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A remark on approximation with polynomials and greedy bases [☆]

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ABSTRACT

We investigate properties of the m-th error of approximation by polynomials with constant coefficients $\mathscr{D}_m(x)$ and with modulus-constant coefficients $\mathscr{D}_m^*(x)$ introduced by Berná and Blasco ([2]) to study greedy bases in Banach spaces. We characterize when $\liminf_m \mathscr{D}_m(x)$ and $\liminf_m \mathscr{D}_m^*(x)$ are equivalent to $\|x\|$ in terms of the democracy and superdemocracy functions, and provide sufficient conditions ensuring that $\lim_m \mathscr{D}_m^*(x) = \lim_m \mathscr{D}_m(x) = \|x\|$, extending previous very particular results.

1. Introduction

Let $(\mathbb{X}, \|\cdot\|)$ be a real Banach space and let $\mathscr{B} = (e_n)_{n=1}^\infty$ be a semi-normalized (Schauder) basis of $\mathbb{X}$ with biorthogonal functionals $(e_n^*)_{n=1}^\infty$, that is:

(i) There exist $a, b > 0$ such that $a \le \|e_n\|, \|e_n^*\| \le b$ for every $n \in \mathbb{N}$,
(ii) $e_k^*(e_n) = \delta_{kn}$ for every $k, n \in \mathbb{N}$,
(iii) The sequence of projections $P_m : \mathbb{X} \longrightarrow \mathbb{X}$ given by

$$P_m(x) = \sum_{n=1}^m e_n^*(x)\, e_n \ , \quad x \in \mathbb{X}$$

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satisfy $\lim_m \|P_m(x) - x\| = 0$ for every $x \in \mathbb{X}$. In this case, the *basis constant* of $\mathscr{B}$ is

$$K_b := \sup_{m\in\mathbb{N}} \|P_m\| < \infty\,.$$

We say that $\mathscr{B}$ is *monotone* whether $K_b = 1$.

Along the paper we will refer to every such $\mathscr{B}$ simply as a *basis*. Of course, as m increases $P_m(x)$ offers a good approximation of x by linear combinations of m-elements of the basis, but it is natural to ask whether a suitable rearrangement can provide better convergence rates. A natural proposal is the *Thresholding Greedy Algorithm* (TGA) introduced by S.V. Konyagin and V.N. Temlyakov ([8]): given $x \in \mathbb{X}$ we first consider the rearranging function $\rho : \mathbb{N} \longrightarrow \mathbb{N}$ satisfying that if $j < k$ then either $|e^*_{\rho(j)}(x)| > |e^*_{\rho(k)}(x)|$ or $|e^*_{\rho(j)}(x)| = |e^*_{\rho(k)}(x)|$ and $\rho(j) < \rho(k)$. The *m-th greedy sum* of x is then

$$\mathscr{G}_m(x) \;=\; \sum_{j=1}^{m} e^*_{\rho(j)}(x)\, e_{\rho(j)} \;=\; \sum_{k\in\Lambda_m(x)} e^*_k(x) e_k\,,$$

where $\Lambda_m(x) = \{\rho(j) : 1 \le j \le m\}$ is the *greedy set* of x with cardinality m. Related to this, S.V. Konyagin and V.N. Temlyakov defined in [8] the concepts of *greedy* and *quasi-greedy* bases.

Definition 1.1. We say that $\mathscr{B}$ is *quasi-greedy* if there exists a positive constant C_q such that

$$\|x - \mathscr{G}_m(x)\| \le C_q\|x\|,\ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}.$$

P. Wojtaszczyk proved in [11] that quasi-greediness is equivalent to the convergence of the algorithm, that is, $\mathscr{B}$ is quasi-greedy if and only if

$$\lim_{m\to+\infty} \|x - \mathscr{G}_m(x)\| = 0,\ \forall x \in \mathbb{X}.$$

Definition 1.2. We say that $\mathscr{B}$ is *greedy* if there exists a positive constant C such that

$$\|x - \mathscr{G}_m(x)\| \le C\sigma_m(x),\ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}, \tag{1}$$

where

$$\sigma_m(x, \mathscr{B})_{\mathbb{X}} = \sigma_m(x) := \inf\left\{ \left\| x - \sum_{n\in A} a_n e_n \right\| : a_n \in \mathbb{R}, A \subset \mathbb{N}, |A| = m \right\}.$$

S.V. Konyagin and V.N. Temlyakov [8] showed that, although every greedy basis is quasi-greedy, the converse does not hold (see also [1, Section 10.2]). They also characterize greedy bases as those which are unconditional and democratic. To define the last notion we have to introduce some notation. For each finite subset $A \subset \mathbb{N}$ and every scalar sequence $\varepsilon = (\varepsilon_n)$ with $|\varepsilon_n| = 1$ for each $n \in \mathbb{N}$ (we will write $|\varepsilon| = 1$, for simplicity) let us denote

$$\mathbf{1}_A := \sum_{n\in A} e_n \qquad \text{and} \qquad \mathbf{1}_{\varepsilon A} := \sum_{n\in A} \varepsilon_n\, e_n\,.$$

As usual, $|A|$ stands for the cardinal of A. We then define the *democracy functions* as

$$h_l(m) = \inf_{|A|=m} \|\mathbf{1}_A\| \quad , \quad h_r(m) = \sup_{|A|=m} \|\mathbf{1}_A\| \qquad (m \in \mathbb{N})\,,$$

and the *superdemocracy functions* as

$$h_l^*(m) = \inf_{|A|=m, |\varepsilon|=1} \|\mathbf{1}_{\varepsilon A}\| \quad , \quad h_r^*(m) = \sup_{|A|=m, |\varepsilon|=1} \|\mathbf{1}_{\varepsilon A}\| \qquad (m \in \mathbb{N}).$$

Definition 1.3. We say that $\mathscr{B}$ is *democratic* (resp. *superdemocratic*) if there exists $C > 0$ such that $h_r(m) \leq C\, h_l(m)$ (resp. $h_r^*(m) \leq C\, h_l^*(m)$) for every $m \in \mathbb{N}$.

More recently, another characterization of greedy bases has been provided by Ó. Blasco and the first author by means of the *best m-th error in the approximation using polynomials of constant* (*resp. modulus-constant*) *coefficients*:

$$\mathscr{D}_m(x, \mathscr{B})_{\mathbb{X}} = \mathscr{D}_m(x) := \inf\{\|x - \alpha \mathbf{1}_A\| : \alpha \in \mathbb{R},\ A \subset \mathbb{N},\ |A| = m\}$$

$$\mathscr{D}_m^*(x, \mathscr{B})_{\mathbb{X}} = \mathscr{D}_m^*(x) := \inf\{\|x - \alpha \mathbf{1}_{\varepsilon A}\| : \alpha \in \mathbb{R},\ A \subset \mathbb{N},\ |A| = m, |\varepsilon| = 1\}$$

Theorem 1.4. *[2, Corollary 1.8] Let $\mathscr{B}$ be a basis of a Banach space $\mathbb{X}$. The following assertions are equivalent:*

(i) *$\mathscr{B}$ is greedy,*
(ii) *There is $C > 0$ such that $\|x - \mathscr{G}_m(x)\| \leq C\, \mathscr{D}_m(x)$ for every $x \in \mathbb{X}$ and $m \in \mathbb{N}$.*
(iii) *There is $C > 0$ such that $\|x - \mathscr{G}_m(x)\| \leq C\, \mathscr{D}_m^*(x)$ for every $x \in \mathbb{X}$ and $m \in \mathbb{N}$.*

The striking feature of this theorem compared to (1) is that, while $\lim_m \sigma_m(x) = 0$ for every $x \in \mathbb{X}$, the terms $\mathscr{D}_m^*(x)$ and $\mathscr{D}_m(x)$ do not necessarily converge to zero if $x \neq 0$. Indeed, we have the following examples:

▷ [2, Theorem 3.2], [3, Theorem 1.4] If $\mathbb{X} = \mathbb{H}$ is a (separable) Hilbert space and $\mathscr{B}$ is an orthonormal basis, then

$$\lim_{m\to\infty} \mathscr{D}_m(x) = \lim_{m\to\infty} \mathscr{D}_m^*(x) = \|x\|, \qquad \text{for every } x \in \mathbb{H}. \tag{2}$$

▷ [2, Proposition 3.4] If $\mathbb{X} = \ell_p$ $(1 < p < \infty)$ and $\mathscr{B}$ is the canonical basis, then

$$\lim_{m\to+\infty} \mathscr{D}_m(\mathbf{1}_B) = \lim_{m\to+\infty} \mathscr{D}_m^*(\mathbf{1}_B) = \|\mathbf{1}_B\|, \qquad \text{for every finite } B \subset \mathbb{N}. \tag{3}$$

In the present paper, we aim to delve into this aspect. Let us briefly explain the structure of the paper. In Section 2 we show that $\mathscr{D}_m^*(x)$ and $\mathscr{D}_m(x)$ do not converge to zero as $m \to +\infty$ for any $x \neq 0$. In Section 3 we prove the main result of the paper (Theorem 3.2), namely a characterization of those bases $\mathscr{B}$ for which there is a positive constant $c > 0$ such that

$$c\|x\| \leq \liminf_{m\to+\infty} \mathscr{D}_m^*(x) \leq \limsup_{m\to+\infty} \mathscr{D}_m^*(x) \leq \|x\| \qquad \text{for every } x \in \mathbb{X},$$

in terms of the democracy and superdemocracy functions. We also provide a quite general condition ensuring that

$$\lim_{m\to+\infty} \mathscr{D}_m^*(x) = \|x\| \quad \text{ for every } x \in \mathbb{X}.$$

In Section 4 we deal with the notion of almost-greedy bases. We study how this property can be also characterized in terms of polynomials of constant or modulus-constant coefficients, extending a recent result of S.J. Dilworth and D. Khurana in [6].

Let us point out [1] as our basic reference for notation and fundamental results on greedy basis.

2. The limit of errors $\mathscr{D}_m^*(x)$ and $\mathscr{D}_m(x)$ is nonzero

Since $\mathscr{D}_m^*(x) \leq \mathscr{D}_m(x) \leq \|x\|$ for every $m \in \mathbb{N}$ and every $x \in \mathbb{X}$, it is only necessary to study lower bounds of $\mathscr{D}_m^*(x)$.

Proposition 2.1. *Let $\mathscr{B} = (e_n)_{n=1}^\infty$ be a basis of a Banach space $\mathbb{X}$. Then, for every $x \in \mathbb{X}$*

$$\frac{1}{4K_b} \sup_{n\in\mathbb{N}} |e_n^*(x)| \leq \liminf_{m\to\infty} \mathscr{D}_m^*(x)\,.$$

Proof. Let $x \in \mathbb{X}$. Note that for every finite set $A \subset \mathbb{N}$, $\alpha \in \mathbb{R}$ and $|\varepsilon| = 1$ it holds that

$$\|x - \alpha \mathbf{1}_{\varepsilon A}\| \geq \sup_{n\in\mathbb{N}} \frac{|e_n^*(x - \alpha \mathbf{1}_{\eta A})|}{\|e_n^*\|} \geq \frac{\sup_{n\in\mathbb{N}} |e_n^*(x - \alpha \mathbf{1}_{\varepsilon A})|}{2K_b} \geq \frac{\sup_{n\in\mathbb{N}} \big||e_n^*(x)| - |\alpha|\big|}{2K_b}\,.$$

Let us also fix $\delta > 0$ and $n_0 \in \mathbb{N}$ with the property that

$$|e_n^*(x)| \leq \delta \quad \text{for every } n \geq n_0\,.$$

If A satisfies $|A| > n_0$, then we can take $j \in A$ with $j > n_0$ and deduce that

$$\|x - \alpha \mathbf{1}_{\varepsilon A}\| \geq \frac{|e_j^*(x) - |\alpha||}{2K_b} \geq \frac{||\alpha| - \delta|}{2K_b}\,.$$

In particular, combining both lower estimations we get that for $|A| > n_0$

$$\|x - \alpha \mathbf{1}_{\varepsilon A}\| \geq \frac{||\alpha| - \delta| + \sup_{n\in\mathbb{N}} \big||e_n^*(x)| - |\alpha|\big|}{4K_b} \geq \sup_{n\in\mathbb{N}} \frac{|e_n^*(x)| - \delta}{4K_b}\,.$$

Therefore, for $m > n_0$

$$\mathscr{D}_m^*(x) \geq \sup_{n\in\mathbb{N}} \frac{|e_n^*(x)| - \delta}{4K_b}\,. \quad \square$$

3. Main result: equivalence with the norm

The issue of when $\liminf_m \mathscr{D}_m^*(x)$ (resp. $\liminf_m \mathscr{D}_m(x)$) is equivalent to $\|x\|$ is going to be determined by the behaviour of the superdemocracy functions (resp. democracy functions) defined in Section 1. Along the present section we are going to focus on proving the results for the superdemocracy case, namely for $h_l^*(m)$, $h_r^*(m)$ and the error $\mathscr{D}_m^*(x)$. The arguments for $h_l(m)$, h_r and the error $\mathscr{D}_m(x)$ are completely analogous. First of all, we recall some well-known estimates of the superdemocray functions valid for every basis in every space:

$$h_l^*(k) \leq K_b\, h_l^*(m)\,, \quad h_r^*(k) \leq K_b\, h_r^*(m) \quad \text{for every } k \leq m\,. \tag{4}$$

These relations together with the trivial inequality $h_l^*(m) \leq h_r^*(m)$ $(m \in \mathbb{N})$ yield that there are three possible cases:

- ▷ $h_l^*(m)$ and $h_r^*(m)$ are bounded.
- ▷ $h_l^*(m)$ is bounded and $h_r^*(m) \to +\infty$ as $m \to +\infty$.
- ▷ $h_l^*(m), h_r^*(m) \to +\infty$ as $m \to +\infty$.

Definition 3.1. The functions $h_l^*(m)$ and $h_r^*(m)$ (resp. $h_l(m)$ and $h_r(m)$) are said to be *comparable* if they are both bounded or divergent to infinity.

The main result of the section is the following theorem.

Theorem 3.2. *Let $\mathscr{B}$ be a basis of a Banach space $\mathbb{X}$. The following assertions are equivalent:*

(i) *There is a positive constant $c > 0$ such that*

$$c\,\|x\| \leq \liminf_{m\to+\infty} \mathscr{D}_m^*(x) \leq \limsup_{m\to+\infty} \mathscr{D}_m^*(x) \leq \|x\| \quad \textit{for every } x \in \mathbb{X}.$$

(ii) *$h_l^*(m)$ and $h_r^*(m)$ are comparable.*

Moreover, if $\mathscr{B}$ is monotone and $h_l^*(m) \to +\infty$ as $m \to +\infty$, then

$$\lim_{m\to+\infty} \mathscr{D}_m^*(x) = \|x\|\,. \tag{5}$$

(*The theorem also holds if we replace $\mathscr{D}_m^*(x)$, $h_l^*(m)$, $h_r^*(m)$ by $\mathscr{D}_m(x)$, $h_l(m)$, $h_r(m)$ respectively.*)

Before going into the proof let us show a few observations and examples:

- ▷ From Theorem 3.2 we can recover (2) and (3). Indeed, if $\mathbb{H}$ is a (separable) Hilbert space and $\mathscr{B}$ is an orthonormal basis of $\mathbb{H}$ then $h_l(m) = h_l^*(m) = m^{1/2}$. On the other hand, if $\mathbb{X} = \ell_p$ with $1 \leq p < \infty$ and $\mathscr{B}$ is the canonical basis, then $h_l(m) = h_l^*(m) = m^{1/p}$.
- ▷ For $\mathbb{X} = L_p[0,1]$ with $1 \leq p < \infty$ we have that the Haar basis $\mathscr{B}$ is monotone (see [7, Theorem 5.18]) and satisfies $h_l^*(m) = h_l(m) \approx m^{1/p}$ (see [1, p. 279]). Hence, it satisfies that $\lim_m \mathscr{D}_m^*(x) = \lim_m \mathscr{D}_m(x) = \|x\|$ for every $x \in \mathbb{X}$.
- ▷ If $\mathscr{B}$ is superdemocratic (resp. democratic), then it is clear that Theorem 3.2.(ii) (resp. the "democratic" version of Theorem 3.2.(ii) with $h_r(m)$ and $h_l(m)$) holds. However, the converse is not true. For instance, if $1 < p < q < \infty$, the canonical basis of $\ell_p \oplus_1 \ell_q$ satisfies that $h_l(m) = h_l^*(m) \approx m^{1/q}$ and $h_r(m) = h_r^*(m) \approx m^{1/p}$ (see [1, Example 10.4.4]).
- ▷ *Example of basis not satisfying Theorem 3.2.(ii)*: Let us consider $\mathbb{X} = \ell_1$ and let $\mathscr{B} = (\mathbf{x}_n)_{n=1}^{\infty}$ be the *difference basis*, which in terms of the canonical basis $(e_n)_{n=1}^{\infty}$ is given by

  $$\mathbf{x}_1 = e_1\,, \quad \mathbf{x}_n = e_n - e_{n-1}\,, \quad n = 2, 3, ...$$

  By [4, Lemma 8.1], it holds that $h_l^*(m) = h_l(m) = 1$ and $h_r^*(m) = h_r(m) = 2m$.
- ▷ *Example of basis satisfying* $\lim_m \mathscr{D}_m(x) = \|x\|$ *for every* $x \in \mathbb{X}$*, but* $\liminf_m \mathscr{D}_m^*(x)$ *is not even equivalent to* $\|x\|$: Let $\mathbb{X} = \mathbf{c}$ be the space of convergent sequences and let $\mathscr{B} = (\mathbf{s}_n)_{n=1}^{\infty}$ be the *summing basis*, defined as

  $$\mathbf{s}_n := (\underbrace{0, \ldots, 0}_{n-1}, 1, 1, \ldots)\,, \quad n \in \mathbb{N}\,.$$

By [4, Lemma 8.1] we know that $h_l^*(m) \approx 1$ and $h_r^*(m) \approx m$, so Theorem 3.2.(ii) does not hold. On the other hand, $\mathscr{B}$ is monotone and $h_l(m) \approx h_r(m) \approx m$ by the same reference. Thus, $\lim_m \mathscr{D}_m(x) = \|x\|$ for every $x \in \mathbb{X}$.

▷ *Condition Theorem 3.2.(ii) is not preserved for dual bases*: If $(e_n)_{n=1}^{\infty}$ is the canonical basis of ℓ_1, let us consider the sequence $\mathbf{x}_n = e_n - (e_{2n+1} + e_{2n+2})/2$, $n \in \mathbb{N}$ and the space

$$\mathbb{X} := \overline{\operatorname{span}\{\mathbf{x}_n : n \in \mathbb{N}\}}^{\ell_1}.$$

This is known as the *Lindenstrauss space* [9] and the sequence $\mathscr{B} = (\mathbf{x}_n)_{n=1}^{\infty}$ is actually a monotone basis for $\mathbb{X}$ (see [10, p. 457]). In [4, Section 8.2] it is shown that $h_l^*(m) \approx m$ and that the dual space $\mathbb{X}^*$ with the corresponding dual basis $\mathscr{B}^*$ satisfies $h_l^*(m) \approx 1$ and $h_r^*(m) \approx \ln(m)$.

3.1. Proof of the main result

Proposition 3.3. *Let $\mathscr{B}$ be a basis of a Banach space $\mathbb{X}$. Then,*

$$\sup_{\substack{A\subset\mathbb{N}\\ finite, |\eta|=1}} \liminf_{m\to+\infty} \mathscr{D}_m^*(\mathbf{1}_{\eta A}) \le (1+K_b) \liminf_{m\to+\infty} h_l^*(m) \le \infty, \tag{6}$$

$$\sup_{\substack{A\subset\mathbb{N}\\ finite}} \liminf_{m\to+\infty} \mathscr{D}_m(\mathbf{1}_{A}) \le (1+K_b) \liminf_{m\to+\infty} h_l(m) \le \infty. \tag{7}$$

Proof. We explain the argument for (6), as the proof of (7) is completely analogous with the obvious replacements. Let us fix a finite set $A \subset \mathbb{N}$ and $\eta \in \{\pm 1\}^A$, and let us take $\lambda \in \mathbb{R}$ satisfying

$$\lambda < \liminf_{m\to+\infty} \mathscr{D}_m^*(\mathbf{1}_{\eta A}). \tag{8}$$

From this condition, we can then find $m_0, n_0 \in \mathbb{N}$ with the following properties:

▷ $\lambda \le \|\mathbf{1}_{\eta A} - \alpha \mathbf{1}_{\varepsilon B}\|$ for every $\alpha \in \mathbb{R}$, every $B \subset \mathbb{N}$ with $|B| \ge m_0$ and every $\varepsilon \in \{\pm 1\}^B$,
▷ $A \subset \{1, \dots, n_0\}$.

Let $C \subset \mathbb{N}$ be a finite set with $|C| \ge m_0 + n_0$. Then,

$$\mathbf{1}_{\varepsilon C} - P_{n_0}(\mathbf{1}_{\varepsilon C}) = \mathbf{1}_{\varepsilon C'}$$

where $C' := C \setminus \{1, \dots, n_0\}$. Notice that $|C'| \ge m_0$, so in particular

$$\lambda \le \|\mathbf{1}_{\eta A} - \mathbf{1}_{(\eta A)\cup(\varepsilon C')}\| = \|\mathbf{1}_{\varepsilon C'}\| \le \|\operatorname{Id} - P_{n_0}\| \, \|\mathbf{1}_{\varepsilon C}\| \le (1+K_b)\, \|\mathbf{1}_{\varepsilon C}\|.$$

Thus, we have the relation

$$\lambda \le (1+K_b) \liminf_{m\to+\infty} h_l^*(m).$$

Taking supremums on λ satisfying (8) we conclude that

$$\liminf_{m\to+\infty} \mathscr{D}_m^*(\mathbf{1}_{\eta A}) \le (1+K_b) \liminf_{m\to+\infty} h_l^*(m). \quad \square$$

Theorem 3.4. *Let $\mathscr{B}$ be a basis of a Banach space $\mathbb{X}$. Assume that there is a constant $C > 0$ satisfying*

$$\sup_{n\in\mathbb{N}} h_r^*(n) \leq C \sup_{n\in\mathbb{N}} h_l^*(n) \leq \infty \,.$$

Then, for every $x \in \mathbb{X}$

$$\frac{1}{C + K_b(1+C)} \|x\| \leq \liminf_m \mathscr{D}_m^*(x) \leq \limsup_m \mathscr{D}_m^*(x) \leq \|x\| \,. \tag{9}$$

Proof. Let us fix $x \in \mathbb{X}$. We just have to show that the left hand-side of (9) holds. For, let $0 < \delta < 1$ and $m_0, n_0 \in \mathbb{N}$ such that

$$\|P_n(x) - x\| \leq \delta \|x\| \quad \text{ for every } n \geq n_0 \,,$$
$$h_r^*(n_0) \leq C\,(1-\delta)\, h_l^*(m_0) \,.$$

Given $\alpha \in \mathbb{R}$, $A \subset \mathbb{N}$ with $|A| \geq m_0 + n_0$ and $\varepsilon \in \{\pm 1\}^A$, we are going to establish two lower bounds for $\|x - \alpha \mathbf{1}_{\varepsilon A}\|$.

- $\triangleright$ Since $|A \cap (n_0, +\infty)| \geq m_0$ we can find $n \geq n_0$ such that $|A \cap (n, +\infty)| = m_0$. Thus, applying the operator $\mathrm{Id} - P_n$ to $x - \alpha \mathbf{1}_{\varepsilon A}$ we have that

$$\|x - \alpha \mathbf{1}_{\varepsilon A}\| \geq \frac{1}{K_b + 1} \|(\mathrm{Id} - P_n)(x) - \alpha \mathbf{1}_{\varepsilon(A\cap(n,+\infty))}\| \geq \frac{1}{K_b+1}\big(|\alpha|\, h_l^*(m_0) - \delta \,\|x\|\big) \,. \tag{10}$$

- $\triangleright$ As $|A| \geq n_0$ we can find $n \geq n_0$ with $|A \cap [1, n]| = n_0$, so that applying P_n to $x - \alpha \mathbf{1}_{\varepsilon A}$

$$\|x - \alpha \mathbf{1}_{\varepsilon A}\| \geq \frac{1}{K_b} \big(\|P_n(x) - \alpha \mathbf{1}_{\varepsilon(A\cap[1,n])}\|\big) \geq \frac{1}{K_b}\big(\|x\|(1-\delta) - |\alpha|\, h_r^*(n_0)\big) \tag{11}$$

$$\geq \frac{1-\delta}{K_b}\big(\|x\| - C\,|\alpha|\, h_l^*(m_0)\big) \tag{12}$$

Note that the lower estimations (10) and (12) are respectively increasing and decreasing linear functions $f(t)$ and $g(t)$ on $t = |\alpha|$. Moreover, these functions have a unique point of intersection $t_0 > 0$ which can be easily checked to satisfy

$$t_0 = \frac{\|x\|}{h_l^*(m_0)} \cdot \frac{(1-\delta)\,(1+K_b) + \delta\, K_b}{C(1-\delta)(1+K_b) + K_b} \,. \tag{13}$$

Thus

$$\|x - \alpha \mathbf{1}_{\varepsilon A}\| \geq \max\{f(|\alpha|), g(|\alpha|)\} \geq f(t_0) = g(t_0) = \frac{\|x\|}{1+K_b}\left[\frac{(1-\delta)\,(1+K_b) + \delta\, K_b}{C(1-\delta)(1+K_b) + K_b} - \delta\right] .$$

Taking the infimum of $\|x - \alpha \mathbf{1}_{\varepsilon A}\|$ on $\alpha \in \mathbb{R}$ and A satisfying the conditions above, we deduce that

$$\liminf_{k\to+\infty} \mathscr{D}_k^*(x) \geq \inf_{k \geq m_0+n_0} \mathscr{D}_k^*(x) \geq \frac{\|x\|}{1+K_b}\left[\frac{(1-\delta)\,(1+K_b) + \delta\, K_b}{C(1-\delta)(1+K_b) + K_b} - \delta\right] .$$

Finally, making $\delta \to 0^+$ we get the desired conclusion. $\square$

Proof of Theorem 3.2. To check (i) $\Rightarrow$ (ii), note that using Proposition 3.3 and the constant c of our hypothesis, we deduce that

$$\sup_{m\in\mathbb{N}} h_r^*(m) = \sup_{\substack{A\subset\mathbb{N}\\ finite, |\eta|=1}} \|\mathbf{1}_{\eta A}\| \leq \frac{1}{c} \sup_{\substack{A\subset\mathbb{N}\\ finite, |\eta|=1}} \liminf_{m\to+\infty} \mathscr{D}_m^*(\mathbf{1}_{\eta A}) \leq \frac{(1+K_b)}{c} \liminf_{m\to+\infty} h_l^*(m) \leq \infty.$$

It is clear from this inequality that $h_l^*(m)$ and $h_r^*(m)$ are then comparable. To see the converse (ii) $\Rightarrow$ (i), note first that if $h_l^*(m)$ and $h_r^*(m)$ are comparable, then there exists $C > 0$ such that

$$\sup_{m\in\mathbb{N}} h_r^*(m) \leq \sup_{m\in\mathbb{N}} C\, h_l^*(m)\,. \tag{14}$$

We can then apply Theorem 3.4 to conclude the result. Finally, the last statement of the theorem follows also from Theorem 3.4 since $\mathscr{B}$ being monotone means that $K_b = 1$, and condition $\lim_m h_l^*(m) = +\infty$ means that (14) holds for every $C > 0$. $\square$

4. Almost-greediness and polynomials with constant coefficients

Definition 4.1. Let $\mathscr{B} = (e_n)_{n=1}^\infty$ be a basis of a Banach space $\mathbb{X}$. We say that $\mathscr{B}$ is *almost-greedy* if there exists a constant $C > 0$ such that

$$\|x - \mathscr{G}_m(x)\| \leq C\,\widetilde{\sigma}_m(x)$$

where

$$\widetilde{\sigma}_m(x,\mathscr{B})_{\mathbb{X}} = \widetilde{\sigma}_m(x) := \inf\{\|x - \sum_{n\in A} e_n^*(x)\,e_n\| : A\subset\mathbb{N}, |A| = m\}.$$

This notion was introduced by S.J. Dilworth, N.J. Kalton, D. Kutzarova and V.N. Temlyakov in [5], together with two characterizations. A first characterization states that a basis is almost-greedy if and only if it is quasi-greedy and democratic. The second one is given in the next theorem.

Theorem 4.2 *([5, Theorem 3.3]). Let $\mathscr{B}$ be a basis of a Banach space $\mathbb{X}$. Then, $\mathscr{B}$ is almost-greedy if and only if for some (resp. every) $\lambda > 1$, there exists a positive constant C_λ such that*

$$\|x - \mathscr{G}_{[\lambda m]}(x)\| \leq C_\lambda \sigma_m(x)\,, \qquad \textit{for every } x\in\mathbb{X} \textit{ and } m\in\mathbb{N}.$$

Indeed, we can take $C_\lambda \approx \frac{1}{\lambda-1}$.

As in the case of greedy basis, we can replace the error $\sigma_m(x)$ by the m-th error of approximation by polynomials with constant (resp. modulus-constant) coefficients.

Theorem 4.3. *Let $\mathscr{B}$ be a basis of a Banach space $\mathbb{X}$ and let $\lambda > 1$. The following assertions are equivalent:*

(i) *$\mathscr{B}$ is almost-greedy.*
(ii) *There is $C_\lambda > 0$ such that $\|x - \mathscr{G}_{[\lambda m]}(x)\| \leq C_\lambda\, \mathscr{D}_m(x)$ for every $x\in\mathbb{X}$ and $m\in\mathbb{N}$.*
(iii) *There is $C_\lambda > 0$ such that $\|x - \mathscr{G}_{[\lambda m]}(x)\| \leq C_\lambda\, \mathscr{D}_m^*(x)$ for every $x\in\mathbb{X}$ and $m\in\mathbb{N}$.*

Proof. Implications (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are clear using Theorem 4.2 and the inequalities $\sigma_m(x) \leq \mathscr{D}_m^*(x) \leq \mathscr{D}_m(x)$. To show that (ii) $\Rightarrow$ (i) we follow ideas from the proof of Theorem 4.2: from the hypothesis we argue that $\mathscr{B}$ is democratic and quasi-greedy.

To see that it is democratic, let $n \in \mathbb{N}$ and let $A, B \subset \mathbb{N}$ with $|A| = n$ and $|B| = [\lambda n]$. Let us also fix a set $E \supset A \cup B$ with $|E| = n + [\lambda n]$, a positive number $\delta > 0$, and consider the element $x := (1+\delta)\mathbf{1}_{E\setminus A} + \mathbf{1}_A$. Hence, since $|E \setminus A| = [\lambda n]$, $|E \setminus B| = n$ and $E \setminus A$ is the greedy set of order $[\lambda n]$ of x, we obtain that

$$\|\mathbf{1}_A\| = \|x - \mathscr{G}_{[\lambda n]}(x)\| \le C_\lambda \|x - \mathbf{1}_{E\setminus B}\| = \|\mathbf{1}_B\| + \delta \|\mathbf{1}_{E\setminus A}\|.$$

As $\delta > 0$ is arbitrary, taking supremum over A and infimum over B we deduce that for every $n \in \mathbb{N}$

$$h_r(n) \le C_\lambda \, h_l([\lambda n])\,. \tag{15}$$

Given $m \in \mathbb{N}$ with $m \ge \lambda$ let us pick $n := [m/\lambda]$, which clearly satisfies $n < m < \lambda(n+1) \le 2\lambda n$. Then, using that $(h_r(m)/m)$ is non-increasing (see [1, Lemma 10.4.(b)]), relation (15) and the obvious relation $h_l(j) \le K_b h_l(k)$ valid for $j \le k$, we conclude that

$$h_r(m) \le \frac{m}{n} h_r(n) \le C_\lambda \frac{m}{n}\, h_l([\lambda n]) \le C_\lambda K_b \frac{m}{n}\, h_l(m) \le 2\,\lambda\, C_\lambda\, K_b\, h_l(m).$$

Let show now that the basis $\mathscr{B}$ is quasi-greedy. First take $m \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$ such that $[\lambda r] \le m < [\lambda(r+1)]$. Then,

$$\|x - \mathscr{G}_m(x)\| \le \|x - \mathscr{G}_{[\lambda r]}(x)\| + \|\mathscr{G}_{[\lambda r]}(x) - \mathscr{G}_m(x)\|\,.$$

Note that $\mathscr{G}_{[\lambda r]}(x) - \mathscr{G}_m(x)$ contains at most $m - [\lambda r] < \lambda$ summands of the form $e_n^*(x)\, e_n$, so that

$$\|\mathscr{G}_{[\lambda r]}(x) - \mathscr{G}_m(x)\| \le \big(\lambda \sup_{n\in\mathbb{N}} \|e_n\| \sup_{n\in\mathbb{N}} \|e_n^*\|\big)\, \|x\|\,.$$

On the other hand, using the hypothesis

$$\|x - \mathscr{G}_{[\lambda r]}(x)\| \le C_\lambda\, \mathscr{D}_r(x) \le C_\lambda\, \|x\|\,.$$

Thus, the basis is quasi-greedy. $\square$

Recently, S.J. Dilworth and D. Khurana provided the following characterization of almost-greedy bases in the same spirit of Theorem 1.4. In order to present it we have to introduce some notation. If $A, B \subset \mathbb{N}$ are finite sets, we will write $A < B$ whether $\max A < \min B$. For every $x \in \mathbb{X}$ and $m \in \mathbb{N}$ define

$$\mathscr{H}_m(x) := \inf\{\|x - \alpha \mathbf{1}_A\| \colon \alpha \in \mathbb{R}\,, |A| = m \text{ and either } A < \Lambda_m(x) \text{ or } A > \Lambda_m(x)\}$$

where recall that $\Lambda_m(x)$ is the m-th greedy set associated to x introduced in Section 1.

Theorem 4.4. *[6] Let $\mathscr{B}$ be a basis of a Banach space $\mathbb{X}$. Then, $\mathscr{B}$ is almost-greedy if and only if there exists $C > 0$ such that*

$$\|x - \mathscr{G}_m(x)\| \le C \inf_{1\le n\le m} \mathscr{H}_n(x) \quad \text{for every } x \in \mathbb{X} \text{ and } m \in \mathbb{N}.$$

Inspiring on the previous theorem, we can prove the following result which is again strinking, since $\mathscr{D}_m(x) \le \mathscr{H}_m(x)$ and so $\liminf \mathscr{H}_m(x) \approx \|x\|$ if $h_l(m)$ and $h_r(m)$ are comparable by Theorem 3.2.

Corollary 4.5. *Let $\mathscr{B}$ be a basis of a Banach space $\mathbb{X}$. Then, $\mathscr{B}$ is almost-greedy if and only if there exists $C > 0$ such that*

$$\|x - \mathscr{G}_m(x)\| \le C\, \mathscr{H}_m(x) \quad \text{for every } x \in \mathbb{X} \text{ and } m \in \mathbb{N}. \tag{16}$$

Proof. If $\mathscr{B}$ is quasi-greedy then (16) holds by Theorem 4.4. To see the converse we use the aforementioned characterization of almost-greedy bases as those being quasi-greedy and democratic. The fact that $\mathscr{B}$ is quasi-greedy follows from the hypothesis and the trivial inequality $\mathscr{H}_m(x) \leq \|x\|$. Let us show that $\mathscr{B}$ is democratic. Let $A, B \subset \mathbb{N}$ be finite subsets of cardinality m, and take $E \subset \mathbb{N}$ satisfying $|E| = m$, $A < E$ and $B < E$. Fixed $\delta > 0$ consider the elements $x = \mathbf{1}_A + (1+\delta)\mathbf{1}_E$ and $y = \mathbf{1}_E + (1+\delta)\mathbf{1}_B$. Then, applying (16)

$$\|\mathbf{1}_A\| = \|x - (1+\delta)\mathbf{1}_E\| = \|x - \mathscr{G}_m(x)\| \leq C\,\mathscr{H}_m(x) \leq C\,\|x - \mathbf{1}_A\| = C\,(1+\delta)\,\|\mathbf{1}_E\|\,.$$

Analogously,

$$\|\mathbf{1}_E\| = \|y - (1+\delta)\mathbf{1}_B\| = \|y - \mathscr{G}_m(y)\| \leq C\,\mathscr{H}_m(y) \leq C\|y - \mathbf{1}_E\| = C\,(1+\delta)\,\|\mathbf{1}_B\|\,.$$

Since $\delta > 0$ was arbitraty, we conclude that $h_r(m) \leq C^2\, h_l(m)$ for every $m \in \mathbb{N}$, and so the basis is democratic. □

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