J. Math. Anal. Appl. 480 (2019) 123357

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Quantum algebra from generalized q-Hermite polynomials

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ARTICLE INFO

Article history:
Received 4 May 2018
Available online 17 July 2019
Submitted by M.J. Schlosser

Keywords:
q-orthogonal polynomials
q-deformed algebras
Harmonic oscillators

ABSTRACT

In this paper, we discuss new results related to the generalized discrete q-Hermite II polynomials $\tilde{h}_{n,\alpha}(x;q)$, introduced by Mezlini et al. in 2014. Our aim is to give a continuous orthogonality relation, a q-integral representation and an evaluation at unity of the Poisson kernel, for these polynomials. Furthermore, we introduce q-Schrödinger operators and we give the raising and lowering operator algebra corresponding to these polynomials. Our results generate a new explicit realization of the quantum algebra $\mathsf{su}_q(1,1)$, using the generators associated with a q-deformed generalized para-Bose oscillator.

1. Introduction

The q-deformed harmonic oscillator algebras have been intensively studied in recent years due to their crucial role in diverse areas of mathematics and physics (see [9,10,15,16]). One of the most important applications of q-deformed algebras based theory arises from a generalization of the fundamental symmetry concept of the classical Lie algebras.

Many algebraic constructions were proposed in the literature to describe assorted generalizations of the quantum harmonic oscillator. However, a common difficulty for most of them is to derive an explicit form of associated Hamiltonian eigenfunctions. It is well known that Hermite polynomials are connected to the realization of classical-harmonic-oscillator algebras. It is worth to mention that generalizations of quantum harmonic oscillators lead to generalizations of q-Hermite polynomials. An explicit realization of q-harmonic oscillator has been explored by many authors (see for instance [4,5,7,15]), where the eigenfunctions of the corresponding Hamiltonian are given explicitly in terms of q-deformed Hermite polynomials. Generators of the corresponding oscillator algebra are realized in terms of first-order difference operators.

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https://doi.org/10.1016/j.jmaa.2019.07.047

This paper investigates the generalized discrete q-Hermite II polynomials $\tilde{h}_{n,\alpha}(x;q)$ to construct a new realization of quantum algebra. From this generalization, we obtain an explicit form of the generators for quantum algebra, in terms of q-difference operators.

The structure of this paper is as follows: Sec. 2 describes briefly the main definitions and properties of some q-basic special functions and operators [2,11]; Sec. 3 recalls some notations and useful results about the generalized discrete q-Hermite II polynomials [12]. Therefore, we obtain continuous orthogonality relations. Moreover, an integral representation of generalized discrete q-Hermite II-polynomials is proposed, and an evaluation at unity of the Poisson kernel for a family of polynomials $\tilde{h}_{n,\alpha}(x;q)$, is also studied. In addition to this, for $\alpha = 1/2$, a formula using q-trigonometric functions $Cos_q(x)$, $Sin_q(x)$, and an expression for the second q-Bessel functions in terms of the generalized discrete q-Hermite II-polynomials, is given. Among other things, by the specialization $x = iq^{\alpha+1/2}$ in the generating function of the even generalized q-Hermite polynomials, a special case of the Heine transformation of ${}_2\phi_1$ series [11, Appendix III, page 359, (III.3)], is recovered. Sec. 4 provides an explicit new realization of quantum algebra, in which the generators are associated with q-deformed generalized para-Bose oscillator.

2. Notations and preliminaries

This section is systematically organized in the following order; Sec. 2.1 introduces some basic notations; Sec. 2.2 recalls the definitions of q-derivatives and q-integrals; Sec. 2.3 recalls the definition of some q-special functions that are important in our paper.

2.1. Basic symbols

For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions, see [11,12]. Throughout this paper, we assume that $0 < q < 1$ and $\alpha > -1$. For each complex number a, we define q-shifted factorials, being

$$(a;q)_0 = 1; \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), n = 1, 2, ...; \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

The q-number and the q-factorial are defined as follows:

$$[\![x]\!]_q = \frac{1-q^x}{1-q}, \qquad x \in \mathbb{C} \quad \text{and} \quad n!_q = [\![1]\!]_q[\![2]\!]_q...[\![n]\!]_q, \qquad 0!_q = 1, \qquad n \in \mathbb{N}.$$

For each real $\alpha > -1$, the generalized q-integers and the generalized q-factorials are defined as:

$$\begin{aligned} [\![2n]\!]_{q,\alpha} &= [\![2n]\!]_q\,, \\ [\![2n+1]\!]_{q,\alpha} &= [\![2n+2\alpha+2]\!]_q\,, \\ n!_{q,\alpha} &= [\![1]\!]_{q,\alpha}\,[\![2]\!]_{q,\alpha}\cdots[\![n]\!]_{q,\alpha}\,, \quad 0!_{q,\alpha} = 1, \end{aligned} \tag{2.1}$$

and the generalized q-shifted factorials are defined as:

$$(q;q)_{n,\alpha} := (1-q)^n n!_{q,\alpha}. \tag{2.2}$$

We can write (2.2) explicitly as:

$$\begin{aligned} (q;q)_{2n,\alpha} &= (q^2;q^2)_n(q^{2\alpha+2};q^2)_n, \\ (q;q)_{2n+1,\alpha} &= (q^2;q^2)_n(q^{2\alpha+2};q^2)_{n+1}. \end{aligned} \tag{2.3}$$

Remark 2.1. The specific value $\alpha = -\frac{1}{2}$ leads to $(q;q)_{n,\alpha} = (q;q)_n$ and $n!_{q,\alpha} = n!_q$.

2.2. q-derivatives and q-integral

Jackson's q-derivative D_q (see [11,13]) is defined by:

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}. \tag{2.4}$$

The variant D_q^+, called forward q-derivative of the (backward) q-derivative $D_q^- = D_q$ (2.4), is defined as:

$$D_q^+ f(z) = \frac{f(q^{-1}z) - f(z)}{(1-q)z}. \tag{2.5}$$

Note that $\lim_{q\to 1^-} D_q f(z) = \lim_{q\to 1^-} D_q^+ f(z) = f'(z)$ whenever f is differentiable at z.

Generalized backward and forward q-derivative operators $D_{q,\alpha}$ and $D_{q,\alpha}^+$ are defined as (see [12])

$$D_{q,\alpha} f(z) = \frac{f(z) - q^{2\alpha+1} f(qz)}{(1-q)z}, \tag{2.6}$$

$$D_{q,\alpha}^+ f(z) = \frac{f(q^{-1}z) - q^{2\alpha+1} f(z)}{(1-q)z}. \tag{2.7}$$

Generalized q-derivatives operators are given by

$$\Delta_{q,\alpha} f = D_q f_e + D_{q,\alpha} f_o, \tag{2.8}$$

$$\Delta_{q,\alpha}^+ f = D_q^+ f_e + D_{q,\alpha}^+ f_o, \tag{2.9}$$

where f_e and f_o are respectively the even and the odd parts of f.

For $\alpha = -\frac{1}{2}$, we have $D_{q,\alpha} = D_q$, $D_{q,\alpha}^+ = D_q^+$, $\Delta_{q,\alpha} = D_q$ and $\Delta_{q,\alpha}^+ = D_q^+$.

We shall need the Jackson q-integral defined by (see [11,13])

$$\int_0^\infty f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n,$$

$$\int_{-\infty}^\infty f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} q^n f(q^n) + (1-q) \sum_{n=-\infty}^{\infty} q^n f(-q^n).$$

One can easily show that these integrals converge for a bounded function f, since the geometric series converges for $0 < q < 1$.

2.3. Some q-analogues of special functions

The two Euler's q-analogues of the exponential function are given by (see [11])

$$E_q(z) = \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}} z^k}{(q;q)_k} = (-z;q)_\infty, \tag{2.10}$$

$$e_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty}, \quad |z| < 1. \tag{2.11}$$

Then q-analogues of the trigonometric functions are defined as

$$Cos_q(z) = \frac{E_q(iz) + E_q(-iz)}{2}, \quad Sin_q(z) = \frac{E_q(iz) - E_q(-iz)}{2i}. \tag{2.12}$$

The generalized q-exponential function is defined as (see [12])

$$E_{q,\alpha}(z) := \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}} z^k}{(q;q)_{k,\alpha}}. \tag{2.13}$$

Using Remark 2.1, the specific value $\alpha = -\frac{1}{2}$ leads to $E_{q,\alpha}(z) = E_q(z)$.

The following q-Bessel functions can be expressed using generalized q-shifted factorials. The Jackson second q-Bessel function is defined as (see [11,14])

$$J_\alpha^{(2)}(x;q^2) = \frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty} \left(\frac{x}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n(n+\alpha)}}{(q;q)_{2n,\alpha}} \left(\frac{x}{2}\right)^{2n}. \tag{2.14}$$

The Hahn-Exton q-Bessel function is defined as (see [14, page 20, Formula (0.7.15)])

$$J_\alpha^{(3)}(x;q^2) = \frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty} x^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(q;q)_{2n,\alpha}} x^{2n}. \tag{2.15}$$

The modified q-Bessel function is defined as

$$j_\alpha(x;q^2) = \frac{(q^2;q^2)_\infty}{(q^{2\alpha+2};q^2)_\infty} x^{-\alpha} J_\alpha^{(3)}(x;q^2) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(q;q)_{2n,\alpha}} x^{2n}. \tag{2.16}$$

3. The generalized discrete q-Hermite II polynomials

The generalized discrete q-Hermite II polynomials $\{\tilde{h}_{n,\alpha}(x;q)\}_{n=0}^{\infty}$ are introduced by the first author et al. [12]. We recall their definition and some of their main properties. They are defined as

$$\tilde{h}_{n,\alpha}(x;q) := (q;q)_n \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k q^{-2nk} q^{k(2k+1)} x^{n-2k}}{(q^2;q^2)_k (q;q)_{n-2k,\alpha}}, \tag{3.1}$$

where $[x]$ denotes the integral part of $x \in \mathbb{R}$.

For $\alpha = -\frac{1}{2}$, $\tilde{h}_{n,\alpha}(x;q)$ reduces to the discrete q-Hermite II polynomial $\tilde{h}_n(x;q)$ (see [14]).

They have the following properties (see [12]):

The generating function:

$$e_{q^2}(-z^2) E_{q,\alpha}(xz) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q;q)_n} \tilde{h}_{n,\alpha}(x;q) z^n. \tag{3.2}$$

The inversion formula:

$$x^n = (q;q)_{n,\alpha} \sum_{k=0}^{[\frac{n}{2}]} \frac{q^{-2nk+3k^2} \tilde{h}_{n-2k,\alpha}(x;q)}{(q^2;q^2)_k (q;q)_{n-2k}}. \tag{3.3}$$

Forward shift operator:

$$\tilde{h}_{n,\alpha}(q^{-1}x;q) - q^{(2\alpha+1)\theta_{n+1}}\tilde{h}_{n,\alpha}(x;q) = q^{-n}(1-q^n)x\tilde{h}_{n-1,\alpha}(x;q), \tag{3.4}$$

where θ_n is defined to be 0 if n is odd and 1 if n is even.

Backward shift operator:

$$\tilde{h}_{n,\alpha}(x;q) - q^{(2\alpha+1)\theta_{n+1}}(1+q^{-2\alpha-1}x^2)\tilde{h}_{n,\alpha}(qx;q) = -q^n\frac{1-q^{-n-1-(2\alpha+1)\theta_n}}{1-q^{-n-1}}x\tilde{h}_{n+1,\alpha}(x;q). \tag{3.5}$$

Three terms recursion formula:

$$x\tilde{h}_{n,\alpha}(x;q) - q^{1-2n}(1-q^n)\tilde{h}_{n-1,\alpha}(x;q) = \frac{1-q^{n+1+(2\alpha+1)\theta_n}}{1-q^{n+1}}\tilde{h}_{n+1,\alpha}(x;q). \tag{3.6}$$

q-Difference equations:

$$(1+q^{-2\alpha-1}x^2)\tilde{h}_{2n,\alpha}(qx;q) - (1+q^{-2\alpha}+q^{2n-2\alpha-1}x^2)\tilde{h}_{2n,\alpha}(x;q) + q^{-2\alpha}\tilde{h}_{2n,\alpha}(q^{-1}x;q) = 0 \tag{3.7}$$

and

$$(1+q^{-2\alpha-1}x^2)\tilde{h}_{2n+1,\alpha}(qx;q) - (q+q^{-2\alpha-1}+q^{2n-2\alpha}x^2)\tilde{h}_{2n+1,\alpha}(x;q) + q^{-2\alpha}\tilde{h}_{2n+1,\alpha}(q^{-1}x;q) = 0. \tag{3.8}$$

The family of generalized discrete q-Hermite II polynomials satisfy two kind of orthogonality relations, a discrete one and a continuous one. As was shown in [12], we have:

A discontinuous orthogonality relation:

$$\int_{-\infty}^{\infty} \tilde{h}_{n,\alpha}(x;q)\tilde{h}_{m,\alpha}(x;q)\omega_\alpha(x;q)|x|^{2\alpha+1}d_qx = \frac{2(1-q)(-q,-q,q^2;q^2)_\infty q^{-n^2}\,(q;q)_n^2}{(-q^{-2\alpha-1},-q^{2\alpha+3},q^{2\alpha+2};q^2)_\infty\,(q;q)_{n,\alpha}}\delta_{n,m}, \tag{3.9}$$

where

$$\omega_\alpha(x;q) = e_{q^2}(-q^{-2\alpha-1}x^2) \tag{3.10}$$

and $\delta_{n,m}$ is the Kronecker symbol.

3.1. A continuous orthogonality relation

Our primary interest in this paper is to prove a continuous orthogonality relation for the family of generalized discrete q-Hermite II polynomials. First, we rewrite the q-Laguerre polynomials (see [14]) by means of the generalized q-shifted factorials as follows:

$$L_n^{(\alpha)}(x;q^2) = (q^{2\alpha+2};q^2)_n\sum_{k=0}^{n}\frac{(-1)^k q^{2k(k+\alpha)}x^k}{(q;q)_{2k,\alpha}(q^2;q^2)_{n-k}}.$$

The discrete q-Hermite II polynomials $\tilde{h}_{n,\alpha}(x;q)$ can also be expressed in terms of q-Laguerre polynomials $L_n^{(\alpha)}(x;q)$ as follows (see [12]):

$$\begin{cases} \tilde{h}_{2n,\alpha}(x;q) &= (-1)^n q^{-n(2n-1)} \frac{(q;q)_{2n}}{(q^{2\alpha+2};q^2)_n} L_n^{(\alpha)}(q^{-2\alpha-1}x^2;q^2), \\ \tilde{h}_{2n+1,\alpha}(x;q) &= (-1)^n q^{-n(2n+1)} \frac{(q;q)_{2n+1}}{(q^{2\alpha+2};q^2)_{n+1}} x L_n^{(\alpha+1)}(q^{-2\alpha-1}x^2;q^2). \end{cases} \tag{3.11}$$

The q-Laguerre polynomials satisfy the following orthogonality relations (see [14]):

$$\int_0^\infty L_n^{(\alpha)}(x;q^2) L_m^{(\alpha)}(x;q^2) x^\alpha e_{q^2}(-x)dx = \Gamma(-\alpha)\Gamma(\alpha+1) \frac{(q^{-2\alpha};q^2)_\infty (q^{2\alpha+2};q^2)_n}{(q^2;q^2)_\infty (q^2;q^2)_n} q^{-2n} \delta_{n,m}. \tag{3.12}$$

Note that the orthogonality measure in (3.12) is not unique.

Theorem 3.1. *The q-polynomials $\{\tilde{h}_{n,\alpha}(x;q)\}_{n=0}^\infty$ satisfy the following continuous orthogonality relations:*

$$\int_{-\infty}^\infty \tilde{h}_{n,\alpha}(x;q)\tilde{h}_{m,\alpha}(x;q)|x|^{2\alpha+1}\omega_\alpha(x;q)dx = d_{n,\alpha}^{-2}\delta_{n,m}, \tag{3.13}$$

where

$$\omega_\alpha(x;q) = e_{q^2}(-q^{-2\alpha-1}x^2) \tag{3.14}$$

and

$$d_{n,\alpha} = C_\alpha q^{\frac{n^2}{2}} \frac{(q;q)_{n,\alpha}^{\frac{1}{2}}}{(q;q)_n}, \quad C_\alpha = \sqrt{\frac{q^{-(\alpha+1)(2\alpha+1)}(q^2;q^2)_\infty}{\Gamma(-\alpha)\Gamma(\alpha+1)(q^{-2\alpha};q^2)_\infty}}. \tag{3.15}$$

Proof. Since the weight function in the integral (3.13) is an even function of the independent variable x and the parity of the q-polynomials $\{\tilde{h}_{n,\alpha}(x;q)\}_{n=0}^\infty$ is the parity of their degrees, it suffices to prove only those cases in (3.13), when degrees of polynomials m and n are either simultaneously even or odd.

First, we consider the even case: it follows from (3.11) that

$$\begin{aligned} &\int_{-\infty}^\infty \tilde{h}_{2n,\alpha}(x;q)\tilde{h}_{2m,\alpha}(x;q)|x|^{2\alpha+1}\omega_\alpha(x;q)dx \\ &= A_{n,m}^\alpha \int_0^\infty L_n^{(\alpha)}(q^{-2\alpha-1}x^2;q^2) L_m^{(\alpha)}(q^{-2\alpha-1}x^2;q^2)\omega_\alpha(x;q)x^{2\alpha+1}dx, \end{aligned} \tag{3.16}$$

where

$$A_{n,m}^\alpha = 2(-1)^{n+m} q^{-n(2n-1)-m(2m-1)} \frac{(q;q)_{2n}(q;q)_{2m}}{(q^{2\alpha+2};q^2)_n (q^{2\alpha+2};q^2)_m}.$$

The change of variable $t = q^{-2\alpha-1}x^2$ in the last integral in (3.16) leads to

$$q^{(\alpha+1)(2\alpha+1)} \frac{A_{n,m}^\alpha}{2} \int_0^\infty L_n^{(\alpha)}(t;q^2) L_m^{(\alpha)}(t;q^2) e_{q^2}(-t) t^\alpha dt.$$

By relation (3.12), it follows that

$$\begin{aligned}&\int_{-\infty}^{\infty}\tilde{h}_{2n,\alpha}(x;q)\tilde{h}_{2m,\alpha}(x;q)|x|^{2\alpha+1}\omega_\alpha(x;q)dx\\&=\Gamma(-\alpha)\Gamma(\alpha+1)q^{(\alpha+1)(2\alpha+1)}\frac{A^\alpha_{n,m}}{2}\frac{(q^{-2\alpha};q^2)_\infty(q^{2\alpha+2};q^2)_n}{(q^2;q^2)_\infty(q^2;q^2)_n}q^{-2n}\delta_{n,m}\\&=\Gamma(-\alpha)\Gamma(\alpha+1)q^{(\alpha+1)(2\alpha+1)}\frac{(q^{-2\alpha};q^2)_\infty(q;q)^2_{2n}}{(q^2;q^2)_\infty(q^{2\alpha+2};q^2)_n(q^2;q^2)_n}q^{-(2n)^2}\delta_{n,m},\end{aligned}$$

then, using (2.3) we obtain the result in the case n even. The odd case is obtained similarly. We have

$$\begin{aligned}&\int_{-\infty}^{\infty}\tilde{h}_{2n+1,\alpha}(x;q)\tilde{h}_{2m+1,\alpha}(x;q)|x|^{2\alpha+1}\omega_\alpha(x;q)dx\\&=B^\alpha_{n,m}\int_0^\infty L_n^{(\alpha+1)}(q^{-2\alpha-1}x^2;q^2)L_m^{(\alpha+1)}(q^{-2\alpha-1}x^2;q^2)\omega_\alpha(x;q)x^{2\alpha+3}dx,\end{aligned}\tag{3.17}$$

where

$$B^\alpha_{n,m}=2(-1)^{n+m}q^{-n(2n+1)-m(2m+1)}\frac{(q;q)_{2n+1}(q;q)_{2m+1}}{(q^{2\alpha+2};q^2)_{n+1}(q^{2\alpha+2};q^2)_{m+1}}.$$

The change of variable $t=q^{-2\alpha-1}x^2$ in the last integral in (3.17) leads to

$$q^{(\alpha+2)(2\alpha+1)}\frac{B^\alpha_{n,m}}{2}\int_0^\infty L_n^{(\alpha+1)}(t;q^2)L_m^{(\alpha+1)}(t;q^2)e_{q^2}(-t)t^{\alpha+1}dt.$$

By relation (3.12), it follows that

$$\begin{aligned}&\int_{-\infty}^{\infty}\tilde{h}_{2n+1,\alpha}(x;q)\tilde{h}_{2m+1,\alpha}(x;q)|x|^{2\alpha+1}\omega_\alpha(x;q)dx\\&=\Gamma(-\alpha-1)\Gamma(\alpha+2)q^{(\alpha+2)(2\alpha+1)}\frac{B^\alpha_{n,m}}{2}\frac{(q^{-2\alpha-2};q^2)_\infty(q^{2\alpha+4};q^2)_n}{(q^2;q^2)_\infty(q^2;q^2)_n}q^{-2n}\delta_{n,m}\\&=\Gamma(-\alpha-1)\Gamma(\alpha+2)q^{(\alpha+1)(2\alpha+1)}\frac{(q^{-2\alpha-2};q^2)_\infty(q^{2\alpha+4};q^2)_n(q;q)^2_{2n+1}}{(q^2;q^2)_\infty(q^{2\alpha+2};q^2)^2_{n+1}(q^2;q^2)_n}q^{-4n(n+1)}\delta_{n,m}.\end{aligned}$$

Using the fact that

$$\begin{gathered}\Gamma(-\alpha-1)\Gamma(\alpha+2)=-\Gamma(-\alpha)\Gamma(\alpha+1),\\(q^{-2\alpha-2};q^2)_\infty=-q^{-2\alpha-2}(1-q^{2\alpha+2})(q^{-2\alpha};q^2)_\infty\end{gathered}$$

and (2.3), we get

$$\begin{aligned}&\int_{-\infty}^{\infty}\tilde{h}_{2n+1,\alpha}(x;q)\tilde{h}_{2m+1,\alpha}(x;q)|x|^{2\alpha+1}\omega_\alpha(x;q)dx\\&=\Gamma(-\alpha)\Gamma(\alpha+1)q^{(\alpha+1)(2\alpha+1)}\frac{(q^{-2\alpha};q^2)_\infty(q;q)^2_{2n+1}}{(q^2;q^2)_\infty(q;q)_{2n+1,\alpha}}q^{-(2n+1)^2}\delta_{n,m},\end{aligned}$$

us desired. □

3.2. The q-integral representation

We start this section by describing the action of some of q-derivatives operators on powers of x and on some q-analogues of Bessel functions.

Proposition 3.1. *The following statements hold:*

$$\Delta_{q,\alpha}^{k} x^{n} = \frac{(q;q)_{n,\alpha}}{(1-q)^{k}(q;q)_{n-k,\alpha}} x^{n-k}, \quad n \geq k. \tag{3.18}$$

$$D_q j_\alpha(\lambda x; q^2) = \frac{-q^2\lambda^2 x}{(1-q)(1-q^{2\alpha+2})} j_{\alpha+1}(q\lambda x; q^2). \tag{3.19}$$

$$\Delta_{q,\alpha}^{2n} j_\alpha(\lambda x; q^2) = \frac{(-1)^n q^{n(n+1)}\lambda^{2n}}{(1-q)^{2n}} j_\alpha(q^n\lambda x; q^2). \tag{3.20}$$

$$\Delta_{q,\alpha}^{2n+1} j_\alpha(\lambda x; q^2) = \frac{(-1)^{n+1} q^{(n+1)(n+2)}\lambda^{2n+2}}{(1-q)^{2n+1}(1-q^{2\alpha+2})} x j_{\alpha+1}(q^{n+1}\lambda x; q^2). \tag{3.21}$$

Proof. From definitions (2.3), (2.4) and by induction, we get (3.18).

Using the fact that

$$(q;q)_{2n,\alpha} = (1-q^{2n})(q;q)_{2n-1,\alpha} \text{ and } (q;q)_{2n-1,\alpha} = (1-q^{2\alpha+2})(q;q)_{2n-2,\alpha+1},$$

we deduce (3.19).

By (2.16) and the result in (3.18) we obtain (3.20).

By definition (2.8), we have $\Delta_{q,\alpha}^{2n+1} j_\alpha(x;q^2) = D_q\left[\Delta_{q,\alpha}^{2n} j_\alpha\right](x;q^2)$. Together with (3.19) and (3.20) we get (3.21). $\square$

We have (see [12, p. 24, Lemma 5.1])

$$\int_0^\infty e_{q^2}(-qy^2) y^{2n+2\alpha+1} d_q y = c_{q,\alpha} q^{-n^2-2\alpha} \left(q^{2\alpha+2}; q^2\right)_n, \tag{3.22}$$

where

$$c_{q,\alpha} = \frac{(1-q)(-q^{2\alpha+3}, -q^{-2\alpha-1}, q^2; q^2)_\infty}{(-q, -q, q^{2\alpha+2}; q^2)_\infty}. \tag{3.23}$$

An important formula used later is

Lemma 3.1.

$$\int_0^\infty e_{q^2}(-qy^2) j_\alpha(xy; q^2) y^{2\alpha+1} d_q y = c_{q,\alpha} e_{q^2}(-q^{-2\alpha-1}x^2). \tag{3.24}$$

Proof. Expand $j_\alpha(xy;q^2)$ as power series and integrate term by term and use (3.22) to conclude (3.24). $\square$

Now we provide a q-integral representation of generalized discrete q-Hermite II polynomials.

Theorem 3.2. *For $n = 0, 1, 2, ...,$ we have*

$$\begin{aligned}\tilde{h}_{2n,\alpha}(x;q) &= \frac{(-1)^n q^{-n^2+n(2\alpha+3)}(q;q)_{2n}}{c_{q,\alpha}(q;q)_{2n,\alpha} e_{q^2}(-q^{-2\alpha-1}x^2)} \\ &\quad \times \int_0^\infty e_{q^2}(-qy^2) j_\alpha(q^n xy;q^2) y^{2n+2\alpha+1} d_q y.\end{aligned} \tag{3.25}$$

$$\begin{aligned}\tilde{h}_{2n+1,\alpha}(x;q) &= \frac{(-1)^{n+1} q^{-n^2+(n+1)(2\alpha+3)}(q;q)_{2n+1} x}{c_{q,\alpha}(1-q^{2\alpha+2})(q;q)_{2n+1,\alpha} e_{q^2}(-q^{-2\alpha-1}x^2)} \\ &\quad \times \int_0^\infty e_{q^2}(-qy^2) j_{\alpha+1}(q^{n+1} xy;q^2) y^{2n+2\alpha+3} d_q y.\end{aligned} \tag{3.26}$$

Proof. We recall the Rodrigues-type formula (see [12])

$$e_{q^2}(-q^{-2\alpha-1}x^2)\tilde{h}_{n,\alpha}(x;q) = \frac{(q-1)^n q^{\frac{-n(n-1)}{2}}(q^{-1};q^{-1})_n}{(q^{-1};q^{-1})_{n,\alpha}} \Delta_{q,\alpha}^n e_{q^2}(-q^{-2\alpha-1}x^2). \tag{3.27}$$

From (3.24), we obtain

$$\int_0^\infty e_{q^2}(-qy^2)\Delta_{q,\alpha}^n j_\alpha(xy;q^2) y^{2\alpha+1} d_q y = c_{q,\alpha}\Delta_{q,\alpha}^n e_{q^2}(-q^{-2\alpha-1}x^2).$$

From (3.20) and (3.27), we get

$$\begin{aligned}&\frac{(-1)^n q^{n(n+1)}}{(1-q)^{2n}} \int_0^\infty e_{q^2}(-qy^2) j_\alpha(q^n xy;q^2) y^{2n+2\alpha+1} d_q y \\ &= c_{q,\alpha} \frac{q^{n(2n-1)}(q^{-1};q^{-1})_{2n,\alpha}}{(q-1)^{2n}(q^{-1};q^{-1})_{2n}} e_{q^2}(-q^{-2\alpha-1}x^2)\tilde{h}_{2n,\alpha}(x;q).\end{aligned}$$

Using the fact that $\frac{(q^{-1};q^{-1})_{2n,\alpha}}{(q^{-1};q^{-1})_{2n}} = q^{-n(2\alpha+1)}\frac{(q;q)_{2n,\alpha}}{(q;q)_{2n}}$, we obtain

$$\begin{aligned}&(-1)^n \int_0^\infty e_{q^2}(-qy^2) j_\alpha(q^n xy;q^2) y^{2n+2\alpha+1} d_q y \\ &= c_{q,\alpha} \frac{q^{n(n-2\alpha-3)}(q;q)_{2n,\alpha}}{(q;q)_{2n}} e_{q^2}(-q^{-2\alpha-1}x^2)\tilde{h}_{2n,\alpha}(x;q).\end{aligned}$$

Hence,

$$\begin{aligned}\tilde{h}_{2n,\alpha}(x;q) &= \frac{(-1)^n q^{-n(n-2\alpha-3)}(q;q)_{2n}}{c_{q,\alpha}(q;q)_{2n,\alpha} e_{q^2}(-q^{-2\alpha-1}x^2)} \\ &\quad \times \int_0^\infty e_{q^2}(-qy^2) j_\alpha(q^n xy;q^2) y^{2n+2\alpha+1} d_q y.\end{aligned}$$

Further,

$$\int_0^\infty e_{q^2}(-qy^2)\Delta_{q,\alpha}^{2n+1} j_\alpha(xy;q^2)y^{2\alpha+1}d_qy = c_{q,\alpha}\Delta_{q,\alpha}^{2n+1}e_{q^2}(-q^{-2\alpha-1}x^2).$$

From (3.21) and (3.27) we get

$$\frac{(-1)^{n+1}q^{(n+1)(n+2)}x}{(1-q)^{2n+1}(1-q^{2\alpha+2})}\int_0^\infty e_{q^2}(-qy^2)j_{\alpha+1}(q^{n+1}xy;q^2)y^{2n+2\alpha+3}d_qy$$
$$= c_{q,\alpha}\frac{q^{n(2n+1)}(q^{-1};q^{-1})_{2n+1,\alpha}}{(q-1)^{2n+1}(q^{-1};q^{-1})_{2n+1}}e_{q^2}(-q^{-2\alpha-1}x^2)\tilde{h}_{2n+1,\alpha}(x;q).$$

Using the fact that $\frac{(q^{-1};q^{-1})_{2n+1,\alpha}}{(q^{-1};q^{-1})_{2n+1}} = q^{-(n+1)(2\alpha+1)}\frac{(q;q)_{2n+1,\alpha}}{(q;q)_{2n+1}}$, we obtain

$$\frac{(-1)^{n+1}q^{(n+1)(n+2)}x}{(1-q)^{2n+1}(1-q^{2\alpha+2})}\int_0^\infty e_{q^2}(-qy^2)j_{\alpha+1}(q^{n+1}xy;q^2)y^{2n+2\alpha+3}d_qy$$
$$= c_{q,\alpha}\frac{q^{n(2n+1)-(n+1)(2\alpha+1)}(q;q)_{2n+1,\alpha}}{(q-1)^{2n+1}(q;q)_{2n+1}}e_{q^2}(-q^{-2\alpha-1}x^2)\tilde{h}_{2n+1,\alpha}(x;q).$$

Thus,

$$\tilde{h}_{2n+1,\alpha}(x;q) = \frac{(-1)^{n+1}q^{-n^2+(n+1)(2\alpha+3)}(q;q)_{2n+1}x}{c_{q,\alpha}(1-q^{2\alpha+2})(q;q)_{2n+1,\alpha}e_{q^2}(-q^{-2\alpha-1}x^2)}$$
$$\times\int_0^\infty e_{q^2}(-qy^2)j_{\alpha+1}(q^{n+1}xy;q^2)y^{2n+2\alpha+3}d_qy.$$

This completes the proof. □

3.3. Evaluation at unity of the Poisson kernel for $\tilde{h}_{n,\alpha}(x;q)$

Theorem 3.3. *The following equation is an evaluation at unity of the Poisson kernel for the generalized discrete q-Hermite II polynomials:*

$$\sum_{n=0}^\infty \frac{q^{n^2}(q;q)_{n,\alpha}}{(q;q)_n^2}\tilde{h}_{n,\alpha}(q^{\alpha+\frac12}x;q)\tilde{h}_{n,\alpha}(q^{\alpha+\frac12}y;q)$$
$$= \frac{(q^2;q^2)_\infty(xy)^{-\alpha}}{(q^{2\alpha+2};q^2)_\infty(x-y)}\left[J_{\alpha+1}^{(2)}(2x;q^2)J_\alpha^{(2)}(2y;q^2) - J_\alpha^{(2)}(2x;q^2)J_{\alpha+1}^{(2)}(2y;q^2)\right]. \tag{3.28}$$

Proof. From the Christoffel–Darboux formula and the limit transition of q-Laguerre polynomials to Jackson's q-Bessel functions (see [18] and [8]), we deduce that

$$\sum_{n=0}^\infty \frac{q^{2n}(q^2;q^2)_n}{(q^{2\alpha+2};q^2)_n}L_n^{(\alpha)}(x^2;q^2)L_n^{(\alpha)}(y^2;q^2)$$
$$= \frac{(q^2;q^2)_\infty(xy)^{-\alpha}}{(q^{2\alpha+2};q^2)_\infty(x^2-y^2)}\left[xJ_{\alpha+1}^{(2)}(2x;q^2)J_\alpha^{(2)}(2y;q^2) - yJ_\alpha^{(2)}(2x;q^2)J_{\alpha+1}^{(2)}(2y;q^2)\right].$$

In the last sum, the $L_n^{(\alpha)}(x^2;q^2)$ can be written in terms of $\tilde{h}_{2n,\alpha}(q^{\alpha+\frac12}x;q)$. Using (3.11), it follows that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^{4n^2}(q;q)_{2n,\alpha}}{(q;q)_{2n}^2} \tilde{h}_{2n,\alpha}(q^{\alpha+\frac{1}{2}}x;q)\tilde{h}_{2n,\alpha}(q^{\alpha+\frac{1}{2}}y;q) \\
&\quad = \sum_{n=0}^{\infty} \frac{q^{2n}(q^2;q^2)_n}{(q^{2\alpha+2};q^2)_n} L_n^{(\alpha)}(x^2;q^2) L_n^{(\alpha)}(y^2;q^2) \\
&\quad = \frac{(q^2;q^2)_\infty (xy)^{-\alpha}}{(q^{2\alpha+2};q^2)_\infty (x^2-y^2)} \left[x J_{\alpha+1}^{(2)}(2x;q^2) J_\alpha^{(2)}(2y;q^2) - y J_\alpha^{(2)}(2x;q^2) J_{\alpha+1}^{(2)}(2y;q^2) \right].
\end{aligned} \tag{3.29}$$

Likewise (again using (3.11)), we have

$$\tilde{h}_{2n+1,\alpha}(q^{\alpha+\frac{1}{2}}x;q) = (-1)^n q^{-n(2n+1)} \frac{(q;q)_{2n+1}}{(q^{2\alpha+2};q^2)_{n+1}} q^{\alpha+\frac{1}{2}} x L_n^{(\alpha+1)}(x^2;q^2),$$

then we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^{(2n+1)^2}(q;q)_{2n+1,\alpha}}{(q;q)_{2n+1}^2} \tilde{h}_{2n+1,\alpha}(q^{\alpha+\frac{1}{2}}x;q)\tilde{h}_{2n+1,\alpha}(q^{\alpha+\frac{1}{2}}y;q) \\
&\quad = \sum_{n=0}^{\infty} \frac{q^{2n+2\alpha+2}(q^2;q^2)_n}{(q^{2\alpha+2};q^2)_{n+1}} xy L_n^{(\alpha+1)}(x^2;q^2) L_n^{(\alpha+1)}(y^2;q^2) \\
&\quad = \frac{q^{2\alpha+2}xy}{(1-q^{2\alpha+2})} \sum_{n=0}^{\infty} \frac{q^{2n}(q^2;q^2)_n}{(q^{2\alpha+2};q^2)_n} L_n^{(\alpha+1)}(x^2;q^2) L_n^{(\alpha+1)}(y^2;q^2) \\
&\quad = \frac{q^{2\alpha+2}(q^2;q^2)_\infty (xy)^{-\alpha}}{(q^{2\alpha+2};q^2)_\infty (x^2-y^2)} \left[x J_{\alpha+2}^{(2)}(2x;q^2) J_{\alpha+1}^{(2)}(2y;q^2) - y J_{\alpha+1}^{(2)}(2x;q^2) J_{\alpha+2}^{(2)}(2y;q^2) \right].
\end{aligned}$$

Using the fact (see [11, p.25])

$$q^{2\alpha+2} x J_{\alpha+2}^{(2)}(2x;q^2) = (1-q^{2\alpha+2}) J_{\alpha+1}^{(2)}(2x;q^2) - x J_\alpha^{(2)}(2x;q^2),$$

we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^{(2n+1)^2}(q;q)_{2n+1,\alpha}}{(q;q)_{2n+1}^2} \tilde{h}_{2n+1,\alpha}(q^{\alpha+\frac{1}{2}}x;q)\tilde{h}_{2n+1,\alpha}(q^{\alpha+\frac{1}{2}}y;q) \\
&\quad = \frac{q^{2\alpha+2}(q^2;q^2)_\infty (xy)^{-\alpha}}{(q^{2\alpha+2};q^2)_\infty (x^2-y^2)} \left[x J_{\alpha+2}^{(2)}(2x;q^2) J_{\alpha+1}^{(2)}(2y;q^2) - y J_{\alpha+1}^{(2)}(2x;q^2) J_{\alpha+2}^{(2)}(2y;q^2) \right] \\
&\quad = \frac{(q^2;q^2)_\infty (xy)^{-\alpha}}{(q^{2\alpha+2};q^2)_\infty (x^2-y^2)} \left[\left((1-q^{2\alpha+2}) J_{\alpha+1}^{(2)}(2x;q^2) - x J_\alpha^{(2)}(2x;q^2) \right) J_{\alpha+1}^{(2)}(2y;q^2) \right. \\
&\quad \left. - J_{\alpha+1}^{(2)}(2x;q^2) \left((1-q^{2\alpha+2}) J_{\alpha+1}^{(2)}(2y;q^2) - y J_\alpha^{(2)}(2y;q^2) \right) \right] \\
&\quad = \frac{(q^2;q^2)_\infty (xy)^{-\alpha}}{(q^{2\alpha+2};q^2)_\infty (x^2-y^2)} \left[y J_{\alpha+1}^{(2)}(2x;q^2) J_\alpha^{(2)}(2y;q^2) - x J_\alpha^{(2)}(2x;q^2) J_{\alpha+1}^{(2)}(2y;q^2) \right].
\end{aligned} \tag{3.30}$$

Add the result in (3.29) to (3.30) to obtain the desired summation. □

Corollary 3.1. *The following equation is an evaluation at unity of the Poisson kernel for the discrete q-Hermite II polynomials:*

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}\tilde{h}_n(x;q)\tilde{h}_n(y;q)$$
$$=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty(x-y)}\left[Sin_q(x)Cos_q(y)-Cos_q(x)Sin_q(y)\right]$$

Proof. For the particular case $\alpha=-\frac{1}{2}$, we have $\tilde{h}_{n,-\frac{1}{2}}(x;q)=\tilde{h}_n(x;q)$ and

$$J^{(2)}_{-\frac{1}{2}}(2x;q^2)=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty\sqrt{x}}Cos_q(x)\ \text{ and }\ J^{(2)}_{\frac{1}{2}}(2x;q^2)=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty\sqrt{x}}Sin_q(x).$$

It is easy now to finish the proof of the Corollary. $\square$

Now, we express the second q-Bessel functions in terms of the generalized discrete q-Hermite II-polynomials:

Proposition 3.2.

$$\sum_{n=0}^{\infty}(-1)^n\frac{q^{n(2n+1)}(q^{2\alpha+2};q^2)_n}{(q;q)_{2n}}\tilde{h}_{2n,\alpha}(q^{\alpha+\frac{1}{2}}x;q)=x^{-\alpha-1}J^{(2)}_{\alpha+1}(2x;q^2). \tag{3.31}$$

Proof. Taking limit as $y\to 0$ in (3.28) and using the two limits

$$\lim_{y\to 0}y^{-\alpha}J^{(2)}_{\alpha}(2y;q^2)=\frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty},\quad \lim_{y\to 0}y^{-\alpha}J^{(2)}_{\alpha+1}(2y;q^2)=0,\quad \alpha\geq-\frac{1}{2}, \tag{3.32}$$

identity (3.31) follows. $\square$

Notice that by taking $x=0$ in (3.31) and by appealing to the first limit in (3.32) and the fact that

$$\tilde{h}_{2n,\alpha}(0;q)=(-1)^nq^{-2n^2+n}(q;q^2)_n\ \text{ and }\ (q;q)_{2n}=(q^2;q^2)_n(q;q^2)_n,$$

the following special case ([2, Theorem 10.2.1] with $q\mapsto q^2$, $x=q^2$ and $a=q^{2\alpha+2}$) of the q-binomial theorem can be recovered:

$$\sum_{n=0}^{\infty}\frac{q^{2n}(q^{2\alpha+2};q^2)_n}{(q^2;q^2)_n}=\frac{(q^{2\alpha+4};q^2)_\infty}{(q^2;q^2)_\infty}. \tag{3.33}$$

The generalized discrete q-Hermite II polynomials can be written in terms of basic hypergeometric functions as:

$$\begin{cases}\tilde{h}_{2n,\alpha}(x;q) &= (-1)^nq^{-n(2n-1)}(q;q^2)_n\,{}_1\phi_1\left(q^{-2n};q^{2\alpha+2};q^2,-q^{2n+1}x^2\right),\\ \tilde{h}_{2n+1,\alpha}(x;q) &= (-1)^nq^{-n(2n+1)}x\frac{(q;q^2)_{n+1}}{1-q^{2\alpha+2}}\,{}_1\phi_1\left(q^{-2n};q^{2\alpha+4};q^2,-q^{2n+3}x^2\right).\end{cases} \tag{3.34}$$

Theorem 3.4.

$$e_{q^2}(-z^2)\sum_{n=0}^{\infty}\frac{(-1)^nq^{2n(n+\alpha)}z^{2n}}{(q^2;q^2)_n(q^{2\alpha+2};q^2)_n}=\sum_{n=0}^{\infty}\frac{(-1)^nz^{2n}}{(q^2;q^2)_n(q^{2\alpha+2};q^2)_n},\quad |z|<1. \tag{3.35}$$

Proof. From (3.34), we have

$$\tilde{h}_{2n,\alpha}(iq^{\alpha+1/2};q) = (-1)^n q^{-n(2n-1)}(q;q^2)_n {}_1\phi_1\left(q^{-2n};q^{2\alpha+2};q^2,q^{2n+2\alpha+2}\right).$$

By the summation formula of ${}_1\phi_1$ series (see [11, Appendix II, (II.5)]):

$${}_1\phi_1\left(a;c;q,c/a\right) = \frac{(c/a;q)_\infty}{(c;q)_\infty},$$

we get

$$\tilde{h}_{2n,\alpha}(iq^{\alpha+1/2};q) = (-1)^n q^{-n(2n-1)}(q;q^2)_n \frac{(q^{2n+2\alpha+2};q^2)_\infty}{(q^{2\alpha+2};q^2)_\infty},$$

which can be written as

$$\tilde{h}_{2n,\alpha}(iq^{\alpha+1/2};q) = (-1)^n q^{-n(2n-1)} \frac{(q;q^2)_n}{(q^{2\alpha+2};q^2)_n}. \tag{3.36}$$

In particular, setting $x = iq^{\alpha+1/2}$ in the even part of the generating function (3.2) and using (3.36), it follows that

$$e_{q^2}(-z^2)\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(2n-1)}q^{n(2\alpha+1)}}{(q^2;q^2)_n(q^{2\alpha+2};q^2)_n}z^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(q^2;q^2)_n(q^{2\alpha+2};q^2)_n}z^{2n}.$$

This identity holds for z in the open unit disk. $\square$

Remark 3.1. Note that formula (3.35) can be deduced from the Heine transformation of ${}_2\phi_1$ series [11, Appendix III, page 359, (III.3)]:

$${}_2\phi_1(a,b;c;q,z) = \frac{(abz/c;q)_\infty}{(z;q)_\infty}{}_2\phi_1(c/a,c/b;c;q,abz/c). \tag{3.37}$$

If one replaces q by q^2, c by $q^{2\alpha+2}$, z by $-z^2q^{2\alpha+2}/ab$, respectively in (3.37), and then send $a \to \infty$ and $b \to \infty$, one obtains the transformation (3.35).

Notice that, setting $x = iq^{\alpha+1/2}$ in the odd part of the generating function (3.2), and using the same method us above, one can derive an equivalent formula to (3.35) in which α is replaced by $\alpha + 1$.

In the next corollary, we recover the two well-known Ramanujan's identities [3, Chapter I, pp. 33–34, identities (1.7.14) and (1.7.17)]:

Corollary 3.2.

$$\sum_{n=0}^{\infty}\frac{q^n}{(q;q)_{2n}} = \frac{(-q^3;q^8)_\infty(-q^5;q^8)_\infty(q^8;q^8)_\infty}{(q;q)_\infty}, \tag{3.38}$$

$$\sum_{n=0}^{\infty}\frac{q^n}{(q;q)_{2n+1}} = \frac{(-q;q^8)_\infty(-q^7;q^8)_\infty(q^8;q^8)_\infty}{(q;q)_\infty}. \tag{3.39}$$

Proof. Setting $\alpha = -\frac{1}{2}$ and $z = iq^{1/2}$ in (3.35), and using the identity (39) in Slater's list [19]:

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q)_{2n}} = \frac{(-q^3;q^8)_\infty(-q^5;q^8)_\infty(q^8;q^8)_\infty}{(q^2;q^2)_\infty},$$

and the fact that $(q;q)_\infty = (q;q^2)_\infty(q^2;q^2)_\infty$, it follows formula (3.38).

Multiplying both sides of (3.35) by $\frac{1}{1-q}$ and setting $\alpha = \frac{1}{2}$ and $z = iq^{1/2}$ in the resulting equation, and then using the identity (38) in Slater's list [19]:

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q)_{2n+1}} = \frac{(-q;q^8)_\infty(-q^7;q^8)_\infty(q^8;q^8)_\infty}{(q^2;q^2)_\infty},$$

it follows the summation formula (3.39). $\square$

Remark 3.2. The equivalence of identities (38) and (39) in Slater's list [19] with the two corresponding identities (3.39) and (3.38) due to Ramanujan is certainly known, as this corresponds to a special case of the Heine transformation formula (3.37).

Proposition 3.3. *For $|qz| < 1$, we have*

$$e_{q^2}(-q^2z^2)\sum_{n=0}^{\infty} \frac{(-1)^n q^{2n(n+\alpha)}}{(q^2;q^2)_n(q^{2\alpha+2};q^2)_n} z^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n}\left(1+q^{2\alpha}-q^{2n+2\alpha}\right)}{(q^2;q^2)_n(q^{2\alpha+2};q^2)_n} z^{2n}. \tag{3.40}$$

Proof. Multiply both sides of (3.35) by $1+z^2$, it follows that

$$\begin{aligned}
e_{q^2}(-q^2z^2)\sum_{n=0}^{\infty} \frac{(-1)^n q^{2n(n+\alpha)}}{(q^2;q^2)_n(q^{2\alpha+2};q^2)_n} z^{2n} &= (1+z^2)\sum_{n=0}^{\infty} \frac{(-1)^n}{(q^2;q^2)_n(q^{2\alpha+2};q^2)_n} z^{2n} \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(q^2;q^2)_n(q^{2\alpha+2};q^2)_n} z^{2n} - \sum_{n=1}^{\infty} \frac{(-1)^n}{(q^2;q^2)_{n-1}(q^{2\alpha+2};q^2)_{n-1}} z^{2n} \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n\left[1-(1-q^{2n})(1-q^{2n+2\alpha})\right]}{(q^2;q^2)_n(q^{2\alpha+2};q^2)_n} z^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n}\left(1+q^{2\alpha}-q^{2n+2\alpha}\right)}{(q^2;q^2)_n(q^{2\alpha+2};q^2)_n} z^{2n},
\end{aligned}$$

which completes the proof. $\square$

Corollary 3.3.

$$\frac{1}{(q^2;q^2)_\infty(q^{2\alpha+2};q^2)_\infty} = \sum_{n=0}^{\infty} \frac{q^{2n}\left(1+q^{2\alpha}-q^{2n+2\alpha}\right)}{(q^2;q^2)_n(q^{2\alpha+2};q^2)_n}. \tag{3.41}$$

Proof. Setting $z = i$ in (3.40), we obtain

$$e_{q^2}(q^2)\sum_{n=0}^{\infty} \frac{q^{2n(n-1)}q^{n(2\alpha+2)}}{(q^2;q^2)_n(q^{2\alpha+2};q^2)_n} = \sum_{n=0}^{\infty} \frac{q^{2n}\left(1+q^{2\alpha}-q^{2n+2\alpha}\right)}{(q^2;q^2)_n(q^{2\alpha+2};q^2)_n}.$$

Applying the following Cauchy's formula (see [2], p. 522)

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)}x^n}{(q;q)_n(x;q)_n}=\frac{1}{(x;q)_\infty},$$

it follows the identity (3.41). □

4. Realization of the quantum algebra $\mathsf{su}_{q^{\frac{1}{2}}}(1,1)$

The quantum algebra $\mathsf{su}_q(1,1)$ is defined as the associative unital algebra generated by the operators $\{K_-,\ K_+,\ K_0\}$ which satisfy the conjugation relations (see [15])

$$(K_0)^*=K_0,\ \ (K_+)^*=K_-,$$

and the commutation relations

$$[K_0,K_\pm]=\pm K_\pm,\quad [K_-,K_+]=[2K_0]_{q^2}\,,$$

where $[x]_q=\frac{q^x-q^{-x}}{q-q^{-1}}$ is a symmetric definition of q-numbers, invariant by $q\leftrightarrow q^{-1}$.

The Casimir operator C, which by definition commutes with the generators $K_\pm$ and K_0 is

$$C=\left[K_0-\frac{1}{2}\right]_{q^2}^2-K_+K_-.$$

Now, we discuss an explicit one-dimensional realization of the quantum algebra $\mathsf{su}_{q^{\frac{1}{2}}}(1,1)$. We give a concrete functional realization of the Hilbert space $\mathfrak{H}$ (defined just below) and an explicit expression of the representation operators K_-, K_+ and K_0 defined in preceding paragraph in terms of q-difference operators.

For this purpose, first we take $\mathfrak{H}=L^2_\alpha(\mathbb{R})$ to be the space of functions $\psi(x)$ such that

$$\int_{-\infty}^{\infty}|\psi(x)|^2|x|^{2\alpha+1}dx<\infty$$

with the scalar product

$$\langle\psi_1,\psi_2\rangle=\int_{-\infty}^{\infty}\psi_1(x)\overline{\psi_2(x)}|x|^{2\alpha+1}dx.$$

Now, we construct a convenient orthonormal basis of $L^2_\alpha(\mathbb{R})$ consisting of (q,α)-deformed Hermite functions defined by

$$\phi_n^\alpha(x;q)=d_{n,\alpha}\sqrt{\omega_\alpha(x;q)}\tilde{h}_{n,\alpha}(x;q), \tag{4.1}$$

where $\tilde{h}_{n,\alpha}(x;q)$, $\omega_\alpha(x;q)$ and $d_{n,\alpha}$ are given by (3.1), (3.14) and (3.15), respectively.

Proposition 4.1. *$\{\phi_n^\alpha(x;q)\}_{n=0}^\infty$ is a complete orthonormal set in $L^2_\alpha(\mathbb{R})$.*

Proof. The continuous orthogonality relation (3.9) for $\tilde{h}_{n,\alpha}(x;q)$ can be written as

$$\int_{-\infty}^{\infty}\phi_n^\alpha(x;q)\phi_m^\alpha(x;q)|x|^{2\alpha+1}dx=\delta_{n,m}.$$

Thus $\{\phi_n^\alpha(x;q)\}_{n=0}^\infty$ is an orthonormal set in $L^2_\alpha(\mathbb{R})$. We prove that it is a complete set. Suppose that there exists $f \in L^2_\alpha(\mathbb{R})$ orthogonal to all $\phi_n^\alpha(x;q)$,

$$\int_{-\infty}^{\infty} \phi_n^\alpha(x;q) f(x)|x|^{2\alpha+1}dx = 0, \quad \text{for all } \ n \in \mathbb{N}.$$

By using the inverse formula (3.3), we obtain

$$\int_{-\infty}^{\infty} \sqrt{\omega_\alpha(x;q)}x^n f(x)|x|^{2\alpha+1}dx = 0, \quad \text{for all } \ n \in \mathbb{N}.$$

Using the technique that appears in [1] (p. 26), we deduce that $f = 0$. □

Let δ_q be the q-dilatation operator in the variable x, i.e. $\delta_q f(x) = f(qx)$. The operator of multiplication by a function g will be denoted also by g.

Let $\mathfrak{S}_{q\alpha}$ be the finite linear span of (q,α)-deformed Hermite functions $\phi_n^\alpha(x;q)$.

It is well known that a solution of the stationary Schrödinger equation is represented by eigenfunctions of the Schrödinger operator.

Definition 4.1. The q-Schrödinger operator H acting on any function f in $L^2_\alpha(\mathbb{R})$ is defined by

$$Hf = \begin{pmatrix} H_e & 0 \\ 0 & H_o \end{pmatrix} \begin{pmatrix} f_e \\ f_o \end{pmatrix} \tag{4.2}$$

where

$$\begin{aligned} H_e &= -\frac{q^{2\alpha+1}}{(1-q)x^2} \\ &\quad \times \left[q^{-2\alpha}\delta_{q^{-1}}\sqrt{1+q^{-2\alpha-1}x^2} + \sqrt{1+q^{-2\alpha-1}x^2}\delta_q - (1+q^{-2\alpha}+q^{-2\alpha-1}x^2)I \right], \\ H_o &= -\frac{q^{2\alpha+1}}{(1-q)x^2} \\ &\quad \times \left[q\delta_{q^{-1}}\sqrt{1+q^{-2\alpha-1}x^2} + q^{2\alpha+1}\sqrt{1+q^{-2\alpha-1}x^2}\delta_q - (1+q^{2\alpha+2}+q^{-2\alpha-1}x^2)I \right], \end{aligned}$$

f_e and f_o are respectively the even and the odd parts of f and I is the identity operator.

Theorem 4.1. *H is a self-adjoint operator in $\mathfrak{S}_{q\alpha}$, with eigenfunctions*

$$\phi_n^\alpha(x;q), \quad n = 0, 1, 2, ...,$$

and we have

$$H\phi_n^\alpha(x;q) = [\![n]\!]_{q,\alpha}\, \phi_n^\alpha(x;q).$$

Proof. Let $f, g \in \mathfrak{S}_{q\alpha}$, $f = f_e + f_o$, $g = g_e + g_o$. Due to the parity of the integrand in (Hf, g), we can write

$$(Hf, g) = (H_e f_e, g_e) + (H_o f_o, g_o),$$

where

$$
\begin{aligned}
(H_e f_e, g_e) = &-\frac{q^{2\alpha+1}}{(1-q)}\int_{-\infty}^{\infty}\frac{q^{-2\alpha}\sqrt{1+q^{-2\alpha-3}x^2}}{x^2}f_e(q^{-1}x)\overline{g_e(x)}|x|^{2\alpha+1}dx\\
&-\frac{q^{2\alpha+1}}{(1-q)}\int_{-\infty}^{\infty}\frac{\sqrt{1+q^{-2\alpha-1}x^2}}{x^2}f_e(qx)\overline{g_e(x)}|x|^{2\alpha+1}dx\\
&+\frac{q^{2\alpha+1}}{(1-q)}\int_{-\infty}^{\infty}\frac{(1+q^{-2\alpha}+q^{-2\alpha-1}x^2)}{x^2}f_e(x)\overline{g_e(x)}|x|^{2\alpha+1}dx,\\
(H_o f_o, g_o) = &-\frac{q^{2\alpha+1}}{(1-q)}\int_{-\infty}^{\infty}\frac{q\sqrt{1+q^{-2\alpha-3}x^2}}{x^2}f_o(q^{-1}x)\overline{g_o(x)}|x|^{2\alpha+1}dx\\
&-\frac{q^{2\alpha+1}}{(1-q)}\int_{-\infty}^{\infty}\frac{q^{2\alpha+1}\sqrt{1+q^{-2\alpha-1}x^2}}{x^2}f_o(qx)\overline{g_o(x)}|x|^{2\alpha+1}dx\\
&+\frac{q^{2\alpha+1}}{(1-q)}\int_{-\infty}^{\infty}\frac{(1+q^{2\alpha+2}+q^{-2\alpha-1}x^2)}{x^2}f_o(x)\overline{g_o(x)}|x|^{2\alpha+1}dx.
\end{aligned}
$$

Using the substitutions $u = q^{-1}x$ in the first integral and $u = qx$ in the second integral, we obtain

$$
\begin{aligned}
(H_e f_e, g_e) = &-\frac{q^{2\alpha+1}}{(1-q)}\int_{-\infty}^{\infty}f_e(u)\overline{\frac{\sqrt{1+q^{-2\alpha-1}u^2}}{u^2}g_e(qu)}|u|^{2\alpha+1}du\\
&-\frac{q^{2\alpha+1}}{(1-q)}\int_{-\infty}^{\infty}f_e(u)\overline{\frac{q^{-2\alpha}\sqrt{1+q^{-2\alpha-3}u^2}}{u^2}g_e(q^{-1}u)}|u|^{2\alpha+1}du\\
&+\frac{q^{2\alpha+1}}{(1-q)}\int_{-\infty}^{\infty}f_e(u)\overline{\frac{(1+q^{-2\alpha}+q^{-2\alpha-1}u^2)}{u^2}g_e(u)}|u|^{2\alpha+1}du\\
= &\,(f_e, H_e g_e).
\end{aligned}
$$

The same argument can prove that $(H_o f_o, g_o) = (f_o, H_o g_o)$. Therefore, we conclude that, H is a self-adjoint operator in $\mathfrak{S}_{q\alpha}$.

We have

$$
\begin{aligned}
H_e(\phi_{2n}^{\alpha}(x;q)) &= d_{2n,\alpha}H_e\left[\sqrt{\omega_\alpha(x;q)}\tilde{h}_{2n,\alpha}(q^{-1}x;q)\right]\\
&= -\frac{q^{2\alpha+1}}{(1-q)x^2}d_{2n,\alpha}\sqrt{\omega_\alpha(x;q)}\left[q^{-2\alpha}\tilde{h}_{2n,\alpha}(q^{-1}x;q)+(1+q^{-2\alpha-1}x^2)\tilde{h}_{2n,\alpha}(qx;q)\right.\\
&\quad\left.-(1+q^{-2\alpha}+q^{-2\alpha-1}x^2)\tilde{h}_{2n,\alpha}(x;q)\right].
\end{aligned}
$$

Using the relation (3.7), it follows that

$$
\begin{aligned}
H_e(\phi_{2n}^{\alpha}(x;q)) &= -\frac{q^{2\alpha+1}}{(1-q)x^2}d_{2n,\alpha}\sqrt{\omega_\alpha(x;q)}\left[(q^{2n-2\alpha-1}-q^{-2\alpha-1})x^2\tilde{h}_{2n,\alpha}(x;q)\right]\\
&= [\![2n]\!]_{q,\alpha}\,\phi_{2n}^{\alpha}(x;q),
\end{aligned}
$$

and

$$H_o(\phi^{\alpha}_{2n+1}(x;q)) = d_{2n+1,\alpha} H_o\left[\sqrt{\omega_\alpha(x;q)}\tilde{h}_{2n+1,\alpha}(q^{-1}x;q)\right]$$
$$= -\frac{q^{2\alpha+1}}{(1-q)x^2} d_{2n+1,\alpha}\sqrt{\omega_\alpha(x;q)}$$
$$\times\left[q\tilde{h}_{2n+1,\alpha}(q^{-1}x;q) + q^{2\alpha+1}(1+q^{-2\alpha-1}x^2)\tilde{h}_{2n,\alpha}(qx;q) - (1+q^{2\alpha+2}+q^{-2\alpha-1}x^2)\tilde{h}_{2n,\alpha}(x;q)\right]$$
$$= -\frac{q^{4\alpha+2}}{(1-q)x^2} d_{2n+1,\alpha}\sqrt{\omega_\alpha(x;q)}$$
$$\times\left[q^{-2\alpha}\tilde{h}_{2n+1,\alpha}(q^{-1}x;q) + (1+q^{-2\alpha-1}x^2)\tilde{h}_{2n+1,\alpha}(qx;q) - (q+q^{-2\alpha-1}+q^{-4\alpha-2}x^2)\tilde{h}_{2n+1,\alpha}(x;q)\right].$$

Using the relation (3.8), it follows that

$$H_o(\phi^{\alpha}_{2n+1}(x;q)) = -\frac{q^{4\alpha+2}}{(1-q)x^2} d_{2n+1,\alpha}\sqrt{\omega_\alpha(x;q)}\left[(q^{2n-2\alpha}-q^{-4\alpha-2})x^2\tilde{h}_{2n+1,\alpha}(x;q)\right]$$
$$= [\![2n+1]\!]_{q,\alpha}\,\phi^{\alpha}_{2n+1}(x;q),$$

us desired. $\square$

Let us note that, due to the regularity of $\phi^{\alpha}_n(x;q)$, the singularity of H at $x=0$ can be omitted when we apply H to the function $f \in \mathfrak{S}_{q\alpha}$.

From the forward and backward shift operators (3.4) and (3.5), we define the operators a and a^+ on $\mathfrak{S}_{q\alpha}$ by means of 2×2 matrix forms:

$$af = \frac{q^{\frac{1}{2}}}{\sqrt{1-q}x}\begin{pmatrix}\delta_{q^{-1}}\sqrt{1+q^{-2\alpha-1}x^2}-1 & 0\\ 0 & \delta_{q^{-1}}\sqrt{1+q^{-2\alpha-1}x^2}-q^{2\alpha+1}\end{pmatrix}\begin{pmatrix}f_e\\ f_o\end{pmatrix}, \tag{4.3}$$
$$a^+f = \frac{q^{2\alpha+\frac{3}{2}}}{\sqrt{1-q}x}\begin{pmatrix}\sqrt{1+q^{-2\alpha-1}x^2}\delta_q-1 & 0\\ 0 & \sqrt{1+q^{-2\alpha-1}x^2}\delta_q-q^{-2\alpha-1}\end{pmatrix}\begin{pmatrix}f_e\\ f_o\end{pmatrix}. \tag{4.4}$$

The reader may verify that these operators are indeed mutually adjoint in the Hilbert space $L^2_\alpha(\mathbb{R})$.

So, the q-Schrödinger operator H can be factorized as

$$H = a^+a.$$

The action of the operators a and a^+ on the basis $\{\phi^{\alpha}_n(x;q)\}_{n=0}^{\infty}$ of $L^2_\alpha(\mathbb{R})$ leads to the explicit results:

Proposition 4.2. *The following statements hold:*

$$a\phi^{\alpha}_0(x;q) = 0, \tag{4.5}$$
$$a\phi^{\alpha}_n(x;q) = \sqrt{[\![n]\!]_{q,\alpha}}\,\phi^{\alpha}_{n-1}(x;q), \quad n\geq 1, \tag{4.6}$$
$$a^+\phi^{\alpha}_n(x;q) = \sqrt{[\![n+1]\!]_{q,\alpha}}\,\phi^{\alpha}_{n+1}(x;q), \tag{4.7}$$
$$\phi^{\alpha}_n(x;q) = (n!_{q,\alpha})^{-\frac{1}{2}}a^{+n}\phi^{\alpha}_0(x;q), \tag{4.8}$$

where $[\![n]\!]_{q,\alpha}$ is defined by (2.1).

Proof. Formula (4.6) follows from the forward and backward shift operators (3.4) and (3.5) and from the fact that

$$d_{n,\alpha} = \frac{q^{n-\frac{1}{2}}\sqrt{[\![n]\!]_{q,\alpha}}}{\sqrt{1-q}\,[\![n]\!]_q} d_{n-1,\alpha}.$$

Formula (4.5) is an immediate consequence of the definition (4.1) and (4.6). Finally (4.8) is a consequence of (4.7). □

From (4.6) and (4.7) one deduces that

$$a^+ a\phi_n^\alpha(x;q) = [\![n]\!]_{q,\alpha}\, \phi_n^\alpha(x;q), \tag{4.9}$$
$$aa^+\phi_n^\alpha(x;q) = [\![n+1]\!]_{q,\alpha}\, \phi_n^\alpha(x;q). \tag{4.10}$$

The number operator N is defined in this case by the relations

$$a^+a = [\![N]\!]_{q,\alpha}\,, \quad aa^+ = [\![N+1]\!]_{q,\alpha} \quad \text{on } \mathfrak{S}_{q\alpha}. \tag{4.11}$$

The formulas (4.11) can be inverted to determine an explicit expression of the operator N as follows:

$$N := \frac{1}{2\log q}\log\left(1-(1-q)aa^+\right) + \frac{1}{2\log q}\log\left(1-(1-q)a^+a\right) - \alpha - 1. \tag{4.12}$$

From (4.9), (4.10) and (4.12), we obtain:

$$N\phi_n^\alpha(x;q) = n\phi_n^\alpha(x;q), \tag{4.13}$$

and

$$[N,a] = -a, \quad [N,a^+] = a^+ \quad \text{on } \mathfrak{S}_{q\alpha}. \tag{4.14}$$

Now, we consider the operators

$$b = q^{-\frac{N+(K+1)(\alpha+\frac{1}{2})}{4}}a, \quad b^+ = a^+q^{-\frac{N+(K+1)(\alpha+\frac{1}{2})}{4}}, \quad K = (-1)^N.$$

Using the relation

$$[x]_{q^{\frac{1}{2}}} = q^{-\frac{x-1}{2}}[\![x]\!]_q,$$

one easily verifies that the actions of the operators b and b^+ on the basis $\{\phi_n^\nu(x;q)\}_{n=0}^\infty$ are given by

$$\begin{aligned}
b\phi_{2n}^\alpha(x;q) &= \sqrt{[2n]_{q^{\frac{1}{2}}}}\,\phi_{2n-1}^\alpha(x;q), \qquad n\geq 1,\\
b\phi_{2n+1}^\alpha(x;q) &= \sqrt{[2n+2\alpha+2]_{q^{\frac{1}{2}}}}\,\phi_{2n}^\alpha(x;q),\\
b^+\phi_{2n}^\alpha(x;q) &= \sqrt{[2n+2\alpha+2]_{q^{\frac{1}{2}}}}\,\phi_{2n+1}^\alpha(x;q),\\
b^+\phi_{2n+1}^\nu(x;q) &= \sqrt{[2n+2]_{q^{\frac{1}{2}}}}\,\phi_{2n+2}^\alpha(x;q).
\end{aligned} \tag{4.15}$$

Now we are ready to construct an explicit realization of the operators K_-, K_+ and K_0 generators of the quantum algebra $su_{q^{\frac{1}{2}}}(1,1)$ in terms of the oscillatorial operators a, a^+ and N by setting,

$$K_- = \gamma\,(b)^2\,, \quad K_+ = \gamma\left(b^+\right)^2, \quad K_0 = \frac{1}{2}(N+\alpha+1), \quad \gamma = ([2]_{q^{\frac{1}{2}}})^{-1}.$$

From (4.15) we derive the actions of these operators on the basis $\{\phi_n^\nu(x;q)\}_{n=0}^\infty$:

$$\begin{aligned} K_0\phi_n^\alpha(x;q) &= \frac{1}{2}(n+\alpha+1)\phi_n^\alpha(x;q), \\ K_+\phi_{2n}^\alpha(x;q) &= \gamma\sqrt{[2n+2]_{q^{\frac{1}{2}}}[2n+2\alpha+2]_{q^{\frac{1}{2}}}}\,\phi_{2n+2}^\alpha(x;q), \\ K_+\phi_{2n+1}^\alpha(x;q) &= \gamma\sqrt{[2n+2]_{q^{\frac{1}{2}}}[2n+2\alpha+4]_{q^{\frac{1}{2}}}}\,\phi_{2n+3}^\alpha(x;q), \\ K_-\phi_{2n}^\alpha(x;q) &= \gamma\sqrt{[2n]_{q^{\frac{1}{2}}}[2n+2\alpha]_{q^{\frac{1}{2}}}}\,\phi_{n-2}^\alpha(x;q), \quad n\geq 1, \\ K_-\phi_{2n+1}^\alpha(x;q) &= \gamma\sqrt{[2n]_{q^{\frac{1}{2}}}[2n+2\alpha+2]_{q^{\frac{1}{2}}}}\,\phi_{2n-1}^\alpha(x;q), \quad n\geq 1. \end{aligned} \tag{4.16}$$

It follows that

$$\begin{aligned} K_-K_+\phi_{2n}^\alpha(x;q) &= \gamma^2\,[2n+2]_{q^{\frac{1}{2}}}[2n+2\alpha+2]_{q^{\frac{1}{2}}}\,\phi_{2n}^\alpha(x;q), \\ K_-K_+\phi_{2n+1}^\alpha(x;q) &= \gamma^2\,[2n+2]_{q^{\frac{1}{2}}}[2n+2\alpha+4]_{q^{\frac{1}{2}}}\,\phi_{2n+1}^\alpha(x;q), \\ K_+K_-\phi_{2n}^\alpha(x;q) &= \gamma^2\,[2n]_{q^{\frac{1}{2}}}[2n+2\alpha]_{q^{\frac{1}{2}}}\,\phi_{2n}^\alpha(x;q), \\ K_+K_-\phi_{2n+1}^\alpha(x;q) &= \gamma^2\,[2n]_{q^{\frac{1}{2}}}[2n+2\alpha+2]_{q^{\frac{1}{2}}}\,\phi_{2n+1}^\alpha(x;q). \end{aligned} \tag{4.17}$$

Using the following identity (see [6] p.58)

$$[a]_q\,[b-c]_q + [b]_q\,[c-a]_q + [c]_q\,[a-b]_q = 0, \tag{4.18}$$

with $a = 2n+2$, $b = -2n-2\alpha$, $c = 2$ and $a = 2n+2$, $b = -2n-2\alpha-2$, $c = 2$ respectively, we obtain

$$\begin{aligned} [2n+2]_{q^{\frac{1}{2}}}[2n+2\alpha+2]_{q^{\frac{1}{2}}} - [2n]_{q^{\frac{1}{2}}}[2n+2\alpha]_{q^{\frac{1}{2}}} &= [2]_{q^{\frac{1}{2}}}[4n+2\alpha+2]_{q^{\frac{1}{2}}}\,, \\ [2n+2]_{q^{\frac{1}{2}}}[2n+2\alpha+4]_{q^{\frac{1}{2}}} - [2n]_{q^{\frac{1}{2}}}[2n+2\alpha+2]_{q^{\frac{1}{2}}} &= [2]_{q^{\frac{1}{2}}}[4n+2\alpha+4]_{q^{\frac{1}{2}}}\,. \end{aligned}$$

By the identity $[2x]_{q^{\frac{1}{2}}} = [2]_{q^{\frac{1}{2}}}[x]_q$, we obtain

$$\begin{aligned} [4n+2\alpha+2]_{q^{\frac{1}{2}}} &= [2]_{q^{\frac{1}{2}}}[2n+\alpha+1]_q\,, \\ [4n+2\alpha+4]_{q^{\frac{1}{2}}} &= [2]_{q^{\frac{1}{2}}}[2n+\alpha+2]_q\,, \end{aligned}$$

which leads to the commutation relations:

$$[K_0, \pm K] = \pm K_\pm, \quad [K_-, K_+] = [2K_0]_q \quad \text{on } \mathfrak{S}_{q\alpha}$$

and the conjugation relations

$$K_0^* = K_0, \quad K_+^* = K_- \quad \text{on } \mathfrak{S}_{q\alpha}.$$

To analyse irreducible representations of the $\mathsf{su}_{q^{\frac{1}{2}}}(1,1)$ algebra, we need the invariant Casimir operator C, which in this case has the form:

$$C = \left[K_0 - \frac{1}{2}\right]_q^2 - K_+K_-.$$

From (4.16) and (4.17) we obtain the action of this operator on the basis $\{\phi_n^\alpha(x;q)\}_{n=0}^\infty$:

$$C\phi_{2n}^{\alpha}(x;q) = \left(\left[n+\frac{\alpha}{2}\right]_q^2 - [n]_q\,[n+\alpha]_q\right)\phi_{2n}^{\alpha}(x;q),$$
$$C\phi_{2n+1}^{\alpha}(x;q) = \left(\left[n+\frac{\alpha+1}{2}\right]_q^2 - [n]_q\,[n+\alpha+1]_q\right)\phi_{2n+1}^{\alpha}(x;q).$$

Using (4.18) with $a = n+\frac{\alpha}{2}$, $b = n$, $c = -\frac{\alpha}{2}$ and $a = n+\frac{\alpha+1}{2}$, $b = n$, $c = -\frac{\alpha+1}{2}$ respectively, we derive

$$\left[n+\frac{\alpha}{2}\right]_q^2 - [n]_q\,[n+\alpha]_q = \left[\frac{\alpha}{2}\right]_q^2,$$
$$\left[n+\frac{\alpha+1}{2}\right]_q^2 - [n]_q\,[n+\alpha+1]_q = \left[\frac{\alpha+1}{2}\right]_q^2.$$

The Casimir operator C has two eigenvalues $\left[\frac{2\alpha+1\mp1}{4}\right]_q^2$ in the subspaces $\mathfrak{S}_{q\alpha}^{\pm}$ formed by the even and odd basis vectors $\{\phi_n^{\alpha}(x;q)\}_{n=0}^{\infty}$, respectively. Thus $\mathfrak{S}_{q\alpha}$ splits into the direct sum of two $\mathsf{su}_{q^{\frac{1}{2}}}(1,1)$-irreducible subspaces $\mathfrak{S}_{q\alpha}^{+}$ and $\mathfrak{S}_{q\alpha}^{-}$.

Remark 4.1. We deduce from (4.15) that the operators b, b^+ and N satisfy the relations

$$bb^+ - q^{\pm\frac{1+2\nu K}{2}}b^+b = [1+2\nu K]_{q^{\frac{1}{2}}}\, q^{\mp\frac{N+\nu-\nu K}{2}} \quad \text{on } \mathfrak{S}_{q\alpha}, \tag{4.19}$$

where $\nu = \alpha + \frac{1}{2}$. This leads to explicit expressions for the generators $\{b\ , b^+\ , N\}$ of the q-deformed Calogero–Vasiliev Oscillator algebra (see [17,16]). In particular, Macfarlane in [17] has shown that if $\nu = \frac{p-1}{2}$, this oscillator realises the q-deformed para-Bose oscillator of order p.

Acknowledgments

The authors would like to express their sincere thanks to the anonymous referee for a very careful reading of the manuscript and useful suggestions, leading to Theorem 3.4, which has grately improved the quality of the paper. This project was supported by King Faisal University under Research Grant #RG186116.

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