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On the global shape of convex functions on locally convex spaces

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A R T I C L E I N F O

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A B S T R A C T

In the recent paper [1] D. Azagra studies the global shape of continuous convex functions defined on a Banach space X. More precisely, when X is separable, it is shown that for every continuous convex function $f : X \to \mathbb{R}$ there exist a unique closed linear subspace Y of X, a convex function $h : X/Y \to \mathbb{R}$ with the property that $\lim_{t\to\infty} h(u + tv) = \infty$ for all $u, v \in X/Y$, $v \neq 0$, and $x^* \in X^*$ such that $f = h \circ \pi + x^*$, where $\pi : X \to X/Y$ is the natural projection. Our aim is to characterize those proper lower semicontinuous convex functions defined on a locally convex space which have the above representation. In particular, we show that the continuity of the function f and the completeness of X can be removed from the hypothesis of Azagra's theorem. For achieving our goal we study general sublinear functions as well as recession functions associated to convex ones.

1. Preliminary notions and results

In the sequel X is a nontrivial real separated locally convex space (lcs for short) with topological dual X^* endowed with its weak* topology (if not explicitly mentioned otherwise); for $x \in X$ and $x^* \in X^*$ we set $\langle x, x^*\rangle := x^*(x)$. In some statements X will be a real normed vector space (nvs for short), or even a Hilbert space, in which case X^* will be identified with X by Riesz theorem. For E a topological vector space and $A \subset E$, we denote by $\overline{A}$ (or $\operatorname{cl} A$) and $\operatorname{span} A$ the closure and the linear hull of A, respectively; moreover, $\overline{\operatorname{span}} A := \overline{\operatorname{span} A}$. In particular, these notations apply for the subsets of X^* which is endowed with the weak-star topology by default; when X is a normed vector space, the norm-closure of $B \subset X^*$ is denoted by $\operatorname{cl}_{\|\cdot\|} B$.

The *domain* of the function $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ is the set $\operatorname{dom} f := \{x \in X \mid f(x) < \infty\}$. The function f is *proper* if $\operatorname{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$; f is *convex* if $\operatorname{epi} f := \{(x, t) \in X \times \mathbb{R} \mid f(x) \leq t\}$ is convex. Hence f is convex if and only if $f(\lambda x+(1-\lambda)x') \leq \lambda f(x)+(1-\lambda)f(x')$ for all $x, x' \in X$ and $\lambda \in \,]0, 1[$ with the convention $(-\infty) + \infty := \infty + (-\infty) := \infty$. A function $g : X \to \overline{\mathbb{R}}$ is *sublinear* if $g(0) = 0$, g is positively homogeneous [that is $g(\lambda x) = \lambda g(x)$ for $\lambda \in \,]0, \infty[$ and $x \in X$] and subadditive

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[that is $g(x+x') \le g(x)+g(x')$ for $x, x' \in X$]. Clearly, any sublinear function $g : X \to \overline{\mathbb{R}}$ is convex; indeed, for $\lambda \in \,]0,1[$ and $x, x' \in X$ one has $g(\lambda x + (1-\lambda)x') \le g(\lambda x) + (1-\lambda)g(x') = \lambda g(x) + (1-\lambda)g(x')$. Of course, f is *lower semicontinuous* (lsc for short) iff $\operatorname{epi} f$ is a closed subset of $X \times \mathbb{R}$ or, equivalently, $\{x \in X \mid f(x) \le \alpha\}$ is closed for every $\alpha \in \mathbb{R}$. By $\Gamma(X)$ we denote the class of proper lsc convex functions $f : X \to \overline{\mathbb{R}}$. Note that g is proper and $\operatorname{epi} g$ is a closed convex cone when $g : X \to \overline{\mathbb{R}}$ is lsc and sublinear.

Having $f : X \to \overline{\mathbb{R}}$, its *conjugate* function is

$$f^* : X^* \to \overline{\mathbb{R}}, \quad f^*(x^*) := \sup\{\langle x, x^*\rangle - f(x) \mid x \in X\} \quad (x^* \in X^*),$$

while its *subdifferential* is the set-valued function $\partial f : X \rightrightarrows X^*$ with

$$\partial f(x) := \{x^* \in X^* \mid \langle x' - x, x^*\rangle \le f(x') - f(x) \ \forall x' \in X\}$$

if $f(x) \in \mathbb{R}$ and $\partial f(x) := \emptyset$ otherwise. By [6, Th. 2.3.3], $f^* \in \Gamma(X^*)$ and $(f^*)^* = f$ (X^* being endowed, as mentioned above, with the weak-star topology w^*) whenever $f \in \Gamma(X)$; in particular $\operatorname{dom} f^* \ne \emptyset$. Moreover, for $f \in \Gamma(X)$ one has $x^* \in \partial f(x)$ iff $x \in \partial f^*(x^*)$ iff $f(x) + f^*(x^*) = \langle x, x^*\rangle$.

A central notion throughout this note is that of recession function. So, having $f \in \Gamma(X)$, its *recession function* f_∞ is (equivalently) defined by

$$f_\infty : X \to \overline{\mathbb{R}}, \quad f_\infty(u) := \lim_{t\to\infty} \frac{f(x_0 + tu) - f(x_0)}{t},$$

where $x_0 \in \operatorname{dom} f$ is arbitrary. The function f_∞ is a proper lsc sublinear function having the property

$$f(x+u) \le f(x) + f_\infty(u) \quad \forall x \in \operatorname{dom} f, \ \forall u \in X \tag{1}$$

(see [6, Eq. (2.28)]); moreover,

$$f_\infty(u) = \sup_{x^* \in \operatorname{dom} f^*} \langle u, x^*\rangle \ \forall u \in X \quad \text{and} \quad \partial f_\infty(0) = \overline{\operatorname{dom} f^*} \tag{2}$$

(see [6, Exer. 2.23 and Th. 2.4.14]). In particular (see also [6, Th. 2.4.14]), if $g : X \to \overline{\mathbb{R}}$ is a (proper) lsc sublinear function one has

$$\partial g(0) = \{x^* \in X^* \mid x^* \le g\}, \quad g^* = \iota_{\partial g(0)}, \quad \text{and} \quad g = g_\infty = \sup\nolimits_{x^* \in \partial g(0)} x^*, \tag{3}$$

where $\iota_A : E \to \overline{\mathbb{R}}$ denotes the *indicator function* of $A \subset E$, being defined by $\iota_A(v) := 0$ for $v \in A$ and $\iota_A(v) := \infty$ for $v \in E \setminus A$. Hence $\partial g(0) \ne \emptyset$.

Recall that the mapping $0 < t \mapsto \frac{f(x_0+tu)-f(x_0)}{t} \in \overline{\mathbb{R}}$ is nondecreasing for $f : X \to \overline{\mathbb{R}}$ a proper convex function, $x_0 \in \operatorname{dom} f$ and $u \in X$. Moreover, for such a function and $x, u \in X$, the mapping $\varphi_{x,u} : \mathbb{R} \to \overline{\mathbb{R}}$ with $\varphi_{x,u}(t) := f(x+tu)$, one of the following alternatives holds: 1) $\varphi_{x,u}$ is nonincreasing on $\mathbb{R}$, 2) $\varphi_{x,u}$ is nondecreasing on $\mathbb{R}$, 3) there exists $t_0 \in \mathbb{R}$ such that $\varphi_{x,u}$ is nonincreasing on $]-\infty, t_0]$ and nondecreasing on $[t_0, \infty[$; moreover, there exists $\gamma_{x,u} := \lim_{t\to\infty} f(x+tu) \in \overline{\mathbb{R}}$.

Lemma 1. *Let $f \in \Gamma(X)$ and $u \in X \setminus \{0\}$. The following assertions are equivalent:*

(a) $\exists x_0 \in \operatorname{dom} f$, $\exists M \in \mathbb{R}$, $\forall t \in [0, \infty[\, : f(x_0 + tu) \le M$;

(b) $\forall x \in \operatorname{dom} f$, $\exists M \in \mathbb{R}$, $\forall t \in [0, \infty[\, : f(x + tu) \le M$;

(c) $f_\infty(u) \le 0$.

Consequently, the following assertions are equivalent:

(a') $\forall x \in \operatorname{dom} f$: $\lim_{t\to\infty} f(x+tu) = \infty$;

(b') $\exists x_0 \in \operatorname{dom} f : \lim_{t\to\infty} f(x_0 + tu) = \infty$;
(c') $f_\infty(u) > 0$.

Proof. (c) $\Rightarrow$ (b) Take $x \in \operatorname{dom} f$; then, by (1), $f(x + tu) \leq f(x) + f_\infty(tu) = f(x) + tf_\infty(u) \leq f(x) =: M$ for $t \geq 0$.

(b) $\Rightarrow$ (a) The implication is obvious.

(a) $\Rightarrow$ (c) Since $t^{-1}[f(x_0 + tu) - f(x_0)] \leq t^{-1}[M - f(x_0)]$ for $t > 0$, one has $f_\infty(u) \leq \lim_{t\to\infty} t^{-1}[M - f(x_0)] = 0$.

Observe that $\rceil$(c') coincides with (c), $\rceil$(b') is equivalent to (b), and $\rceil$(a') is equivalent to (a). Hence, from the first part, we get (a') $\Leftrightarrow$ (b') $\Leftrightarrow$ (c'). $\square$

Having in view the statements of Theorems 5 and 6 in [1], it is worth observing that for $x_0 \in \operatorname{dom} f$, $u \in X$ and $u^* \in X^*$ one has

$$f_\infty(\pm u) = \langle \pm u, u^* \rangle \iff [f(x_0 + tu) - f(x_0) - \langle tu, u^* \rangle = 0 \quad \forall t \in \mathbb{R}]. \tag{4}$$

Indeed, the implication "$\Leftarrow$" is obvious. Assume that $f_\infty(\pm u) = \langle \pm u, u^* \rangle$ [$\Leftrightarrow f_\infty(tu) = \langle tu, u^* \rangle$ for all $t \in \mathbb{R}$]. Using (1) we get

$$f(x_0 + tu) \leq f(x_0) + f_\infty(tu) = f(x_0) + \langle tu, u^* \rangle, \quad f(x_0 - tu) \leq f(x_0) - \langle tu, u^* \rangle \quad \forall t \in \mathbb{R}.$$

Since $x_0 = \frac{1}{2}(x_0 + tu) + \frac{1}{2}(x_0 - tu)$, from the convexity of f and the previous inequalities we get

$$f(x_0) \leq \tfrac{1}{2} f(x_0 + tu) + \tfrac{1}{2} f(x_0 - tu) \leq \tfrac{1}{2}[f(x_0) + \langle tu, u^* \rangle] + \tfrac{1}{2}[f(x_0) - \langle tu, u^* \rangle] = f(x_0),$$

and so $f(x_0 + tu) = f(x_0) + \langle tu, u^* \rangle$ for every $t \in \mathbb{R}$. Hence (4) holds.

Taking $u \neq 0$ and $u^* = 0$, from (4) with $x_0 \in \operatorname{dom} f$ we have that

$$f_\infty(\pm u) = 0 \iff [f(x_0 + tu) = f(x_0) \quad \forall t \in \mathbb{R}] \iff f|_{x_0 + \mathbb{R}u} \text{ is constant.} \tag{5}$$

Moreover, it is worth observing that $f_\infty \geq 0$ if f is bounded from below; indeed, if $f_\infty(u) < 0$, from (1) we have that $f(x + tu) \leq f(x) + tf_\infty(u)$, and so $\lim_{t\to\infty} f(x + tu) = -\infty$, for every $x \in \operatorname{dom} f$.

In the sequel, for $\varphi, \psi : E \to \overline{\mathbb{R}}$ and $\rho \in \{\leq, <, =\}$ we set $[\varphi\ \rho\ \psi] := \{x \in E \mid \varphi(x)\ \rho\ \psi(x)\}$. For example $[\varphi \leq 0] := \{x \in X \mid \varphi(x) \leq 0\}$.

As in [1, Def. 3], we say that f is *directionally coercive* if $\lim_{t\to\infty} f(x + tu) = \infty$ for all $x \in X$ and $u \in X \setminus \{0\}$, and f is *essentially directionally coercive* if $f - x^*$ is directionally coercive for some $x^* \in X^*$.

From the equivalence of assertions (a'), (b') and (c') of Lemma 1 we get the next result.

Corollary 2. *Let $f \in \Gamma(X)$; then* (a) *f is directionally coercive if and only if $[f_\infty \leq 0] = \{0\}$, and* (b) *$f$ is essentially directionally coercive if and only if there exists $x^* \in X^*$ such that $[f_\infty \leq x^*] = \{0\}$.*

The previous result motivates a deeper study of proper lsc sublinear functions; several properties of such functions are mentioned in [6, Th. 2.4.14].

Recall that the orthogonal spaces of the nonempty subsets $A \subset X$ and $B \subset X^*$ are defined by

$$A^\perp := \{x^* \in X^* \mid \langle x, x^* \rangle = 0\ \forall x \in A\} \quad \text{and} \quad B^\perp := \{x \in X \mid \langle x, x^* \rangle = 0\ \forall x^* \in B\},$$

respectively; clearly, $A^\perp$ is a w^*-closed linear subspace of X^*, $B^\perp$ is a closed linear subspace of X, $A^\perp = (\overline{\operatorname{span}} A)^\perp$, $B^\perp = (\overline{\operatorname{span}} B)^\perp$, $(A^\perp)^\perp = \overline{\operatorname{span}} A$, $(B^\perp)^\perp = \overline{\operatorname{span}} B$. Also recall that the *quasi-interior* and the *quasi-relative interior* of the nonempty convex set $A \subset X$ are

$$\operatorname{qi} A := \{a \in A \mid \overline{\mathbb{R}_+(A-a)} = X\}, \quad \operatorname{qri} A := \{a \in A \mid \overline{\mathbb{R}_+(A-a)} \text{ is a linear space}\},$$

respectively, where $\mathbb{R}_+ := [0, \infty[$. Having in view that

$$\mathbb{R}_+(A-a) \subset \operatorname{span}(A-a) = \operatorname{span}(A-A) = \mathbb{R}_+(A-A) \quad \forall a \in A,$$

for $\emptyset \neq A \subset X$ a convex set, one obtains (see e.g. [7]) that

$$\operatorname{qri} A = \{a \in A \mid \overline{\mathbb{R}_+(A-a)} = \overline{\operatorname{span}}(A-A)\} = \big\{a \in A \mid \overline{\mathbb{R}_+(A-a)} = \overline{\mathbb{R}_+(A-A)}\big\}, \tag{6}$$

$$\operatorname{qri} A = A \cap \operatorname{qri} \overline{A}, \quad \operatorname{qi} A = \begin{cases} \operatorname{qri} A & \text{if } \overline{\mathbb{R}_+(A-A)} = X, \\ \emptyset & \text{otherwise.} \end{cases} \tag{7}$$

2. Some results related to sublinear functions

Throughout this section $g \in \Gamma(X)$ is assumed to be sublinear.

Lemma 3. *Let us set $K := [g \leq 0]$ and $L := K \cap (-K)$. Then K is a closed convex cone and L is a closed linear subspace of X. Moreover,*

$$L = \{x \in X \mid g(x) = g(-x) = 0\} = [\partial g(0)]^{\perp}, \tag{8}$$

$$g(x+u) = g(x) \quad \forall x \in X, \ \forall u \in L. \tag{9}$$

Proof. Because g is a lsc sublinear function, $[g \leq 0]$ is a closed convex cone. The set L is a closed convex cone as the intersection of (two) closed convex cones. Since $L = -L$, L is also a linear subspace of X.

Take $x \in L$; because $0 = g\big(x + (-x)\big) \leq g(x) + g(-x) \leq 0 + 0 = 0$, we get $g(x) = 0 = g(-x)$, and so $L \subset \{x \in X \mid g(x) = g(-x) = 0\}$. The reverse inclusion being obvious, the first equality in (8) holds.

Set $B := \partial g(0)$. Taking into account the formula for g from (3), for $x \in X$ one has

$$x \in L \iff g(\pm x) \leq 0 \iff [\pm \langle x, x^* \rangle \leq 0 \ \forall x^* \in B] \iff [\langle x, x^* \rangle = 0 \ \forall x^* \in B] \iff x \in B^{\perp},$$

and so the second equality in (8) holds, too.

Take now $x \in X$ and $u \in L$. Using the sublinearity of g one has

$$g(x+u) \leq g(x) + g(u) = g(x) = g\big((x+u) + (-u)\big) \leq g(x+u) + g(-u) = g(x+u),$$

and so $g(x+u) = g(x)$. □

Proposition 4. *For $x^* \in X^*$ set $L_{x^*} := \{x \in X \mid g(\pm x) = \langle \pm x, x^* \rangle$. The following assertions hold:*

(a) *If $x^* \in X^*$, then L_{x^*} is a closed linear subspace of X, and*

$$L_{x^*} = \{x \in X \mid g(\pm x) \leq \langle \pm x, x^* \rangle\} = [\partial g(0) - x^*]^{\perp}, \tag{10}$$

$$g(x+u) = g(x) + \langle u, x^* \rangle \quad \forall x \in X, \ \forall u \in L_{x^*}. \tag{11}$$

(b) *If $u^* \in \partial g(0)$, then $L_{u^*} = [\partial g(0) - \partial g(0)]^{\perp}$. Consequently, $L_{x^*} \subset L_{u^*}$ for all $x^* \in X^*$ and $u^* \in \partial g(0)$; in particular $L_{u^*} = L_{v^*}$ for all $u^*, v^* \in \partial g(0)$.*

Proof. (a) Clearly, $h := g - x^*$ is a proper lsc sublinear function. Using Lemma 3 for g replaced by h we obtain that L_{x^*} is a closed linear subspace of X and the formulas for L_{x^*} hold by the definition of L and because $\partial h(0) = \partial g(0) - x^*$. Moreover, $g(x+u) - \langle x+u, x^*\rangle = g(x) - \langle x, x^*\rangle$ for all $x \in X$ and $u \in L_{x^*}$, and so (11) holds, too.

(b) Take now $u^* \in \partial g(0) =: B$. Then $B - u^* \subset B - B$, whence $Y := \operatorname{span}(B - u^*) \subset \operatorname{span}(B - B) =: Z$. Since $B - B = (B - u^*) - (B - u^*) \subset Y$, we get $Z \subset Y$, and so $Y = Z$. Using (a) one has $L_{u^*} = B^\perp = Y^\perp = Z^\perp = (B - B)^\perp = [\partial g(0) - \partial g(0)]^\perp$.

Let $x^* \in X^*$. Because $B - x^* \subset \operatorname{span}(B - x^*)$, as above one has $B - B \subset \operatorname{span}(B - x^*)$, and so $L_{x^*} = (B - x^*)^\perp = [\operatorname{span}(B - x^*)]^\perp \subset (B - B)^\perp = L_{u^*}$. $\square$

As seen in Proposition 4 (b), the set $\{L_{u^*} \mid u^* \in \partial g(0)\}$ is a singleton; its element will be denoted by L_g in the sequel. It follows that

$$u \in L_g \iff g(-u) = -g(u) \iff \mathbb{R} \cdot \big(u, g(u)\big) \subset \operatorname{epi} g. \tag{12}$$

Proposition 5. *Let $x^* \in X^*$. The following assertions are equivalent:* (a) $x^* \in \operatorname{qri} \partial g(0)$; (b) $[g \le x^*]$ *is a linear space;* (b') $L_{x^*} = [g \le x^*]$; (c) $x^* \in \partial g(0)$ *and* $[g = x^*]$ *is a linear space;* (c') $x^* \in \partial g(0)$ *and* $L_{x^*} = [g = x^*]$.

Proof. Because $x^* \in \operatorname{qri} \partial g(0)$ if and only if $0 \in \operatorname{qri} [\partial g(0) - x^*]$ and $\partial(g - x^*)(0) = \partial g(0) - x^*$, we may (and do) assume that $x^* = 0$. Let us set $B := \partial g(0)$ and $K := [g \le 0]$; K is a (closed) convex cone and $l(K) := K \cap (-K)$ is a linear space.

Because L_{x^*} is a linear space, the equivalences (b') $\Leftrightarrow$ (b) and (c') $\Leftrightarrow$ (c) follow immediately from (10).

(c) $\Rightarrow$ (b) Because $0 \in \partial g(0)$, one has $g \ge 0$, and so $[g \le 0] = [g = 0]$. Hence (b) holds.

(b) $\Rightarrow$ (c) Because K $(= [g \le 0])$ is a linear space, taking $x \in K$ $(= -K)$ we get $g(\pm x) = 0$ by Lemma 3. It follows that $g \ge 0$ $(\Leftrightarrow 0 \in B)$ and $K = [g = 0]$. Hence $[g = 0]$ is a linear space.

(b) $\Rightarrow$ (a) We have to show that $\overline{\mathbb{R}_+(B - B)} \subset \overline{\mathbb{R}_+ B}$, the reverse inclusion being obvious. For this assume that $\overline{x}^* \in X^* \setminus \overline{\mathbb{R}_+ B}$. Then, by a separation theorem, there exist $\overline{x} \in X$ and $\alpha \in \mathbb{R}$ such that $\langle \overline{x}, \overline{x}^*\rangle > \alpha \ge \langle \overline{x}, tu^*\rangle$ for all $t \in \mathbb{R}_+$ and $u^* \in B$, whence $\alpha \ge 0 \ge \langle \overline{x}, u^*\rangle$ for $u^* \in B$, that is $\alpha \ge 0 \ge g(\overline{x})$. Hence $0 \ne \overline{x} \in K$ $(= -K)$, and so $g(\pm\overline{x}) = 0$. It follows that $\langle \pm x, u^*\rangle \le g(\pm x) = 0$, whence $\langle x, u^*\rangle = 0$, for all $u^* \in B$. Hence $\langle \overline{x}, t(u^* - v^*)\rangle = 0 < \langle \overline{x}, \overline{x}^*\rangle$ for all $t \in \mathbb{R}_+$ and $u^*, v^* \in B$, proving that $\overline{x} \notin \overline{\mathbb{R}_+(B - B)}$. Therefore, $(x^* =)\ 0 \in \operatorname{qri} B$.

(a) $\Rightarrow$ (b) Because $0 \in \operatorname{qri} B$, $0 \in B$ and $\operatorname{cl}(\mathbb{R}_+ B) = \overline{\mathbb{R}_+(B - B)}$. Take $x \in K$; then $\langle x, u^*\rangle \le g(x) \le 0$ for $u^* \in B$, and so $\langle -x, u^*\rangle \ge 0$ for all $u^* \in B$, whence $-x \in [\operatorname{cl}(\mathbb{R}_+ B)]^+ = \big(\overline{\mathbb{R}_+(B - B)}\big)^+$. It follows that $\langle x, v^*\rangle = \langle -x, 0 - v^*\rangle \ge 0$, that is $\langle -x, v^*\rangle \le 0$, for all $v^* \in B$, whence $g(-x) \le 0$. Hence $x \in -K$, and so K is a linear space. $\square$

Corollary 6. *Let $x^* \in X^*$. Then $x^* \in \operatorname{qi} \partial g(0)$ if and only if $[g \le x^*] = \{0\}$.*

Proof. Set $B := \partial g(0)$. Assume that $x^* \in \operatorname{qi} B$. From (6) and (7) we have that $x^* \in \operatorname{qri} B$ and $\overline{\mathbb{R}_+(B - B)} = X^*$. Using the equivalence (a) $\Leftrightarrow$ (b') of Proposition 5 and Proposition 4 (b) we obtain that $[g \le x^*] = L_{x^*} = (X^*)^\perp = \{0\}$.

Conversely, assume that $[g \le x^*] = \{0\}$. Using the implication (b) $\Rightarrow$ (a) $\wedge$ (b') of Proposition 5, we get $x^* \in \operatorname{qri} B$ $(\subset B)$ and $(L_g =)\ L_{x^*} = \{0\}$. Using now Proposition 4 (b) we obtain that $X^* = \{0\}^\perp = L_{x^*}^\perp = \big([B - B]^\perp\big)^\perp = \overline{\mathbb{R}_+(B - B)}$. Using again (7) we get $x^* \in \operatorname{qi} B$. $\square$

Proposition 7. *Assume that X is a separable normed vector space. Then w^*-$\operatorname{qri} \partial g(0) \ne \emptyset$, and so there exists $x^* \in X^*$ such that the set $[g \le x^*]$ is a linear space.*

Proof. In order to get the conclusion we apply [2, Th. 2.19 (b)] which states that for any weakly* cs-closed[1] subset $C \subset (X^*, w^*)$, X being a separable nvs, one has $w^*\text{-qri}\, C \neq \emptyset$. So, consider $(\alpha_n)_{n\geq 1} \subset \mathbb{R}_+$ with $\sum_{n=1}^{\infty} \alpha_n = 1$ and $(x_n^*)_{n\geq 1} \subset C := \partial g(0)$ such that $w^*\text{-lim} \sum_{k=1}^{n} \alpha_k x_k^* = x^* \in X^*$. We need to prove that $x^* \in C$. For this, observe first that there exists $n_0 \geq 1$ such that $\alpha_{n_0} > 0$. Then, for $n \geq n_0$ we have that $\beta_n := \sum_{k=1}^{n} \alpha_n > 0$ and $u_n^* := \beta_n^{-1} \sum_{k=1}^{n} \alpha_n x_n^* \in C$. Since $\beta_n \to 1$, we obtain that $C \ni w^*\text{-lim}\, u_n^* = x^*$. The proof is complete. $\square$

Remark 8. Notice that the separability of the nvs X in Proposition 7 is essential. For example, the space of square summable real-valued functions $X := \ell_2(\Gamma)$, endowed with the norm $\|\cdot\|$ defined by $\|x\| := \left(\sum_{\gamma\in\Gamma} |x(\gamma)|^2\right)^{1/2}$, is a Hilbert space, while $X_+ := \{x \in X \mid x(\gamma) \geq 0\ \forall \gamma \in \Gamma\}$ is a closed convex cone such that $X_+ - X_+ = X$. If Γ is at most countable, then $\text{qri}\, X_+ = \text{qi}\, X_+ = \{x \in X \mid x(\gamma) > 0\ \forall \gamma \in \Gamma\}$. If Γ is uncountable then, as in [2, Ex. 3.11 (iii)], $\text{qri}\, X_+ = \emptyset$.

Considering the quotient space $\widehat{X} := X/L_g := \{\widehat{x} \mid x \in X\}$ of X with respect to L_g endowed with the quotient topology, $\widehat{X}$ becomes a separated locally convex space such that the natural projection $\pi : X \to \widehat{X}$, defined by $\pi(x) := \widehat{x}$, is a continuous open linear operator; moreover $A \subset \widehat{X}$ is closed if and only if $\pi^{-1}(A)$ is closed.

Fixing $x^* \in \partial g(0)$ one has $L_{x^*} = L_g$; using (11), we obtain that

$$\widehat{g}_{x^*} : \widehat{X} \to \overline{\mathbb{R}}, \quad \widehat{g}_{x^*}(\widehat{x}) := g(x) - \langle x, x^* \rangle \quad (x \in X) \tag{13}$$

is well defined.

Proposition 9. *Assume that $x^* \in \partial g(0)$. Then $\widehat{g}_{x^*}$ defined by (13) is a proper lsc sublinear function such that $\widehat{g}_{x^*} \geq 0$ and $L_{\widehat{g}_{x^*}} = \{\widehat{0}\}$. Moreover, $x^* \in \text{qri}\, \partial g(0)$ if and only if $0 \in \text{qi}\, \partial \widehat{g}_{x^*}(\widehat{0})$.*

Proof. The fact that $\widehat{g}_{x^*}$ is proper, sublinear and takes nonnegative values follows immediately from its definition. For $\alpha \in \mathbb{R}$ one has

$$[\widehat{g}_{x^*} \leq \alpha] = \{\widehat{x} \in \widehat{X} \mid \widehat{g}_{x^*}(\widehat{x}) \leq \alpha\} = \pi(\{x \in X \mid g(x) - \langle x, x^* \rangle \leq \alpha\}) = \pi([g - x^* \leq \alpha]); \tag{14}$$

using (11) we have that $\pi^{-1}([\widehat{g}_{x^*} \leq \alpha]) = [g - x^* \leq \alpha]$. Since g is lsc, $g - x^*$ is so; it follows that $[g - x^* \leq \alpha]$ is closed, and so $[\widehat{g}_{x^*} \leq \alpha]$ is closed in $\widehat{X}$ for every $\alpha \in \mathbb{R}$, whence $\widehat{g}_{x^*}$ is lsc.

Because $\widehat{g}_{x^*} \geq 0$ one has $0 \in \partial \widehat{g}_{x^*}(0)$, and so $L_{\widehat{g}_{x^*}} = \{\widehat{x} \mid \widehat{g}_{x^*}(\widehat{x}) = \widehat{g}_{x^*}(-\widehat{x}) = 0\}$. Take $x \in X$ with $\widehat{x} \in L_{\widehat{g}_{x^*}}$; from the definition of $\widehat{g}_{x^*}$ we have that $g(\pm x) - \langle \pm x, x^* \rangle = 0$, and so $x \in L_{x^*} = L_g$. It follows that $\widehat{x} = 0$, and so $L_{\widehat{g}_{x^*}} = \{\widehat{0}\}$. Taking $\alpha := 0$ in (14) and in the equality on the line below it we obtain that $[\widehat{g}_{x^*} \leq 0] = \pi(K_{x^*})$ and $\pi^{-1}([\widehat{g}_{x^*} \leq 0]) = K_{x^*}$; hence K_{x^*} is a linear space if and only if $[\widehat{g}_{x^*} \leq 0]$ is a linear space. Using Proposition 5 we obtain that $x^* \in \text{qri}\, \partial g(0)$ if and only if $0 \in \text{qri}\, \widehat{g}_{x^*}(0)$. Because $L_{\widehat{g}_{x^*}} = \{\widehat{0}\}$ we have that $\text{cl}\left[\mathbb{R}_+\left(\partial \widehat{g}_{x^*}(\widehat{0}) - \partial \widehat{g}_{x^*}(\widehat{0})\right)\right] = (\widehat{X})^*$, and so $\text{qri}\, \widehat{g}_{x^*}(0) = \text{qi}\, \widehat{g}_{x^*}(0)$ by (7). $\square$

In this context it is natural to know sufficient conditions for having $[g \leq 0] = \{0\}$. Some sufficient conditions are provided in the next result. Recall that the *core* of the subset A of the real linear space E is $\text{core}\, A := \{x \in E \mid \forall u \in E, \exists \delta > 0, \forall t \in [0, \delta] : x + tu \in A\}$.

Proposition 10. *Let $x^* \in X^*$. Consider the following assertions:*

(i) $[g \leq x^*] = \{0\}$;

[1] Having (Y, τ) a locally convex space, the set $C \subset Y$ is (τ-) *cs-closed* if for any sequences $(\lambda_n)_{n\geq 1} \subset \mathbb{R}_+$ with $\sum_{n=1}^{\infty} \lambda_n = 1$ and $(y_n)_{n\geq 1} \subset C$ for which $\sum_{k=1}^{n} \alpha_k y_k \to^{\tau} y \in Y$, $y \in C$ (see [2, p. 21] or [6, p. 9]).

(ii) $x^* \in \operatorname{qi} \partial g(0)$;

(iii) $x^* \in \operatorname{core} \partial g(0)$;

(iv) *there exists a linear topology* τ *on* X^* *such that* $x^* \in \operatorname{int}_\tau \partial g(0)$;

(v) *the topology of* X *is defined by the norm* $\|\cdot\|$ *and* $x^* \in \operatorname{int}_{\|\cdot\|_*} (\partial g(0))$, *where* $\|\cdot\|_*$ *is the dual norm on* X^*;

(vi) *the topology of* X *is defined by the norm* $\|\cdot\|$ *and there exists* $\alpha > 0$ *such that* $g(x) - \langle x, x^* \rangle \geq \alpha \|x\|$ *for all* $x \in X$.

Then (vi) $\Leftrightarrow$ (v) $\Rightarrow$ (iv) $\Leftrightarrow$ (iii) $\Rightarrow$ (ii) $\Leftrightarrow$ (i); *moreover, if* $\dim X < \infty$ *then* (i) $\Rightarrow$ (vi).

Proof. Because $\partial (g - x^*)(0) = \partial g(0) - x^*$, we may (and do) assume that $x^* = 0$. We set $B := \partial g(0)$.

(vi) $\Leftrightarrow$ (v) This assertion follows immediately from the equivalence of assertions (e) and (f) of [6, Exer. 2.41].

(v) $\Rightarrow$ (iv) This assertion is true because the topology generated by any norm on a linear space is a linear topology.

(iv) $\Rightarrow$ (iii) It is well known that $\operatorname{core} A = \operatorname{int}_\sigma A$ when A is a convex subset of topological vector space (Y, σ) with $\operatorname{int}_\sigma A \neq \emptyset$. The set $B \subset X^*$ being convex, the implication is true.

(iii) $\Rightarrow$ (iv) Consider the core convex topology, that is the finest locally convex topology, τ_c on X^* (see [3, Exer. 2.10] as well as [4, Sect. 6.3] for a short presentation of this topology); then $\operatorname{core} \partial g(0) = \operatorname{int}_{\tau_c} \partial g(0)$.

(iii) $\Rightarrow$ (ii) Because $0 \in \operatorname{core} B$, we have that $0 \in B$ and $\mathbb{R}_+ B = X^*$, and so $\operatorname{cl}(\mathbb{R}_+ B) = X^*$. Therefore, $0 \in \operatorname{qi} B$.

(ii) $\Leftrightarrow$ (i) This equivalence is provided by Corollary 6.

(i) $\Rightarrow$ (vi) (if $\dim X < \infty$). Assume that $\dim X < \infty$. It is well known that all the norms on a finite dimensional linear space are equivalent, and any separated linear topology on such a space is normable. So, let $\|\cdot\|$ be a norm on X. Because $S_X := \{x \in X \mid \|x\| = 1\}$ is compact and g is lsc, there exists $\overline{x} \in S_X$ such that $g(x) \geq g(\overline{x}) =: \alpha \ (> 0)$. Taking $x \in X \setminus \{0\}$, $x' := \|x\|^{-1} x \in S_X$, and so $g(x) = \|x\| \cdot g(x') \geq \alpha \|x\|$. Hence (vi) holds. □

Observe that the reverse implication of (i) $\Rightarrow$ (vi) from Proposition 10 is not true even when X is an infinite dimensional separable Hilbert space. Indeed, take $X := \ell_2$ endowed with its usual norm $\|\cdot\|_2$ and $g : X \to \overline{\mathbb{R}}$ defined by $g(x) := \left(\sum_{n \geq 1} |x_n|^q\right)^{1/q}$ $(= \|x\|_q)$ for $x := (x_n)_{n \geq 1} \in X$, where $q \in \,]2, \infty[$. Because $\ell_p \subset \ell_{p'}$ for $1 \leq p < p' \leq \infty$ with $\|x\|_{p'} \leq \|x\|_p$ for $x \in \ell_p$, $g(x) \leq \|x\|_2$ for all $x \in X$, and so g is a finitely valued continuous sublinear function verifying (i). Assuming that (vi) holds, there exists $\alpha > 0$ such that $\|x\|_q \geq \alpha \|x\|_2$ for all $x \in \ell_2$. Consider the sequence $x = (n^{-1/2})_{n \geq 1} \subset \mathbb{R}$; then $\xi_n := (1, 2^{-1/2}, ..., n^{-1/2}, 0, 0, ...) \in \ell_2$, and so

$$\left(\sum\nolimits_{k=1}^{n} \frac{1}{k}\right)^{1/2} = \|\xi_n\|_2 \leq \alpha^{-1} \|\xi_n\|_q = \alpha^{-1} \left(\sum\nolimits_{k=1}^{n} \frac{1}{k^{q/2}}\right)^{1/q} \quad \forall n \geq 1,$$

whence the contradiction $\infty = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k}\right)^{1/2} \leq \lim_{n\to\infty} \alpha^{-1} \left(\sum_{k=1}^{n} \frac{1}{k^{q/2}}\right)^{1/q} < \infty$.

3. Applications to the shape of convex functions

The following results are motivated by the notions and results from [1].

Throughout this section $f \in \Gamma(X)$; to f we associate $L_f := L_{f_\infty}$. As seen in (2), $\partial f_\infty(0) = \overline{\operatorname{dom} f^*}$, and so, by Proposition 4, we have that

$$L_f = \{u \in X \mid f_\infty(\pm u) = \langle \pm u, x^* \rangle\} \text{ for some (any) } x^* \in \overline{\operatorname{dom} f^*}. \tag{15}$$

Because $(\operatorname{epi} f)_\infty = \operatorname{epi} f_\infty$ (see e.g. [6, p. 74]), from (12) we have that the epigraph of f does not contain lines, that is $\operatorname{epi} f$ is sharp in the sense of [5], if and only if $L_f = \{0\}$. Observe that $\operatorname{Im} \partial f \subset \operatorname{dom} f^* \subset \overline{\operatorname{dom} f^*}$ and so

$$\overline{\operatorname{span}}\,(\operatorname{Im} \partial f - \operatorname{Im} \partial f) \subset \overline{\operatorname{span}}\,(\operatorname{dom} f^* - \operatorname{dom} f^*) = \overline{\operatorname{span}}\left(\overline{\operatorname{dom} f^*} - \overline{\operatorname{dom} f^*}\right) = (L_f)^\perp,$$

where all the closures are taken wrt the weak-star topology w^* on X^*; moreover, the first inclusion becomes equality if X is a Banach space because in this case $\operatorname{dom} f^* \subset \operatorname{cl}_{\|\cdot\|} \operatorname{Im} \partial f$ by Brøndsted–Rockafellar theorem (see e.g. [6, Th. 3.1.2]), and so $\operatorname{cl}_{\|\cdot\|} \operatorname{dom} f^* = \operatorname{cl}_{\|\cdot\|} \operatorname{Im} \partial f$.

Corollary 11. (a) *The function f is directionally coercive if and only if $0 \in \operatorname{qi} \overline{\operatorname{dom} f^*}$.*

(b) *The function f is essentially directionally coercive if and only if $\operatorname{qi} \overline{\operatorname{dom} f^*} \neq \emptyset$.*

Proof. Having $x^* \in X^*$, one has $(f - x^*)_\infty = f_\infty - x^*$ and $\operatorname{dom}(f - x^*)^* = \operatorname{dom} f^* - x^*$, and so $\operatorname{qi} \overline{\operatorname{dom}(f - x^*)} = \operatorname{qi} \overline{\operatorname{dom} f^*} - x^*$. Hence (a) $\Rightarrow$ (b) by Corollary 2.

(a) By Corollary 2 one has that f is directionally coercive if and only if $[f_\infty \leq 0] = \{0\}$, and the latter is equivalent to $0 \in \operatorname{qi} \overline{\operatorname{dom} f^*}$ by (2) and the equivalence (i) $\Leftrightarrow$ (ii) of Proposition 10. □

The representations of the (continuous) convex function f from Theorems 4–6 of [1] motivate the next result.

Proposition 12. *Assume that $f = h \circ A + x^*$, where $x^* \in X^*$, $h \in \Gamma(Y)$ with Y a separated locally convex space, and $A : X \to Y$ is a continuous linear operator. Then the following assertions hold:*

(a) *$f_\infty = h_\infty \circ A + x^*$ and $\ker A \subset [f_\infty = x^*]$;*

(b) *if $h_\infty \geq 0$, then $x^* \in \partial f_\infty(0)$ $(= \overline{\operatorname{dom} f^*})$, the reverse implication being true if, moreover, $\operatorname{Im} A = Y$;*

(c) *if $[h_\infty \leq 0] = \{0\}$ then $\ker A = [f_\infty \leq x^*]$ and $x^* \in \operatorname{qri} \overline{\operatorname{dom} f^*}$; conversely, if $\operatorname{Im} A = Y$ and $\ker A = [f_\infty \leq x^*]$, then $[h_\infty \leq 0] = \{0\}$.*

(d) *if h is bounded from below, then $h_\infty \geq 0$ and $x^* \in \operatorname{dom} f^*$; if $\operatorname{Im} A = Y$ then $\inf h := \inf_{y \in Y} h(y) = -f^*(x^*)$, and so h is bounded from below if and only if $x^* \in \operatorname{dom} f^*$.*

(e) *Assume that $\operatorname{Im} A = Y$. Then h attains its infimum on Y if and only if $x^* \in \operatorname{Im} \partial f$.*

Proof. (a) Let $x_0 \in \operatorname{dom} f$; then $Ax_0 \in \operatorname{dom} h$ and

$$\begin{aligned} f_\infty(u) &= \lim_{t \to \infty} \frac{f(x_0 + tu) - f(x_0)}{t} = \lim_{t \to \infty} \frac{h(Ax_0 + tAu) - h(Ax_0) + t\langle u, x^* \rangle}{t} \\ &= h_\infty(Au) + \langle u, x^* \rangle \quad \forall u \in X, \end{aligned}$$

and so $f_\infty = h_\infty \circ A + x^*$. The desired inclusion follows now immediately.

(b) Assume that $h_\infty \geq 0$. Then $f_\infty - x^* = h_\infty \circ A \geq 0$, and so $x^* \in \partial f_\infty(0)$ $[= \overline{\operatorname{dom} f^*}$ by (2)]. Assume now that $x^* \in \overline{\operatorname{dom} f^*}$ and $\operatorname{Im} A = Y$. Clearly $x^* \in \partial f_\infty(0)$, and so $f_\infty \geq x^*$. Taking $y \in Y = \operatorname{Im} A$, there exists $u \in X$ with $Au = y$. Hence $h_\infty(y) = h_\infty(Au) = f_\infty(u) - \langle u, x^* \rangle \geq 0$. Therefore $h_\infty \geq 0$.

(c) Assume that $[h_\infty \leq 0] = \{0\}$. Then $h_\infty \geq 0$, and so $x^* \in \overline{\operatorname{dom} f^*}$ by (b); moreover, by (a), $\ker A \subset [f_\infty = x^*]$. Take $u \in [f_\infty = x^*]$; then $h_\infty(Au) = f_\infty(u) - \langle u, x^* \rangle = 0$, and so $Au = 0$, that is $u \in \ker A$. Hence $\ker A = [f_\infty = x^*]$; this shows that $[f_\infty = x^*]$ is a linear space and so, using the implications (c) $\Rightarrow$ (b) $\Rightarrow$ (a) of Proposition 5, we obtain that $\ker A = [f_\infty \leq x^*]$ and $x^* \in \operatorname{qri} \overline{\operatorname{dom} f^*}$.

Assume now that $\operatorname{Im} A = Y$ and $\ker A = [f_\infty \leq x^*]$. Using the implication (b) $\Rightarrow$ (c) of Proposition 5 we obtain that $x^* \in \overline{\operatorname{dom} f^*}$ and $\ker A = [f^* = x^*]$. From (b) we have that $h_\infty \geq 0$. Take $y \in [h_\infty \leq 0]$. Because $\operatorname{Im} A = Y$, there exists $u \in X$ such that $y = Au$, and so $f_\infty(u) = h_\infty(Au) + \langle u, x^* \rangle \leq \langle u, x^* \rangle$. Hence $u \in \ker A$, and so $y = Au = 0$.

(d) Assume that h is bounded from below. Then $f - x^* \geq \inf h \in \mathbb{R}$, and so, $f^*(x^*) = \sup_{x \in X}[\langle x, x^* \rangle - f(x)] < \infty$, whence $x^* \in \operatorname{dom} f^*$.

Assume now that $\operatorname{Im} A = Y$. Then $\inf h = \inf h \circ A = \inf(f - x^*) = -f^*(x^*)$. Hence, h is bounded from below if and only if $x^* \in \operatorname{dom} f^*$.

(e) Assume $\operatorname{Im} A = Y$. Suppose that h attains its infimum at $\overline{y} \in Y$ and take $\overline{x} \in X$ such that $A\overline{x} = \overline{y}$. Then $f(x) - \langle x, x^* \rangle = h(Ax) \geq h(A\overline{x}) = f(\overline{x}) - \langle \overline{x}, x^* \rangle$, whence $x^* \in \partial f(\overline{x}) \subset \operatorname{Im} \partial f$. Conversely, assume that $x^* \in \partial f(\overline{x})$ ($\subset \operatorname{dom} f^*$). Then, as seen in (d), $\inf h = -f^*(x^*) = f(\overline{x}) - \langle \overline{x}, x^* \rangle = h(A\overline{x})$. $\square$

Lemma 13. *Let $x^* \in X^*$ and $L := L_{x^*} := \{u \in X \mid f_\infty(\pm u) = \pm \langle u, x^* \rangle\}$. Then L is a closed linear subspace of X, $\operatorname{dom} f + L = \operatorname{dom} f$, $(X \setminus \operatorname{dom} f) + L = X \setminus \operatorname{dom} f$, and*

$$f(x + u) = f(x) + \langle u, x^* \rangle \quad \forall x \in X, \ \forall u \in L. \tag{16}$$

Proof. Applying Lemma 3 for $g := f_\infty$, we have that L is a closed linear subspace of X. Because $0 \in L$ the inclusions $\operatorname{dom} f + L \supset \operatorname{dom} f$ and $(X \setminus \operatorname{dom} f) + L \supset X \setminus \operatorname{dom} f$ are obvious. Take $x \in \operatorname{dom} f$ and $u \in L$. Then $f(x+u) \leq f(x) + f_\infty(u) = f(x) + \langle u, x^* \rangle < \infty$, and so $\operatorname{dom} f + L \subset \operatorname{dom} f$; hence $\operatorname{dom} f + L = \operatorname{dom} f$. Assuming that for some $x \in X \setminus \operatorname{dom} f$ and $u \in L$ one has $x' := x + u \in \operatorname{dom} f$ we get the contradiction $x = x' + (-u) \in \operatorname{dom} f$. Hence $(X \setminus \operatorname{dom} f) + L = X \setminus \operatorname{dom} f$.

From the previous equality it is clear that $f(x+u) = f(x) + \langle u, x^* \rangle$ ($= \infty$) for $x \in X \setminus \operatorname{dom} f$ and $u \in L$. Take now $x \in \operatorname{dom} f$ and $u \in L$. Then $x + u \in \operatorname{dom} f$ and, as seen above, $f(x + u) \leq f(x) + \langle u, x^* \rangle$. Hence

$$f(x+u) \leq f(x) + \langle u, x^* \rangle \leq f(x+u) + \langle -u, x^* \rangle + \langle u, x^* \rangle = f(x+u),$$

and so $f(x + u) = f(x) + \langle u, x^* \rangle$. Therefore, (16) holds. $\square$

In the conditions and notation of Lemma 13 we have that $f(x+u) - \langle x+u, x^* \rangle = f(x) - \langle x, x^* \rangle$ for all $x \in X$ and $u \in L$, which shows that

$$h_{x^*} : X/L \to \overline{\mathbb{R}}, \quad h_{x^*}(\widehat{x}) := f(x) - \langle x, x^* \rangle \quad (x \in X) \tag{17}$$

is well defined and $f = h_{x^*} \circ \pi + x^*$, where $\pi : X \to X/L$ is the (natural) projection defined by $\pi(x) := \widehat{x}$. The convexity and properness of h follow immediately from the corresponding properties of f.

Proposition 14. *Let $x^* \in X^*$, $L := L_{x^*} := \{u \in X \mid f_\infty(\pm u) = \pm \langle u, x^* \rangle\}$, and $h := h_{x^*}$ be defined in (17). Then the following assertions hold:*

(a) *$h \in \Gamma(X/L)$, $h_\infty(\widehat{u}) = f_\infty(u) - \langle u, x^* \rangle$ for all $u \in X$, and $\{\widehat{u} \in X/L \mid h_\infty(\widehat{u}) = h_\infty(-\widehat{u}) = 0\} = \{\widehat{0}\}$;*

(b) *$h_\infty \geq 0$ if and only if $x^* \in \overline{\operatorname{dom} f^*}$;*

(c) *if $x^* \in \overline{\operatorname{dom} f^*}$ (consequently $L = L_f$), then*

$$[h_\infty \leq 0] = \{\widehat{0}\} \iff x^* \in \operatorname{qri} \overline{\operatorname{dom} f^*} \iff L = [f_\infty \leq x^*] \iff L = [f_\infty = x^*];$$

(d) *$\inf h = -f^*(x^*)$, and so h is bounded from below, if and only if $x^* \in \operatorname{dom} f^*$;*

(e) *h attains its infimum on X/L if and only if $x^* \in \operatorname{Im} \partial f$.*

Proof. (a) As seen above, h is well defined, proper and convex, and $f = h \circ \pi + x^*$, where $\pi : X \to X/L$ with $\pi(x) := \widehat{x}$. For $\alpha \in \mathbb{R}$ and $x \in X$ one has

$$\widehat{x} \in [h \leq \alpha] \iff f(x) - \langle x, x^* \rangle \leq \alpha \iff x \in [f - x^* \leq \alpha],$$

and so $\pi^{-1}([h \le \alpha]) = [f - x^* \le \alpha]$. Since $f - x^* \in \Gamma(X)$, $[f - x^* \le \alpha]$ is closed. Hence $[h \le \alpha]$ is closed. Because $\alpha \in \mathbb{R}$ is arbitrary, it follows that h is lsc. Therefore, $h \in \Gamma(X/L)$. The expression of h_∞ is obtained using Proposition 12 (a). Take $u \in X$; from the expression of h_∞ we have that

$$h_\infty(\pm\widehat{u}) = 0 \iff h_\infty(\widehat{\pm u}) = 0 \iff f_\infty(\pm u) - \langle \pm u, x^*\rangle = 0 \iff u \in L \iff \widehat{u} = \widehat{0}.$$

Because π is onto, the assertions (b), (d), (e) follow from assertions (b), (d) and (e) of Proposition 12, respectively.

(c) Because π is onto and $\ker \pi = L$, using Proposition 12 (c) we get the equivalence $[h_\infty \le 0] = \{\widehat{0}\} \Leftrightarrow L = [f \le x^*]$; the other equivalences follow from (a) $\Leftrightarrow$ (b') $\Leftrightarrow$ (c') of Proposition 5 because $x^* \in \overline{\operatorname{dom} f^*} = \partial f_\infty(0)$. $\square$

Our main result is the following theorem; in its statement, for the closed linear subspace Y of X, $\pi : X \to X/Y$ is the natural projection of X onto Y, that is $\pi(x) := \widehat{x}$.

Theorem 15. *Let $f \in \Gamma(X)$. The following assertions hold:*

(i) *For every $x^* \in X^*$, there exist a closed linear subspace Y of X and $h \in \Gamma(X/Y)$ such that h is not constant on any line $\widehat{x} + \mathbb{R}\widehat{u}$ with $\widehat{x} \in \operatorname{dom} h$ and $\widehat{u} \ne \widehat{0}$ such that $f = h \circ \pi + x^*$. Moreover, for $x^* \in \operatorname{dom} f^*$, h is bounded from below, while for $x^* \in \operatorname{Im} \partial f$, h attains its infimum on X/Y; in both cases $Y = L_f$, where, by (15),*

$$L_f = \{u \in X \mid f_\infty(\pm u) = \langle \pm u, u^*\rangle\} \text{ for some (any) } u^* \in \overline{\operatorname{dom} f^*}.$$

(ii) *There exist a closed linear subspace Y of X, a directionally coercive function $h \in \Gamma(X/Y)$ and $x^* \in X^*$ such that $f = h \circ \pi + x^*$ if and only if $\operatorname{qri} \overline{\operatorname{dom} f^*} \ne \emptyset$. In such a case, $x^* \in \operatorname{qri} \overline{\operatorname{dom} f^*}$ and $Y = L_f$.*

(iii) *Assume that $(X, \langle\cdot,\cdot\rangle)$ is a Hilbert space and $\operatorname{qri} \overline{\operatorname{dom} f^*} \ne \emptyset$. Then there exist a unique closed linear subspace Y of X, a unique essentially directionally coercive function $c \in \Gamma(Z)$ with $Z := Y^\perp$, and a unique $v \in Y$ such that $f = c \circ \operatorname{Pr}_Z + \langle\cdot, v\rangle$, where Pr_Z is the orthogonal projection of X onto Z. More precisely, $Y = L_f$, $c = h|_Z$ and $v := \operatorname{Pr}_Y(x^*)$ for some (any) $x^* \in \overline{\operatorname{dom} f^*}$, where $\operatorname{Pr}_Y = \operatorname{Id}_X - \operatorname{Pr}_Z$.*

Proof. (i) Take $x^* \in X^*$ and consider $Y := L_{x^*} := \{u \in X \mid f_\infty(\pm u) = \langle \pm u, x^*\rangle\}$. Then Y is closed linear subspace of X by Lemma 13. Using Proposition 14 (a) we get $h \in \Gamma(X/Y)$ such that $f = h \circ \pi + x^*$ and $h_\infty(\pm\widehat{u}) = 0 \Rightarrow \widehat{u} = \widehat{0}$; hence h is not constant on any line by (5). The other conclusions follow from Proposition 14 (d) and (e).

(ii) The assertion is a consequence of (i) and Proposition 14 (c).

(iii) We identify X^* with X by Riesz theorem; then, for Y a closed linear subspace of X, the natural projection π of X onto X/Y becomes the orthogonal projection of X onto $Y^\perp$.

Assuming that $f = c \circ \operatorname{Pr}_Z + \langle\cdot, v\rangle$ with $c \in \Gamma(Z)$ essentially directionally coercive and $v \in Y$ (less the uniqueness), then $c = h + \langle\cdot, z\rangle$ with $h \in \Gamma(Z)$ directionally coercive ($\Leftrightarrow [h_\infty \le 0] = \{0\}$) and $z \in Z$, whence

$$f = h \circ \operatorname{Pr}_Z + \langle\cdot, z + v\rangle .$$

Having in view Proposition 12, because Pr_Z is onto, one must have $x^* := \langle\cdot, z + v\rangle \in \operatorname{qri} \overline{\operatorname{dom} f^*}$ and $(Z^\perp =) \ker \operatorname{Pr}_Z = [f_\infty \le x^*]$. Using Proposition 5, one must have $(Y =)\ Z^\perp = L_{x^*} = L_f$, whence $Z = L_f^\perp = \overline{\operatorname{span}}\left(\operatorname{dom} f^* - \operatorname{dom} f^*\right)$ by Proposition 4 (b); in particular, we got the uniqueness of Y. In order to get the uniqueness of v, let us consider $x_1^*, x_2^* \in \overline{\operatorname{dom} f^*}$. Then $x_i^* = u_i + v_i$ with $u_i \in Z$ and $v_i \in Y$ for $i = 1, 2$. It follows that $Z \ni x_1^* - x_2^* = (u_1 - u_2) + (v_1 - v_2)$. Because $u_1 - u_2 \in Z$, $v_1 - v_2 \in Y$ and $Z \cap Y = \{0\}$, we obtain that $v_1 = v_2$. This shows that $\operatorname{Pr}_Y(\overline{\operatorname{dom} f^*})$ is a singleton $\{v\}$. Because $v \in Y$ and

$c \circ \mathrm{Pr}_Z = f - \langle \cdot, v \rangle$, we have that $c(z) = c(\mathrm{Pr}_Z(z)) = f(z) - \langle z, v \rangle = f(z)$ for $z \in Z$, that is $c = f|_Z$. This proves the uniqueness of c in the representation $f = c \circ \mathrm{Pr}_Z + \langle \cdot, v \rangle$ with the desired properties.

In what concerns the existence of Y, c and v with the desired properties, we proceed as follows: Consider $x^* \in \operatorname{qri} \overline{\operatorname{dom} f^*}$ and $Y = L_f$ $(= L_{x^*})$; set $Z := Y^\perp$ $(= X/Y)$. By (ii) there exist $h \in \Gamma(Z)$ directionally coercive and $x^* \in X^*$ $(= X)$ such that $f = h \circ \pi + x^*$ $(= h \circ \mathrm{Pr}_Z + x^*)$. Take $c := f|_Z$, $v := \mathrm{Pr}_Y(x^*) \in Y$ and $z := x^* - v \in Z$. Then

$$c(z') = f(z') = (h \circ \mathrm{Pr}_Z)(z') + \langle z', x^* \rangle = h(z') + \langle z', z + v \rangle = h(z') + \langle z', z \rangle \quad \forall z' \in Z,$$

that is $c = h + \langle \cdot, z \rangle$. Hence c is essentially directionally coercive and $f = c \circ \mathrm{Pr}_Z + \langle \cdot, v \rangle$. $\square$

Remark 16. As in Azagra's paper [1], consider X a Banach space and $f : X \to \mathbb{R}$ a continuous convex function; when X is separable one has $\operatorname{qri} \overline{\operatorname{dom} f^*} \neq \emptyset$ by Proposition 7. So, from assertions (i), (ii) and (iii) of Theorem 15 one obtains Theorems 5, 6 and 4 of [1], respectively.

We end this note with an example which could be useful for providing (counter-) examples.

Example 17. Let X be a normed vector space and $C \subset X^*$ be a nonempty w^*-closed convex set. Then $\varphi_C := (\frac{1}{2} \|\cdot\|^2) \square s_C$ with $s_C(x) := \sup_{x^* \in C} \langle x, x^* \rangle$ for $x \in X$ is a real-valued continuous convex function such that $\operatorname{dom} \varphi_C^* = C$ and $(\varphi_C)_\infty = s_C$. Here $h_1 \square h_2$ denotes the convolution of the functions $h_1, h_2 : X \to \overline{\mathbb{R}}$ and is defined by $(h_1 \square h_2)(x) := \inf\{h_1(x_1) + h_2(x_2) \mid x_1, x_2 \in X, \ x_1 + x_2 = x\}$.

Proof. Clearly, s_C is a proper sublinear lsc function with $\psi^* = \iota_C$. By [6, Exer. 3.11 1)] we have that φ_C is a continuous convex function such that $\varphi_C \leq \frac{1}{2} \|\cdot\|^2$, while from [6, Th. 2.3.1 (ix)], $\varphi_C^* = (\frac{1}{2} \|\cdot\|^2)^* + s_C^* = \frac{1}{2} \|\cdot\|^2 + \iota_C$. Hence $\operatorname{dom} \varphi_C^* = C$, whence $(\varphi_C)_\infty = s_C$ by (2). $\square$

Notice that taking $X := \ell_2(\Gamma)$ and $C := X_+$ as defined in Remark 8, and f the function defined in [1, Ex. 7], then $f = 2\varphi_C$, where φ_C is defined in Example 17. Then $\operatorname{dom} f^* = X_+$. So $L_f = (X_+ - X_+)^\perp = \{0\}$ which shows that f is not constant on any line [by Theorem 15 (i)]; moreover, if Γ is uncountable, then $\operatorname{qri} \overline{\operatorname{dom} f^*} = \operatorname{qi} \overline{\operatorname{dom} f^*} = \emptyset$ by Remark 8, and so f is not essentially directionally coercive by Theorem 15 (ii). So, the conclusions of [1, Ex. 7] are confirmed.

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