infty}(U_g)_{\mathbb{R},\infty} \quad (F \subset G \ \text{ is a finite set, } \ A_g \in \mathfrak{A}^0 \ \text{ for all } \ g \in F). \tag{6.11}$$

Let $\mathfrak{A}^0_{\mathbb{R},\infty} := \{A_{\mathbb{R},\infty} : A \in \mathfrak{A}^0\}$ and $\mathfrak{B}^0_{\mathbb{R},\infty} := \{B_{\mathbb{R},\infty} : B \in \mathfrak{B}^0\}$. For any set $\Gamma \subset \mathbb{R}$, we define the sets

$$\Omega_{\Gamma,\infty} := \bigcup_{t\in\Gamma} M_t(SO^\diamond) \times M_\infty(SO^\diamond), \quad \Delta^\circ_{\Gamma,\infty} := \Omega_{\Gamma,\infty} \times \mathbb{R}, \quad \dot{\Delta}_{\Gamma,\infty} := \Omega_{\Gamma,\infty} \times \dot{\mathbb{R}}, \quad \Delta_{\Gamma,\infty} := \Omega_{\Gamma,\infty} \times \{\infty\}.$$

Lemma 6.4. *If $B \in \mathfrak{B}^0$, Γ is a finite set of $\mathbb{R}$ and $V_\Gamma := \sum_{t\in\Gamma} V_t \in \mathfrak{A}$, then*

$$\|B_{\mathbb{R},\infty}(V_\Gamma)_{\mathbb{R},\infty}\|_{\mathfrak{B}_{\mathbb{R},\infty}} = \|\Phi_1(B)\Pi_\Gamma vI\|_{\mathcal{B}(\mathcal{H}_{\mathbb{R},\infty,1})} \quad \text{and} \quad \Pi_\Gamma := \mathrm{diag}\,\{\chi_\Gamma(t)I_2\}_{t\in\mathbb{R}}, \tag{6.12}$$

where χ_Γ is the characteristic function of Γ, $\Phi_1(B)$ is given by (3.9)–(3.10), $v(x) = -i/\cosh(\pi x)$ for $x \in \mathbb{R}$.

Proof. Fix a finite set $\Gamma \subset \mathbb{R}$ and consider the operator $B_{\mathbb{R},\infty} \in \mathfrak{B}^0_{\mathbb{R},\infty}$ given by (6.11). Take the finite subset $\widetilde{\Gamma} := \{g^{-1}(t) : \ t \in \Gamma, \ g \in F\}$ of $\mathbb{R}$. As $U_g V_t \simeq V_{g^{-1}(t)} U_g$ and $A_g V_t \simeq V_t A_g$ for $t \in \dot{\mathbb{R}}, \ g \in G, \ A_g \in \mathfrak{A}$, we get

$$(BV_\Gamma)_{\mathbb{R},\infty} = \sum_{g\in F}\sum_{t\in\Gamma}(A_g)_{\mathbb{R},\infty}(U_g)_{\mathbb{R},\infty}(V_t)_{\mathbb{R},\infty} = \sum_{g\in F}\sum_{t\in\Gamma}(V_{g^{-1}(t)})_{\mathbb{R},\infty}(A_g)_{\mathbb{R},\infty}(U_g)_{\mathbb{R},\infty}. \tag{6.13}$$

Making use of Theorem 6.3, we deduce for every $A \in \mathfrak{A}$ and every $t \in \mathbb{R}$ that

$$\|(AV_t)_{\mathbb{R},\infty}\|_{\mathfrak{B}_{\mathbb{R},\infty}} = \|P_\varphi(\Delta^\circ_{t,\infty})(AV_t)_{\mathbb{R},\infty}\|_{\mathcal{B}(P_\varphi(\Delta^\circ_{t,\infty})\mathcal{H}_{\varphi,\mathbb{R},\infty})} = \|[\Psi(AV_t)]|_{\widetilde{\Omega}_{t,\infty}} I\|_{\mathcal{B}(l^2(\widetilde{\Omega}_{t,\infty},\mathbb{C}^2))}. \tag{6.14}$$

Applying Lemma 5.3 and (6.14), we infer from the second equality in (6.13) similarly to [5, Subsection 8.1] and [5, Lemma 10.5] that $P_\varphi(\dot{\Delta}_{\mathbb{R}\setminus\widetilde{\Gamma},\infty})\,(BV_\Gamma)_{\mathbb{R},\infty} = 0$ and $P_\varphi(\Delta_{\widetilde{\Gamma},\infty} \cup \Delta_{\infty,\infty})\,(BV_\Gamma)_{\mathbb{R},\infty} = 0$. Hence, because

$$P_\varphi(\Delta^\circ_{\widetilde{\Gamma},\infty}) + P_\varphi(\dot{\Delta}_{\mathbb{R}\setminus\widetilde{\Gamma},\infty}) + P_\varphi(\Delta_{\widetilde{\Gamma},\infty} \cup \Delta_{\infty,\infty}) = P_\varphi(\widetilde{\Delta}_{\mathbb{R},\infty}) = I_{\mathbb{R},\infty}$$

in view of the partition $\widetilde{\Delta}_{\mathbb{R},\infty} = \Delta^\circ_{\widetilde{\Gamma},\infty} \cup \dot{\Delta}_{\mathbb{R}\setminus\widetilde{\Gamma},\infty} \cup \big(\Delta_{\widetilde{\Gamma},\infty} \cup \Delta_{\infty,\infty}\big)$, we conclude from Lemma 6.1 that

$$\|(BV_\Gamma)_{\mathbb{R},\infty}\|_{\mathfrak{B}_{\mathbb{R},\infty}} = \|P_\varphi(\Delta^\circ_{\widetilde{\Gamma},\infty})(BV_\Gamma)_{\mathbb{R},\infty}\|_{\mathcal{B}(\mathcal{H}_{\varphi,\mathbb{R},\infty})} = \|P_\varphi(\Delta^\circ_{\widetilde{\Gamma},\infty})(BV_\Gamma)_{\mathbb{R},\infty}P_\varphi(\Delta^\circ_{\Gamma,\infty})\|_{\mathcal{B}(\mathcal{H}_{\varphi,\mathbb{R},\infty})}. \tag{6.15}$$

Let G_B be the subgroup of G generated by the finite set F in (6.11) and let $\mathcal{O}_{B,\Gamma}$ be the finite set of G_B-orbits ω of all points $t \in \Gamma$. Then $\Gamma_\omega := \Gamma \cap \omega$ is a finite subset of $\omega \in \mathcal{O}_{B,\Gamma}$. Since

$$\|(BV_\Gamma)_{\mathbb{R},\infty}\|_{\mathfrak{B}_{\mathbb{R},\infty}} = \max_{\omega\in\mathcal{O}_{B,\Gamma}} \|(BV_{\Gamma_\omega})_{\mathbb{R},\infty}\|_{\mathfrak{B}_{\mathbb{R},\infty}}, \quad \|\Phi_1(B)\Pi_\Gamma vI\|_{\mathcal{B}(\mathcal{H}_{\mathbb{R},\infty,1})} = \max_{\omega\in\mathcal{O}_{B,\Gamma}} \|\Phi_1(B)\Pi_{\Gamma_\omega} vI\|_{\mathcal{B}(\mathcal{H}_{\mathbb{R},\infty,1})},$$

we only need to prove (6.12) for $(V_\Gamma)_{\mathbb{R},\infty}$ replaced by any $(V_{\Gamma_\omega})_{\mathbb{R},\infty}$. In what follows we assume without loss of generality that $\Gamma, \widetilde{\Gamma} \subset \omega$ and $\omega = G_B(t_0)$. As the group G_B is at most countable, so is the G_B-orbit ω.

We now define the Hilbert space $\mathcal{H}_{t_0} := \bigoplus_{t\in\omega} P_\varphi(\Delta^\circ_{t_0,\infty})\mathcal{H}_{\varphi,\mathbb{R},\infty}$ and the isomorphism

$$\sigma_\omega : \ P_\varphi(\Delta^\circ_{\omega,\infty})\mathcal{H}_{\varphi,\mathbb{R},\infty} \to \bigoplus_{t\in\omega} P_\varphi(\Delta^\circ_{t_0,\infty})\mathcal{H}_{\varphi,\mathbb{R},\infty}, \quad P_\varphi(\Delta^\circ_{\omega,\infty})f \mapsto \big(P_\varphi(\Delta^\circ_{t_0,\infty})(U_{g_t})_{\mathbb{R},\infty} f\big)_{t\in\omega}, \tag{6.16}$$

where $f \in \mathcal{H}_{\varphi,\mathbb{R},\infty}$ and $g_t = g_{1,t-t_0} \in Y_{t_0,t}$ for every $t \in \omega$. Taking the isometric C^*-algebra homomorphism

$$\Upsilon_\omega : \ \mathcal{B}(P_\varphi(\Delta^\circ_{\omega,\infty})\mathcal{H}_{\varphi,\mathbb{R},\infty}) \to \mathcal{B}\Big(\bigoplus_{t\in\omega} P_\varphi(\Delta^\circ_{t_0,\infty})\mathcal{H}_{\varphi,\mathbb{R},\infty}\Big), \quad T \mapsto \sigma_\omega T \sigma_\omega^{-1}, \tag{6.17}$$

with σ_ω given by (6.16), and applying for $t, \tau \in \omega$, $g \in G$ and $s \in \Gamma$ the relations

$$(A_{g,t})_{\mathbb{R},\infty} := (U_{g_t} A_g U_{g_t}^{-1})_{\mathbb{R},\infty} \in \mathfrak{A}_{\mathbb{R},\infty}, \quad (U_{g_t} U_g U_{g_\tau}^{-1})_{\mathbb{R},\infty} = (U_{\widetilde{g}_{t,\tau}})_{\mathbb{R},\infty}, \quad U_{g_\tau} V_s U_{g_\tau}^{-1} \simeq V_{g_\tau^{-1}(s)},$$

where $\widetilde{g}_{t,\tau} = g_t g g_\tau^{-1} \in Y_{t_0,t_0}$ if $g(t) = \tau$ (see (3.4)), we infer from (6.13) and (6.17) that

$$\begin{aligned}
&\Upsilon_\omega\big(P_\varphi(\Delta^\circ_{\widetilde{\Gamma},\infty})(BV_\Gamma)_{\mathbb{R},\infty}P_\varphi(\Delta^\circ_{\Gamma,\infty})\big) = \Upsilon_\omega\Big(P_\varphi(\Delta^\circ_{\widetilde{\Gamma},\infty})\sum_{g\in F}\sum_{s\in\Gamma}(A_g U_g V_s)_{\mathbb{R},\infty}P_\varphi(\Delta^\circ_{\Gamma,\infty})\Big) \\
&= \Pi_\omega^{\widetilde{\Gamma}}\Big(P_\varphi(\Delta^\circ_{t_0,\infty})\sum_{g\in F}\sum_{s\in\Gamma}(U_{g_t} A_g U_{g_t}^{-1})_{\mathbb{R},\infty}(U_{\widetilde{g}_{t,\tau}})_{\mathbb{R},\infty}(U_{g_\tau} V_s U_{g_\tau}^{-1})_{\mathbb{R},\infty}P_\varphi(\Delta^\circ_{t_0,\infty})\Big)_{t,\tau\in\omega}\Pi_\omega^{\Gamma} \\
&= \Pi_\omega^{\widetilde{\Gamma}}\Big(\sum_{g\in F}\delta_g(t,\tau)P_\varphi(\Delta^\circ_{t_0,\infty})(A_{g,t}U_{\widetilde{g}_{t,\tau}}V_{t_0})_{\mathbb{R},\infty}\Big)_{t,\tau\in\omega}\Pi_\omega^{\Gamma},
\end{aligned} \tag{6.18}$$

where $\Pi_\omega^{\widetilde{\Gamma}} := \operatorname{diag}\{\chi_{\widetilde{\Gamma}}(t)\}_{t\in\omega} I$ and $\Pi_\omega^{\Gamma} := \operatorname{diag}\{\chi_\Gamma(t)\}_{t\in\omega} I$. It follows from (6.17) that

$$\big\|P_\varphi(\Delta^\circ_{\widetilde{\Gamma},\infty})(BV_\Gamma)_{\mathbb{R},\infty}P_\varphi(\Delta^\circ_{\Gamma,\infty})\big\|_{\mathcal{B}(\mathcal{H}_{\varphi,\mathbb{R},\infty})} = \big\|\Upsilon_\omega\big(P_\varphi(\Delta^\circ_{\widetilde{\Gamma},\infty})(BV_\Gamma)_{\mathbb{R},\infty}P_\varphi(\Delta^\circ_{\Gamma,\infty})\big)\big\|_{\mathcal{B}(\mathcal{H}_{t_0})}. \tag{6.19}$$

Hence, taking into account the finiteness of the sets $\Gamma, \widetilde{\Gamma} \subset \omega$ in (6.19), we infer from (6.12), (6.19) (6.18) and Lemma 5.3, by analogy with Theorem 6.3, that

$$\begin{aligned}
&\big\|P_\varphi(\Delta^\circ_{\widetilde{\Gamma},\infty})(BV_\Gamma)_{\mathbb{R},\infty}P_\varphi(\Delta^\circ_{\Gamma,\infty})\big\|_{\mathcal{B}(\mathcal{H}_{\varphi,\mathbb{R},\infty})} = \Big\|\Pi_\omega^{\widetilde{\Gamma}}\Big(\sum_{g\in F}\delta_g(t,\tau)P_\varphi(\Delta^\circ_{t_0,\infty})(A_{g,t}U_{\widetilde{g}_{t,\tau}}V_{t_0})_{\mathbb{R},\infty}\Big)_{t,\tau\in\omega}\Pi_\omega^{\Gamma}\Big\|_{\mathcal{B}(\mathcal{H}_{t_0})}\\
&= \Big\|\Pi_\omega^{\widetilde{\Gamma}}\Big(\sum_{g\in F}\delta_g(t,\tau)\big[\Psi(A_{g,t}U_{\widetilde{g}_{t,\tau}}V_{t_0})\big]\big|_{\Delta^\circ_{t_0,\infty}}I\Big)_{t,\tau\in\omega}\Pi_\omega^{\Gamma}\Big\|_{\mathcal{B}(l^2(\Delta^\circ_{t_0,\infty},l^2(\omega,\mathbb{C}^2)))}\\
&= \Big\|\Pi_{\widetilde{\Gamma}}\Big(\sum_{g\in F}\delta_g(t,\tau)\big[\Psi(A_{g,t}U_{\widetilde{g}_{t,\tau}}V_{t_0})\big]\big|_{\Delta^\circ_{t_0,\infty}}I\Big)_{t,\tau\in\mathbb{R}}\Pi_\Gamma I\Big\|_{\mathcal{B}(\mathcal{H}_{\mathbb{R},\infty,1})} = \|\Phi_1(B)\Pi_\Gamma vI\|_{\mathcal{B}(\mathcal{H}_{\mathbb{R},\infty,1})}.
\end{aligned} \tag{6.20}$$

Finally, combining (6.15), (6.19) and (6.20), we obtain the first equality in (6.12). □

Lemma 6.4 is the key to proving the continuity of the algebraic homomorphism Φ_1. Applying this lemma and directly following the proof of [9, Theorem 8.3], we establish the following estimate.

Theorem 6.5. *If $B \in \mathfrak{B}^0$, then*

$$\|\Phi_1(B)\|_{\mathcal{B}(\mathcal{H}_{\mathbb{R},\infty,1})} \le \|B_{\mathbb{R},\infty}\|_{\mathfrak{B}_{\mathbb{R},\infty}} \le \|B^\pi\|. \tag{6.21}$$

Making use of Theorem 6.5 and (3.9)–(3.10), we obtain the following.

Theorem 6.6. *The algebraic homomorphism Φ_1 given by* (3.9)–(3.10) *extends to a representation $\Phi_1 : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_{\mathbb{R},\infty,1})$, such that* (6.21) *holds for every $B \in \mathfrak{B}$. Given an operator $B \in \mathfrak{B}$, the operator $\Phi_1(B)$ is invertible on the Hilbert space $\mathcal{H}_{\mathbb{R},\infty,1}$ if and only if condition* (i) *of Theorem* 3.2 *holds.*

By (3.9)–(3.10) and Theorems 6.5 and 6.6, the map

$$\Phi_{\mathbb{R},\infty,1} : \ \mathfrak{B}_{\mathbb{R},\infty} \to \mathcal{B}(\mathcal{H}_{\mathbb{R},\infty,1}), \quad B_{\mathbb{R},\infty} \mapsto \Phi_1(B), \tag{6.22}$$

is a homomorphism. Since the set $\big\{(BV_\Gamma)_{\mathbb{R},\infty} : B \in \mathfrak{B}^0,\ \Gamma \text{ runs through finite subsets of } \mathbb{R}\big\}$ is dense in the ideal $\mathfrak{H} = \mathfrak{H}_{t_0}$ of the C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty}$, we immediately obtain the following result from Lemma 6.4.

Theorem 6.7. *The restriction of the homomorphism* (6.22) *to the closed two-sided ideal $\mathfrak{H} = \mathfrak{H}_{t_0}$ is an isometric *-isomorphism of $\mathfrak{H}$ onto the closed two-sided ideal $\Phi_{\mathbb{R},\infty,1}(\mathfrak{H})$ of the C^*-algebra $\Phi_1(\mathfrak{B}) \subset \mathcal{B}(\mathcal{H}_{\mathbb{R},\infty,1})$.*

6.3. Invertibility in the C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty}/\mathfrak{H}$ and the homomorphism Φ_2

Given $t \in \mathbb{R}$, let $\mathcal{J}_t$ denote the closed two-sided ideal of the C^*-algebra $\mathfrak{S}$ generated by the operator V_t and the ideal $\mathcal{K}$, and let $\mathcal{J}$ stand for the closed two-sided ideal of $\mathfrak{S}$ generated by all the ideals $\mathcal{J}_t$ for $t \in \dot{\mathbb{R}}$.

Given $b \in PSO^\diamond$, fix functions $b_\infty^\pm \in SO_\infty$ such that $b_\infty^\pm(\eta) = b(\eta^\mp)$ for all $\eta \in M_\infty(SO^\diamond)$. We then get

$$[W^0(b)]_{\mathbb{R},\infty} = [W^0(b_\infty^-\chi_- + b_\infty^+\chi_+)]_{\mathbb{R},\infty} = [W^0(b_\infty^-)]_{\mathbb{R},\infty}[P_+]_{\mathbb{R},\infty} + [W^0(b_\infty^+)]_{\mathbb{R},\infty}[P_-]_{\mathbb{R},\infty}, \tag{6.23}$$

where $\chi_\pm$ are the characteristic functions of $\mathbb{R}_\pm$, $P_\pm := (I \pm S_\mathbb{R})/2$, and $S_\mathbb{R}$ is given by (5.2). Analogously, for every $t \in \mathbb{R}$ and every $a \in PSO^0$ there exist functions $a_t^\pm \in SO_t$ such that $a_t^\pm(\xi^\pm) = a(\xi^\pm)$ for every

$\xi \in M_t(SO^\diamond)$. Applying (6.23), we see that for all $a, b \in PSO^\diamond$ the local behavior of the commutators $[aW^0(b) - W^0(b)aI]_{\mathbb{R},\infty} \in \mathfrak{A}_{\mathbb{R},\infty}$ at the point $(t, \infty) \in \dot{\mathbb{R}} \times \dot{\mathbb{R}}$ for $t \in \mathbb{R}$ coincides with the local behavior at this point of the commutators $[(a_t^- \chi_t^- + a_t^+ \chi_t^+)I,\ (W^0(b_\infty^-)P_+ + W^0(b_\infty^+)P_-)]_{\mathbb{R},\infty} \in \mathfrak{A}_{\mathbb{R},\infty}$, where χ_t^- and χ_t^+ are, respectively, the characteristic functions of the intervals $(t-1, t)$ and $(t, t+1)$. Since

$$
\begin{aligned}
&[(a_t^- \chi_t^- + a_t^+ \chi_t^+)I, (W^0(b_\infty^-)P_+ + W^0(b_\infty^+)P_-)] \\
&\quad \simeq a_t^+ W^0(b_\infty^+)[\chi_t^+ I, P_-] + a_t^+ W^0(b_\infty^-)[\chi_t^+ I, P_+] + a_t^- W^0(b_\infty^+)[\chi_t^- I, P_-] + a_t^- W^0(b_\infty^-)[\chi_t^- I, P_+]
\end{aligned}
$$

and since the commutators $[\chi_t^\pm I, S_\mathbb{R}]$ belong, respectively, to the ideal $\mathcal{J}_t + \mathcal{J}_{t\pm1}$ similarly to [5, Lemma 5.3] (also see [26, Sections 2.3–2.4]), we conclude that for all $a, b \in PSO^\diamond$ the commutators $[aW^0(b) - W^0(b)aI]_{\mathbb{R},\infty}$ belong to the closed two-sided ideal $\mathcal{J}_{\mathbb{R},\infty}$ of the C^*-algebra $\mathfrak{A}_{\mathbb{R},\infty}$, which is generated by all operators $(V_t)_{\mathbb{R},\infty}$ $(t \in \mathbb{R})$. Thus, the quotient C^*-algebra $\mathfrak{A}_{\mathbb{R},\infty}/\mathcal{J}_{\mathbb{R},\infty}$ is commutative.

Since the closed two-sided ideal $\mathfrak{H} = \mathfrak{H}_{t_0}$ of the C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty}$ is generated by the operator $(V_{t_0})_{\mathbb{R},\infty}$, we conclude that $\mathcal{J}_{\mathbb{R},\infty} \subset \mathfrak{H}$ and, moreover, $\mathfrak{H} \cap \mathfrak{A}_{\mathbb{R},\infty} = \mathcal{J}_{\mathbb{R},\infty}$. Consider the quotient C^*-algebras

$$
\mathfrak{B}_{\mathbb{R},\infty,\mathfrak{H}} := \mathfrak{B}_{\mathbb{R},\infty}/\mathfrak{H} \quad \text{and} \quad \mathfrak{A}_{\mathbb{R},\infty,\mathfrak{H}} := (\mathfrak{A}_{\mathbb{R},\infty} + \mathfrak{H})/\mathfrak{H} \cong \mathfrak{A}_{\mathbb{R},\infty}/\mathcal{J}_{\mathbb{R},\infty}. \tag{6.24}
$$

The C^*-algebra $\mathfrak{A}_{\mathbb{R},\infty,\mathfrak{H}}$ is commutative along with $\mathfrak{A}_{\mathbb{R},\infty}/\mathcal{J}_{\mathbb{R},\infty}$ and is generated by the cosets

$$
[aW^0(b)]_{\mathbb{R},\infty,\mathfrak{H}} = [aW^0(b_\infty^-)P_+]_{\mathbb{R},\infty} + [aW^0(b_\infty^+)P_-]_{\mathbb{R},\infty} + \mathfrak{H} \tag{6.25}
$$

for all $a, b \in PSO^\diamond$, where the functions $b_\infty^\pm \in SO_\infty$ possess the property $b_\infty^\pm(\eta) = b(\eta^\mp)$ for all $\eta \in M_\infty(SO^\diamond)$. In particular, for given $a_1, a_2, b_1, b_2 \in PSO^\diamond$, we obtain

$$
[a_1 W^0(b_1)]_{\mathbb{R},\infty,\mathfrak{H}} [a_2 W^0(b_2)]_{\mathbb{R},\infty,\mathfrak{H}} = [a_1 a_2 W^0(b_1 b_2)]_{\mathbb{R},\infty,\mathfrak{H}}. \tag{6.26}
$$

By (6.25)–(6.26), the maximal ideal space of the C^*-algebra $\mathfrak{A}_{\mathbb{R},\infty,\mathfrak{H}}$ is homeomorphic to the compact set

$$
\widehat{\mathfrak{N}}_{\mathbb{R},\infty} := (M(SO^\diamond) \times \{0,1\}) \times (M_\infty(SO^\diamond) \times \{0,1\}) \tag{6.27}
$$

whose topology is induced by the product topology of $M(PSO^\diamond) \times M(PSO^\diamond)$, and the topology of $M(PSO^\diamond)$ is given by (2.3). The Gelfand transform $\mathfrak{A}_{\mathbb{R},\infty,\mathfrak{H}} \to C(\widehat{\mathfrak{N}}_{\mathbb{R},\infty})$, $A_{\mathbb{R},\infty,\mathfrak{H}} \mapsto A_{\mathbb{R},\infty,\mathfrak{H}}(\cdot,\cdot,\cdot,\cdot)$ is defined on the generators $A_{\mathbb{R},\infty,\mathfrak{H}} = [aW^0(b)]_{\mathbb{R},\infty,\mathfrak{H}}$ $(a, b \in PSO^\diamond)$ of the C^*-algebra $\mathfrak{A}_{\mathbb{R},\infty,\mathfrak{H}}$ by

$$
A_{\mathbb{R},\infty,\mathfrak{H}}(\xi, \mu, \eta, \nu) = a(\xi, \mu) b(\eta, \nu) \quad \text{for all} \quad (\xi, \mu, \eta, \nu) \in \widehat{\mathfrak{N}}_{\mathbb{R},\infty}, \tag{6.28}
$$

where $a(\xi, 0) = a(\xi^-)$, $a(\xi, 1) = a(\xi^+)$, $b(\eta, 0) = b(\eta^-)$, $b(\eta, 1) = b(\eta^+)$.

Applying the local-trajectory method described in Section 4, we will obtain here an invertibility criterion for the cosets $B_{\mathbb{R},\infty,\mathfrak{H}} \in \mathfrak{B}_{\mathbb{R},\infty,\mathfrak{H}}$, where $B_{\mathbb{R},\infty,\mathfrak{H}} := B_{\mathbb{R},\infty} + \mathfrak{H}$ for $B_{\mathbb{R},\infty} \in \mathfrak{B}_{\mathbb{R},\infty}$.

By (6.3) and (6.4), we conclude that for every $g \in G$ the map $\widetilde{\alpha}_g :\ A_{\mathbb{R},\infty,\mathfrak{H}} \mapsto (U_g A U_g^{-1})_{\mathbb{R},\infty,\mathfrak{H}}$ is a *-automorphism of the commutative C^*-algebra $\mathfrak{A}_{\mathbb{R},\infty,\mathfrak{H}}$. Indeed, for all $a, b \in PSO^\diamond$ we infer in view of (6.3)–(6.5) that $[U_g a W^0(b) U_g^{-1}]_{\mathbb{R},\infty,\mathfrak{H}} = [(a \circ g) W^0(b)]_{\mathbb{R},\infty,\mathfrak{H}}$. Hence the C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty,\mathfrak{H}}$ is the closure of the algebra $\mathfrak{B}^0_{\mathbb{R},\infty,\mathfrak{H}}$ consisting of the cosets $\sum_{g\in F} (A_g)_{\mathbb{R},\infty,\mathfrak{H}} (U_g)_{\mathbb{R},\infty,\mathfrak{H}}$, where $(A_g)_{\mathbb{R},\infty} \in \mathfrak{A}^0_{\mathbb{R},\infty}$ and g runs through finite subsets $F \subset G$. For each $g \in G$, the *-automorphism $\widetilde{\alpha}_g$ of the C^*-algebra $\mathfrak{A}_{\mathbb{R},\infty,\mathfrak{H}}$ induces on the maximal ideal space $\widehat{\mathfrak{N}}_{\mathbb{R},\infty}$ defined by (6.27) the homeomorphism

$$
\widetilde{\beta}_g : (\xi, \mu, \eta, \nu) \mapsto (g(\xi), \mu, \eta, \nu) \quad \text{for all} \quad (\xi, \mu, \eta, \nu) \in \widehat{\mathfrak{N}}_{\mathbb{R},\infty} \tag{6.29}
$$

by the rule $A_{\mathbb{R},\infty,\mathfrak{H}}\big[\widetilde{\beta}_g(\xi, \mu, \eta, \nu)\big] = \big[\widetilde{\alpha}_g(A_{\mathbb{R},\infty,\mathfrak{H}})\big](\xi, \mu, \eta, \nu)$ for all $(\xi, \mu, \eta, \nu) \in \widehat{\mathfrak{N}}_{\mathbb{R},\infty}$ and all $g \in G$.

Since the homeomorphism $\xi \mapsto g(\xi)$ given by (6.7) sends the fibers $M_t(SO^\diamond)$ onto the fibers $M_{g(t)}(SO^\diamond)$ for all $t \in \dot{\mathbb{R}}$, it follows from the proof of [5, Theorem 6.4] that $g(\xi) = \xi$ for every $\xi \in M_\infty(SO^\diamond)$. This in view of (6.29) gives the following.

Lemma 6.8. $\widehat{\mathfrak{N}}_{\infty,\infty} := (M_\infty(SO^\diamond) \times \{0,1\}) \times (M_\infty(SO^\diamond) \times \{0,1\})$ *is the set of all common fixed points of all homeomorphisms* $\widetilde{\beta}_g$ $(g \in G)$ *on the compact set* $\widehat{\mathfrak{N}}_{\mathbb{R},\infty}$.

Since G acts topologically freely on $\dot{\mathbb{R}}$, we easily deduce from Lemma 6.8 and the Gelfand topology on $\widehat{\mathfrak{N}}_{\mathbb{R},\infty}$ that the group G acts topologically freely on $\widehat{\mathfrak{N}}_{\mathbb{R},\infty}$ as well. Moreover, since the open set

$$\mathfrak{N}_{\mathbb{R},\infty} := \bigcup_{t\in\mathbb{R}} \big(M_t(SO^\diamond) \times \{0,1\} \times M_\infty(SO^\diamond) \times \{0,1\}\big)$$

is dense in $\widehat{\mathfrak{N}}_{\mathbb{R},\infty}$, we see that for every nonempty open set $W \subset \widehat{\mathfrak{N}}_{\mathbb{R},\infty}$ and every finite set $G_0 \subset G$ there exists a point $(\xi_0, \mu_0, \eta_0, \nu_0) \in W \cap \mathfrak{N}_{\mathbb{R},\infty}$ such that $\widetilde{\beta}_g(\xi_0, \mu_0, \eta_0, \nu_0) \neq (\xi_0, \mu_0, \eta_0, \nu_0)$ for all $g \in G_0 \setminus \{e\}$. Due to this fact and the amenability of the solvable group G, we infer that all conditions of the local-trajectory method (see [22], [24]) for the C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty,\mathfrak{H}}$ are fulfilled.

Since $G(t_0) = \mathbb{R}$ for $t_0 \in \mathbb{R}$, it follows from (6.7) and (6.29) that the set

$$\mathfrak{N}_{t_0,\infty} := M_{t_0}(SO^\diamond) \times \{0,1\} \times M_\infty(SO^\diamond) \times \{0,1\} \tag{6.30}$$

contains exactly one point in each G-orbit of every point in $\mathfrak{N}_{\mathbb{R},\infty}$. Consider the Hilbert space $l^2(G)$ consisting of all complex-valued functions defined on G and having at most countable sets of non-zero values, and with every point $(\xi, \mu, \eta, \nu) \in \mathfrak{N}_{t_0,\infty}$ we associate the representation

$$\Pi_{\xi,\mu,\eta,\nu} : \ \mathfrak{B}_{\mathbb{R},\infty,\mathfrak{H}} \to \mathcal{B}(l^2(G)), \quad B_{\mathbb{R},\infty,\mathfrak{H}} \mapsto \widetilde{B}_{\xi,\mu,\eta,\nu} := \Pi_{\xi,\mu,\eta,\nu}(B_{\mathbb{R},\infty,\mathfrak{H}}) \tag{6.31}$$

given for $B_{\mathbb{R},\infty,\mathfrak{H}} = \sum_{g\in F}(A_g)_{\mathbb{R},\infty,\mathfrak{H}}(U_g)_{\mathbb{R},\infty,\mathfrak{H}}$, where F is a finite subset of G and $(A_g)_{\mathbb{R},\infty,\mathfrak{H}} \in \mathfrak{A}^0_{\mathbb{R},\infty,\mathfrak{H}}$, by

$$\big(\widetilde{B}_{\xi,\mu,\eta,\nu} f\big)(h) = \sum_{g\in F} \big(\big[\widetilde{\alpha}_h((A_g)_{\mathbb{R},\infty,\mathfrak{H}})\big](\xi,\mu,\eta,\nu)\big) f(hg) \quad \text{for all} \ \ f \in l^2(G) \ \ \text{and all} \ \ h \in G. \tag{6.32}$$

Then Theorem 4.2 immediately implies the following invertibility criterion by analogy with [8, Theorem 2.7].

Theorem 6.9. *A coset* $B_{\mathbb{R},\infty,\mathfrak{H}} \in \mathfrak{B}_{\mathbb{R},\infty,\mathfrak{H}}$ *is invertible in the* C^**-algebra* $\mathfrak{B}_{\mathbb{R},\infty,\mathfrak{H}}$ *if and only if the operators* $\widetilde{B}_{\xi,\mu,\eta,\nu}$ *are invertible on the space* $l^2(G)$ *for all* $(\xi, \mu, \eta, \nu) \in \mathfrak{N}_{t_0,\infty}$ *and*

$$\sup_{(\xi,\mu,\eta,\nu)\in\mathfrak{N}_{t_0,\infty}} \Big\| \big(\widetilde{B}_{\xi,\mu,\eta,\nu}\big)^{-1} \Big\|_{\mathcal{B}(l^2(G))} < \infty.$$

Applying Theorem 6.9 to the coset $B_{\mathbb{R},\infty,\mathfrak{H}} B^*_{\mathbb{R},\infty,\mathfrak{H}} \in \mathfrak{B}_{\mathbb{R},\infty,\mathfrak{H}}$ and using spectral radii $r(\cdot)$, we get

$$\begin{aligned} \|B_{\mathbb{R},\infty,\mathfrak{H}}\| &= \|B_{\mathbb{R},\infty,\mathfrak{H}} B^*_{\mathbb{R},\infty,\mathfrak{H}}\|^{1/2} = \big[r\big(B_{\mathbb{R},\infty,\mathfrak{H}} B^*_{\mathbb{R},\infty,\mathfrak{H}}\big)\big]^{1/2} \\ &= \sup_{(\xi,\mu,\eta,\nu)\in\mathfrak{N}_{t_0,\infty}} \big[r\big(\widetilde{B}_{\xi,\mu,\eta,\nu}\widetilde{B}^*_{\xi,\mu,\eta,\nu}\big)\big]^{1/2} = \sup_{(\xi,\mu,\eta,\nu)\in\mathfrak{N}_{t_0,\infty}} \|\widetilde{B}_{\xi,\mu,\eta,\nu}\|_{\mathcal{B}(l^2(G))}. \end{aligned}$$

Thus, we obtained the following assertion for the C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty,\mathfrak{H}}$ given by (6.24).

Corollary 6.10. *The representation*

$$\bigoplus_{(\xi,\mu,\eta,\nu)\in\mathfrak{N}_{t_0,\infty}} \Pi_{\xi,\mu,\eta,\nu}:\ \mathfrak{B}_{\mathbb{R},\infty,\mathfrak{H}} \to \mathcal{B}\Big(\bigoplus_{(\xi,\mu,\eta,\nu)\in\mathfrak{N}_{t_0,\infty}} l^2(G)\Big),$$

where $\Pi_{\xi,\mu,\eta,\nu}$ and $\mathfrak{N}_{t_0,\infty}$ are given by (6.31)–(6.32) and (6.30), is an isometric C^*-algebra homomorphism.

Along with C^*-algebra homomorphisms $\Pi_{\xi,\mu,\eta,\nu}$ defined for $(\xi,\mu,\eta,\nu)\in\mathfrak{N}_{t_0,\infty}$ by (6.31) and (6.32), we consider the C^*-algebra homomorphisms $\Pi_{\xi,\mu,\eta,\nu}:\mathfrak{B}_{\mathbb{R},\infty,\mathfrak{H}}\to\mathcal{B}(l^2(G))$ given by (6.31) and (6.32) for every $(\xi,\mu,\eta,\nu)\in(M_\infty(SO^\diamond)\times\{0,1\})^2$, where the expressions $\big[\widetilde{\alpha}_h((A_g)_{\mathbb{R},\infty,\mathfrak{H}})\big](\xi,\mu,\eta,\nu)$ in (6.32) are replaced by $\big[(A_g)_{\mathbb{R},\infty,\mathfrak{H}}\big](\xi,\mu,\eta,\nu)$. We then infer the following corollary from Theorem 6.9.

Corollary 6.11. *If a coset $B_{\mathbb{R},\infty,\mathfrak{H}}\in\mathfrak{B}_{\mathbb{R},\infty,\mathfrak{H}}$ is invertible in the C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty,\mathfrak{H}}$, then the operators $\widetilde{B}_{\xi,\mu,\eta,\nu}=\Pi_{\xi,\mu,\eta,\nu}(B_{\mathbb{R},\infty,\mathfrak{H}})$ are invertible on the space $l^2(G)$ for all $(\xi,\mu,\eta,\nu)\in(M_\infty(SO^\diamond)\times\{0,1\})^2$.*

The Hilbert space $\mathcal{H}_{\mathbb{R},\infty,2}:=l^2(\widehat{\Omega}_{t_0,\infty},l^2(G,\mathbb{C}^2))$ given by (3.7) is isometrically isomorphic to the space $\bigoplus_{(\xi,\mu,\eta,\nu)\in\mathfrak{N}_{t_0,\infty}} l^2(G)$. Identifying these Hilbert spaces, we conclude that the algebraic *-homomorphism

$$\Phi_{\mathbb{R},\infty,2}:\ \mathfrak{B}^0_{\mathbb{R},\infty}\to\mathcal{B}(\mathcal{H}_{\mathbb{R},\infty,2}),\quad B_{\mathbb{R},\infty}\mapsto\Phi_2(B), \tag{6.33}$$

defined initially on the generators of $\mathfrak{B}_{\mathbb{R},\infty}$, where Φ_2 is given by (3.9) and (3.11), can be rewritten for $B_{\mathbb{R},\infty}\in\mathfrak{B}^0_{\mathbb{R},\infty}$ and $(\xi,\eta,\pm\infty)\in\widehat{\Omega}_{t_0,\infty}$ in the following equivalent form:

$$\begin{aligned}
&[\mathrm{Sym}_2(B)](\xi,\eta,+\infty)I\sim\mathrm{diag}\{\Pi_{\xi,1,\eta,1}(B_{\mathbb{R},\infty,\mathfrak{H}}),\ \Pi_{\xi,0,\eta,0}(B_{\mathbb{R},\infty,\mathfrak{H}})\},\\
&[\mathrm{Sym}_2(B)](\xi,\eta,-\infty)I\sim\mathrm{diag}\{\Pi_{\xi,1,\eta,0}(B_{\mathbb{R},\infty,\mathfrak{H}}),\ \Pi_{\xi,0,\eta,1}(B_{\mathbb{R},\infty,\mathfrak{H}})\}.
\end{aligned} \tag{6.34}$$

Hence, we infer from (6.34) and Corollary 6.10 that

$$\begin{aligned}
&\|\Phi_{\mathbb{R},\infty,2}(B_{\mathbb{R},\infty})\|_{\mathcal{B}(\mathcal{H}_{\mathbb{R},\infty,2})}=\sup_{(\xi,\eta,x)\in\widehat{\Omega}_{t_0,\infty}}\|[\mathrm{Sym}_2(B)](\xi,\eta,x)I\|_{\mathcal{B}(l^2(G,\mathbb{C}^2))}\\
&=\sup_{(\xi,\mu,\eta,\nu)\in\mathfrak{N}_{t_0,\infty}}\|\Pi_{\xi,\mu,\eta,\nu}(B_{\mathbb{R},\infty,\mathfrak{H}})\|_{\mathcal{B}(l^2(G))}=\|B_{\mathbb{R},\infty,\mathfrak{H}}\|_{\mathfrak{B}_{\mathbb{R},\infty,\mathfrak{H}}}\le\|B_{\mathbb{R},\infty}\|_{\mathfrak{B}_{\mathbb{R},\infty}}
\end{aligned}$$

for all $B_{\mathbb{R},\infty}\in\mathfrak{B}_{\mathbb{R},\infty}$. This immediately implies the following.

Theorem 6.12. *The algebraic *-homomorphism (6.33) given on generators of the C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty}$ by formulas (3.11) extends by continuity to a representation $\Phi_{\mathbb{R},\infty,2}:\mathfrak{B}_{\mathbb{R},\infty}\to\mathcal{B}(\mathcal{H}_{\mathbb{R},\infty,2})$ such that*

$$\|\Phi_{\mathbb{R},\infty,2}(B_{\mathbb{R},\infty})\|_{\mathcal{B}(\mathcal{H}_{\mathbb{R},\infty,2})}=\|B_{\mathbb{R},\infty,\mathfrak{H}}\|_{\mathfrak{B}_{\mathbb{R},\infty,\mathfrak{H}}}\le\|B_{\mathbb{R},\infty}\|_{\mathfrak{B}_{\mathbb{R},\infty}} \tag{6.35}$$

for all $B_{\mathbb{R},\infty}\in\mathfrak{B}_{\mathbb{R},\infty}$, and hence $\ker\Phi_{\mathbb{R},\infty,2}=\mathfrak{H}$.

By (6.34) and (6.35), $\Phi_{\mathbb{R},\infty,2}(\mathfrak{B}_{\mathbb{R},\infty})\cong\mathfrak{B}_{\mathbb{R},\infty,\mathfrak{H}}$. Hence, because $\Phi_2(B)=\Phi_{\mathbb{R},\infty,2}(B_{\mathbb{R},\infty})$ for all $B\in\mathfrak{B}$, we infer from Theorems 6.9 and 6.12 the following.

Theorem 6.13. *Given an operator $B\in\mathfrak{B}$, the operator $\Phi_2(B)$ is invertible on the Hilbert space $\mathcal{H}_{\mathbb{R},\infty,2}$ if and only if condition* (ii) *of Theorem 3.2 holds. The map Φ_2 given by (3.7), (3.9) and (3.11) is a C^*-algebra homomorphism, and* $\ker\Phi_2=\{B\in\mathfrak{B}:B_{\mathbb{R},\infty}\in\mathfrak{H}\}$.

6.4. Invertibility in the C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty}$

Theorem 6.14. *An operator $B_{\mathbb{R},\infty} \in \mathfrak{B}_{\mathbb{R},\infty}$ is invertible in the C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty}$ if and only if conditions* (i)–(ii) *of Theorem 3.2 are fulfilled.*

Proof. It follows from Theorem 6.12 and Theorem 6.7 that $\ker \Phi_{\mathbb{R},\infty,1} \cap \ker \Phi_{\mathbb{R},\infty,2} = \ker \Phi_{\mathbb{R},\infty,1} \cap \mathfrak{H} = \{0\}$. Hence, the map $\Phi_{\mathbb{R},\infty,1} \oplus \Phi_{\mathbb{R},\infty,2}$ is a faithful representation of the C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty}$ in the Hilbert space $\mathcal{H}_{\mathbb{R},\infty,1} \oplus \mathcal{H}_{\mathbb{R},\infty,2}$. Consequently, an operator $B_{\mathbb{R},\infty} \in \mathfrak{B}_{\mathbb{R},\infty}$ is invertible in $\mathfrak{B}_{\mathbb{R},\infty}$ if and only if for $i = 1, 2$ the operator $\Phi_i(B) = \Phi_{\mathbb{R},\infty,i}(B_{\mathbb{R},\infty})$ is invertible on the Hilbert spaces $\mathcal{H}_{\mathbb{R},\infty,i}$, which is equivalent to the fulfillment of conditions (i)–(ii) of Theorem 3.2 by Theorems 6.6 and 6.13. $\square$

7. The C^*-algebra $\mathfrak{B}_{\infty,\mathbb{R}}$

7.1. Another form of the C^*-algebra $\mathfrak{B}_{\infty,\mathbb{R}}$

Along with $\mathfrak{A} = \text{alg}(aI, W^0(b) : \ a, b \in PSO^\diamond)$, we consider the C^*-algebras

$$\widetilde{\mathfrak{A}} := \text{alg}\left\{aI, W^0(b), U_{g_{1,h}} : \ a, b \in PSO^\diamond, \ h \in \mathbb{R}\right\}, \quad \widetilde{\mathfrak{A}}_{\infty,\mathbb{R}} := P_\varphi(\dot{\Delta}_{\infty,\mathbb{R}})\varphi(\widetilde{\mathfrak{A}}^\pi).$$

Let $e_h(x) := e^{ihx}$ for all $x \in \mathbb{R}$ and all $h \in \mathbb{R}$. Since $U_{g_{1,h}} = W^0(e_{-h})$ for all $h \in \mathbb{R}$, we conclude that the C^*-algebra $\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}}$ is generated by the operators $A_{\infty,\mathbb{R}} := P_\varphi(\dot{\Delta}_{\infty,\mathbb{R}})\varphi(A^\pi)$, where $A \in \{aI, W^0(b), W^0(e_{-h}) : a, b \in PSO^\diamond, h \in \mathbb{R}\}$. Then $\mathfrak{B} = \text{alg}\left\{A, U_{g_{k,0}} : \ A \in \widetilde{\mathfrak{A}}, \ k \in \mathbb{R}_+\right\}$, and $\mathfrak{B}_{\infty,\mathbb{R}} := P_\varphi(\dot{\Delta}_{\infty,\mathbb{R}})\varphi(\mathfrak{B}^\pi)$.

According to the equality $U_{g_{1,h}} = W^0(e_{-h})$, we define the 2×2 matrices

$$\Psi_{\xi,\eta,x}(U_{g_{1,h}}) := e^{-ih\eta} I_2 \ \text{ for all } \ h \in \mathbb{R} \ \text{ and all } \ (\xi, \eta, x) \in \widetilde{\Omega}_{\infty,\mathbb{R}}, \tag{7.1}$$

where $e^{ih\eta} = e^{ih\tau}$ for $\eta \in M_\tau(SO^\diamond)$. This allows us to extend the mappings $\Psi_{\xi,\eta,x}$ for every $(\xi, \eta, x) \in \widetilde{\Omega}_{\infty,\mathbb{R}}$ to the C^*-algebra $\widetilde{\mathfrak{A}}$ properly containing $\mathfrak{A}$. Consider the space $\mathcal{H}_{\phi,\infty,\mathbb{R}}$ given by (5.21) and the mapping

$$\begin{aligned} &\psi_{\infty,\mathbb{R}} : \widetilde{\mathfrak{A}}_{\infty,\mathbb{R}} \to \bigoplus_{(\xi,\eta,x) \in \widetilde{\Omega}_{\infty,\mathbb{R}}} \Psi_{\xi,\eta,x}(\widetilde{\mathfrak{A}}) I \subset \mathcal{B}(\mathcal{H}_{\phi,\infty,\mathbb{R}}), \\ &\psi_{\infty,\mathbb{R}}\Big(\sum_{h\in F} (A_h U_{g_{1,h}})_{\infty,\mathbb{R}}\Big) := \bigoplus_{(\xi,\eta,x) \in \widetilde{\Omega}_{\infty,\mathbb{R}}} \sum_{h \in F} \Psi_{\xi,\eta,x}(A_h) \Psi_{\xi,\eta,x}(U_{g_{1,h}}) I, \end{aligned} \tag{7.2}$$

where F is a finite subset of $\mathbb{R}$, $A_h \in \mathfrak{A}$ for $h \in F$, and the matrices $\Psi_{\xi,\eta,x}(A_h)$ and $\Psi_{\xi,\eta,x}(U_{g_{1,h}})$ for $h \in F$ are defined by (2.6) and (7.1), respectively. Similarly to [30, Theorem 10.1], the mapping

$$P_\varphi(\dot{\Delta}_{\infty,\mathbb{R}})\varphi\Big(\sum_{h\in F} A_h^\pi U_{g_{1,h}}^\pi\Big) \mapsto \psi_{\infty,\mathbb{R}}\Big(\sum_{h\in F} (A_h U_{g_{1,h}})_{\infty,\mathbb{R}}\Big),$$

where F is a finite subset of $\mathbb{R}$, extends to a C^*-algebra isomorphism

$$\psi_{\infty,\mathbb{R}} : \widetilde{\mathfrak{A}}_{\infty,\mathbb{R}} \to \widehat{\mathfrak{A}}_{\infty,\mathbb{R}} := \psi_{\infty,\mathbb{R}}(\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}}) \subset \mathcal{B}(\mathcal{H}_{\phi,\infty,\mathbb{R}}). \tag{7.3}$$

Since the C^*-algebras $\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}}$ and $\widehat{\mathfrak{A}}_{\infty,\mathbb{R}}$ consist of the operators

$$A_{\infty,\mathbb{R}} = P_\varphi(\dot{\Delta}_{\infty,\mathbb{R}})\varphi(A^\pi) \ \text{ for } \ A \in \widetilde{\mathfrak{A}}, \quad \widehat{A}_{\infty,\mathbb{R}} := \psi_{\infty,\mathbb{R}}(A_{\infty,\mathbb{R}}) = \bigoplus_{(\xi,\eta,x) \in \widetilde{\Omega}_{\infty,\mathbb{R}}} \Psi_{\xi,\eta,x}(A) I \ \text{ for } \ A \in \widetilde{\mathfrak{A}}, \tag{7.4}$$

we deduce the following in view of the C^*-algebra isomorphism (7.3).

Theorem 7.1. *For every $A \in \widetilde{\mathfrak{A}}$, the operator $A_{\infty,\mathbb{R}} \in \widetilde{\mathfrak{A}}_{\infty,\mathbb{R}}$ is invertible on the space $\mathcal{H}_{\varphi,\infty,\mathbb{R}}$ if and only if for all $(\xi,\eta,x) \in \widetilde{\Omega}_{\infty,\mathbb{R}}$ the operators $\Psi_{\xi,\eta,x}(A)I$ are invertible on the space $\mathbb{C}^2$ and*

$$\sup_{(\xi,\eta,\mu)\in\widetilde{\Omega}_{\infty,\mathbb{R}}} \left\|(\Psi_{\xi,\eta,x}(A)I)^{-1}\right\|_{\mathcal{B}(\mathbb{C}^2)} < \infty.$$

For the C^*-algebra $B(\widetilde{\Omega}_{\infty,\mathbb{R}}, \mathbb{C}^{2\times 2})$ of bounded functions $\widetilde{\Omega}_{\infty,\mathbb{R}} \to \mathbb{C}^{2\times 2}$, Theorem 7.1 gives the following.

Corollary 7.2. *The mapping $\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}} \to B(\widetilde{\Omega}_{\infty,\mathbb{R}}, \mathbb{C}^{2\times 2})$, $P_\varphi(\dot{\Delta}_{\infty,\mathbb{R}})\varphi(A^\pi) \mapsto \Psi(A)|_{\widetilde{\Omega}_{\infty,\mathbb{R}}}$ is an isometric C^*-algebra homomorphism.*

7.2. The spectral measure associated with the C^*-algebra $\mathfrak{B}_{\infty,\mathbb{R}}$

Let $\widetilde{\mathcal{Z}}_{\infty,\mathbb{R}} := P_\varphi(\dot{\Delta}_{\infty,\mathbb{R}})\varphi(\widetilde{\mathcal{Z}}^\pi)$, where the C^*-algebra $\widetilde{\mathcal{Z}} \subset \mathcal{B}(L^2(\mathbb{R}))$ is given by (5.7). Since the quotient C^*-algebra $\widetilde{\mathcal{Z}}^\pi$ is a central subalgebra of $\mathfrak{A}^\pi$ and

$$U_{g_{1,h}}(aI)U_{g_{1,h}}^{-1} = (a \circ g_{1,h})I, \quad U_{g_{1,h}}W^0(b)U_{g_{1,h}}^{-1} = W^0(b), \quad (a \circ g_{1,h})(\xi^\pm) = a(\xi^\pm)$$

for all $a, b \in PSO^\diamond$, all $h \in \mathbb{R}$ and all $\xi \in M_\infty(SO^\diamond)$, we conclude that $\widetilde{\mathcal{Z}}_{\infty,\mathbb{R}}$ is a central C^*-subalgebra of $\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}}$. The maximal ideal space $M(\widetilde{\mathcal{Z}}_{\infty,\mathbb{R}})$ of $\widetilde{\mathcal{Z}}_{\infty,\mathbb{R}}$ is homeomorphic to the compact Hausdorff space

$$\widetilde{\Delta}_{\infty,\mathbb{R}} := \dot{\Delta}_{\infty,\mathbb{R}} \cup \Delta_{\infty,\infty} \tag{7.5}$$

equipped with topology induced by the Gelfand topology of $\widetilde{\Delta}$ (see (5.13)).

For all $a, b \in PSO^\diamond$, all $h \in \mathbb{R}$, all $k \in \mathbb{R}_+$ and all $P \in \mathcal{P}$, we infer that

$$\begin{gathered} U_{g_{k,0}}(aI)U_{g_{k,0}}^{-1} = (a \circ g_{k,0})I, \quad U_{g_{k,0}}W^0(b)U_{g_{k,0}}^{-1} = W^0(b \circ g_{k^{-1},0}), \quad U_{g_{k,0}}S_{\mathbb{R}}U_{g_{k,0}}^{-1} = S_{\mathbb{R}}, \\ U_{g_{k,0}}W^0(e_h)U_{g_{k,0}}^{-1} = W^0(e_{h/k}), \quad U_{g_{k,0}}\widetilde{V}_0U_{g_{k,0}}^{-1} \simeq \widetilde{V}_0, \quad U_{g_{k,0}}\widetilde{H}_{P,0}U_{g_{k,0}}^{-1} \simeq \widetilde{H}_{P,0}, \end{gathered} \tag{7.6}$$

where $b \circ g_{k^{-1},0} \in PSO^\diamond$ along with b (cf. [4, Lemma 4.2]). Hence, for $k \in \mathbb{R}_+$,

$$(U_{g_{k,0}})_{\infty,\mathbb{R}}\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}}((U_{g_{k,0}})_{\infty,\mathbb{R}})^{-1} = \widetilde{\mathfrak{A}}_{\infty,\mathbb{R}}, \quad (U_{g_{k,0}})_{\infty,\mathbb{R}}\widetilde{\mathcal{Z}}_{\infty,\mathbb{R}}((U_{g_{k,0}})_{\infty,\mathbb{R}})^{-1} = \widetilde{\mathcal{Z}}_{\infty,\mathbb{R}}, \tag{7.7}$$

where $(U_{g_{k,0}})_{\infty,\mathbb{R}} = P_\varphi(\dot{\Delta}_{\infty,\mathbb{R}})\varphi(U^\pi_{g_{k,0}})$. As a consequence of (7.7), for each $k \in \mathbb{R}_+$, the mapping

$$\widehat{\alpha}_k : A_{\infty,\mathbb{R}} \mapsto (U_{g_{k,0}})_{\infty,\mathbb{R}}A_{\infty,\mathbb{R}}((U_{g_{k,0}})_{\infty,\mathbb{R}})^{-1} \tag{7.8}$$

is a *-automorphism of the C^*-algebra $\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}}$ and its central C^*-subalgebra $\widetilde{\mathcal{Z}}_{\infty,\mathbb{R}}$. The *-automorphisms $\widehat{\alpha}_k$ ($k \in \mathbb{R}_+$) in view of (7.6) and the equalities $z[\widehat{\beta}_k(m)] = [\widehat{\alpha}_k(z)](m)$ ($z \in \widetilde{\mathcal{Z}}_{\infty,\mathbb{R}},\ m \in \widetilde{\Delta}_{\infty,\mathbb{R}},\ k \in \mathbb{R}_+$), where $z(\cdot) \in C(\widetilde{\Delta}_{\infty,\mathbb{R}})$ is the Gelfand transform of the operator $z \in \widetilde{\mathcal{Z}}_{\infty,\mathbb{R}}$, induce on $\widetilde{\Delta}_{\infty,\mathbb{R}}$ the homeomorphisms

$$\widehat{\beta}_k : \widetilde{\Delta}_{\infty,\mathbb{R}} \to \widetilde{\Delta}_{\infty,\mathbb{R}}, \quad (\xi,\eta,x) \mapsto (\xi, g_{k^{-1},0}(\eta), x), \quad k \in \mathbb{R}_+. \tag{7.9}$$

The maps $\eta \mapsto g_{k,0}(\eta)$ are homeomorphisms of $M(SO^\diamond)$ onto itself given by

$$b(g_{k,0}(\eta)) = (b \circ g_{k,0})(\eta) \quad \text{for all} \;\; b \in SO^\diamond \;\; \text{and all} \;\; \eta \in M(SO^\diamond) \tag{7.10}$$

(as usual $b(\eta) := \eta(b)$). Since 0 and ∞ are the only fixed points on $\dot{\mathbb{R}}$ for the shifts $g_{k,0} \in \widetilde{G} \setminus \{e\}$, we infer that $g_{k,0}(\eta) = \eta$ for all $\eta \in M_0(SO^\diamond) \cup M_\infty(SO^\diamond)$ and all $k \in \mathbb{R}_+ \setminus \{1\}$ (see, e.g., the proof of [5, Theorem 6.4]). Hence, we obtain from (7.9) the following assertion similarly to [7, Lemma 4.2].

Lemma 7.3. *The set $\dot{\Delta}_{\infty,0} \cup \Delta_{\infty,\infty}$ consists of all fixed points of the homeomorphisms $\widehat{\beta}_k$ $(k \in \mathbb{R}_+ \setminus \{1\})$.*

Let $\mathfrak{R}(\widetilde{\Delta}_{\infty,\mathbb{R}})$ be the σ-algebra of all Borel subsets of the compact set $\widetilde{\Delta}_{\infty,\mathbb{R}}$ given by (7.5). Taking

$$\mathfrak{R}_{\widetilde{G}}(\widetilde{\Delta}_{\infty,\mathbb{R}}) := \big\{\Delta \in \mathfrak{R}(\widetilde{\Delta}_{\infty,\mathbb{R}}) : \widehat{\beta}_k(\Delta) = \Delta \ \text{ for all } \ g_{k,0} \in \widetilde{G}\big\},$$

where the group $\widetilde{G}$ is given by (3.2), we conclude from [24] that $P_\varphi(\Delta)B_{\infty,\mathbb{R}} = B_{\infty,\mathbb{R}}P_\varphi(\Delta)$ for each $\Delta \in \mathfrak{R}_{\widetilde{G}}(\widetilde{\Delta}_{\infty,\mathbb{R}})$ and each $B_{\infty,\mathbb{R}} \in \mathfrak{B}_{\infty,\mathbb{R}}$. For every $t \in \dot{\mathbb{R}}$ and every $g_{k,0} \in \widetilde{G}$, the homeomorphism $\eta \mapsto g_{k,0}(\eta)$ defined by (7.10) sends the fibers $M_t(SO^\diamond)$ onto the fibers $M_{g_{k^{-1},0}(t)}(SO^\diamond)$. Setting then

$$\begin{aligned} &\Omega_{\infty,\mathbb{R}\setminus\{0\}} := \Omega_{\infty,\mathbb{R}} \setminus \Omega_{\infty,0}, \quad && \widetilde{\Omega}_{\infty,\mathbb{R}\setminus\{0\}} := \Omega_{\infty,\mathbb{R}\setminus\{0\}} \times \overline{\mathbb{R}}, \\ &\dot{\Delta}_{\infty,\mathbb{R}\setminus\{0\}} := \Omega_{\infty,\mathbb{R}\setminus\{0\}} \times \dot{\mathbb{R}}, \quad && \Delta^\circ_{\infty,\mathbb{R}\setminus\{0\}} := \Omega_{\infty,\mathbb{R}\setminus\{0\}} \times \mathbb{R}, \end{aligned} \tag{7.11}$$

we obtain the partition $\widetilde{\Delta}_{\infty,\mathbb{R}} = \dot{\Delta}_{\infty,\mathbb{R}\setminus\{0\}} \cup \Delta^\circ_{\infty,0} \cup \Delta_{\infty,0} \cup \Delta_{\infty,\infty}$, where $\dot{\Delta}_{\infty,\mathbb{R}\setminus\{0\}}$ and $\Delta^\circ_{\infty,0}$ are open sets in $\mathfrak{R}_{\widetilde{G}}(\widetilde{\Delta}_{\infty,\mathbb{R}})$, while $\Delta_{\infty,\infty}$ and $\Delta_{\infty,0}$ are closed subsets of $\mathfrak{R}_{\widetilde{G}}(\widetilde{\Delta}_{\infty,\mathbb{R}})$. We now introduce the C^*-algebras

$$\begin{aligned} &\widetilde{\mathcal{Z}}_{\infty,\mathbb{R}\setminus\{0\}} := P_\varphi(\dot{\Delta}_{\infty,\mathbb{R}\setminus\{0\}})\widetilde{\mathcal{Z}}_{\infty,\mathbb{R}}, \quad && \widetilde{\mathcal{Z}}^\circ_{\infty,0} := P_\varphi(\Delta^\circ_{\infty,0})\widetilde{\mathcal{Z}}_{\infty,\mathbb{R}}, \\ &\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}\setminus\{0\}} := P_\varphi(\dot{\Delta}_{\infty,\mathbb{R}\setminus\{0\}})\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}}, \quad && \widetilde{\mathfrak{A}}^\circ_{\infty,0} := P_\varphi(\Delta^\circ_{\infty,0})\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}}, \\ &\mathfrak{B}_{\infty,\mathbb{R}\setminus\{0\}} := P_\varphi(\dot{\Delta}_{\infty,\mathbb{R}\setminus\{0\}})\mathfrak{B}_{\infty,\mathbb{R}}, \quad && \mathfrak{B}^\circ_{\infty,0} := P_\varphi(\Delta^\circ_{\infty,0})\mathfrak{B}_{\infty,\mathbb{R}}. \end{aligned} \tag{7.12}$$

8. The C^*-algebra $\mathfrak{B}_{\infty,\mathbb{R}\setminus\{0\}}$

The maximal ideal space $M(\widetilde{\mathcal{Z}}_{\infty,\mathbb{R}\setminus\{0\}})$ of the central subalgebra $\widetilde{\mathcal{Z}}_{\infty,\mathbb{R}\setminus\{0\}}$ of the C^*-algebra $\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}\setminus\{0\}}$ is homeomorphic to the compact set $\widetilde{\Delta}_{\infty,\mathbb{R}\setminus\{0\}} := \dot{\Delta}_{\infty,\mathbb{R}\setminus\{0\}} \cup \Delta_{\infty,\infty} \cup \Delta_{\infty,0}$. The restriction of the automorphism $\widehat{\alpha}_k$ $(k \in \mathbb{R}_+)$ given by (7.8) to the C^*-algebras $\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}\setminus\{0\}}$ and $\widetilde{\mathcal{Z}}_{\infty,\mathbb{R}\setminus\{0\}}$ are *-automorphisms of these C^*-algebras. Thus, assumption (A1) of Section 4 is fulfilled for the C^*-algebras $\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}\setminus\{0\}}$ and $\widetilde{\mathcal{Z}}_{\infty,\mathbb{R}\setminus\{0\}}$. Since the commutative group $\widetilde{G}$ given by (3.2) is amenable, assumption (A2) of Section 4 is also fulfilled.

Let $J_{\xi,\eta,x}$ denote the closed two-sided ideal of the C^*-algebra $\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}\setminus\{0\}}$ generated by the maximal ideal of the C^*-algebra $\widetilde{\mathcal{Z}}_{\infty,\mathbb{R}\setminus\{0\}}$ associated with the point $(\xi,\eta,x) \in \widetilde{\Delta}_{\infty,\mathbb{R}\setminus\{0\}}$. Applying Theorem 7.1 and Corollary 7.2, we deduce the following three assertions.

(i) For each $(\xi,\eta,x) \in \Delta^\circ_{\infty,\mathbb{R}\setminus\{0\}}$, where $\Delta^\circ_{\infty,\mathbb{R}\setminus\{0\}}$ is given by (7.11), the mapping

$$\widetilde{\pi}_{\xi,\eta,x} : \ P_\varphi(\dot{\Delta}_{\infty,\mathbb{R}\setminus\{0\}})(A_{\infty,\mathbb{R}}) + J_{\xi,\eta,x} \mapsto \Psi_{\xi,\eta,x}(A)$$

is a *-isomorphism of the C^*-algebra $\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}\setminus\{0\}}/J_{\xi,\eta,x}$ onto the C^*-subalgebra $\big\{\Psi_{\xi,\eta,x}(A) : A \in \widetilde{\mathfrak{A}}\big\}$ of $\mathbb{C}^{2\times 2}$.

(ii) For each $(\xi,\eta,x) \in \Delta_{\infty,\mathbb{R}}$, where $\Delta_{\infty,\mathbb{R}}$ is given by (5.10), the mapping

$$\widetilde{\pi}_{\xi,\eta,x} : \ P_\varphi(\dot{\Delta}_{\infty,\mathbb{R}\setminus\{0\}})(A_{\infty,\mathbb{R}}) + J_{\xi,\eta,x} \mapsto \operatorname{diag}\big\{\Psi_{\xi,\eta,-\infty}(A),\ \Psi_{\xi,\eta,+\infty}(A)\big\},$$

is a C^*-algebra isomorphism of the quotient C^*-algebra $\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}\setminus\{0\}}/J_{\xi,\eta,x}$ onto the C^*-subalgebra

$$\big\{\operatorname{diag}\big\{\Psi_{\xi,\eta,-\infty}(A),\ \Psi_{\xi,\eta,+\infty}(A)\big\} : A \in \widetilde{\mathfrak{A}}\big\} \quad \text{of} \quad \mathbb{C}^{4\times 4}.$$

(iii) For each $(\xi,\eta,x) \in \Delta_{\infty,\infty}$, where $\Delta_{\infty,\infty}$ is given by (5.10), the mapping

$$\widetilde{\pi}_{\xi,\eta,x} : \ P_\varphi(\dot{\Delta}_{\infty,\mathbb{R}\setminus\{0\}})\Big(\sum_{h\in F}(A_h U_{g_{1,h}})_{\infty,\mathbb{R}}\Big) + J_{\xi,\eta,x} \mapsto \sum_{h\in F}\operatorname{diag}\big\{\Psi_{\xi,\eta,-\infty}(A_h)e_{-h},\ \Psi_{\xi,\eta,+\infty}(A_h)e_{-h}\big\},$$

where $A_h \in \mathfrak{A}$ for all $h \in F$, and F runs through finite subsets of $\mathbb{R}$, extends to a C^*-algebra isomorphism of the quotient C^*-algebra $\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}\setminus\{0\}}/J_{\xi,\eta,x}$ onto the C^*-subalgebra $\widetilde{\pi}_{\xi,\eta,x}(\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}\setminus\{0\}}/J_{\xi,\eta,x})$ of 4×4 diagonal matrices with entries in the C^*-algebra AP of uniformly almost periodic functions on $\mathbb{R}$. As is known, AP is the C^*-subalgebra of $L^\infty(\mathbb{R})$ generated by all functions e_λ ($\lambda \in \mathbb{R}$).

One can prove that for every pure state $\nu \in \mathcal{P}_{\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}\setminus\{0\}}}$ of the C^*-algebra $\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}\setminus\{0\}}$, which satisfy the condition $m_\nu = \ker \nu \cap \widetilde{\mathcal{Z}}_{\infty,\mathbb{R}\setminus\{0\}} \in \Delta_{\infty,\infty} \cup \Delta_{\infty,0}$, and every open neighborhood $W_\nu \subset \mathcal{P}_{\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}\setminus\{0\}}}$ of ν there exists a state $w \in W_\nu$ such that the point $m_w = \ker w \cap \widetilde{\mathcal{Z}}_{\infty,\mathbb{R}\setminus\{0\}}$ belongs to $\mathbb{R}_+ \cup \mathbb{R}_-$. This means that assumption (A3) of Section 4 is also fulfilled for the C^*-algebra

$$\mathfrak{B}_{\infty,\mathbb{R}\setminus\{0\}} = P_\varphi(\dot{\Delta}_{\infty,\mathbb{R}\setminus\{0\}})\,\mathrm{alg}\,\{\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}}, (U_{g_{k,0}})_{\infty,\mathbb{R}} :\ k \in \mathbb{R}_+\}.$$

Indeed, if $(\xi,\eta,\infty) \in \Delta_{\infty,0}$, then the set of all pure states of the C^*-algebra $\widetilde{\mathfrak{A}}$ at the point (ξ,η,∞) consists of four elements, which are given for $A \in \widetilde{\mathfrak{A}}$ by the (k,k)-entries $[\Psi_{\xi,\eta,\pm\infty}(A)]_{k,k}$ for $k = 1,2$ of the matrix $\Psi_{\xi,\eta,\pm\infty}(A)$, where

$$[\Psi_{\xi,\eta,\pm\infty}(aW^0(b)U_{g_{1,h}})]_{1,1} = a(\xi^+)b(\eta^\pm), \quad [\Psi_{\xi,\eta,\pm\infty}(aW^0(b)U_{g_{1,h}})]_{2,2} = a(\xi^-)b(\eta^\mp)$$

for the generator $aW^0(b)U_{g_{1,h}}$ ($a,b \in PSO^\diamond,\ h \in \mathbb{R}$) of the C^*-algebra $\widetilde{\mathfrak{A}}$. It is obvious that for every $\varepsilon > 0$ there exist points $\tau_\pm \in \mathbb{R}_\pm$ close to 0 and points $(\xi,\eta_\pm,\infty) \in M_\infty(SO^\diamond) \times M_{\tau_\pm}(SO^\diamond) \times \{\infty\}$ such that

$$\begin{aligned} \big|[\Psi_{\xi,\eta,\pm\infty}(aW^0(b)U_{g_{1,h}})]_{1,1} - [\Psi_{\xi,\eta_\pm,\pm\infty}(aW^0(b)U_{g_{1,h}})]_{1,1}\big| &= \big|a(\xi^+)b(\eta^\pm) - a(\xi^+)b(\eta_\pm^\pm)e^{-ih\tau_\pm}\big| < \varepsilon, \\ \big|[\Psi_{\xi,\eta,\pm\infty}(aW^0(b)U_{g_{1,h}})]_{2,2} - [\Psi_{\xi,\eta_\mp,\pm\infty}(aW^0(b)U_{g_{1,h}})]_{2,2}\big| &= \big|a(\xi^-)b(\eta^\mp) - a(\xi^-)b(\eta_\mp^\mp)e^{-ih\tau_\mp}\big| < \varepsilon, \end{aligned} \tag{8.1}$$

where $b(\eta_\pm^\pm)$ means either $b(\eta_+^+)$, or $b(\eta_-^-)$. Moreover, we can choose $\tau_\pm$ in the set $\mathbb{R}_\pm \setminus \mathcal{T}$, where $\mathcal{T}$ is the at most countable set of all discontinuity points of $b \in SO^\diamond$, and then replace $b(\eta_\pm^\pm)$ by $b(\tau_\pm)$, respectively. Hence, by (8.1), for every $A \in \widetilde{\mathfrak{A}}$, the pure state values $[\Psi_{\xi,\eta,\pm\infty}(A)]_{k,k}$ can be approximated by the pure state values $[\Psi_{\xi,\eta_\pm,\pm\infty}(A)]_{1,1}$ if $k = 1$, and by the pure state values $[\Psi_{\xi,\eta_\mp,\pm\infty}(A)]_{2,2}$ if $k = 2$, where $\eta_\pm \in M_{\tau_\pm}(SO^\diamond)$ and the points $\tau_\pm \in \mathbb{R}_\pm$ are close to 0. On the other hand, $g_{k,0}(\tau_\pm) \neq \tau_\pm$ for every $k \in \mathbb{R}_+ \setminus \{1\}$, which proves assumption (A3) for all points $(\xi,\eta,\infty) \in \Delta_{\infty,0}$.

We now suppose that $(\xi,\eta,\infty) \in \Delta_{\infty,\infty}$. Then the set of all pure states of the C^*-algebra $\widetilde{\mathfrak{A}}$ at the point (ξ,η,∞) is given on the generators $aW^0(b)U_{g_{1,h}}$ ($a,b \in PSO^\diamond,\ h \in \mathbb{R}$) of the C^*-algebra $\widetilde{\mathfrak{A}}$ by

$$\widetilde{\Psi}_{\xi,\eta,z,\pm\infty,1}(aW^0(b)U_{g_{1,h}}) = a(\xi^+)b(\eta^\pm)e^{-ihz}, \quad \widetilde{\Psi}_{\xi,\eta,z,\pm\infty,2}(aW^0(b)U_{g_{1,h}}) = a(\xi^-)b(\eta^\mp)e^{-ihz},$$

where $(\xi,\eta,z) \in \Omega_{\infty,\infty} \times M_\infty(AP)$, $M_\infty(AP)$ is the fiber over ∞ of the maximal ideal space $M(AP)$ of AP, and $e^{ihz} = z(e_h)$ for every $h \in \mathbb{R}$ and every $z \in M_\infty(AP)$. As is well known (see, e.g., [34]), $M_\infty(AP) = M(AP) = \mathbb{R}_B$, where $\mathbb{R}_B$ is the Bohr compactification of the real line $\mathbb{R}$, and the C^*-algebras SO_∞ and AP are asymptotically independent, that is, $M_\infty(\mathrm{alg}(SO_\infty, AP)) = M_\infty(SO_\infty) \times M_\infty(AP)$. Hence, by (2.1), $M_\infty(\mathrm{alg}(SO^\diamond, AP)) = M_\infty(SO^\diamond) \times M_\infty(AP)$.

Modifying the proof of [34, Lemma 1], we obtain the following.

Lemma 8.1. *If $\{b_k : k = 1,\dots,N\} \subset PSO^\diamond$ and $\{g_k : k = 1,\dots,N\} \subset AP$, then for every pair $(\eta,z) \in M_\infty(SO^\diamond) \times M_\infty(AP)$ there exist sequences $\{\tau_n^\pm\}_{n\in\mathbb{N}} \subset \mathbb{R}\setminus\{0\}$ such that $\tau_n^\pm$ are points of continuity for functions b_k for all $k = 1,\dots,N$, $\lim_{n\to\infty} \tau_n^\pm = \pm\infty$ and*

$$\lim_{n\to\infty} (b_k(\tau_n^\pm)g_k(\tau_n^\pm)) = b_k(\eta^\pm)g_k(z). \tag{8.2}$$

By Lemma 8.1, for arbitrary sets $\{a_k, b_k : k = 1, \dots, N\} \subset PSO^\diamond$ and $\{g_k : k = 1, \dots, N\} \subset AP$ and for every point $\eta \in M_\infty(SO^\diamond)$ and every point $z \in M_\infty(AP)$ there exist sequences $\{\tau_n^\pm\}_{n\in\mathbb{N}} \subset \mathbb{R} \setminus \{0\}$ such that $\lim_{n\to\infty} \tau_n^\pm = \pm\infty$ and, for all $k = 1, \dots, N$, $\tau_n^\pm$ are points of continuity for functions b_k and

$$\lim_{n\to\infty}(a_k(\xi^+)b_k(\tau_n^\pm)g_k(\tau_n^\pm)) = a_k(\xi^+)b_k(\eta^\pm)g_k(z), \quad \lim_{n\to\infty}(a_k(\xi^-)b_k(\tau_n^\pm)g_k(\tau_n^\pm)) = a_k(\xi^-)b_k(\eta^\pm)g_k(z).$$

Since $g_{k,0}(\tau_n^\pm) \neq \tau_n^\pm$ for all $k \in \mathbb{R}_+ \setminus \{1\}$, we conclude that assumption (A3) for the C^*-algebra $\widetilde{\mathfrak{A}}$ is also fulfilled for the points $(\xi, \eta, \infty) \in \Delta_{\infty,\infty}$.

Hence, to study the invertibility of operators $B_{\infty,\mathbb{R}\setminus\{0\}}$, we can apply Theorem 4.2, with the set $M_0 = \dot{\Delta}_{\infty,\mathbb{R}\setminus\{0\}} \subset M(\widetilde{\mathcal{Z}}_{\infty,\mathbb{R}\setminus\{0\}})$ chosen in assumption (A3) (see Lemma 8.1 and arguments before that lemma).

Fix $t_\pm \in \mathbb{R}_\pm$, put $\widehat{\Omega}_{\infty,0} := \Omega_{\infty,0} \times \{\pm\infty\}$, take the sets $\widetilde{\Omega}_{\infty,t_\pm}$ given by (3.5) and, for each point (ξ, η, x) in the set $\widetilde{\Omega}_{\infty,t_+} \cup \widetilde{\Omega}_{\infty,t_-} \cup \widehat{\Omega}_{\infty,0}$, we introduce the representation

$$\pi_{\xi,\eta,x} : \mathfrak{B}_{\infty,\mathbb{R}\setminus\{0\}} \to \mathcal{B}(l^2(\mathbb{R}_+, \mathbb{C}^2)) \tag{8.3}$$

given on the generators of the C^*-algebra $\mathfrak{B}_{\infty,\mathbb{R}\setminus\{0\}}$ in view of (7.6) by

$$\begin{aligned}
\big[\pi_{\xi,\eta,x}\big((aI)_{\infty,\mathbb{R}\setminus\{0\}}\big)f\big](t) &= [\Psi_{\xi,\eta,x}(aI)]f(t),\\
\big[\pi_{\xi,\eta,x}\big((W^0(b))_{\infty,\mathbb{R}\setminus\{0\}}\big)f\big](t) &= [\Psi_{\xi,\eta,x}(W^0(b \circ g_{t^{-1},0}))]f(t),\\
\big[\pi_{\xi,\eta,x}\big((U_{g_{k,h}})_{\infty,\mathbb{R}\setminus\{0\}}\big)f\big](t) &= [\Psi_{\xi,\eta,x}(U_{g_{1,h/(kt)}})]f(kt),
\end{aligned} \tag{8.4}$$

where $a, b \in PSO^\diamond$, $g_{k,h} \in G$, $f \in l^2(\mathbb{R}_+, \mathbb{C}^2)$ and $t \in \mathbb{R}_+$.

Applying now Theorem 4.2, we establish the following criterion.

Theorem 8.2. *For each $B \in \mathfrak{B}$, the operator $B_{\infty,\mathbb{R}\setminus\{0\}}$ is invertible on the space $\mathcal{H}_{\varphi,\infty,\mathbb{R}\setminus\{0\}}$ if and only if for all $(\xi, \eta, x) \in \widetilde{\Omega}_{\infty,t_+} \cup \widetilde{\Omega}_{\infty,t_-}$ the operators $\pi_{\xi,\eta,x}(B_{\infty,\mathbb{R}\setminus\{0\}})$ are invertible on the space $l^2(\mathbb{R}_+, \mathbb{C}^2)$ and*

$$\sup_{(\xi,\eta,x)\in\widetilde{\Omega}_{\infty,t_+}\cup\widetilde{\Omega}_{\infty,t_-}} \big\|(\pi_{\xi,\eta,x}(B_{\infty,\mathbb{R}\setminus\{0\}}))^{-1}\big\|_{\mathcal{B}(l^2(\mathbb{R}_+,\mathbb{C}^2))} < \infty.$$

Proof. The set $\widetilde{\Omega}_{\infty,t_+} \cup \widetilde{\Omega}_{\infty,t_-}$ contains exactly one point in each G-orbit defined on the set $\widetilde{\Omega}_{\infty,\mathbb{R}\setminus\{0\}}$ by the group $\{\widehat{\beta}_k : k \in \mathbb{R}_+\}$ of homeomorphisms given by (7.9)–(7.10). Thus, following (4.1)–(4.2), we obtain the family of representations (8.3) indexed by the points $(\xi, \eta, x) \in \widetilde{\Omega}_{\infty,t_+} \cup \widetilde{\Omega}_{\infty,t_-}$ and given by (8.4). Since assumptions (A1)–(A3) for the C^*-algebra $\mathfrak{B}_{\infty,\mathbb{R}\setminus\{0\}}$ are fulfilled, we infer the assertion of the theorem from Theorem 4.2. □

For every $B \in \mathfrak{B}$ and every $(\xi, \eta, x) \in \widetilde{\Omega}_{\infty,t_\pm}$, we put

$$[\mathrm{Sym}_\pm(B)](\xi, \eta, x) := \pi_{\xi,\eta,x}\big(B_{\infty,\mathbb{R}\setminus\{0\}}\big), \tag{8.5}$$

where the representations $\pi_{\xi,\eta,x}$ are given by (8.3) and (8.4). One can see from (8.4) and (3.3) that formulas (8.5) coincide with (3.13) on the generators of the C^*-algebra $\mathfrak{B}$. Consequently, Theorem 8.2 combined with (3.9) and (8.3)–(8.5) implies the following corollary by analogy with Corollary 6.10.

Corollary 8.3. *For each $B \in \mathfrak{B}$, the operator $B_{\infty,\mathbb{R}\setminus\{0\}}$ is invertible on the space $\mathcal{H}_{\varphi,\infty,\mathbb{R}\setminus\{0\}}$ if and only if condition* (iv) *of Theorem* 3.2 *holds. The maps $\Phi_\pm : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_{\infty,\mathbb{R}_\pm})$ given by* (3.7), (3.9) *and* (3.13) *are C^*-algebra homomorphisms, and* $\ker \Phi_+ \cap \ker \Phi_- = \{B \in \mathfrak{B} : B_{\infty,\mathbb{R}\setminus\{0\}} = 0\}$.

Applying representations $\pi_{\xi,\eta,x}$ for $(\xi,\eta,x) \in \widehat{\Omega}_{\infty,0}$, we get the following.

Corollary 8.4. *If an operator $B_{\infty,\mathbb{R}\setminus\{0\}}$ is invertible on the space $\mathcal{H}_{\varphi,\infty,\mathbb{R}\setminus\{0\}}$, then for every $(\xi,\eta,x) \in \widehat{\Omega}_{\infty,0}$ the operator $\pi_{\xi,\eta,x}(B_{\infty,\mathbb{R}\setminus\{0\}})$ is invertible on the space $l^2(\mathbb{R}_+,\mathbb{C}^2)$.*

For $(\xi,\eta) \in \Omega_{\infty,0}$, the generators of the C^*-algebras $\pi_{\xi,\eta,\pm\infty}(\mathfrak{B}_{\infty,\mathbb{R}\setminus\{0\}}) \subset \mathcal{B}(l^2(\mathbb{R}_+,\mathbb{C}^2))$ are given, respectively, by

$$\begin{aligned}
\big[\pi_{\xi,\eta,\pm\infty}\big((aI)_{\infty,\mathbb{R}\setminus\{0\}}\big)f\big](t) &= \operatorname{diag}\{a(\xi^+),\, a(\xi^-)\}f(t),\\
\big[\pi_{\xi,\eta,+\infty}\big((W^0(b))_{\infty,\mathbb{R}\setminus\{0\}}\big)f\big](t) &= \operatorname{diag}\{b(\eta^+),\, b(\eta^-)\}f(t),\\
\big[\pi_{\xi,\eta,-\infty}\big((W^0(b))_{\infty,\mathbb{R}\setminus\{0\}}\big)f\big](t) &= \operatorname{diag}\{b(\eta^-),\, b(\eta^+)\}f(t),\\
\big[\pi_{\xi,\eta,\pm\infty}\big((U_{g_{k,h}})_{\infty,\mathbb{R}\setminus\{0\}}\big)f\big](t) &= f(kt),
\end{aligned} \tag{8.6}$$

where $a,b \in PSO^\diamond$, $g_{k,h} \in G$, $f \in l^2(\mathbb{R}_+,\mathbb{C}^2)$ and $t \in \mathbb{R}_+$. By (8.6), for every $(\xi,\eta) \in \Omega_{\infty,0}$ the C^*-algebras $\pi_{\xi,\eta,\pm\infty}(\mathfrak{B}_{\infty,\mathbb{R}\setminus\{0\}})$ are commutative and *-isomorphic to the C^*-algebras $\operatorname{diag}\{AP, AP\}$, the isomorphisms are given on the operators $B_{\infty,\mathbb{R}\setminus\{0\}} = \sum_{k\in F}(A_k U_{g_{k,0}})_{\infty,\mathbb{R}\setminus\{0\}}$ with $A_k \in \widetilde{\mathfrak{A}}$ and finite sets $F \subset \mathbb{R}_+$ by

$$\pi_{\xi,\eta,\pm\infty}\big(B_{\infty,\mathbb{R}\setminus\{0\}}\big) \mapsto \sum_{k\in F} \Psi_{\xi,\eta,\pm\infty}(A_k)\, e_{-\ln k}\, I_2. \tag{8.7}$$

9. The C^*-algebra $\mathfrak{B}^\circ_{\infty,0}$

Let us study the invertibility of the operators $B^\circ_{\infty,0} := P_\varphi(\Delta^\circ_{\infty,0})B_{\infty,\mathbb{R}}$ in the C^*-algebra $\mathfrak{B}^\circ_{\infty,0}$ given by (7.12). As $e_h(0) = 1$ for all $h \in \mathbb{R}$, we infer that

$$\widetilde{\mathfrak{A}}^\circ_{\infty,0} = P_\varphi(\Delta^\circ_{\infty,0})\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}} = P_\varphi(\Delta^\circ_{\infty,0})\mathfrak{A}_{\infty,\mathbb{R}} =: \mathfrak{A}^\circ_{\infty,0}. \tag{9.1}$$

The maximal ideal space of the central C^*-algebra $\widetilde{\mathcal{Z}}^\circ_{\infty,0} = P_\varphi(\Delta^\circ_{\infty,0})\widetilde{\mathcal{Z}}_{\infty,\mathbb{R}}$ of the C^*-algebra $\widetilde{\mathfrak{A}}^\circ_{\infty,0}$ is homeomorphic to the closure $\dot{\Delta}_{\infty,0}$ of the set $\Delta^\circ_{\infty,0}$. Since the set $\Delta^\circ_{\infty,0}$ is open and the C^*-algebras $\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}}$ and $\widehat{\mathfrak{A}}_{\infty,\mathbb{R}}$ are *-isomorphic (see (7.3)), we get the following result similarly to Theorem 6.2.

Theorem 9.1. *The mapping $P_\varphi(\Delta^\circ_{\infty,0})A_{\infty,\mathbb{R}} \mapsto P_\phi(\Delta^\circ_{\infty,0})\widehat{A}_{\infty,\mathbb{R}}$, where the operators $A_{\infty,\mathbb{R}} \in \widetilde{\mathfrak{A}}_{\infty,\mathbb{R}}$ and $\widehat{A}_{\infty,\mathbb{R}}$ are given by (7.4), is a C^*-algebra isomorphism of the C^*-algebra $\widetilde{\mathfrak{A}}^\circ_{\infty,0} = P_\varphi(\Delta^\circ_{\infty,0})\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}}$ onto the C^*-algebra $\widehat{\mathfrak{A}}^\circ_{\infty,0} := P_\phi(\Delta^\circ_{\infty,0})\widehat{\mathfrak{A}}_{\infty,\mathbb{R}}$.*

Let $\Psi(A)|_{\widetilde{\Omega}_{\infty,0}}$ denote the matrix function $(\xi,\eta,x) \mapsto \Psi_{\xi,\eta,x}(A)$ defined on $\widetilde{\Omega}_{\infty,0}$ by (2.6), (7.1) and (7.2) for $A \in \widetilde{\mathfrak{A}}$, where $\widetilde{\Omega}_{\infty,0}$ is given by (3.5) for $t = 0$. Applying Theorem 9.1 and (2.8), we obtain the following invertibility criterion for the operators in the C^*-algebra $\widetilde{\mathfrak{A}}^\circ_{\infty,0}$ by analogy with Theorem 6.3.

Theorem 9.2. *The mapping*

$$\mathrm{Sym}^\circ_{\infty,0}:\ \widetilde{\mathfrak{A}}^\circ_{\infty,0} \to \widehat{\mathfrak{A}}^\circ_{\infty,0} \to \mathcal{B}(l^2(\widetilde{\Omega}_{\infty,0},\mathbb{C}^2)), \quad P_\varphi(\Delta^\circ_{\infty,0})A_{\infty,\mathbb{R}} \mapsto P_\phi(\Delta^\circ_{\infty,0})\widehat{A}_{\infty,\mathbb{R}} \mapsto \Psi(A)|_{\widetilde{\Omega}_{\infty,0}} I$$

is an isometric C^*-algebra homomorphism. For any $A \in \widetilde{\mathfrak{A}}$, the operator $A^\circ_{\infty,0} = P_\varphi(\Delta^\circ_{\infty,0})A_{\infty,\mathbb{R}}$ is invertible on the Hilbert space $\mathcal{H}^\circ_{\varphi,\infty,0} = P_\varphi(\Delta^\circ_{\infty,0})\mathcal{H}_\varphi$ if and only if $\det[\Psi_{\xi,\eta,x}(A)] \neq 0$ for all $(\xi,\eta,x) \in \widetilde{\Omega}_{\infty,0}$.

Since $\widetilde{V}_0 = W^0(V_0)$ by (5.6), since $U_{g_{k^{-1},0}} V_0 \in \mathfrak{A}$ for $k \in \mathbb{R}_+$ by Lemma 5.3 and since $e_{-h}V_0 \simeq V_0$ for $h \in \mathbb{R}$, we deduce that $U_{g_{1,h}} \mathcal{F}^{-1} V_0 \mathcal{F} = \mathcal{F}^{-1} e_{-h} V_0 \mathcal{F} \simeq \mathcal{F}^{-1} V_0 \mathcal{F}$, whence, for all $k \in \mathbb{R}_+$ and all $h \in \mathbb{R}$,

$$U_{g_{k,h}} \widetilde{V}_0 = U_{g_{k,0}} U_{g_{1,h}} \mathcal{F}^{-1} V_0 \mathcal{F} \simeq U_{g_{k,0}} \mathcal{F}^{-1} V_0 \mathcal{F} = \mathcal{F}^{-1} U_{g_{k^{-1},0}} V_0 \mathcal{F} = W^0(U_{g_{k^{-1},0}} V_0) \in \mathfrak{A}. \tag{9.2}$$

Taking into account (9.2), we infer the following assertion from Lemma 5.3 and (2.6).

Lemma 9.3. *If $k \in \mathbb{R}_+$, $h \in \mathbb{R}$ and $v(x) = -i/\cosh(\pi x)$ for $x \in \overline{\mathbb{R}}$, then $U_{g_{k,h}} \widetilde{V}_0 \in \mathfrak{A}$ and*

$$\Psi_{\xi,\eta,x}(U_{g_{k,h}} \widetilde{V}_0) = e^{-ix \ln k} v(x) I_2 \quad \text{if} \quad (\xi,\eta,x) \in \widetilde{\Omega}_{\infty,0}, \quad \Psi_{\xi,\eta,x}(U_{g_{k,h}} \widetilde{V}_0) = 0_{2\times 2} \quad \text{if} \quad (\xi,\eta,x) \in \widetilde{\Omega} \setminus \widetilde{\Omega}_{\infty,0}.$$

Consider the Hilbert spaces $\mathcal{H}^\circ_{\phi,\infty,0} := P_\phi(\Delta^\circ_{\infty,0})\mathcal{H}_\phi = \bigcup_{(\xi,\eta,x)\in\Delta^\circ_{\infty,0}} \mathbb{C}^2$ and introduce the C^*-algebra

$$\widehat{\mathfrak{B}}^\circ_{\infty,0} := \operatorname{alg}\left\{\widehat{A}^\circ_{\infty,0},\, (\widehat{U}_{g_{k,0}})^\circ_{\infty,0} :\ A \in \widetilde{\mathfrak{A}},\ k \in \mathbb{R}_+\right\} \subset \mathcal{B}(\mathcal{H}^\circ_{\phi,\infty,0}) \tag{9.3}$$

generated by the operators

$$\widehat{A}^\circ_{\infty,0} := \bigoplus_{(\xi,\eta,x)\in\Delta^\circ_{\infty,0}} \Psi_{\xi,\eta,x}(A) I \quad (A \in \widetilde{\mathfrak{A}}), \quad (\widehat{U}_{g_{k,0}})^\circ_{\infty,0} := \bigoplus_{(\xi,\eta,x)\in\Delta^\circ_{\infty,0}} e^{-ix\ln k} I_2 \quad (k \in \mathbb{R}_+). \tag{9.4}$$

The mapping $k \mapsto (\widehat{U}_{g_{k,0}})^\circ_{\infty,0}$ is a unitary representation of the group $\mathbb{R}_+$ in the Hilbert space $\mathcal{H}^\circ_{\phi,\infty,0}$, $((\widehat{U}_{g_{k,0}})^\circ_{\infty,0})^* = (\widehat{U}_{g_{k^{-1},0}})^\circ_{\infty,0}$ and, by (9.4),

$$(\widehat{U}_{g_{k,0}})^\circ_{\infty,0} \widehat{A}^\circ_{\infty,0} ((\widehat{U}_{g_{k,0}})^\circ_{\infty,0})^* = \widehat{A}^\circ_{\infty,0} \quad \text{for all} \quad k \in \mathbb{R}_+ \quad \text{and all} \quad A \in \widetilde{\mathfrak{A}}.$$

Hence, the C^*-algebra $\widehat{\mathfrak{B}}^\circ_{\infty,0}$ is the closure of the set of all finite sums $\sum_k (\widehat{A}_k)^\circ_{\infty,0} (\widehat{U}_{g_{k,0}})^\circ_{\infty,0}$, where $A_k \in \widetilde{\mathfrak{A}}$.

Theorem 9.4. *The mapping*

$$\sum_{k\in F} (A_k U_{g_{k,0}})^\circ_{\infty,0} \mapsto \bigoplus_{(\xi,\eta,x)\in\Delta^\circ_{\infty,0}} \sum_{k\in F} \Psi_{\xi,\eta,x}(A_k) e^{-ix\ln k} I_2, \tag{9.5}$$

where $A_k \in \widetilde{\mathfrak{A}}$ for $k \in F$ and F runs through finite subsets of $\mathbb{R}_+$, extends to a C^*-algebra isomorphism of the C^*-algebra $\mathfrak{B}^\circ_{\infty,0}$ onto the C^*-algebra $\widehat{\mathfrak{B}}^\circ_{\infty,0}$ given by (9.3).

Proof. Let $B^\circ_{\infty,0} = P_\varphi(\Delta^\circ_{\infty,0}) B_{\infty,\mathbb{R}}$ for every $B \in \mathfrak{B}$. Since the set $\Delta^\circ_{\infty,0}$ is open, we infer similarly to [5, Lemma 3.5] that, for every $B_{\infty,\mathbb{R}} \in \mathfrak{B}_{\infty,\mathbb{R}}$,

$$\|B^\circ_{\infty,0}\|_{\mathcal{B}(\mathcal{H}^\circ_{\varphi,\infty,\mathbb{R}})} = \|P_\varphi(\Delta^\circ_{\infty,0}) B_{\infty,\mathbb{R}}\|_{\mathcal{B}(\mathcal{H}_{\varphi,\infty,\mathbb{R}})} = \sup_{Z_{\infty,\mathbb{R}} \in \widetilde{\mathcal{Z}}_{\infty,\mathbb{R}}(\Delta^\circ_{\infty,0})} \|Z_{\infty,\mathbb{R}} B_{\infty,\mathbb{R}}\|_{\mathcal{B}(\mathcal{H}_{\varphi,\infty,\mathbb{R}})}, \tag{9.6}$$

where the set $\widetilde{\mathcal{Z}}_{\infty,\mathbb{R}}(\Delta^\circ_{\infty,0})$ consists of the operators $Z_{\infty,\mathbb{R}} \in \widetilde{\mathcal{Z}}_{\infty,\mathbb{R}}$ for which the Gelfand transform is a function in $C(\widetilde{\Delta}_{\infty,\mathbb{R}})$ with values in $[0,1]$ and with support in $\dot{\Delta}_{\infty,0}$.

By analogy with [5, Lemma 6.1] and Lemma 9.3, it follows from (2.6) that for every operator $\widetilde{H}_{P,0}$ given by (5.5) and (5.6), where $P \in \mathcal{P}$ is a polynomial,

$$\Psi_{\xi,\eta,x}(\widetilde{H}_{P,0}) = P(u(x))v(x) I_2 \quad \text{if} \quad (\xi,\eta,x) \in \widetilde{\Omega}_{\infty,0}, \quad \Psi_{\xi,\eta,x}(\widetilde{H}_{P,0}) = 0_{2\times 2} \quad \text{if} \quad (\xi,\eta,x) \in \widetilde{\Omega} \setminus \widetilde{\Omega}_{\infty,0},$$

with $u(x) = \tanh(\pi x)$ and $v(x) = -i/\cosh(\pi x)$. Hence, the set $\widetilde{\mathcal{Z}}_{\infty,\mathbb{R}}(\Delta^\circ_{\infty,0})$ is the closure of the set of all operators $(\widetilde{H}_{P,0})_{\infty,\mathbb{R}}$ with polynomials $P \in \mathcal{P}$ such that $\{P(u(x))v(x) : x \in \mathbb{R}\} \subset [0,1]$. For every $k > 0$

and every $h \in \mathbb{R}$, we infer from (9.2), (5.5) and (5.6) that $U_{g_{k,h}} \widetilde{H}_{P,0} \in \mathfrak{A}$. Then, for every operator $B_{\infty,\mathbb{R}}$ and every operator $Z_{\infty,\mathbb{R}} \in \widetilde{\mathcal{Z}}_{\infty,\mathbb{R}}(\Delta^\circ_{\infty,0})$, the operator $Z_{\infty,\mathbb{R}} B_{\infty,\mathbb{R}}$ belongs to the C^*-algebra $\mathfrak{A}^\circ_{\infty,0} = \widetilde{\mathfrak{A}}^\circ_{\infty,0}$ defined by (9.1). Hence, applying Theorem 9.1, we infer that

$$\|Z_{\infty,\mathbb{R}} B_{\infty,\mathbb{R}}\|_{\mathcal{B}(\mathcal{H}_{\varphi,\infty,\mathbb{R}})} = \|P_\varphi(\Delta^\circ_{\infty,0}) Z_{\infty,\mathbb{R}} B_{\infty,\mathbb{R}}\|_{\mathcal{B}(\mathcal{H}_{\varphi,\infty,\mathbb{R}})} = \|P_\phi(\Delta^\circ_{\infty,0}) \psi_{\infty,\mathbb{R}}(Z_{\infty,\mathbb{R}} B_{\infty,\mathbb{R}})\|_{\mathcal{B}(\mathcal{H}_{\phi,\infty,\mathbb{R}})}, \quad (9.7)$$

where the operators $P_\phi(\Delta^\circ_{\infty,0}) \psi_{\infty,\mathbb{R}}(Z_{\infty,\mathbb{R}} B_{\infty,\mathbb{R}})$ are in the C^*-algebra $\widehat{\mathfrak{A}}^\circ_{\infty,0}$. Let

$$B^\circ_{\infty,0} = \sum_{k\in F} (A_k U_{g_{k,0}})^\circ_{\infty,\mathbb{R}} \in \mathfrak{B}^\circ_{\infty,0}, \quad \widehat{B}^\circ_{\infty,0} := \sum_{k\in F} (\widehat{A}_k \widehat{U}_{g_{k,0}})^\circ_{\infty,0} \in \widehat{\mathfrak{B}}^\circ_{\infty,0} \quad (9.8)$$

for $B_{\infty,\mathbb{R}} = \sum_{k\in F} (A_k U_{g_{k,0}})_{\infty,\mathbb{R}}$, where F is a finite subset of $\mathbb{R}_+$ and $A_k \in \widetilde{\mathfrak{A}}$. Then we deduce that

$$P_\phi(\Delta^\circ_{\infty,0}) \psi_{\infty,\mathbb{R}}(Z_{\infty,\mathbb{R}} B_{\infty,\mathbb{R}}) = \psi_{\infty,\mathbb{R}}(Z_{\infty,\mathbb{R}}) \widehat{B}^\circ_{\infty,0}, \quad (9.9)$$

where the operator $\widehat{B}^\circ_{\infty,0}$ is given by (9.8). It is easily seen that

$$\|\widehat{B}^\circ_{\infty,0}\|_{\mathcal{B}(\mathcal{H}^\circ_{\phi,\infty,\mathbb{R}})} = \sup_{Z_{\infty,0} \in \widetilde{\mathcal{Z}}_{\infty,\mathbb{R}}(\Delta^\circ_{\infty,0})} \|\psi_{\infty,\mathbb{R}}(Z_{\infty,\mathbb{R}}) \widehat{B}^\circ_{\infty,0}\|_{\mathcal{B}(\mathcal{H}_{\phi,\infty,\mathbb{R}})}. \quad (9.10)$$

Combining (9.6), (9.7), (9.9) and (9.10), we infer for the operators (9.8) that

$$\|B^\circ_{\infty,0}\|_{\mathcal{B}(\mathcal{H}^\circ_{\varphi,\infty,\mathbb{R}})} = \|\widehat{B}^\circ_{\infty,0}\|_{\mathcal{B}(\mathcal{H}^\circ_{\phi,\infty,\mathbb{R}})}. \quad (9.11)$$

Since the sets of such operators are dense in the C^*-algebras $\mathfrak{B}^\circ_{\infty,0}$ and $\widehat{\mathfrak{B}}^\circ_{\infty,0}$, respectively, we infer from (9.11) that the mapping (9.5) uniquely extends to a C^*-algebra isomorphism of $\mathfrak{B}^\circ_{\infty,0}$ onto $\widehat{\mathfrak{B}}^\circ_{\infty,0}$. □

Thus, for every $(\xi, \eta, x) \in \Delta^\circ_{\infty,0}$, we obtain the representation

$$\sigma_{\xi,\eta,x} : \mathfrak{B}^\circ_{\infty,0} \to \mathcal{B}(\mathbb{C}^2) \quad (9.12)$$

given on the generators of the C^*-algebra $\mathfrak{B}^\circ_{\infty,0}$ by

$$\begin{gathered}[\sigma_{\xi,\eta,x}((aI)^\circ_{\infty,0})]f = [\Psi_{\xi,\eta,x}(aI)]f, \quad [\sigma_{\xi,\eta,x}((W^0(b))^\circ_{\infty,0})] = [\Psi_{\xi,\eta,x}(W^0(b))]f, \\ [\sigma_{\xi,\eta,x}((U_{g_{k,h}})^\circ_{\infty,0})]f = e^{-ix\ln k} f \quad (a, b \in PSO^\diamond,\ g_{k,h} \in G,\ f \in \mathbb{C}^2).\end{gathered} \quad (9.13)$$

Applying Theorem 9.4 and (9.12)–(9.13), we immediately obtain the following invertibility criterion.

Theorem 9.5. *For each $B \in \mathfrak{B}$, the operator $B^\circ_{\infty,0} \in \mathfrak{B}^\circ_{\infty,0}$ is invertible on the space $\mathcal{H}^\circ_{\varphi,\infty,\mathbb{R}}$ if and only if for all $(\xi, \eta, x) \in \Delta^\circ_{\infty,0}$ the operators $\sigma_{\xi,\eta,x}(B^\circ_{\infty,0})$ are invertible on the space $\mathbb{C}^2$ and*

$$\sup_{(\xi,\eta,x)\in\Delta^\circ_{\infty,0}} \big\|(\sigma_{\xi,\eta,x}(B^\circ_{\infty,0}))^{-1}\big\|_{\mathcal{B}(\mathbb{C}^2)} < \infty.$$

For every $B \in \mathfrak{B}$ and every $(\xi, \eta, x) \in \Delta^\circ_{\infty,0}$, we put

$$[\mathrm{Sym}_0(B)](\xi, \eta, x) := \sigma_{\xi,\eta,x}\big(B^\circ_{\infty,0}\big), \quad (9.14)$$

where the representations $\sigma_{\xi,\eta,x}$ are given by (9.12) and (9.13). One can see from (9.13) that formulas (9.14) coincide with (3.12) on the generators of the C^*-algebra $\mathfrak{B}$. Consequently, Theorem 9.5 combined with (3.9) and (9.12)–(9.14) implies the following corollary by analogy with Corollary 6.10.

Corollary 9.6. *For each $B \in \mathfrak{B}$, the operator $B^\circ_{\infty,0} \in \mathfrak{B}^\circ_{\infty,0}$ is invertible on the space $\mathcal{H}^\circ_{\varphi,\infty,\mathbb{R}}$ if and only if condition* (iii) *of Theorem 3.2 holds. The map $\Phi_0 : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_{\infty,0})$ given by (3.7), (3.9) and (3.12) is a C^*-algebra homomorphism, and* $\ker \Phi_0 = \{B \in \mathfrak{B} : B^\circ_{\infty,0} = 0\}$.

10. The C^*-algebras $\mathfrak{B}_{\infty,0}$ and $\mathfrak{B}_{\infty,\infty}$

Let us study the invertibility in the C^*-algebras $\mathfrak{B}_{\infty,0}$ and $\mathfrak{B}_{\infty,\infty}$, where

$$\mathfrak{B}_{\infty,0} = P_\varphi(\Delta_{\infty,0})\varphi(\mathfrak{B}^\pi), \quad \mathfrak{A}_{\infty,0} := P_\varphi(\Delta_{\infty,0})\varphi(\mathfrak{A}^\pi), \quad \widetilde{\mathcal{Z}}_{\infty,0} := P_\varphi(\Delta_{\infty,0})\varphi(\widetilde{\mathcal{Z}}^\pi),$$
$$\mathfrak{B}_{\infty,\infty} = P_\varphi(\Delta_{\infty,\infty})\varphi(\mathfrak{B}^\pi), \quad \mathfrak{A}_{\infty,\infty} := P_\varphi(\Delta_{\infty,\infty})\varphi(\mathfrak{A}^\pi), \quad \widetilde{\mathcal{Z}}_{\infty,\infty} := P_\varphi(\Delta_{\infty,\infty})\varphi(\widetilde{\mathcal{Z}}^\pi),$$

$\Delta_{\infty,0} = \Omega_{\infty,0} \times \{\infty\}$ and $\Delta_{\infty,\infty} = \Omega_{\infty,\infty} \times \{\infty\}$. The C^*-algebras $\widetilde{\mathcal{Z}}_{\infty,0}$ and $\widetilde{\mathcal{Z}}_{\infty,\infty}$ are central subalgebras of the C^*-algebras $\mathfrak{A}_{\infty,0}$ and $\mathfrak{A}_{\infty,\infty}$, respectively. By [24, Subsection 5.1], $M(\widetilde{\mathcal{Z}}_{\infty,0}) = \Delta_{\infty,0}$ and $M(\widetilde{\mathcal{Z}}_{\infty,\infty}) = \Delta_{\infty,\infty}$. Moreover, $\widetilde{\mathcal{Z}}_{\infty,0}$ and $\widetilde{\mathcal{Z}}_{\infty,\infty}$ are also central subalgebras of $\mathfrak{B}_{\infty,0}$ and $\mathfrak{B}_{\infty,\infty}$, respectively.

Since $P_\varphi(\Delta_{\infty,0})\mathfrak{A}_{\infty,\mathbb{R}} = P_\varphi(\Delta_{\infty,0})\widetilde{\mathfrak{A}}_{\infty,\mathbb{R}}$ similarly to (9.1) and since $\Delta_{\infty,0} \in \mathfrak{R}_{\widetilde{G}}(\widetilde{\Delta}_{\infty,\mathbb{R}})$ consists of fixed points for all $\widehat{\beta}_k$ $(k \in \mathbb{R}_+)$ by Lemma 7.3, we infer that the C^*-algebra $\mathfrak{B}_{\infty,0}$ is commutative. Let $\mathcal{J}_{\xi,\eta}$ be the closed two-sided ideal of the C^*-algebra $\mathfrak{B}_{\infty,0}$ generated by the maximal ideal $(\xi, \eta, \infty) \in \Delta_{\infty,0}$ of $\widetilde{\mathcal{Z}}_{\infty,0}$. By the Allan-Douglas local principle related to $\widetilde{\mathcal{Z}}_{\infty,0}$ (see Theorem 4.1), we obtain the following.

Lemma 10.1. *An operator $B_{\infty,0} \in \mathfrak{B}_{\infty,0}$ is invertible on the space $\mathcal{H}_{\varphi,\infty,0}$ if and only if for every $(\xi, \eta) \in \Omega_{\infty,0}$ the coset $B_{\infty,0} + \mathcal{J}_{\xi,\eta}$ is invertible in the quotient algebra $\mathfrak{B}_{\infty,0}/\mathcal{J}_{\xi,\eta}$.*

With every $(\xi, \eta) \in \Omega_{\infty,0}$, every finite set $F \subset \mathbb{R}_+$ and every operator

$$B_{\infty,\mathbb{R}} = \sum_{k \in F} (A_k)_{\infty,\mathbb{R}}(U_{g_{k,0}})_{\infty,\mathbb{R}}, \quad \text{with} \quad (A_k)_{\infty,\mathbb{R}} \in \widetilde{\mathfrak{A}}_{\infty,\mathbb{R}}, \tag{10.1}$$

we associate four functional operators with constant coefficients, which are given by

$$T_{\xi,\eta,\pm\infty,i} := \sum_{k \in F} [\Psi_{\xi,\eta,\pm\infty}(A_k)]_{i,i} U_{g_{k,0}} \in \mathcal{B}(L^2(\mathbb{R})) \quad (i = 1, 2). \tag{10.2}$$

Lemma 10.2. *If the operator $B_{\infty,\mathbb{R}\setminus\{0\}} = P_\varphi(\dot{\Delta}_{\infty,\mathbb{R}\setminus\{0\}})B_{\infty,\mathbb{R}}$, where $B_{\infty,\mathbb{R}}$ is given by (10.1), is invertible on the space $\mathcal{H}_{\varphi,\infty,\mathbb{R}\setminus\{0\}}$, then for every $(\xi, \eta) \in \Omega_{\infty,0}$ and every $i = 1, 2$ the functional operators $T_{\xi,\eta,\pm\infty,i}$ given by (10.2) are invertible on the Hilbert space $L^2(\mathbb{R})$.*

Proof. By Corollary 8.4, the invertibility of the operator $B_{\infty,\mathbb{R}\setminus\{0\}}$ on the Hilbert space $\mathcal{H}_{\varphi,\infty,\mathbb{R}\setminus\{0\}}$ implies the invertibility on the space $l^2(\mathbb{R}_+, \mathbb{C}^2)$ of all the operators $\pi_{\xi,\eta,x}(B_{\infty,\mathbb{R}\setminus\{0\}})$ for $(\xi, \eta, x) \in \widehat{\Omega}_{\infty,0}$. Put

$$D_1 := \mathrm{diag}\,\big\{ \mathrm{diag}\{1, 0\} \big\}_{t \in \mathbb{R}_+}, \quad D_2 := \mathrm{diag}\,\big\{ \mathrm{diag}\{0, 1\} \big\}_{t \in \mathbb{R}_+}.$$

It is easily seen from (8.4), (2.6) and (10.2) that, for every $(\xi, \eta) \in \Omega_{\infty,0}$,

$$\pi_{\xi,\eta,+\infty}(B_{\infty,\mathbb{R}\setminus\{0\}}) = D_1 \pi_{\xi,\eta,+\infty}\big([T_{\xi,\eta,+\infty,1}]_{\infty,\mathbb{R}\setminus\{0\}}\big) D_1 I + D_2 \pi_{\xi,\eta,+\infty}\big([T_{\xi,\eta,+\infty,2}]_{\infty,\mathbb{R}\setminus\{0\}}\big) D_2 I,$$
$$\pi_{\xi,\eta,-\infty}(B_{\infty,\mathbb{R}\setminus\{0\}}) = D_1 \pi_{\xi,\eta,-\infty}\big([T_{\xi,\eta,-\infty,1}]_{\infty,\mathbb{R}\setminus\{0\}}\big) D_1 I + D_2 \pi_{\xi,\eta,-\infty}\big([T_{\xi,\eta,-\infty,2}]_{\infty,\mathbb{R}\setminus\{0\}}\big) D_2 I.$$

Hence, the invertibility of the operators $\pi_{\xi,\eta,\pm\infty}(B_{\infty,\mathbb{R}\setminus\{0\}})$ implies the invertibility of the operators

$$D_1 \pi_{\xi,\eta,\pm\infty}\big([T_{\xi,\eta,\pm\infty,1}]_{\infty,\mathbb{R}\setminus\{0\}}\big) D_1 I, \quad D_2 \pi_{\xi,\eta,\pm\infty}\big([T_{\xi,\eta,\pm\infty,2}]_{\infty,\mathbb{R}\setminus\{0\}}\big) D_2 I$$

on the spaces $D_1 l^2(\mathbb{R}_+, \mathbb{C}^2)$ and $D_2 l^2(\mathbb{R}_+, \mathbb{C}^2)$, respectively. Since the C^*-algebras of these operators are commutative and *-isomorphic to the C^*-algebra AP in view of (8.7), since the C^*-algebras $\mathcal{A}_{\xi,\eta,\pm\infty,i}$ $(i = 1, 2)$ generated by the operators (10.2) also are commutative and *-isomorphic to the C^*-algebra AP, and since the images in AP of such operators coincide in view of (8.6), we conclude that for every $(\xi, \eta) \in \Omega_{\infty,0}$ and every $i = 1, 2$ the invertibility of the operators $D_i \pi_{\xi,\eta,\pm\infty}\big([T_{\xi,\eta,\pm\infty,i}]_{\infty,\mathbb{R}\setminus\{0\}}\big) D_i I$ on the space $D_i l^2(\mathbb{R}_+, \mathbb{C}^2)$ is equivalent to the invertibility of the operators $T_{\xi,\eta,\pm\infty,i}$ on the space $L^2(\mathbb{R})$. $\square$

Applying spectral radii $r(\cdot)$, we infer from Lemma 10.2 that, for every invertible operator $B_{\infty,\mathbb{R}\setminus\{0\}}$,

$$\|T_{\xi,\eta,\pm\infty,i}\|^2_{\mathcal{B}(L^2(\mathbb{R}))} = r\big(T_{\xi,\eta,\pm\infty,i} T^*_{\xi,\eta,\pm\infty,i}\big) \le r(B_{\infty,\mathbb{R}\setminus\{0\}} B^*_{\infty,\mathbb{R}\setminus\{0\}}) = \|B_{\infty,\mathbb{R}\setminus\{0\}}\|^2_{\mathcal{B}(\mathcal{H}_{\varphi,\infty,\mathbb{R}\setminus\{0\}})} \tag{10.3}$$

for all $(\xi, \eta) \in \Omega_{\infty,0}$ and all $i = 1, 2$. Hence the maps $B_{\infty,\mathbb{R}\setminus\{0\}} \mapsto T_{\xi,\eta,\pm\infty,i}$ extend by continuity to C^*-algebra homomorphisms $\nu_{\xi,\eta,\pm,i} : \mathfrak{B}_{\infty,\mathbb{R}\setminus\{0\}} \to \mathcal{A}_{\xi,\eta,\pm\infty,i}$. Lemma 10.2 implies the following.

Corollary 10.3. *If an operator $B_{\infty,\mathbb{R}\setminus\{0\}} \in \mathfrak{B}_{\infty,\mathbb{R}\setminus\{0\}}$ is invertible on the space $\mathcal{H}_{\varphi,\infty,\mathbb{R}\setminus\{0\}}$, then for every $(\xi, \eta) \in \Omega_{\infty,0}$ and every $i = 1, 2$ the functional operators $T_{\xi,\eta,\pm\infty,i} = \nu_{\xi,\eta,\pm,i}(B_{\infty,\mathbb{R}\setminus\{0\}})$ are invertible on the Hilbert space $L^2(\mathbb{R})$.*

Theorem 10.4. *If $B \in \mathfrak{B}$ and the operator $B_{\infty,\mathbb{R}\setminus\{0\}}$ is invertible on the space $\mathcal{H}_{\varphi,\infty,\mathbb{R}\setminus\{0\}}$, then the operator $B_{\infty,0}$ is invertible on the space $\mathcal{H}_{\varphi,\infty,0}$.*

Proof. One can see that, for every operator $B \in \mathfrak{B}$ and every $(\xi, \eta) \in \Omega_{\infty,0}$,

$$\begin{aligned} B_{\infty,0} + \mathcal{J}_{\xi,\eta} = &\big[\chi_- W^0(\chi_+) T_{\xi,\eta,+\infty,1} + \chi_- W^0(\chi_-) T_{\xi,\eta,-\infty,1} \\ &+ \chi_+ W^0(\chi_-) T_{\xi,\eta,+\infty,2} + \chi_+ W^0(\chi_+) T_{\xi,\eta,-\infty,2}\big]_{\infty,0} + \mathcal{J}_{\xi,\eta}, \end{aligned} \tag{10.4}$$

where $\chi_\pm$ are the characteristic functions of $\mathbb{R}_\pm$. By Corollary 10.3, the invertibility of the operator $B_{\infty,\mathbb{R}\setminus\{0\}}$ on the Hilbert space $\mathcal{H}_{\varphi,\infty,\mathbb{R}\setminus\{0\}}$ implies the invertibility on the space $L^2(\mathbb{R})$ of the operators $T_{\xi,\eta,\pm\infty,i}$, which in turn implies the invertibility of the cosets $[T_{\xi,\eta,\pm\infty,i}]_{\infty,0} + \mathcal{J}_{\xi,\eta}$ for all $i = 1, 2$ and all $(\xi, \eta) \in \Omega_{\infty,0}$.

Taking a sequence of open sets $\Delta_n \subset \widetilde{\Delta}_{\infty,\mathbb{R}}$ such that $\bigcap_n \Delta_n = \Delta_{\infty,0}$, one can easily prove that for all $k \in \mathbb{R}_+$ the operators $[\chi_\pm I]_{\infty,0}$, $[W^0(\chi_\pm)]_{\infty,0}$ and $[U_{g_{k,0}}]_{\infty,0}$ pairwise commute and the operators $[\chi_- W^0(\chi_\pm)]_{\infty,0}$ and $[\chi_+ W^0(\chi_\pm)]_{\infty,0}$ are pairwise orthogonal projections on the space $\mathcal{H}_{\varphi,\infty,0}$. Hence, for every $(\xi, \eta) \in \Omega_{\infty,0}$, the inverse to the coset (10.4) has the form

$$\begin{aligned} &\big[\chi_- W^0(\chi_+)(T_{\xi,\eta,+\infty,1})^{-1} + \chi_- W^0(\chi_-)(T_{\xi,\eta,-\infty,1})^{-1} \\ &\quad + \chi_+ W^0(\chi_-)(T_{\xi,\eta,+\infty,2})^{-1} + \chi_+ W^0(\chi_+)(T_{\xi,\eta,-\infty,2})^{-1}\big]_{\infty,0} + \mathcal{J}_{\xi,\eta}. \end{aligned}$$

Finally, applying Lemma 10.1, we obtain the invertibility of the operator $B_{\infty,0}$ on the space $\mathcal{H}_{\varphi,\infty,0}$. $\square$

Since the set $\Delta_{\infty,\infty} \in \mathfrak{R}_G(\widetilde{\Delta})$ consists of fixed points of all homeomorphisms $\gamma_{k,h}$ $(k \in \mathbb{R}_+,\ h \in \mathbb{R})$ given by (5.15), we infer that the C^*-algebra $\widetilde{\mathcal{Z}}_{\infty,\infty} := P_\varphi(\Delta_{\infty,\infty})\varphi(\widetilde{\mathcal{Z}}^\pi)$ is invariant under the transform $Z_{\infty,\infty} \mapsto (U_{g_{k,h}})_{\infty,\infty} Z_{\infty,\infty} (U_{g_{k,h}})^{-1}_{\infty,\infty}$. Hence $\widetilde{\mathcal{Z}}_{\infty,\infty}$ is a central subalgebra of the C^*-algebra $\mathfrak{B}_{\infty,\infty}$. Let $\widetilde{\mathcal{J}}_{\xi,\eta}$ be the closed two-sided ideal of the C^*-algebra $\mathfrak{B}_{\infty,\infty}$ generated by the maximal ideal $(\xi, \eta, \infty) \in \Delta_{\infty,\infty}$ of $\widetilde{\mathcal{Z}}_{\infty,\infty}$. By the Allan-Douglas local principle related to the central algebra $\widetilde{\mathcal{Z}}_{\infty,\infty}$, we obtain the following.

Lemma 10.5. *An operator $B_{\infty,\infty}$ is invertible on the space $\mathcal{H}_{\varphi,\infty,\infty}$ if and only if for every $(\xi, \eta) \in \Omega_{\infty,\infty}$ the coset $B_{\infty,\infty} + \widetilde{\mathcal{J}}_{\xi,\eta}$ is invertible in the quotient algebra $\mathfrak{B}_{\infty,\infty}/\widetilde{\mathcal{J}}_{\xi,\eta}$.*

For every operator $B = \sum_{g\in F} A_g U_g \in \mathfrak{B}$, where $A_g \in \mathfrak{A}$ and F is a finite subset of G, and every point $(\xi, \eta) \in \Omega_{\infty,\infty}$, we define four functional operators with constant coefficients, which are given by

$$\mathcal{T}_{\xi,\eta,\pm\infty,i} := \sum_{g\in F} [\Psi_{\xi,\eta,\pm\infty}(A_g)]_{i,i} U_g \in \mathcal{B}(L^2(\mathbb{R})) \quad (i = 1, 2). \tag{10.5}$$

Lemma 10.6. *If the coset $B_{\mathbb{R},\infty,\mathfrak{H}} = \sum_{g\in F}(A_g U_g)_{\mathbb{R},\infty,\mathfrak{H}}$, where $A_g \in \mathfrak{A}$ and F is a finite subset of G, is invertible in the quotient C^*-algebra $\mathfrak{B}_{\mathbb{R},\infty,\mathfrak{H}}$, then for every $(\xi, \eta) \in \Omega_{\infty,\infty}$ and every $i = 1, 2$ the functional operators $\mathcal{T}_{\xi,\eta,\pm\infty,i}$ given by (10.5) are invertible on the Hilbert space $L^2(\mathbb{R})$.*

Proof. By Corollary 6.11, the invertibility of the coset $B_{\mathbb{R},\infty,\mathfrak{H}} \in \mathfrak{B}_{\mathbb{R},\infty,\mathfrak{H}}$ implies the invertibility of the operators $\widetilde{B}_{\xi,\mu,\eta,\nu} = \Pi_{\xi,\mu,\eta,\nu}(B_{\mathbb{R},\infty,\mathfrak{H}})$ on the space $l^2(G)$ for all $(\xi, \mu, \eta, \nu) \in (M_\infty(SO^\diamond) \times \{0, 1\})^2$. The latter holds if and only if the operators $B_{\xi,\mu,\eta,\nu} = \sum_{g\in F}[(A_g)_{\mathbb{R},\infty,\mathfrak{H}}](\xi, \mu, \eta, \nu)U_g$ are invertible on the space $L^2(\mathbb{R})$ for all $(\xi, \mu, \eta, \nu) \in (M_\infty(SO^\diamond) \times \{0, 1\})^2$. It is easily seen from (6.28) and (2.6) that

$$B_{\xi,1,\eta,1} = \sum_{g\in F}[\Psi_{\xi,\eta,+\infty}(A_g)]_{1,1}U_g, \quad B_{\xi,1,\eta,0} = \sum_{g\in F}[\Psi_{\xi,\eta,-\infty}(A_g)]_{1,1}U_g,$$
$$B_{\xi,0,\eta,0} = \sum_{g\in F}[\Psi_{\xi,\eta,+\infty}(A_g)]_{2,2}U_g, \quad B_{\xi,0,\eta,1} = \sum_{g\in F}[\Psi_{\xi,\eta,-\infty}(A_g)]_{2,2}U_g$$

for every $(\xi, \eta) \in \Omega_{\infty,\infty}$, which implies the invertibility of operators (10.5) on the space $L^2(\mathbb{R})$. $\square$

We now infer from Lemma 10.6 similarly to (10.3) that $\|\mathcal{T}_{\xi,\eta,\pm\infty,i}\|_{\mathcal{B}(L^2(\mathbb{R}))} \le \|B_{\mathbb{R},\infty,\mathfrak{H}}\|_{\mathfrak{B}_{\mathbb{R},\infty,\mathfrak{H}}}$ for every invertible coset $B_{\mathbb{R},\infty,\mathfrak{H}}$, all $(\xi, \eta) \in \Omega_{\infty,\infty}$ and all $i = 1, 2$. Hence the maps $B_{\mathbb{R},\infty} \mapsto B_{\mathbb{R},\infty,\mathfrak{H}} \mapsto \mathcal{T}_{\xi,\eta,\pm\infty,i}$ extend by continuity to C^*-algebra homomorphisms $\widetilde{\nu}_{\xi,\eta,\pm,i} : \mathfrak{B}_{\mathbb{R},\infty} \to \widetilde{\mathcal{A}}_{\xi,\eta,\pm\infty,i}$, where the C^*-algebras $\widetilde{\mathcal{A}}_{\xi,\eta,\pm\infty,i}$ $(i = 1, 2)$ are generated by the operators (10.5). Consequently, Lemma 10.6 implies the following.

Corollary 10.7. *If an operator $B_{\mathbb{R},\infty}$ is invertible on the space $\mathcal{H}_{\varphi,\mathbb{R},\infty}$, then for every $(\xi, \eta) \in \Omega_{\infty,\infty}$ and each $i = 1, 2$ the functional operators $\mathcal{T}_{\xi,\eta,\pm\infty,i} = \widetilde{\nu}_{\xi,\eta,\pm,i}(B_{\mathbb{R},\infty})$ are invertible on the Hilbert space $L^2(\mathbb{R})$.*

Theorem 10.8. *If $B \in \mathfrak{B}$ and the operator $B_{\mathbb{R},\infty}$ is invertible on the space $\mathcal{H}_{\varphi,\mathbb{R},\infty}$, then the operator $B_{\infty,\infty}$ is invertible on the space $\mathcal{H}_{\varphi,\infty,\infty}$.*

Proof. For every operator $B \in \mathfrak{B}$ and every $(\xi, \eta) \in \Omega_{\infty,\infty}$, the coset $B_{\infty,\infty} + \widetilde{\mathcal{J}}_{\xi,\eta}$ has the form

$$\begin{aligned} B_{\infty,\infty} + \widetilde{\mathcal{J}}_{\xi,\eta} = & \big[\chi_- W^0(\chi_-)\mathcal{T}_{\xi,\eta,+\infty,1} + \chi_- W^0(\chi_+)\mathcal{T}_{\xi,\eta,-\infty,1} \\ & + \chi_+ W^0(\chi_+)\mathcal{T}_{\xi,\eta,+\infty,2} + \chi_+ W^0(\chi_-)\mathcal{T}_{\xi,\eta,-\infty,2}\big]_{\infty,\infty} + \widetilde{\mathcal{J}}_{\xi,\eta}, \end{aligned} \tag{10.6}$$

where $\chi_\pm$ are the characteristic functions of $\mathbb{R}_\pm$. By Corollary 10.7, the invertibility of the operator $B_{\mathbb{R},\infty}$ on the Hilbert space $\mathcal{H}_{\varphi,\mathbb{R},\infty}$ implies the invertibility on the space $L^2(\mathbb{R})$ of the operators $\mathcal{T}_{\xi,\eta,\pm\infty,i}$ for all $(\xi, \eta) \in \Omega_{\infty,\infty}$ and all $i = 1, 2$. On the other hand, the latter implies the invertibility of the cosets $P_\varphi(\Delta_{\infty,\infty})\big([W^0(\mathcal{T}_{\xi,\eta,\pm\infty,i})]_{\infty,\mathbb{R}}\big) + \widetilde{\mathcal{J}}_{\xi,\eta}$ for $i = 1, 2$ and all $(\xi, \eta) \in \Omega_{\infty,\infty}$.

Taking a sequence of open sets $\Delta_n \subset \widetilde{\Delta}_{\mathbb{R},\infty}$ such that $\bigcap_n \Delta_n = \Delta_{\infty,\infty}$, one can easily infer from [19, Lemma 7.1] that the operators $[\chi_\pm I]_{\infty,\infty}$ and $[W^0(\chi_\pm)]_{\infty,\infty}$ pairwise commute and commute with each operator $[U_{g_{k,h}}]_{\infty,\infty}$ for $k > 0$ and $h \in \mathbb{R}$, and the operators $[\chi_- W^0(\chi_\pm)]_{\infty,\infty}$ and $[\chi_+ W^0(\chi_\pm)]_{\infty,\infty}$ are pairwise orthogonal projections on the space $\mathcal{H}_{\varphi,\infty,\infty}$. Hence, similarly to Theorem 10.4, the coset

$$\begin{aligned} & \big[\chi_- W^0(\chi_-)(\mathcal{T}_{\xi,\eta,+\infty,1})^{-1} + \chi_- W^0(\chi_+)(\mathcal{T}_{\xi,\eta,-\infty,1})^{-1} \\ & \quad + \chi_+ W^0(\chi_+)(\mathcal{T}_{\xi,\eta,+\infty,2})^{-1} + \chi_+ W^0(\chi_-)(\mathcal{T}_{\xi,\eta,-\infty,2})^{-1}\big]_{\infty,\infty} + \widetilde{\mathcal{J}}_{\xi,\eta} \end{aligned}$$

is the inverse to the coset (10.6) for every $(\xi, \eta) \in \Omega_{\infty,\infty}$. Finally, applying Lemma 10.5, we obtain the invertibility of the operator $B_{\infty,\infty}$ on the space $P_\varphi(\Delta_{\infty,\infty})\mathcal{H}_{\varphi,\infty,\mathbb{R}}$. $\square$

11. Proofs of the main theorems for the C^*-algebra $\mathfrak{B}$

Applying results of previous sections, we can now complete the proofs of the main results of the paper presented in Section 3.

First, we conclude from Theorems 10.4 and 10.8 that assertions (iv) and (v) of Theorem 5.4 follow, respectively, from assertions (ii) and (i) of this theorem, and therefore are superfluous. Theorem 3.1 directly follows from Theorems 6.6, 6.13 and Corollaries 8.3 and 9.6. Applying Theorem 5.4, we immediately infer Theorem 3.2 from Theorem 6.14 and Corollaries 8.3 and 9.6. Finally, we get Corollary 3.3 from Theorem 3.2.

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