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Zeros of accretive operators and asymptotics of a second order difference inclusion in Banach spaces

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ABSTRACT

By developing new methods, we investigate the asymptotic behavior of solutions to a general second order difference inclusion of accretive type, and apply the results to approximate zeros of accretive operators in Banach spaces, and to optimization problems. Our results extend some previously known results in Hilbert and Banach spaces.

1. Introduction

Second-order difference equations of the form

$$\begin{cases} u_{i+1} - (1+\theta_i)u_i + \theta_i u_{i-1} \in c_i A u_i + f_i, & 1 \leqslant i \in \mathbb{N}, \\ u_0 = x, \quad \sup\{\|u_i\| : i \geqslant 0\} < \infty, \end{cases} \tag{1.1}$$

where A is a nonlinear m-accretive (possibly multivalued) operator in a real Banach space $(X, \|\cdot\|)$, $c_i > 0$ and $\theta_i > 0$, correspond to the discrete version of the following second-order evolution equation:

$$\begin{cases} p(t)u''(t) + r(t)u'(t) \in Au(t) + f(t), & \text{a.e. on } \mathbb{R}^+, \\ u(0) = u_0, \quad \sup\{\|u(t)\| : t \geqslant 0\} < \infty. \end{cases} \tag{1.2}$$

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The theory of second-order evolution equations of monotone (accretive) type has been investigated by many authors. We refer the reader in particular to the books by Barbu [9,10], Brézis [11] and Morosanu [28], as well as to Refs. [31–33,7,8,13,37,38,1–5,23].

Morosanu [27] investigated the difference inclusion (1.1), for the existence and asymptotic behavior of solutions, and obtained the convergence of $\{u_i\}$ to an element of $A^{-1}(0)$, whenever A is a maximal monotone operator in a Hilbert space, $0 \in R(A)$, $\theta_i \equiv 1$ and $f_i \equiv 0$ (the homogeneous case). In Hilbert spaces, nonlinear maximal monotone operators coincide with m-accretive operators. Investigations on the existence and asymptotic behavior of solutions to (1.1) were followed by many authors; see [32,31,33,25,20,14,6,34].

In the Banach space setting, to the best of our knowledge, few papers can be found in the literature dealing with the problem (1.1). Poffald and Reich [32] extended Morosanu's result and proved the same result in Banach spaces having a strongly monotone duality mapping; the same results were extended to the nonhomogeneous case in [33,34], under the additional condition of A being coercive. The problem (1.1) was studied by Apreutesei in [6] with $f_i \equiv 0$, $\theta_i \geqslant 1$, $\{\theta_i\}$ nonincreasing and the duality mapping of the Banach space X being strongly monotone. As is known [32], a Banach space has a strongly monotone duality mapping if and only if it is uniformly convex with a modulus of convexity of power type 2; this is the case, say, when X is a Hilbert space or one of the Lebesgue spaces L^p, $1 < p \leq 2$. It would be desirable to study the asymptotic behavior of solutions of (1.1) when X is a more general Banach space, as well as when θ_i is not necessarily nonincreasing and $f_i \neq 0$.

In this paper, we investigate the asymptotic behavior of solutions to (1.1). We improve some of the previous results in [6,33,32,34] by assuming much weaker conditions on $\{\theta_i\}$, and without requiring the duality mapping of X to be strongly monotone. Then we apply our results to provide, in the context of Banach spaces, new approximation methods for zeros of A, as well as for finding a minimum point of a proper, convex and lower semicontinuous function $\psi : X \to (-\infty, +\infty]$ through a recently implemented tool in [35].

2. Preliminaries

Let X be a real Banach space with norm $\|\cdot\|$, and let X^* be the dual space of X. We denote the pairing between X and X^* by $(\cdot,\cdot)$. When $\{x_n\}$ is a sequence in X, we denote the strong convergence of $\{x_n\}$ to $x \in X$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. A Banach space X is said to be strictly convex if $\|x + y\| < 2$, for all $x, y \in X$ with $\|x\| \leq 1, \|y\| \leq 1$ and $x \neq y$. The modulus δ of convexity of X is defined by

$$\delta(\epsilon) = \inf\{1 - \frac{\|x+y\|}{2} : \ \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\},$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space X is said to be uniformly convex if $\delta(\epsilon) > 0$, for every $\epsilon > 0$. It is known that L_p and l_p spaces, $1 < p < \infty$, are uniformly convex. Uniformly convex Banach spaces include Hilbert spaces. The duality mapping J from X into 2^{X^*} is defined by $J(x) = \{x^* \in X^* : (x, x^*) = \|x\|^2 = \|x^*\|^2\}$ for every $x \in X$. By the Hahn-Banach theorem, $J(x) \neq \emptyset$ for each $x \in X$. Note that in a Hilbert space, the duality mapping is the identity operator. X is said to be smooth, if J is single-valued. In this case, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} = (y, J(x))$$

exists, for each $x, y \in S(X) := \{x \in X : \|x\| = 1\}$. The space X is said to be uniformly smooth if the limit is attained uniformly for $x, y \in S(X)$.

It is well known that a uniformly convex Banach space is strictly convex and reflexive; X is reflexive if and only if J is surjective, and X is strictly convex if and only if J is one-to-one. So, if X is reflexive, strictly convex and smooth, then J is a single-valued bijection, and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* from X^* onto $X^{**} \cong X$. The duality mapping J has the following properties that will be used throughout the paper.

Lemma 2.1. (See [10]) *Let X be a Banach space and let $J : X \to 2^{X^*}$ be the normalized duality mapping. Then:*

(1) $(x - y, j_x - j_y) \geq (\|x\| - \|y\|)^2$, *for all $x, y \in X, j_x \in J(x)$ and $j_y \in J(y)$, and consequently J is monotone;*
(2) $\|x\|^2 - \|y\|^2 \geq 2(x - y, j_y)$, *for all $x, y \in X$ and $j_y \in J(y)$;*
(3) $(x, j_y) \leq \|x\|\|y\| \leq \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2$, *for all $x, y \in X$ and $j_y \in J(y)$.*

It is known that a Banach space X is uniformly smooth if and only if X^* is uniformly convex. Further, we know the following result, which characterizes uniformly convex Banach spaces.

Lemma 2.2. (See [39]) *Let $r > 0$ and let X be a Banach space. Then X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$, such that $(x - y, j_x - j_y) \geqslant g(\|x - y\|)$, for all $x, y \in \{z \in X : \|z\| \leq r\}, j_x \in J(x)$ and $j_y \in J(y)$.*

A subset A of $X \times X$ with domain $D(A)$ and range $R(A)$ is called accretive, if for all $y_i \in Ax_i$, $i = 1, 2$, there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that

$$(y_1 - y_2, j(x_1 - x_2)) \geq 0.$$

The accretive operator $A \subseteq X \times X$ is called m-accretive if $R(I + A) = X$, where I is the identity operator of X. It follows that $R(I + \lambda A) = X, \forall \lambda > 0$.

For an accretive operator A, the resolvent and the Yosida approximation of A, are defined by

$$J_\lambda x = (I + \lambda A)^{-1} x, \qquad x \in R(I + \lambda A);$$
$$A_\lambda x = \frac{I - J_\lambda}{\lambda} x, \qquad x \in R(I + \lambda A),$$

respectively. We state below some of the main properties of J_λ and A_λ.

Lemma 2.3. *(See [9,22]). Let A be m-accretive in $X \times X$. Then,*

(1) $\|J_\lambda x - J_\lambda y\| \leq \|x - y\|$ *for all $x, y \in X$;*
(2) $\|J_\lambda x - x\| \leq \lambda\|A_\lambda x\| \leq \lambda \inf\{\|y\| \ ; \ y \in Ax\}$, *for all $x \in D(A)$;*
(3) A_λ *is $m-$accretive on X and $\|A_\lambda x - A_\lambda y\| \leq (2/\lambda)\|x - y\|$, for all $\lambda > 0$, $x, y \in X$;*
(4) $A_\lambda x \in AJ_\lambda x$ *and $x = J_\lambda x + \lambda A_\lambda x$, for all $x \in X$.*

The operator $A \subseteq X \times X$ is said to be $\alpha-$strongly accretive ($\alpha > 0$) if for all $y_i \in Ax_i$, $i = 1, 2$, there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that

$$(y_1 - y_2, j(x_1 - x_2)) \geq \alpha\|x_1 - x_2\|^2.$$

Let C be a nonempty, closed and convex subset of a uniformly convex Banach space X. Then we know that, for any $x \in X$, there exists a unique element $z \in C$ (called the nearest point projection of x onto C) such that $\|x - z\| \leq \|x - y\|$, for all $y \in C$. Denoting $z = P_C(x)$, P_C is called the nearest point projection (or metric projection) map of X onto C. If in addition, X is assumed to be smooth, then $z \in C$ is the nearest point projection of $x \in X$ onto C, if and only if

$$(y - z, J(x - z)) \leq 0, \quad \forall y \in C. \tag{2.1}$$

We also know [17] that if X is uniformly convex, then P_C is continuous.

Let X be uniformly convex and smooth, and A be m-accretive, and assume that $0 \in R(A)$, or equivalently $A^{-1}0 \neq \emptyset$. Let $P : X \to A^{-1}0$ be the nearest point projection map onto the (closed and convex) zero set of A. Then we shall say that A satisfies the convergence condition [29] if $(x_i, y_i) \in A, \|x_i\| \leq K, \|y_i\| \leq K$, and $\lim_{i\to\infty}(y_i, J(x_i - Px_i)) = 0$ imply that $\liminf_{i\to\infty} \|x_i - Px_i\| = 0$. It is obvious that every strongly accretive operator A satisfies the convergence condition.

It is worth pointing out that in the case that X is a uniformly convex Banach space and $A \subseteq X \times X$ is m-accretive, $A^{-1}0 \neq \emptyset$ if and only if $\liminf_{\lambda\to\infty} \|J_\lambda x\| < \infty$ for some $x \in X$ (see [21, Theorem 1]). Some studies have been made on the existence of zeros of accretive, or m-accretive, operators. The reader is referred to the papers [21,26,19,16,24].

A Banach space X is said to satisfy Opial's condition if

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|,$$

for all $y \in X$ with $y \neq x$, where $x_n \rightharpoonup x$. It is well known that Hilbert spaces and $l_p(1 < p < \infty)$ satisfy Opial's condition. One of the fundamental and celebrated results in the theory of nonexpansive mappings is Browder's demiclosedness principle [12] which states that if X is a uniformly convex Banach space, C is a nonempty closed and convex subset of X, and $T : C \to X$ is a nonexpansive mapping, then $I - T$ is demiclosed at each $x \in X$; that is, for any sequence $\{x_n\}$ in C satisfying $x_n \rightharpoonup x$ and $(I - T)x_n \to y$, we have $(I - T)x = y$. This principle also holds in a Banach space satisfying Opial's condition.

The following lemmas will be used throughout the paper.

Lemma 2.4. ([20]) *Let $\{a_i\}$ be a sequence of positive real numbers with $\Sigma_{i=1}^{\infty} a_i^{-1} = \infty$. If $\{b_i\}$ is a bounded sequence, then $\liminf_{i\to\infty} a_i(b_{i+1} - b_i) \leq 0$.*

Lemma 2.5. ([36]) *Suppose that $\{a_i\}$ and $\{\epsilon_i\}$ are two sequences of nonnegative real numbers such that $a_{i+1} \leq a_i + \epsilon_i$, for all $i \geq 0$, and $\sum_{i=0}^{\infty} \epsilon_i < \infty$. Then, $\lim_{i\to\infty} a_i$ exists.*

Lemma 2.6. ([14]) *Let $\{a_i\}$ and $\{b_i\}$ be two sequences of real positive numbers. If $\{a_i\}$ is nonincreasing and convergent to zero and $\sum_{i=1}^{\infty} a_i b_i < \infty$, then $(\sum_{i=1}^{n} b_i)a_n \to 0$, as $n \to \infty$.*

Lemma 2.7. ([30]) (Opial) *Let X be a uniformly convex Banach space satisfying Opial's condition, $(x_n)_{n\geq 1}$ a sequence in X, and $F \subseteq X$ nonempty. Assume*

(1) *$\|x_n - u\|$ has a limit as $n \to \infty$, for each $u \in F$; and*
(2) *the weak limit of each weakly convergent subsequence of $(x_n)_{n\geq 1}$ belongs to F.*

Then, $(x_n)_{n\geq 1}$ converges weakly to some $x \in F$.

3. The nonhomogeneous case

Let us consider the second order difference equation (1.1), as well as the auxiliary sequence $(a_i)_{i\geqslant 1}$ given by

$$a_0 = 1 \quad , \quad a_i = \frac{1}{\theta_1\theta_2 \ldots \theta_i}, \quad i \geq 1. \tag{3.1}$$

Observe that

$$a_i\theta_i = a_{i-1}, \quad i \geq 1, \tag{3.2}$$

and denote

$$h_k = \Sigma_{i=1}^{k} \frac{1}{\theta_k\theta_{k-1}\ldots\theta_i}, \quad \forall k \geq 1. \tag{3.3}$$

Remark. From now on we assume that the difference inclusion (1.1) has a solution for an initial value $u_0 = x$ in X. It is clear that in general (1.1) has no solution even if $A = 0$, $\theta_i \equiv 1$ and $(f_i)_{i\geq 1} \in \ell^1(X)$; we refer to Poffald and Reich [32]. The existence of solution for (1.1), in the framework Banach spaces, has been studied in [18]. One result worth mentioning is [18, Theorem 4.4]: Let X be a uniformly smooth and uniformly convex Banach space. Let $A \subseteq X \times X$ be $m-$accretive with $A^{-1}0 \neq \emptyset$ and $c_i, \theta_i > 0, \forall i \geq 1$, such that $\sum_{i=1}^{\infty} \frac{1}{h_i} = \infty$ holds. If $(f_i)_{i\geq 1}$ is a sequence in X satisfying $\Sigma_{i=1}^{\infty} h_i\|f_i\| < \infty$, then (1.1) has a unique solution for every initial point $x \in X$.

In the following, we prove some new weak and strong convergence theorems for the solutions to (1.1), and provide new approximation results for the zeros of A in the context of Banach spaces. Our results extend previous corresponding results by Poffald and Reich [32], where $\theta_i \equiv 1$ and X is assumed to have a strongly monotone duality mapping, as well as the results of Apreutesei [6], where $((\theta_i)$ was assumed to be a nonincreasing sequence in $[1, \infty)$, and X was assumed to have a strongly monotone duality mapping. We assume neither X to have a strongly monotone duality mapping, nor (θ_i) to be nonincreasing.

We present our results separately for $0 < \theta_i < 1$, $\forall i$, and for $\theta_i \geq 1$, $\forall i$.

3.1. The case $0 < \theta_i < 1$, $\forall i$

Suppose that $\theta_i \in (0, 1), \forall i \geq 1$, and in addition

$$\sum_{i=1}^{\infty} a_i^{-1} = \sum_{i=1}^{\infty} \theta_1\theta_2\ldots\theta_i = \infty. \tag{3.4}$$

Our results for $0 < \theta_i < 1$ are new in the framework of Banach spaces.

Lemma 3.1. *Let X be uniformly convex and let $A \subseteq X \times X$ be accretive such that $A^{-1}0 \neq \emptyset$. Suppose that $\sum_{i=1}^{\infty} a_i\|f_i\| < \infty$. Then $\liminf_{i\to\infty} a_{i-1}\|u_i - u_{i-1}\| = 0$.*

Proof. Let $p \in A^{-1}0$. From the accretivity of A and (1.1), we have

$$(u_{i+1} - (1 + \theta_i)u_i + \theta_i u_{i-1} - f_i, j(u_i - p)) \geq 0, \quad \forall i \geq 1. \tag{3.5}$$

Thus

$$(u_{i+1} - u_i, j(u_i - p)) - \theta_i(u_i - u_{i-1}, j(u_i - p)) - (f_i, j(u_i - p)) \geq 0, \quad \forall i \geq 1. \tag{3.6}$$

Multiplying both sides by a_i and using (3.2), we obtain

$$a_i(u_{i+1} - u_i, j(u_i - p)) - a_{i-1}(u_i - u_{i-1}, j(u_i - p)) - a_i(f_i, j(u_i - p)) \geq 0, \quad \forall i \geq 1. \tag{3.7}$$

Since X is uniformly convex, by using Lemma 2.2, we have

$$(a_{i-1}u_i - a_{i-1}u_{i-1}, j(a_{i-1}(u_i - p)) - j(a_{i-1}(u_{i-1} - p))) \geq g(a_{i-1}\|u_i - u_{i-1}\|),$$

for all $i \geq 1$. Hence

$$a_{i-1}(u_i - u_{i-1}, j(u_i - p)) - a_{i-1}(u_i - u_{i-1}, j(u_{i-1} - p)) \geq a_{i-1}^{-1} g(a_{i-1}\|u_i - u_{i-1}\|), \tag{3.8}$$

for all $i \geq 1$. Using (3.7) in (3.8), we deduce

$$\begin{aligned} &a_{i-1}^{-1} g(a_{i-1}\|u_i - u_{i-1}\|) \leq \\ &\qquad a_i(u_{i+1} - u_i, j(u_i - p)) - a_{i-1}(u_i - u_{i-1}, j(u_{i-1} - p)) - a_i(f_i, j(u_i - p)), \end{aligned} \tag{3.9}$$

for all $i \geq 1$. Summing up from $i = k$ to m and using Lemma 2.1, we arrive at

$$\begin{aligned} &\Sigma_{i=k}^m a_{i-1}^{-1} g(a_{i-1}\|u_i - u_{i-1}\|) \\ &\qquad \leq \Sigma_{i=k}^m (a_i(u_{i+1} - u_i, j(u_i - p)) - a_{i-1}(u_i - u_{i-1}, j(u_{i-1} - p))) \\ &\qquad - \Sigma_{i=k}^m a_i(f_i, j(u_i - p)) \\ &\qquad = a_m(u_{m+1} - u_m, j(u_m - p)) - a_{k-1}(u_k - u_{k-1}, j(u_{k-1} - p)) \\ &\qquad - \Sigma_{i=k}^m a_i(f_i, j(u_i - p)) \\ &\qquad \leq \frac{a_m}{2}(\|u_{m+1} - p\|^2 - \|u_m - p\|^2) - a_{k-1}(u_k - u_{k-1}, j(u_{k-1} - p)) \\ &\qquad + \Sigma_{i=k}^m a_i\|f_i\|\|u_i - p\|. \end{aligned} \tag{3.10}$$

Taking liminf in (3.10) as $m \to \infty$, using (3.4) and Lemma 2.4, we see that

$$\Sigma_{i=k}^\infty a_{i-1}^{-1} g(a_{i-1}\|u_i - u_{i-1}\|) \leq -a_{k-1}(u_k - u_{k-1}, j(u_{k-1} - p)) + \Sigma_{i=k}^\infty a_i\|f_i\|\|u_i - p\|.$$

Since u_i is bounded, there exist positive constants α, β such that

$$\Sigma_{i=k}^\infty a_{i-1}^{-1} g(a_{i-1}\|u_i - u_{i-1}\|) \leq \alpha + \beta(\Sigma_{i=k}^\infty a_i\|f_i\|). \tag{3.11}$$

The assumption implies that $\Sigma_{i=k}^\infty a_{i-1}^{-1} g(a_{i-1}\|u_i - u_{i-1}\|) < \infty$. Using again (3.4), we arrive at

$$\liminf_{i\to\infty} g(a_{i-1}\|u_i - u_{i-1}\|) = 0.$$

Since g is continuous and strictly increasing, and $g(0) = 0$, we conclude that

$$\liminf_{i\to\infty} a_{i-1}\|u_i - u_{i-1}\| = 0. \quad \square$$

Lemma 3.2. *With the same assumptions as in Lemma 3.1, $(u_i - u_{i-1})$ converges to zero, as $i \to \infty$.*

Proof. Let $p \in A^{-1}0$. Since X is uniformly convex, by using Lemma 2.2, we have

$$(u_i - u_{i-1}, j(u_i - p) - j(u_{i-1} - p)) \geq g(\|u_i - u_{i-1}\|), \qquad \forall i \geq 1.$$

Hence

$$(u_i - u_{i-1}, j(u_i - p)) - (u_i - u_{i-1}, j(u_{i-1} - p)) \geq g(\|u_i - u_{i-1}\|), \qquad \forall i \geq 1.$$

Multiplying both sides by θ_i, we get

$$\theta_i(u_i - u_{i-1}, j(u_i - p)) - \theta_i(u_i - u_{i-1}, j(u_{i-1} - p)) \geq \theta_i g(\|u_i - u_{i-1}\|), \tag{3.12}$$

for all $i \geq 1$. Using (3.6) in (3.12), we can write

$$(u_{i+1} - u_i, j(u_i - p)) - (f_i, j(u_i - p)) - \theta_i(u_i - u_{i-1}, j(u_{i-1} - p)) \geq \theta_i g(\|u_i - u_{i-1}\|),$$

for all $i \geq 1$. Multiplying both sides by a_i and using (3.2), we obtain

$$a_i(u_{i+1} - u_i, j(u_i - p)) - a_i(f_i, j(u_i - p)) - a_{i-1}(u_i - u_{i-1}, j(u_{i-1} - p)) \geq a_{i-1} g(\|u_i - u_{i-1}\|),$$

for all $i \geq 1$. Summing up from $i = k$ to m, and then letting $m \to \infty$, in a similar way as in Lemma 3.1, we conclude that

$$\Sigma_{i=k}^{\infty} a_{i-1} g(\|u_i - u_{i-1}\|) \leq -a_{k-1}(u_k - u_{k-1}, j(u_{k-1} - p)) + \Sigma_{i=k}^{\infty} a_i \|f_i\| \|u_i - p\|.$$

Since $0 < \theta_i < 1$, by (3.1) the sequence $\{a_i\}$ is increasing. Therefore

$$a_{k-1} \Sigma_{i=k}^{\infty} g(\|u_i - u_{i-1}\|) \leq -a_{k-1}(u_k - u_{k-1}, j(u_{k-1} - p)) + \Sigma_{i=k}^{\infty} a_i \|f_i\| \|u_i - p\|.$$

Dividing both sides by a_{k-1}, we get

$$\Sigma_{i=k}^{\infty} g(\|u_i - u_{i-1}\|) \leq -(u_k - u_{k-1}, j(u_{k-1} - p)) + \Sigma_{i=k}^{\infty} \frac{a_i}{a_{k-1}} \|f_i\| \|u_i - p\|.$$

Since u_i is bounded, there exist positive constants α, β such that

$$\Sigma_{i=k}^{\infty} g(\|u_i - u_{i-1}\|) \leq \alpha + \beta(\Sigma_{i=k}^{\infty} \frac{a_i}{a_{k-1}} \|f_i\|). \tag{3.13}$$

Since $0 < \theta_i < 1, \forall i \geq 1$, by (3.1), we have $\dfrac{1}{a_{k-1}} \leq 1, \forall k \geq 1$. Therefore it follows from the assumption that

$$\Sigma_{i=k}^{\infty} \frac{a_i}{a_{k-1}} \|f_i\| \leq \Sigma_{i=k}^{\infty} a_i \|f_i\| < \infty. \tag{3.14}$$

Using (3.14) in (3.13), we get $\Sigma_{i=k}^{\infty} g(\|u_i - u_{i-1}\|) < \infty$, so $\lim_{i\to\infty} g(\|u_i - u_{i-1}\|) = 0$. Since g is continuous, strictly increasing and $g(0) = 0$, we conclude that

$$\lim_{i\to\infty} \|u_i - u_{i-1}\| = 0$$

and we obtain the desired conclusion. □

Lemma 3.3. *With the same assumptions as in Lemma 3.1, if moreover* $\sum_{i=1}^{\infty} h_i \|f_i\| < \infty$, *then* $\lim_{i\to\infty} \|u_i - p\|$ *exists, for each* $p \in A^{-1}0$.

Proof. From (3.5), we have

$$(u_{i+1} - (1 + \theta_i)u_i + \theta_i u_{i-1} - f_i, j(u_i - p)) \geq 0, \qquad \forall i \geq 1.$$

Hence

$$(\|u_{i+1} - p\| - \|u_i - p\|) - \theta_i(\|u_i - p\| - \|u_{i-1} - p\|) + \|f_i\| \geq 0,$$

for all $i \geq 1$. Therefore

$$\begin{aligned}
&\|u_i - p\| - \|u_{i-1} - p\| \\
&\quad \leq \frac{1}{\theta_i}(\|u_{i+1} - p\| - \|u_i - p\|) + \frac{\|f_i\|}{\theta_i} \\
&\quad \leq \frac{1}{\theta_{i+1}\theta_i}(\|u_{i+2} - p\| - \|u_{i+1} - p\|) + \frac{\|f_{i+1}\|}{\theta_{i+1}\theta_i} + \frac{\|f_i\|}{\theta_i} \\
&\quad \vdots \\
&\quad \leq \frac{1}{\theta_{i+j}\cdots\theta_{i+1}\theta_i}(\|u_{i+j+1} - p\| - \|u_{i+j} - p\|) + \sum_{k=i}^{i+j} \frac{\|f_k\|}{\theta_k\theta_{k-1}\cdots\theta_i} \\
&\quad \leq a_{i-1}^{-1}a_{i+j}(\|u_{i+j+1} - p\| - \|u_{i+j} - p\|) + \sum_{k=i}^{i+j} \frac{\|f_k\|}{\theta_k\theta_{k-1}\cdots\theta_i},
\end{aligned}$$

for all $i \geq 1, j \geq 0$. Taking liminf as $j \to \infty$, by our assumption (3.4) and Lemma 2.4, we get

$$\|u_i - p\| - \|u_{i-1} - p\| \leq \sum_{k=i}^{\infty} \frac{\|f_k\|}{\theta_k\theta_{k-1}...\theta_i},$$

for all $i \geq 1$. Set $\epsilon_i = \sum_{k=i}^{\infty} \frac{\|f_k\|}{\theta_k\theta_{k-1}...\theta_i}$; then we have:

$$\|u_i - p\| \leq \|u_{i-1} - p\| + \epsilon_i, \quad \forall i \geq 1.$$

On the other hand

$$\sum_{i=1}^{\infty} \epsilon_i = \sum_{i=1}^{\infty}\sum_{k=i}^{\infty} \frac{\|f_k\|}{\theta_k\theta_{k-1}...\theta_i} = \sum_{i=1}^{\infty} h_i\|f_i\| < \infty, \tag{3.15}$$

since

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{k=i}^{\infty}\frac{\|f_k\|}{\theta_k\theta_{k-1}...\theta_i} = \\
&\quad \sum_{k=1}^{\infty}\frac{\|f_k\|}{\theta_k\theta_{k-1}...\theta_1} + \sum_{k=2}^{\infty}\frac{\|f_k\|}{\theta_k\theta_{k-1}...\theta_2} + \sum_{k=3}^{\infty}\frac{\|f_k\|}{\theta_k\theta_{k-1}...\theta_3} + ... + \sum_{k=n}^{\infty}\frac{\|f_k\|}{\theta_k\theta_{k-1}...\theta_n} + ... \\
&\quad = (\frac{1}{\theta_1}\|f_1\| + \frac{1}{\theta_2\theta_1}\|f_2\| + \frac{1}{\theta_3\theta_2\theta_1}\|f_3\| + \frac{1}{\theta_4\theta_3\theta_2\theta_1}\|f_4\| \\
&\quad + ... + \frac{1}{\theta_n\theta_{n-1}...\theta_1}\|f_n\| + ...) \\
&\quad + (\frac{1}{\theta_2}\|f_2\| + \frac{1}{\theta_3\theta_2}\|f_3\| + \frac{1}{\theta_4\theta_3\theta_2}\|f_4\| + ... + \frac{1}{\theta_n\theta_{n-1}...\theta_2}\|f_n\| + ...) \\
&\quad + (\frac{1}{\theta_3}\|f_3\| + \frac{1}{\theta_4\theta_3}\|f_4\| + ... + \frac{1}{\theta_n\theta_{n-1}...\theta_3}\|f_n\| + ...)
\end{aligned}$$

$$
\begin{aligned}
&+ ... + (\frac{1}{\theta_n}\|f_n\| + \frac{1}{\theta_{n+1}\theta_n}\|f_{n+1}\| + ...) \\
&= \frac{1}{\theta_1}\|f_1\| + (\frac{1}{\theta_2} + \frac{1}{\theta_2\theta_1})\|f_2\| + (\frac{1}{\theta_3} + \frac{1}{\theta_3\theta_2} + \frac{1}{\theta_3\theta_2\theta_1})\|f_3\| + ... \\
&+ (\frac{1}{\theta_n} + ... + \frac{1}{\theta_n\theta_{n-1}...\theta_2} + \frac{1}{\theta_n\theta_{n-1}...\theta_1})\|f_n\| + ... \\
&= h_1\|f_1\| + h_2\|f_2\| + h_3\|f_3\| + ... + h_n\|f_n\| + ... \\
&= \sum_{i=1}^{\infty} h_i\|f_i\|.
\end{aligned}
$$

Now the conclusion follows from Lemma 2.5. □

We can now state our main results for the case $0 < \theta_i < 1$.

Theorem 3.4. *Let X be a uniformly convex Banach space satisfying Opial's condition, and let $A \subseteq X \times X$ be $m-$accretive such that $A^{-1}0 \neq \emptyset$. Assume that $\liminf_{i\to\infty} c_i > 0$ and $\sum_{i=1}^{\infty} h_i\|f_i\| < \infty$. Then $u_i \rightharpoonup p \in A^{-1}0$.*

Proof. We use Lemma 2.7 for the nonempty set $F = A^{-1}(0)$. First we verify hypothesis (1). In fact, for any $q \in A^{-1}(0)$, it follows from Lemma 3.3 that $\lim_{i\to\infty}\|u_i - q\|$ exists. To check hypothesis (2), let p be a weak limit of a weakly convergent subsequence $(u_{i_n})_{n\geq 1}$ of $(u_i)_{i\geq 1}$. We need to prove that $p \in A^{-1}(0)$. From equation (1.1), we have

$$v_i := \frac{1}{c_i}((u_{i+1} - u_i) - \theta_i(u_i - u_{i-1}) - f_i) \in Au_i, \qquad \forall i \geq 1. \tag{3.16}$$

Since $\sum_{i=1}^{\infty} h_i\|f_i\| < \infty$, $0 < \theta_i < 1$ and $h_i > 1$ (by (3.3)), we deduce that $\lim_{i\to\infty}\|f_i\| = 0$. On the other hand, since

$$h_i = \frac{1}{\theta_i\theta_{i-1}...\theta_1} + \frac{1}{\theta_i\theta_{i-1}...\theta_2} + ... + \frac{1}{\theta_i} \geq \frac{1}{\theta_i\theta_{i-1}...\theta_1} = a_i,$$

we have

$$\sum_{i=1}^{\infty} a_i\|f_i\| \leq \sum_{i=1}^{\infty} h_i\|f_i\| < \infty. \tag{3.17}$$

Then, using Lemma 3.2, we get $c_iv_i \to 0$, as $i \to \infty$. Since $\liminf_{i\to\infty} c_i > 0$, we have $\lim_{i\to\infty}\|v_i\| = 0$. Let $J_\lambda = (I + \lambda A)^{-1}$ be the resolvent of A. By Lemma 2.3, we have $\|J_\lambda u_i - u_i\| \leq \lambda\|v_i\|$. Now since $u_{i_n} \rightharpoonup p$, and X is uniformly convex, by using Browder's demiclosedness principle, we conclude that p is a fixed point of J_λ, hence a zero of A. □

Theorem 3.5. *With the same assumptions as in Theorem 3.4, if moreover X is smooth and A satisfies the convergence condition, then u_i converges strongly to a zero of A.*

Proof. Let $P : X \to A^{-1}0$ be the nearest point projection map of X onto the zero set of A. From accretivity of A, (1.1), (2.1), (3.16) and Lemma 2.1, for all $i \geq 1$, we have

$$0 \leq c_i(v_i, J(u_i - Pu_i)) = (u_{i+1} - (1 + \theta_i)u_i + \theta_iu_{i-1} - f_i, J(u_i - Pu_i))$$

$$
\begin{aligned}
=&(u_{i+1}-Pu_{i+1},J(u_i-Pu_i))-(1+\theta_i)(u_i-Pu_i,J(u_i-Pu_i))\\
&+\theta_i(u_{i-1}-Pu_{i-1},J(u_i-Pu_i))-(f_i,J(u_i-Pu_i))\\
&+(Pu_{i+1}-(1+\theta_i)Pu_i+\theta_iPu_{i-1},J(u_i-Pu_i))\\
\le&\frac{1}{2}\|u_{i+1}-Pu_{i+1}\|^2+\frac{1}{2}\|u_i-Pu_i\|^2-(1+\theta_i)\|u_i-Pu_i\|^2\\
&+\frac{\theta_i}{2}\|u_{i-1}-Pu_{i-1}\|^2+\frac{\theta_i}{2}\|u_i-Pu_i\|^2+\|f_i\|\|u_i-Pu_i\|\\
=&\frac{1}{2}(\|u_{i+1}-Pu_{i+1}\|^2-\|u_i-Pu_i\|^2)\\
&-\frac{\theta_i}{2}(\|u_i-Pu_i\|^2-\|u_{i-1}-Pu_{i-1}\|^2)+M\|f_i\|,
\end{aligned}
$$

where $M=\sup_{i\ge1}\|u_i-Pu_i\|$. Multiplying both sides of the above inequality by a_i, using (3.2) and summing up from $i=1$ to m, we obtain:

$$
\begin{aligned}
0\le\sum_{i=1}^{m}c_ia_i(v_i,J(u_i-Pu_i))\le&\frac{a_m}{2}(\|u_{m+1}-Pu_{m+1}\|^2-\|u_m-Pu_m\|^2)\\
&-\frac{a_0}{2}(\|u_1-Pu_1\|^2-\|u_0-Pu_0\|^2)+M\sum_{i=1}^{m}a_i\|f_i\|.
\end{aligned}
$$

Taking liminf as $m\to\infty$, by using (3.4), (3.17), our assumption and Lemma 2.4, we get

$$\sum_{i=1}^{\infty}c_ia_i(v_i,J(u_i-Pu_i))<\infty. \tag{3.18}$$

Since $a_i>1$, we have

$$\sum_{i=1}^{\infty}c_i(v_i,J(u_i-Pu_i))<\infty.$$

On the other hand, since $\liminf\limits_{i\to\infty}c_i>0$, it follows that $\lim\limits_{i\to\infty}(v_i,J(u_i-Pu_i))=0$. Hence by the convergence condition, we get

$$\liminf_{i\to\infty}\|u_i-Pu_i\|=0. \tag{3.19}$$

By a similar proof as in Lemma 3.3, we can show that

$$\|u_{i+1}-Pu_{i+1}\|\le\|u_{i+1}-Pu_i\|\le\|u_i-Pu_i\|+\sum_{k=i+1}^{\infty}\frac{\|f_k\|}{\theta_k\theta_{k-1}...\theta_i}.$$

Thus by (3.15) and Lemma 2.5, $\lim\limits_{i\to\infty}\|u_i-Pu_i\|$ exists, and therefore by (3.19) we have

$$\lim_{i\to\infty}\|u_i-Pu_i\|=0.$$

On the other hand

$$\begin{aligned}\|u_{i+m}-u_i\| &\le \|u_{i+m}-Pu_i\|+\|u_i-Pu_i\| \\ &\le 2\|u_i-Pu_i\|+\sum_{n=1}^{m}\sum_{k=i+n}^{\infty}\frac{\|f_k\|}{\theta_k\theta_{k-1}...\theta_{i+n}} \\ &\le 2\|u_i-Pu_i\|+\sum_{k=i+1}^{\infty}h_k\|f_k\| \to 0,\end{aligned}$$

as $i \to \infty$, uniformly in $m \ge 0$. So the convergence of u_i follows. $\square$

Here is another result in this direction.

Theorem 3.6. *Let X be uniformly convex, and let $A \subseteq X \times X$ be strongly accretive such that $A^{-1}0 \neq \emptyset$. Assume that $\liminf_{i\to\infty} ic_i > 0$ and $\sum_{i=1}^{\infty} h_i\|f_i\| < \infty$. Then $u_i \to p \in A^{-1}0$.*

Proof. Let p be the unique element of $A^{-1}(0)$. By (1.1) and the strong accretivity of A, we have

$$(u_{i+1}-(1+\theta_i)u_i+\theta_i u_{i-1}-f_i, j(u_i-p)) \ge \alpha c_i\|u_i-p\|^2, \qquad \forall i \ge 1.$$

Hence

$$(\|u_{i+1}-p\|-\|u_i-p\|)-\theta_i(\|u_i-p\|-\|u_{i-1}-p\|)+\|f_i\| \ge \alpha c_i\|u_i-p\|,$$

for all $i \ge 1$. Multiplying both sides by a_i and using (3.2), we obtain:

$$\begin{aligned}a_i(\|u_{i+1}-p\|-\|u_i-p\|)-a_{i-1}(\|u_i-p\|-\|u_{i-1}-p\|)+a_i\|f_i\| \\ \ge \alpha a_i c_i\|u_i-p\|.\end{aligned}$$

Summing up from $i=k$ to $i=m$, we get:

$$\begin{aligned}&\alpha\sum_{i=k}^{m}a_ic_i\|u_i-p\| \le \\ &\sum_{i=k}^{m}(a_i(\|u_{i+1}-p\|-\|u_i-p\|)-a_{i-1}(\|u_i-p\|-\|u_{i-1}-p\|))+\sum_{i=k}^{m}a_i\|f_i\| \\ &= a_m(\|u_{m+1}-p\|-\|u_m-p\|)-a_{k-1}(\|u_k-p\|-\|u_{k-1}-p\|)+\sum_{i=k}^{m}a_i\|f_i\|.\end{aligned}$$

Taking liminf as $m \to \infty$, by using (3.4), the assumption and Lemma 2.4, we obtain:

$$\alpha\sum_{i=k}^{\infty}a_ic_i\|u_i-p\| \le -a_{k-1}(\|u_k-p\|-\|u_{k-1}-p\|)+\sum_{i=k}^{\infty}a_i\|f_i\|. \tag{3.20}$$

Since $0 < \theta_i < 1$, by (3.1) the sequence $\{a_i\}$ is increasing. Therefore

$$\alpha a_{k-1}\sum_{i=k}^{\infty}c_i\|u_i-p\| \le -a_{k-1}(\|u_k-p\|-\|u_{k-1}-p\|)+\sum_{i=k}^{\infty}a_i\|f_i\|.$$

Dividing both sides by a_{k-1}, we get

$$\alpha \sum_{i=k}^{\infty} c_i \|u_i - p\| \leq -(\|u_k - p\| - \|u_{k-1} - p\|) + \sum_{i=k}^{\infty} \frac{a_i}{a_{k-1}} \|f_i\|.$$

Summing up from $k = 1$ to n, we obtain:

$$\begin{aligned} \alpha \sum_{k=1}^{n} \sum_{i=k}^{\infty} c_i \|u_i - p\| &\leq \|u_0 - p\| - \|u_n - p\| + \sum_{k=1}^{n} \sum_{i=k}^{\infty} \frac{a_i}{a_{k-1}} \|f_i\| \\ &\leq \|u_0 - p\| + \sum_{k=1}^{n} \sum_{i=k}^{\infty} \frac{a_i}{a_{k-1}} \|f_i\|, \end{aligned}$$

and letting $n \to \infty$, we get

$$\alpha \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} c_i \|u_i - p\| \leq \|u_0 - p\| + \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} \frac{a_i}{a_{k-1}} \|f_i\|. \tag{3.21}$$

Using (3.1), (3.3), (3.15) and the assumption, we have

$$\sum_{k=1}^{\infty} \sum_{i=k}^{\infty} \frac{a_i}{a_{k-1}} \|f_i\| = \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} \frac{1}{\theta_i \theta_{i-1} \ldots \theta_k} \|f_i\| = \sum_{i=1}^{\infty} h_i \|f_i\| < \infty. \tag{3.22}$$

Now (3.21) and (3.22), imply that

$$\sum_{i=1}^{\infty} i c_i \|u_i - p\| < \infty.$$

Since $\liminf\limits_{i \to \infty} i c_i > 0$, then $\lim\limits_{i \to \infty} \|u_i - p\| = 0$. The proof is now complete. $\square$

Remark 3.7. The above proof actually shows that if in Theorem 3.6, the assumption "$\liminf_{i \to \infty} i c_i > 0$" is replaced by "$\liminf_{i \to \infty} c_i > 0$", then $i\|u_i - p\|$ converges to zero as $i \to \infty$; i.e., $\|u_i - p\| = o(1/i)$.

3.2. The case $\theta_i \geq 1$, $\forall i$

The remainder of this section will be devoted to the case $\theta_i \geq 1$, for all $i \geq 1$. We state several extensions of the results in [6] and [32] to the nonhomogeneous case. In our results, we neither require X to have a strongly monotone duality mapping, nor (θ_i) to be nonincreasing.

Lemma 3.8. *Let X be uniformly convex, $A \subseteq X \times X$ be accretive such that $A^{-1}0 \neq \emptyset$, and assume that $\Sigma_{i=1}^{\infty} a_i \|f_i\| < \infty$. Then $a_{i-1}(u_i - u_{i-1})$ converges to zero, as $i \to \infty$.*

Proof. Let $p \in A^{-1}0$. From (3.11), we have

$$\Sigma_{i=k}^{\infty} a_{i-1}^{-1} g(a_{i-1} \|u_i - u_{i-1}\|) < \infty, \tag{3.23}$$

since when $\theta_i \geq 1$, the condition (3.4) automatically holds. On the other hand, by $\theta_i \geq 1$ and (3.1), we have $a_{i-1}^{-1} \geq 1, \forall i \geq 1$. Combining this with (3.23) yields

$$\Sigma_{i=k}^{\infty} g(a_{i-1} \|u_i - u_{i-1}\|) < \infty.$$

Hence

$$\lim_{i\to\infty} g(a_{i-1}\|u_i - u_{i-1}\|) = 0.$$

Since g is continuous, strictly increasing and $g(0) = 0$, we conclude that

$$\lim_{i\to\infty} a_{i-1}\|u_i - u_{i-1}\| = 0. \quad \square$$

Lemma 3.9. *Let X be uniformly convex, $A \subseteq X \times X$ be accretive and $\sum_{i=1}^{\infty} h_i\|f_i\| < \infty$. If $p \in A^{-1}0$, then $\lim_{i\to\infty} \|u_i - p\|$ exists.*

Proof. The proof is done along similar lines as that of Lemma 3.3, by noting that (3.4) is automatically satisfied when $\theta_n \geq 1$. Therefore we omit it here. $\square$

Theorem 3.10. (cf. [6, Theorem 3.1] and [32, Theorem 4.3]) *Let X be a uniformly convex Banach space satisfying Opial's condition, and let $A \subseteq X \times X$ be $m-$accretive such that $A^{-1}0 \neq \emptyset$. Assume $\sum_{i=1}^{\infty} h_i\|f_i\| < \infty$ and $\liminf_{i\to\infty} a_i c_i > 0$. Then $u_i \rightharpoonup p \in A^{-1}0$.*

Proof. We use Lemma 2.7 for the nonempty set $F = A^{-1}(0)$. The first part of the proof is similar to that in Theorem 3.4. From (3.16) and (3.2), we have

$$v_i = \frac{1}{a_i c_i}(a_i(u_{i+1} - u_i) - a_{i-1}(u_i - u_{i-1}) - a_i f_i) \in Au_i, \qquad \forall i \geq 1.$$

By (3.17) and the assumption, we have:

$$\lim_{i\to\infty} a_i\|f_i\| = 0.$$

The rest of the proof is similar to that of Theorem 3.4, by using the assumption and Lemma 3.8. $\square$

Theorem 3.11. (cf. [32, Theorem 4.4]) *With the same assumptions as in Theorem 3.10, if moreover X is smooth and A satisfies the convergence condition, then u_i converges strongly to a zero of A.*

Proof. By a similar argument as in Theorem 3.5, we get (3.18), and subsequently we have $\lim_{i\to\infty} (v_i, J(u_i - Pu_i)) = 0$. The rest of the proof is similar to that of Theorem 3.5, and we omit it here. $\square$

Theorem 3.12. *Let X be uniformly convex, $A \subseteq X \times X$ be strongly accretive such that $A^{-1}0 \neq \emptyset$, and $\sum_{i=1}^{\infty} h_i\|f_i\| < \infty$. If $\liminf_{i\to\infty} ia_i c_i > 0$, then $u_i \to p \in A^{-1}0$.*

Proof. Let p be the unique element of $A^{-1}(0)$. By (3.20),

$$\begin{aligned}
\alpha \sum_{i=k}^{\infty} a_i c_i\|u_i - p\| &\leq -a_{k-1}(\|u_k - p\| - \|u_{k-1} - p\|) + \sum_{i=k}^{\infty} a_i\|f_i\| \\
&= a_{k-1}\|u_{k-1} - p\| - a_{k-1}\|u_k - p\| + \sum_{i=k}^{\infty} a_i\|f_i\| \qquad (3.24) \\
&\leq a_{k-1}\|u_{k-1} - p\| - a_k\|u_k - p\| + \sum_{i=k}^{\infty} a_i\|f_i\|,
\end{aligned}$$

since in this case a_i is nonincreasing. Summing up (3.24) from $k = 1$ to m and letting $m \to \infty$, we get

$$\alpha \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} a_i c_i \|u_i - p\| \leq \|u_0 - p\| + \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} a_i \|f_i\|. \tag{3.25}$$

On the other hand, by the assumption,

$$\sum_{k=1}^{\infty} \sum_{i=k}^{\infty} a_i \|f_i\| = \sum_{k=1}^{\infty} k a_k \|f_k\| \leq \sum_{k=1}^{\infty} h_k \|f_k\| < \infty, \tag{3.26}$$

since $\theta_i \geq 1, \forall i \geq 1$, and by (3.3), we have

$$\begin{aligned} h_k &= \frac{1}{\theta_k \theta_{k-1} \dots \theta_1} + \frac{1}{\theta_k \theta_{k-1} \dots \theta_2} + \dots + \frac{1}{\theta_k} \\ &\geq \frac{1}{\theta_k \theta_{k-1} \dots \theta_1} + \frac{1}{\theta_k \theta_{k-1} \dots \theta_1} + \dots + \frac{1}{\theta_k \theta_{k-1} \dots \theta_1} = k a_k. \end{aligned}$$

Using (3.25) and (3.26), we deduce that

$$\sum_{i=1}^{\infty} i a_i c_i \|u_i - p\| < \infty. \tag{3.27}$$

Since $\liminf\limits_{i \to \infty} i a_i c_i > 0$, we conclude that $\lim\limits_{i \to \infty} \|u_i - p\| = 0$. $\square$

Remark 3.13. From (3.27), it follows that if in Theorem 3.12, the assumption "$\liminf_{i \to \infty} i a_i c_i > 0$" is replaced by "$\liminf_{i \to \infty} a_i c_i > 0$", then $i\|u_i - p\|$ converges to zero, as $i \to \infty$; i.e., $\|u_i - p\| = o(1/i)$.

4. The homogeneous case

In this section, we investigate the asymptotic behavior of solutions to homogeneous case of (1.1), i.e., the difference equation

$$\begin{cases} u_{i+1} - (1 + \theta_i) u_i + \theta_i u_{i-1} \in c_i A u_i, & i \geqslant 1, \\ u_0 = x, \quad \sup\{\|u_i\| : i \geqslant 0\} < \infty. \end{cases} \tag{4.1}$$

We provide estimates for the rate of convergence of the solution.

We will need the following lemmas.

Lemma 4.1. (see [18]) *Let $A \subseteq X \times X$ be accretive and $\{u_i\}$ be a solution to (4.1) with $c_i, \theta_i > 0, \forall i \geq 1$. Then $a_{i-1}\|u_i - u_{i-1}\|$ is nonincreasing or eventually increasing.*

Lemma 4.2. *If $(b_i)_{i \geq 1}$ is a bounded sequence of positive numbers satisfying*

$$b_i \leq \frac{1}{1 + \theta_i} b_{i+1} + \frac{\theta_i}{1 + \theta_i} b_{i-1},$$

where θ_i is a positive sequence such that $\sum\limits_{i=1}^{\infty} \theta_1 \theta_2 \dots \theta_i = \infty$, then b_i is nonincreasing.

Proof. Suppose for a contradiction, that there exists $j \geq 1$ such that $b_{j-1} < b_j$. Since

$$b_j \leq \frac{1}{1+\theta_j} b_{j+1} + \frac{\theta_j}{1+\theta_j} b_{j-1},$$

it follows that

$$b_{j+1} \geq b_j + \theta_j (b_j - b_{j-1}).$$

Now by induction, we may easily prove that for all $i > j$,

$$b_i \geq b_j + \theta_j (b_j - b_{j-1}) + \theta_{j+1}\theta_j (b_j - b_{j-1}) + \cdots + \theta_{i-1}\theta_{i-2} \cdots \theta_j (b_j - b_{j-1}),$$

or equivalently,

$$b_i \geq b_j + (\Sigma_{k=j}^{i-1} \theta_k \theta_{k-1} \cdots \theta_j)(b_j - b_{j-1}),$$

for all $i > j$. Thus, denoting $\alpha = b_j - b_{j-1} > 0$, and then taking the liminf when $i \to \infty$, we obtain

$$\liminf_{i \to \infty} b_i \geq b_j + (\Sigma_{k=j}^{\infty} \theta_k \theta_{k-1} \cdots \theta_j)\alpha.$$

Since $\liminf_{i \to \infty} b_i < \infty$, the above inequality contradicts our assumption that $\sum\limits_{i=1}^{\infty} \theta_1 \theta_2 \dots \theta_i = \infty$. $\square$

It is worth mentioning that Lemma 4.2 improves upon [15, Lemma 3.1], where it is assumed that $\sum_{i=1}^{\infty} \dfrac{1}{h_i} = \infty$. Indeed, since

$$h_i = \frac{1}{\theta_i \theta_{i-1} \dots \theta_1} + \frac{1}{\theta_i \theta_{i-1} \dots \theta_2} + \dots + \frac{1}{\theta_i} \geq \frac{1}{\theta_i \theta_{i-1} \dots \theta_1}, \tag{4.2}$$

it follows that

$$\sum_{i=1}^{\infty} \theta_i \theta_{i-1} \dots \theta_1 \geq \sum_{i=1}^{\infty} \frac{1}{h_i}. \tag{4.3}$$

Lemma 4.3. *With the same assumptions as in Lemma 4.1, $\|u_i - p\|$ is nonincreasing or eventually increasing, for any $p \in A^{-1}0$. If moreover (3.4) holds, then $\|u_i - p\|$ is nonincreasing.*

Proof. Let $p \in A^{-1}0$. From the accretivity of A and (4.1), we have

$$(u_{i+1} - (1+\theta_i)u_i + \theta_i u_{i-1}, j(u_i - p)) \geq 0, \qquad \forall i \geq 1. \tag{4.4}$$

Hence

$$(\|u_{i+1} - p\| - \|u_i - p\|) - \theta_i (\|u_i - p\| - \|u_{i-1} - p\|) \geq 0, \tag{4.5}$$

for all $i \geq 1$. If $(\|u_i - p\|)_{i \geq 1}$ is not nonincreasing, then there exists $j \geq 1$ such that $\|u_{j-1} - p\| < \|u_j - p\|$. Now (4.5) with $i = j$ implies that $(\|u_i - p\|)_{i \geq j}$ is increasing. We prove the second part of the lemma. Using (4.5), we have

$$\|u_i - p\| \leq \frac{1}{1+\theta_i}\|u_{i+1} - p\| + \frac{\theta_i}{1+\theta_i}\|u_{i-1} - p\|,$$

for all $i \geq 1$. Since $(\|u_i - p\|)_{i\geq 1}$ is bounded and (3.4) holds, it must be nonincreasing by Lemma 4.2. □

Lemma 4.4. *With the same assumptions as in Lemma 4.1, if moreover X is uniformly convex, then $a_{i-1}\|u_i - u_{i-1}\|$ is nonincreasing. Moreover, $a_{i-1}(u_i - u_{i-1})$ converges to zero, as $i \to \infty$.*

Proof. We note that the assertion of Lemma 3.1 holds for general $\theta_i > 0$. Thus, by Lemma 3.1, we have $\liminf_{i\to\infty} a_{i-1}\|u_i - u_{i-1}\| = 0$. Then Lemma 4.1 implies that $a_{i-1}\|u_i - u_{i-1}\|$ is nonincreasing or eventually increasing, and therefore $a_{i-1}(u_i - u_{i-1})$ converges to zero as $i \to \infty$. □

Theorem 4.5. *Let X be uniformly convex, and $A \subseteq X \times X$ be strongly accretive such that $A^{-1}0 \neq \emptyset$. If $0 < \theta_n < 1$, (3.4) holds and $\sum_{i=1}^{\infty} ic_i = \infty$, then $u_i \to p \in A^{-1}0$, and $\|u_i - p\| = o((\sum_{n=1}^{i} nc_n)^{-1})$.*

Proof. By the same argument as in Theorem 3.6, we have:

$$\sum_{i=1}^{\infty} ic_i\|u_i - p\| < \infty. \tag{4.6}$$

From (4.6) and the assumption $\sum_{i=1}^{\infty} ic_i = \infty$, we deduce that $\liminf_{i\to\infty} \|u_i - p\| = 0$. On the other hand, by Lemma 4.3, $\|u_i - p\|$ is nonincreasing. Therefore $u_i \to p$, as $i \to \infty$, as well as $\|u_i - p\| = o((\sum_{n=1}^{i} nc_n)^{-1})$, by Lemma 2.6. □

The following theorem extends Theorem 3.3 of Apreutesei [6], as well as Proposition 4.5 of [32].

Theorem 4.6. *Let X be uniformly convex, and $A \subseteq X \times X$ be strongly accretive such that $A^{-1}0 \neq \emptyset$. If $\theta_n \geq 1$ and $\sum_{i=1}^{\infty} ic_ia_i = \infty$, then $u_i \to p \in A^{-1}0$, and $\|u_i - p\| = o((\sum_{n=1}^{i} nc_na_n)^{-1})$.*

Proof. The proof is similar to that of Theorem 4.5, by using (3.27) instead of (4.6). □

The following remains an open question.

Open question. Does Theorem 3.4 (resp. Theorem 3.10) still hold if the assumption $\liminf_{i\to\infty} c_i > 0$ (resp. $\liminf_{i\to\infty} a_ic_i > 0$) in that theorem is replaced by $\sum_{i=1}^{\infty} c_i = \infty$ (resp. $\sum_{i=1}^{\infty} a_ic_i = \infty$)?

This open question is motivated by the fact that we provide an affirmative answer for it in the next section, when A is a generalized subdifferential.

5. Applications to optimization

Let X be a smooth Banach space and let $\psi : X \to (-\infty, \infty]$ be a proper function. Following Saeidi and Kim [35], we define the (possibly empty) set

$$S_\psi(x) := \{z \in X : \ \psi(x) - \psi(y) \leq \langle z, J(x - y)\rangle, \text{ for all } y \in X\}, \ x \in X. \tag{5.1}$$

For $X = H$, a real Hilbert space, it follows that $S_\psi = \partial\psi$, the subdifferential of ψ. The domain of S_ψ is denoted and defined by $D(S_\psi) = \{x \in X : \ S_\psi(x) \neq \emptyset\}$. It is easy to check that ψ attains its minimum at x if and only if $0 \in S_\psi(x)$. Moreover, $D(S_\psi) \subseteq D(\psi)$. It is known that S_ψ is single-valued if ψ is a convex differentiable function, and X is a strictly convex, smooth and reflexive Banach space (see [35]).

Remark 5.1. In the remainder of this section, we assume that $\psi : X \to \mathbb{R}$ is a proper, convex and lower semicontinuous function which attains its minimum at some point.

We shall now verify that S_ψ is accretive. Let $x_1, x_2 \in X$ and $y_i \in S_\psi(x_i)$, $i = 1, 2$. Then

$$\psi(x_1) - \psi(x_2) \leq (y_1, J(x_1 - x_2))$$

and

$$\psi(x_2) - \psi(x_1) \leq (y_2, J(x_2 - x_1)) = (-y_2, J(x_1 - x_2)).$$

Adding the above inequalities, we get:

$$(y_1 - y_2, J(x_1 - x_2)) \geq 0,\ \forall x_1, x_2 \in X.$$

A zero of S_ψ is a minimum point of ψ as stated. We now prove some weak convergence results for solutions to (1.1) when $A = S_\psi$. We also give the rate of asymptotic convergence of $\psi(u_i)$. It is worth mentioning that in the following two theorems, we neither assume S_ψ to be $m-$accretive, nor S_ψ to be strongly accretive.

For convenience, we denote $S_\psi(u_i) = A(u_i)$ by the element

$$\frac{u_{i+1} - (1 + \theta_i)u_i + \theta_i u_{i-1} - f_i}{c_i},$$

in X.

Theorem 5.2. *Let X be a smooth and uniformly convex Banach space satisfying Opial's condition. Assume that u_i is a solution to (4.1) with $A = S_\psi$. If $0 < \theta_i < 1$, $\sum_{i=1}^{\infty} ic_i = \infty$ and (3.4) holds, then $u_i \rightharpoonup p \in S_\psi^{-1}0$, which is a minimum point of ψ, and $(\sum_{i=1}^{n} ic_i)(\psi(u_n) - \psi(p)) \to 0$, as $n \to \infty$.*

Proof. We use Lemma 2.7 for the nonempty set $F = S_\psi^{-1}(0)$. First we verify hypothesis (1) of Lemma 2.7; let $p \in S_\psi^{-1}(0)$. By Lemma 3.3, $\lim\limits_{i\to\infty} \|u_i - p\|$ exists. To check the second hypothesis of Lemma 2.7, by (5.1), (3.2) and Lemma 4.4 we have

$$\begin{aligned}
\psi(u_i) - \psi(u_{i-1}) &\leq (S_\psi(u_i), J(u_i - u_{i-1})) \\
&= \frac{1}{c_i}(u_{i+1} - (1 + \theta_i)u_i + \theta_i u_{i-1}, J(u_i - u_{i-1})) \\
&= \frac{1}{c_i}(u_{i+1} - u_i, J(u_i - u_{i-1})) - \frac{\theta_i}{c_i}(u_i - u_{i-1}, J(u_i - u_{i-1})) \\
&\leq \frac{1}{c_i}\|u_{i+1} - u_i\|\|u_i - u_{i-1}\| - \frac{\theta_i}{c_i}\|u_i - u_{i-1}\|^2 \\
&= \frac{1}{c_i a_i}(a_i\|u_{i+1} - u_i\| - a_{i-1}\|u_i - u_{i-1}\|)\|u_i - u_{i-1}\| \leq 0,
\end{aligned}$$

for all $i \geq 1$. So $\psi(u_i)$ is nonincreasing. Using again (5.1), (3.2) and Lemma 2.1, one obtains

$$\begin{aligned}
c_i a_i(\psi(u_i) - \psi(p)) &\leq a_i(u_{i+1} - (1 + \theta_i)u_i + \theta_i u_{i-1}, J(u_i - p)) \\
&\leq \frac{a_i}{2}(\|u_{i+1} - p\|^2 - \|u_i - p\|^2) - \frac{a_{i-1}}{2}(\|u_i - p\|^2 - \|u_{i-1} - p\|^2).
\end{aligned}$$

Summing up from $i = k$ to m, taking liminf as $m \to \infty$, by the assumption and Lemma 2.4, we arrive at

$$\sum_{i=k}^{\infty} c_i a_i(\psi(u_i) - \psi(p)) \leq \frac{a_{k-1}}{2}(\|u_{k-1} - p\|^2 - \|u_k - p\|^2). \tag{5.2}$$

Since a_i is increasing, we get

$$a_{k-1} \sum_{i=k}^{\infty} c_i(\psi(u_i) - \psi(p)) \leq \frac{a_{k-1}}{2}(\|u_{k-1} - p\|^2 - \|u_k - p\|^2).$$

So

$$\sum_{i=k}^{\infty} c_i(\psi(u_i) - \psi(p)) \leq \frac{1}{2}(\|u_{k-1} - p\|^2 - \|u_k - p\|^2).$$

Summing up again from $k = 1$ to m, then letting $m \to \infty$, in the same way as in the proof of Theorem 3.6, we deduce that

$$\sum_{i=1}^{\infty} i c_i(\psi(u_i) - \psi(p)) \leq \frac{1}{2}\|u_0 - p\|^2 < \infty. \tag{5.3}$$

Since by assumption $\sum_{i=1}^{\infty} i c_i = \infty$, we deduce that $\liminf\limits_{i\to\infty}(\psi(u_i) - \psi(p)) = 0$. Since $\psi(u_i)$ is nonincreasing, it follows that $\lim\limits_{i\to\infty} \psi(u_i) = \psi(p)$. If $u_{i_j} \rightharpoonup u$, then $\psi(u) \leq \liminf\limits_{j\to\infty} \psi(u_{i_j}) = \psi(p)$. Since p is a minimum point of ψ, one obtains $u \in S_\psi^{-1}0$. The proof is now completed by using (5.3) and Lemma 2.6. □

Theorem 5.3. *Let X be a smooth and uniformly convex Banach space satisfying Opial's condition. Assume that u_i is a solution to (4.1) with $A = S_\psi$. If $\theta_i \geq 1$ and $\sum_{i=1}^{\infty} i c_i a_i = \infty$, then $u_i \rightharpoonup p \in S_\psi^{-1}0$, which is a minimum point of ψ, and $(\sum_{i=1}^{n} i c_i a_i)(\psi(u_n) - \psi(p)) \to 0$, as $n \to \infty$.*

Proof. We follow the proof of Theorem 5.2 by using Lemma 3.9 instead of Lemma 3.3, to show that $\lim\limits_{i\to\infty} \|u_i - p\|$ exists for any $p \in S_\psi^{-1}0$. Summing up (5.2) from $k = 1$ to n, and letting $n \to \infty$, in the same way as in the proof of Theorem 3.12, we find (since $a_i \leq 1$)

$$\sum_{i=1}^{\infty} i c_i a_i(\psi(u_i) - \psi(p)) \leq \frac{1}{2}\|u_0 - p\|^2 < \infty. \tag{5.4}$$

Now, the assumption implies that $\liminf\limits_{i\to\infty}(\psi(u_i) - \psi(p)) = 0$. The rest of the proof is similar to that of Theorem 5.2. □

Finally, we mention the following result which was proved in [18].

Theorem 5.4. ([18]) *Let X be a uniformly smooth and uniformly convex Banach space. Let $A \subseteq X \times X$ be $m-$accretive with $A^{-1}(0) \neq \emptyset$. Assume that $c_i, \theta_i > 0$, $\forall i \geq 1$, and $\sum_{i=1}^{\infty} \frac{1}{h_i} = \infty$. Let $(u_i)_{i\geq 1}$ be the solution to (4.1). If $\liminf_{i\to\infty} c_i a_i > 0$, then $u_i \rightharpoonup p \in A^{-1}(0)$.*

From Theorem 5.4 and by using (5.3) and (5.4), we have the following result.

Corollary 5.5. *Let X be a uniformly smooth and uniformly convex Banach space. Assume that S_ψ is $m-$accretive with $A^{-1}(0) \neq \emptyset$, and $c_i, \theta_i > 0$, $\forall i \geq 1$, and $\sum_{i=1}^{\infty} \frac{1}{h_i} = \infty$. Let $(u_i)_{i\geq 1}$ be the solution to (4.1) with $A = S_\psi$. If $\liminf_{i\to\infty} c_i a_i > 0$, then $u_i \rightharpoonup p \in S_\psi^{-1}(0)$, which is a minimum point of ψ. Moreover, if $0 < \theta_i < 1$, then $(\sum_{i=1}^{n} i c_i)(\psi(u_n) - \psi(p)) \to 0$, and if $\theta_i \geq 1$, then $(\sum_{i=1}^{n} i c_i a_i)(\psi(u_n) - \psi(p)) \to 0$, as $n \to \infty$.*

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