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Global dynamics below the ground state for the focusing semilinear Schrödinger equation with a linear potential

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A B S T R A C T

We study global dynamics of the solution to the Cauchy problem for the focusing semilinear Schrödinger equation with a linear potential on the real line $\mathbb{R}$:

$$\begin{cases} i\partial_t u + \partial_x^2 u - Vu + |u|^{p-1}u = 0, & (t,x) \in I \times \mathbb{R}, \\ u(0) = u_0 \in \mathcal{H}, \end{cases} \tag{NLS$_V$}$$

where $u = u(t,x)$ is a complex-valued unknown function of $(t,x) \in I \times \mathbb{R}$, I denotes the maximal existence time interval of u, $V = V(x)$ is non-negative and in $L^1(\mathbb{R}) + L^\infty(\mathbb{R})$, p belongs to the so-called mass-supercritical case, i.e. $p > 5$, and $\mathcal{H}$ is a Hilbert space connected to the Schrödinger operator $-\partial_x^2 + V$ and is called energy space. It is well known that (NLS$_V$) is locally well-posed in $\mathcal{H}$. Our aim in the present paper is to study global behavior of the solution and prove a scattering result and a blow-up result for (NLS$_V$) with the data u_0 whose mass-energy is less than that of the ground state Q, where the function $Q = Q(x)$ is the unique radial positive solution to the stationary Schrödinger equation without the potential:

$$-Q'' + Q = |Q|^{p-1}Q, \quad \text{in } H^1(\mathbb{R}).$$

The similar result for NLS without potential ($V \equiv 0$), which is invariant of translation and scaling transformation, in one space dimension was obtained by Akahori–Nawa. Lafontaine treated the defocusing version of (NLS$_V$), that is, (NLS$_V$) with a replacement of $+|u|^{p-1}u$ into $-|u|^{p-1}u$, and prove that the solution scatters as $t \to \pm\infty$ in $H^1(\mathbb{R})$ for an arbitrary data in $H^1(\mathbb{R})$ by Kenig-Merle's argument with a profile decomposition. However, the method to the defocusing case cannot be applicable to our focusing case because the energy is positive in the defocusing case, on the other hand, the energy may be negative in the focusing case. To overcome this difficulty, we use a variational argument. Our proof of the blow-up result is based on the argument of Du–Wu–Zhang. The difficulty of our case lies

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in deriving a uniform bound of a functional related to Virial Identity because of existence of the potential.

Contents

1. Introduction

1.1. Background

In the present paper, we study global dynamics of the solution to the Cauchy problem for the focusing semilinear Schrödinger equation with a linear potential on the real line $\mathbb{R}$:

$$\begin{cases} i\partial_t u + \partial_x^2 u - Vu + |u|^{p-1}u = 0, & (t,x) \in I \times \mathbb{R}, \\ u(0) = u_0 \in \mathcal{H}, \end{cases} \tag{NLS$_V$}$$

where $u = u(t,x)$ is a complex-valued unknown function of (t,x), $(0 \in)$ I denotes a existence time interval of the function u, $(0 \neq)$ $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R}) := \{f : f = f_1 + f_2, f_1 \in L^1(\mathbb{R}), f_2 \in L^\infty(\mathbb{R})\}$ is a non-negative function of $x \in \mathbb{R}$, where $L^q = L^q(\mathbb{R})$ $(1 \leq q \leq \infty)$ denotes a usual Lebesgue space, $p > 5$ belongs to the mass-supercritical region, $u_0 = u_0(x)$ is a complex-valued prescribed function of $x \in \mathbb{R}$, and $\mathcal{H}$ is a Hilbert space associated with the Schrödinger operator $-\partial_x^2 + V$ and is called energy space. The precise definition of $\mathcal{H}$ is given by (1.3).

The Cauchy problem (NLS$_V$) is locally well-posed in the energy space $\mathcal{H}$ (see Proposition 1.1, or Theorem 3.7.1 in [4] for more general setting). Our aim in the present paper is to study global dynamics of the solution to (NLS$_V$) and prove a scattering result and a blow-up result of (NLS$_V$) with the initial data whose mass-energy is less than that of the ground state Q_1, where for $\omega > 0$, $Q_\omega = Q_\omega(x)$ is the unique radial positive solution to the stationary Schrödinger equation without the potential:

$$-Q_\omega'' + \omega Q_\omega = |Q_\omega|^{p-1} Q_\omega, \text{ in } H^1(\mathbb{R}), \tag{1.1}$$

where $H^s = H^s(\mathbb{R})$ denotes the s-th ordered L^2-based Sobolev space. We note that $e^{i\omega t} Q_\omega(x)$ is a non-scattering global solution to (NLS) with $u_0 = Q_\omega$. Our result extends the results of NLS without potentials ($V \equiv 0$) obtained in [1,8] (see also [11,12,7] for three dimensional cases) to that of (NLS_V) with a potential $V \geq 0$. The similar results for (NLS) with another potential were obtained in [15,20,22,13], where [15] treats the focusing mass-supercritical NLS with a repulsive Dirac delta potential ($V = \delta_0$) in one space dimension (see also [3] for the defocusing case) and [20,22] studied the focusing mass-supercritical NLS with an inverse square potential in three spatial dimensions ($V(x) = \frac{a}{|x|^2}$ with $a > -\frac{1}{4}$). Lafontaine [21] studied the defocusing version of (NLS_V), that is, (NLS_V) with a replacement of $+|u|^{p-1}u$ into $-|u|^{p-1}u$, and proved that the local solution can be extended globally and it tends to a free one as $t \to \pm\infty$ in the energy space $H^1(\mathbb{R})$ for an arbitrary data in $H^1(\mathbb{R})$, if $V \in L^1_1(\mathbb{R}) := \{f \in L^1(\mathbb{R}) : (1 + |\cdot|)f \in L^1(\mathbb{R})\}$. However, about study of classification of global behaviors of solutions, the focusing case is more difficult than the defocusing one, because the sign of the energy functional of the solution to the focusing problem may change by the initial data, and there exists a blow-up solution.

When $V \equiv 0$, the Cauchy problem (NLS_V) in the energy space $\mathcal{H}$ is

$$\begin{cases} i\partial_t u + \partial_x^2 u + |u|^{p-1} u = 0, & (t, x) \in I \times \mathbb{R}, \\ u(0) = u_0 \in H^1(\mathbb{R}). \end{cases} \tag{NLS}$$

NLS with the power type nonlinearity arises in various physical contexts such as nonlinear optics and plasma physics (see [32,29,5] for example). The nonlinearity enters due to the effect of changes in the field intensity on the wave propagation characteristics of the medium. There are large amount of literature for Mathematical results about local or global well-posedness, blow-up, scattering to a free solution, and stability of special solutions like solitary waves for (NLS). We mention the results about global dynamics for NLS without a potential after introducing several notations.

We roughly explain why the exponent p is restricted to $p > 5$. NLS without the potential ($V = 0$) is invariant under the scale transformation

$$u(t, x) \mapsto u_\gamma(t, x) := \gamma^{\frac{2}{p-1}} u(\gamma^2 t, \gamma x), \quad \text{for } \gamma > 0.$$

We note that (NLS_V) with a potential ($V \neq 0$) does not have this scaling invariant property. Moreover, a simple computation gives

$$\|u_\gamma(0, \cdot)\|_{L^2} = \gamma^{\frac{2}{p-1} - \frac{1}{2}} \|u(0, \cdot)\|_{L^2}.$$

From this identity, we see that if the exponent $p \geq 1$ satisfies

$$\frac{2}{p-1} - \frac{1}{2} = 0, \ i.e.\ p = 5,$$

then L^2-norm of the initial data is also invariant. In this sense, the case of $p = 5$ is called L^2 or mass-critical case. And the case of $p > 5$ (resp. $p < 5$) is called L^2 or mass-supercritical case (resp. L^2 or mass-subcritical case).

When $V \neq 0$, the potential can be thought of as modeling inhomogeneities in the medium. In [28], Equation (NLS_V) with $V \in L^\infty(\mathbb{R})$ is studied as a model proposed to describe the local dynamics at a nucleation site. We note that studies of equation (NLS_V) are essential to those of the Schrödinger equation with an electromagnetic potential in one spatial dimension:

$$i\partial_t v + \left(\frac{\partial}{\partial x} - iA\right)^2 v - Vv + |v|^{p-1}v = 0, \qquad (t,x) \in I \times \mathbb{R}, \tag{1.2}$$

where $v = v(t,x)$ is a complex valued wave function of $(t,x) \in I \times \mathbb{R}$, and $A = A(x)$ is a real-valued continuous bounded function of $x \in \mathbb{R}$. Indeed, when for any $x_0 \in \mathbb{R}$, we introduce the following gauge transformation $v \mapsto u$ as

$$u(t,x) := \exp\left(i\int_{x_0}^{x} A(y)dy\right) v(t,x),$$

we see that v is a solution to (1.2) with the initial condition $v(0) = v_0 \in H^1(\mathbb{R})$, if and only if u satisfies (NLS_V) with $u_0(x) := \exp\left(i\int_{x_0}^{x} A(y)dy\right) v_0(x) \in H^1(\mathbb{R})$.

Next we introduce several notations associated with the Schrödinger operator $-\partial_x^2 + V$. We assume that the function $V = V(x)$ satisfies

$$V \text{ is real-valued and } V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R}). \tag{A.1}$$

H_V and L_V denote the Schrödinger operators defined by

$$H_V := -\frac{d^2}{dx^2} + V, \quad \text{and} \quad L_V := 1 + H_V,$$

respectively, with the domain

$$D(H_V) = D(L_V) := \left\{ f \in H^2(\mathbb{R}) : \|f\|_{L_V} < \infty \right\},$$

where the norm $\|\cdot\|_{L_V}$ is defined by

$$\|f\|_{L_V}^2 := \|f\|_{H^2}^2 + \int_{-\infty}^{\infty} V(x)|f(x)|^2 dx.$$

For $s \geq 0$, $H^s = H^s(\mathbb{R})$ denotes the usual s-th order L^2-based Sobolev spaces. Then both operators H_V and L_V are self-adjoint on $L^2(\mathbb{R})$. Thus by Stone's theorem, the Schrödinger evolution group $\{e^{-itH_V}\}_{t\in\mathbb{R}}$ is generated by H_V on $L^2(\mathbb{R})$. Moreover, if V is non-negative, i.e. $V \geq 0$, then both H_V and L_V are non-negative, i.e. the estimates

$$(H_V f, f)_{L^2} = \|\partial_x f\|_{L^2}^2 + \int_{-\infty}^{\infty} V(x)|f(x)|^2 dx > 0$$

$$(L_V f, f)_{L^2} = \|f\|_{L^2}^2 + (H_V f, f) > \|f\|_{L^2}^2 > 0$$

hold for any $(0 \neq)\ f \in D(H_V)$. Therefore, the fractional operators $H_V^{\frac{1}{2}}$ and $L_V^{\frac{1}{2}}$ are well defined with the domain

$$\mathcal{H} := D(H_V^{\frac{1}{2}}) = D(L_V^{\frac{1}{2}}) := \left\{ f \in H^1(\mathbb{R}) : \|f\|_{L_V^{\frac{1}{2}}} < \infty \right\}, \tag{1.3}$$

where the norms $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{L_V^{\frac{1}{2}}}$ are defined by

$$\|f\|_{\mathcal{H}}^2 := \|f\|_{L_V^{\frac{1}{2}}}^2 := \|f\|_{H^1}^2 + \int_{-\infty}^{\infty} V(x)|f(x)|^2 dx.$$

We note that the norm $\|\cdot\|_{\mathcal{H}}$ is equivalent to the usual Sobolev norm $\|\cdot\|_{H^1}$, since $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ is non-negative and the embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$ holds due to one spatial dimension. We also introduce the norm $\|\cdot\|_{H_V^{\frac{1}{2}}}$ defined by

$$\|f\|_{H_V^{\frac{1}{2}}}^2 := \|\partial_x f\|_{L^2}^2 + \int_{-\infty}^{\infty} V(x)|f(x)|^2 dx, \tag{1.4}$$

which satisfies the estimate

$$\|\partial_x f\|_{L^2} \le \|f\|_{H_V^{\frac{1}{2}}} \le \|f\|_{L_V^{\frac{1}{2}}}.$$

Then we see that $H_V^{\frac{1}{2}}$ and $L_V^{\frac{1}{2}}$ are also non-negative and self-adjoint operator on $L^2(\mathbb{R})$.

By using the above properties, we see that if $u_0 \in \mathcal{H}$, then $e^{-itH_V}u_0$ belongs to

$$C(\mathbb{R} : \mathcal{H}) \cap C^1(\mathbb{R} : \mathcal{H}^{-1}),$$

where $\mathcal{H}^{-1} := \{f \in \mathcal{S}'(\mathbb{R}) : \|L_V^{-\frac{1}{2}} f\|_{L^2} < \infty\}$ and $\mathcal{S}'(\mathbb{R})$ denotes a function space of tempered distributions, and is the unique global solution to the Cauchy problem for the linear Schrödinger equation with the potential:

$$\begin{cases} i\partial_t u - H_V u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0) = u_0 \in \mathcal{H}, \end{cases} \tag{1.5}$$

(see [27] for more details for example).

Next we state the local well-posedness result of (NLS_V) in the energy space $\mathcal{H}$.

Proposition 1.1 *(Local well-posedness of (NLS_V) in $\mathcal{H}$). Let $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ be non-negative and $p \ge 1$. Then the Cauchy problem (NLS_V) is locally well-posed in the energy space $\mathcal{H}$ for arbitrary initial data $u_0 \in \mathcal{H}$. More precisely, the following statements hold:*

- *(Existence) For any $\varrho > 0$ and $u_0 \in \mathcal{H}$ satisfying $\|u_0\|_{\mathcal{H}} \le \varrho$, there exists a positive time $T = T(\varrho)$ such that there exists a unique solution $u \in C(I_T; \mathcal{H}) \cap C^1(I_T; \mathcal{H}^{-1})$ to (NLS_V), where $I_T := (-T, T)$.*
- *(Uniqueness) Let u be the solution to (NLS_V) obtained in the Existence part. Let $T_1 \in (0, T(\varrho)]$ and $v \in C(I_{T_1}; \mathcal{H}) \cap C^1(I_{T_1}; \mathcal{H}^{-1})$ be another solution to (NLS_V) on I_{T_1}. If $v(0) = u_0$, then $u|_{I_{T_1}} = v$ on I_{T_1}.*
- *(Continuity of the flow map) Let $\varrho > 0$ and $T = T(\varrho)$ be same as in the Existence part. Then the flow map*

$$\Xi : \{f \in \mathcal{H} : \|f\|_{\mathcal{H}} \le \varrho\} \mapsto L_t^\infty(I_T : \mathcal{H}), \quad \Xi[u_0](t) = u(t)$$

is Lipschitz continuous.

By the existence result and the uniqueness result, the maximal existence times $T_\pm$ of the solution are well defined as

$$T_+ := \sup\{T \in (0,\infty] : \textit{there exists a unique solution } u \textit{ to } (\text{NLS}_V) \textit{ on } [0,T)\}$$
$$T_- := \sup\{T \in (0,\infty] : \textit{there exists a unique solution } u \textit{ to } (\text{NLS}_V) \textit{ on } (-T,0]\}.$$

Moreover, the conservation laws and the blow-up criterion hold:

- *(Conservation Laws) The energy E and the mass M are conserved by the flow, i.e.*

$$E(u(t)) = E(u_0),\ M(u(t)) = M(u_0),\ \textit{for any } t \in I_T,$$

where the functionals $E : \mathcal{H} \mapsto \mathbb{R}$ and $M : L^2(\mathbb{R}) \mapsto \mathbb{R}$ are defined as

$$E(\phi) = E_V(\phi) := \frac{1}{2}\|\phi\|^2_{H_V^{\frac{1}{2}}} - \frac{1}{p+1}\|\phi\|^{p+1}_{L^{p+1}}, \tag{1.6}$$
$$M(\phi) := \|\phi\|^2_{L^2}. \tag{1.7}$$

- *(Blow-up criterion) If $T_\pm < \infty$, then*

$$\lim_{t\to\pm T_\pm \mp 0} \|\partial_x u(t)\|_{L^2} = \infty,$$

where double-sign corresponds.

For the proof, see Theorem 3.7.1 in [4] for example, and for the convenience of readers, we give a proof of the proposition in Appendix A, which is based on the contraction argument with the commutation relation $[L_V^{\frac{1}{2}}, e^{itH_V}] = 0$, the equivalency between the norms $\|\cdot\|_{L_V^{\frac{1}{2}}}$ and $\|\cdot\|_{H^1}$ and the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$.

Next we are interested in whether the local solution to (NLS_V) obtained in the proposition can be extended globally or not, and how global solution behaves like, if global solutions exist. Let us recall the definitions of blow-up, glow-up and scattering. Let u be the solution to (NLS_V) on the maximal existence time interval $(-T_-, T_+)$.

Definition 1.1 *(Blow-up).* We say that the solution u to (NLS_V) blows up in positive time (resp. negative time) if and only if $T_+ < \infty$ (resp. $T_- < \infty$). Then the blow-up criterion implies

$$\lim_{t\to\pm T_\pm \mp 0} \|\partial_x u(t)\|_{L^2} = \infty.$$

Definition 1.2 *(Grow-up).* We say that the solution u to (NLS_V) grows up in positive time if and only if

$$T_+ = \infty \quad \text{and} \ \limsup_{t\to+\infty} \|\partial_x u(t)\|_{L^2} = \infty.$$

Glow-up in negative time is defined in the similar manner.

Definition 1.3 *(Scattering).* We say that solution u to (NLS_V) scatters in the energy space $\mathcal{H}$ if and only if $T_\pm = \infty$ and there exist scattering states $u_\pm \in \mathcal{H}$ such that the following holds:

$$\left\|u(t) - e^{-itH_V}u_\pm\right\|_{\mathcal{H}} \to 0,\ \text{as } t \to \pm\infty.$$

Scattering solution to (NLS_V) behaves like the solution to the Cauchy problem (1.5) with the initial data $u_\pm$ as $t \to \pm\infty$.

Next we recall the results about global dynamics for (NLS). Since a pioneer work by Kenig and Merle [19], global dynamics of solutions to focusing nonlinear Schrödinger equations with the initial condition below a ground state have been studied. Holmer and Roudenko [11] studied global dynamics of solutions to the focusing cubic semilinear Schrödinger equation in three spatial dimensions. They proved that if the initial data $u_0 \in H_r^1(\mathbb{R}^3)$, where $H_r^1(\mathbb{R}^d) := \{f \in H^1(\mathbb{R}^d) : f \text{ is radially symmetric}\}$, and satisfies the following condition

$$M(u_0)E_0(u_0) < M(Q_1)E_0(Q_1), \tag{1.8}$$

where Q_1 is the corresponding ground state, then the following relations hold:

$$\begin{cases} \|u_0\|_{L^2} \|\nabla u_0\|_{L^2} < \|Q_1\|_{L^2} \|\nabla Q_1\|_{L^2} \iff & u \text{ scatters in } H_r^1(\mathbb{R}^3) \text{ as } t \to \pm\infty, \\ \|u_0\|_{L^2} \|\nabla u_0\|_{L^2} > \|Q_1\|_{L^2} \|\nabla Q_1\|_{L^2} \iff & u \text{ blows up in both time directions} \end{cases}$$

Duyckaerts, Holmer, and Roudenko [7] extended the scattering result to non-radial data in $H^1(\mathbb{R}^3)$, and Holmer and Roudenko [12] treated non-radial data in $H^1(\mathbb{R}^3)$ which belongs to the above blow-up region and proved that the local solution blows up in finite time or grows up at infinite time. Fang, Xie, and Cazenave [8] extended the scattering result of [7] and Akahori and Nawa [1] extended both scattering and blow-up results of [7,12] to the mass-supercritical and energy-subcritical Schrödinger equations in general dimensions, i.e. $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$ if $d \geq 3$ or $1 + \frac{4}{d} < p$ if $d = 1, 2$.

Next let us recall several related results for NLS with a linear potential. Banica–Visciglia [3] studied the mass-supercritical defocusing Schrödinger equation with a repulsive Dirac delta potential in one space dimension and proved that all solutions scatter in the energy space $H^1(\mathbb{R})$ as $t \to \pm\infty$. On the other hand, the author and Inui [15] studied the focusing Schrödinger equation with a repulsive Dirac delta potential in one space dimension, and proved the similar result obtained in [1], that is, a scattering result and a blow-up result below the ground state without the potential. They also classified global dynamics of the solution up to twice times above the ground state without the potential, if the initial data is in $H_r^1(\mathbb{R})$. We note that Inui [16] studied global dynamics of solutions to some extent above the ground state without the potential to NLS with the initial data which have several symmetries (see also [23,24] and the references for above ground states). Hong [13] studied global dynamics of solutions for the cubic focusing Schrödinger equation with a linear potential in three spatial dimensions and proved the similar result obtained in [20,15] if the potential belongs to the Kato-class.

1.2. Main results

We state our main results in the present paper. The following theorem gives dichotomy between an upper bound and a lower bound of solutions below the ground state Q_1, where Q_ω is the unique positive radial solution to (1.1) and can be written explicitly by

$$Q_\omega(x) := \left\{ \frac{(p+1)\omega}{2} \operatorname{sech}^2 \left(\frac{(p-1)\sqrt{\omega}}{2} |x| \right) \right\}^{\frac{1}{p-1}}, \tag{1.9}$$

and uniform bounds connected to a functional $P_0 : H^1(\mathbb{R}) \mapsto \mathbb{R}$. Here P_0 is defined by

$$P_0(\phi) := \|\partial_x \phi\|_{L^2}^2 - \frac{p-1}{2(p+1)} \|\phi\|_{L^{p+1}}^{p+1},$$

is related to Virial Identity (Lemma 3.1).

Theorem 1.2 *(Dichotomy between an upper bound and a lower bound of solutions and uniform bounds connected to the functional P_0). Let $p > 5$, $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ be non-negative, $u_0 \in \mathcal{H}$ and u be the unique solution to (NLS$_V$) on the maximal interval $I = (-T_-, T_+)$. Moreover we assume that u_0 satisfies*

$$M(u_0)^\sigma E_V(u_0) < M(Q_1)^\sigma E_0(Q_1), \tag{1.10}$$

where $\sigma = \sigma(p) := \frac{p+3}{p-5}$ and $Q_1 \in H^1(\mathbb{R})$ is defined by (1.9) with $\omega = 1$. Then the following holds:

(1) *If u_0 satisfies*

$$\|u_0\|_{L^2}^\sigma \left\| H_V^{\frac{1}{2}} u_0 \right\|_{L^2} < \|Q_1\|_{L^2}^\sigma \|\partial_x Q_1\|_{L^2}\,, \tag{1.11}$$

then $T_+ = T_- = \infty$ and u satisfies

$$\|u_0\|_{L^2}^\sigma \left\| H_V^{\frac{1}{2}} u(t) \right\|_{L^2} < \|Q_1\|_{L^2}^\sigma \|\partial_x Q_1\|_{L^2}\,, \tag{1.12}$$

$$2E_V(u_0) \le \|H_V^{\frac{1}{2}} u(t)\|_{L^2}^2 \le \frac{2(p-1)}{p-5} E_V(u_0) \tag{1.13}$$

for any $t \in \mathbb{R}$. Moreover, if $u_0 \neq 0$ and $V \in L^1(\mathbb{R})$, then there exists $\delta_0 > 0$ depending only on $p, \|V\|_{L^1}, u_0, Q_1$ such that the estimate

$$P_0(u(t)) \ge \delta_0 \tag{1.14}$$

holds for any $t \in \mathbb{R}$.

(2) *If u_0 satisfies*

$$\|u_0\|_{L^2}^\sigma \left\| H_V^{\frac{1}{2}} u_0 \right\|_{L^2} > \|Q_1\|_{L^2}^\sigma \|\partial_x Q_1\|_{L^2}\,, \tag{1.15}$$

then u satisfies

$$\|u_0\|_{L^2}^\sigma \left\| H_V^{\frac{1}{2}} u(t) \right\|_{L^2} > \|Q_1\|_{L^2}^\sigma \|\partial_x Q_1\|_{L^2}\,, \tag{1.16}$$

for any $t \in I$. Moreover, there exists $\delta_1 > 0$ depending only on p, V, u_0, Q_1 such that the estimate

$$P_0(u(t)) + \int_{-\infty}^{\infty} V(x)|u(t,x)|^2 dx < -\delta_1 \tag{1.17}$$

holds for any $t \in I$.

(3) *The equality*

$$\|u_0\|_{L^2}^\sigma \|H_V^{\frac{1}{2}} u_0\|_{L^2} = \|Q_1\|_{L^2}^\sigma \|\partial_x Q_1\|_{L^2},$$

is impossible.

Next we state a scattering result and a blow-up or grow-up result of solutions to (NLS$_V$) under an additional assumption on the potential V, that is

$$V \text{ is differentiable in a distribution sense and } xV' \in L^1(\mathbb{R}) + L^\infty(\mathbb{R}). \tag{A}$$

Theorem 1.3 *(Scattering and blow-up or grow-up results below the ground state). In addition to the assumptions of Theorem 1.2, we assume that the potential V satisfies (A). Then the following holds:*

(1) *If u_0 satisfies (1.11) and V and V' belong to $L^1_1(\mathbb{R})$, where $L^1_1(\mathbb{R})$ denotes the weighted Lebesgue space given by*

$$L^1_1(\mathbb{R}) := \{f \in L^1(\mathbb{R}) : \|(1 + |\cdot|)f\|_{L^1} < \infty\},$$

and is repulsive, i.e.

$$xV'(x) \leq 0 \quad \text{for a.e. } x \in \mathbb{R},$$

then the global solution u scatters in $\mathcal{H}$ as $t \to \pm\infty$. Moreover, there exists $\psi_\pm \in H^1(\mathbb{R})$ such that

$$\lim_{t\to\pm\infty} \|u(t) - e^{it\partial_x^2}\psi_\pm\|_{H^1} = 0. \tag{1.18}$$

(2) *If u_0 satisfies (1.15) and the potential V satisfies the inequality*

$$-xV'(x) - 2V(x) \leq 0 \quad \text{for a.e. } x \in \mathbb{R}, \tag{1.19}$$

then one of the following four cases occurs:
(a) *The solution u blows up in both time directions.*
(b) *The solution u blows up in a positive time, and u grows up in negative time.*
(c) *The solution u blows up in a negative time, and u grows up in positive time.*
(d) *The solution u glows up in both negative and positive time.*

Theorem 1.4. *In (2) in the above theorem, we further assume that $xu_0 \in L^2(\mathbb{R})$. Then we can prove that the solution blows up in a finite time in both time directions, namely we can exclude the possibility that (b)–(d) (grow-up) occurs.*

Remark 1.1. The scattering result to the free solution (1.18) in $L^2(\mathbb{R})$-topology can be proved in the similar manner as the proof of the scattering result (1-30) in [30], since the wave operator $\lim_{t\to\infty} e^{itH_V} e^{it\partial_x^2}$ is well defined and bounded in $L^p(\mathbb{R})$ due to $V \in L^1_1(\mathbb{R})$ (see [10]). Moreover the convergence can be extended to H^1-topology via a density argument and the Sobolev inequality $\|f\|_{H^1} \leq C\|f\|_{H^2}^{\frac{1}{2}}\|f\|_{L^2}^{\frac{1}{2}}$ for $f \in H^2(\mathbb{R})$.

Remark 1.2. In (2) in the above theorem, even if u_0 and V are restricted to radially symmetric functions, it is not known whether we can exclude the possibility that (b)–(d) (grow-up) occurs or not, because we consider one spatial dimension (see [25,26]).

Remark 1.3. We define a function $V : \mathbb{R} \to \mathbb{R}$ as

$$V(x) := \begin{cases} e^{-x}, & x \geq 0, \\ e^{x}, & x < 0. \end{cases}$$

Then V satisfies the all assumptions in (1) in Theorem 1.3.

Remark 1.4. Condition (1.19) and $V \geq 0$ are equivalent to

$$V(x) \begin{cases} \geq \frac{V(\operatorname{sgn} x)}{x^2}, & |x| \geq 1, \\ \leq \frac{V(\operatorname{sgn} x)}{x^2}, & |x| \leq 1. \end{cases} \tag{1.20}$$

Thus from this inequality, we see that

$$\int_1^\infty xV(x)dx \geq V(1)[\log x]_{x=1}^{x=\infty} = \infty,$$

which implies that $V \notin L_1^1(\mathbb{R})$.

In the mass-critical case, i.e. $p = 5$, we can also prove the following blow-up or grow-up result in the same manner as the proof (2) in Theorem 1.3, though a scattering result for large data is not known in the mass-critical case.

Corollary 1.5 *(Unboundedness of solutions in the mass-critical case $p = 5$). Let $p = 5$, V be non-negative and satisfy (A) and (1.19) and $u_0 \in \mathcal{H}$ satisfy $E_V(u_0) < 0$, then the same conclusion as (2) in Theorem 1.3 holds.*

Theorem 1.2 and Theorem 1.3 can be written into another equivalent form. To state the results, we introduce several notations. Let ω be a positive parameter and denote the frequency. We introduce the action $S_{\omega,V} : \mathcal{H} \mapsto \mathbb{R}$ and the Nehari functional $I_{\omega,V} : \mathcal{H} \mapsto \mathbb{R}$ defined by

$$\begin{aligned} S_\omega(\phi) = S_{\omega,V}(\phi) &:= E(\phi) + \frac{\omega}{2}M(\phi) = \frac{1}{2}\|\phi\|^2_{H_V^{\frac{1}{2}}} + \frac{\omega}{2}\|\phi\|^2_{L^2} - \frac{1}{p+1}\|\phi\|^{p+1}_{L^{p+1}} \qquad (1.21)\\ &= \frac{1}{2}\|\partial_x\phi\|^2_{L^2} + \frac{1}{2}\int_{-\infty}^{\infty} V(x)|\phi(x)|^2dx + \frac{\omega}{2}\|\phi\|^2_{L^2} - \frac{1}{p+1}\|\phi\|^{p+1}_{L^{p+1}}, \end{aligned}$$

$$\begin{aligned} I_\omega(\phi) = I_{\omega,V}(\phi) &:= \|\phi\|^2_{H_V^{\frac{1}{2}}} + \omega\|\phi\|^2_{L^2} - \|\phi\|^{p+1}_{L^{p+1}} \qquad (1.22)\\ &= \|\partial_x\phi\|^2_{L^2} + \int_{-\infty}^{\infty} V(x)|\phi(x)|^2dx + \omega\|\phi\|^2_{L^2} - \|\phi\|^{p+1}_{L^{p+1}}. \end{aligned}$$

We often omit the index V, if it does not cause a confusion. We sometimes insert 0 into V, such as $S_{\omega,0}$ and $I_{\omega,0}$, to employ known results for the nonlinear Schrödinger equation without the potential.

We study the following two minimizing problems

$$n_\omega = n_{\omega,V} := \inf\{S_\omega(\phi) : \phi \in \mathcal{H}\setminus\{0\}, I_{\omega,V}(\phi) = 0\}, \quad \text{for } V \neq 0, \qquad (1.23)$$

$$l_\omega := n_{\omega,0} = \inf\{S_{\omega,0}(\phi) : \phi \in H^1(\mathbb{R})\setminus\{0\}, I_{\omega,0}(\phi) = 0\}. \qquad (1.24)$$

l_ω is the minimizing problem for the nonlinear Schrödinger equation without the potential and it is well known that l_ω is positive and is attained by the ground state Q_ω (see [31] for example). On the other hand, we can prove the following properties for n_ω.

Proposition 1.6. *Let $\omega > 0$ and $V \in L^1(\mathbb{R})$ be non-negative and $p > 5$. Then the following statements are valid.*

(1) $n_\omega = l_\omega$.
(2) n_ω *is not attained, if* $\mu(\{x \in \mathbb{R} : V(x) \neq 0\}) > 0$,

where μ denotes the Lebesgue measure.

Next, in order to rewrite Theorem 1.2 and Theorem 1.3 into another form, we introduce the following subsets in $\mathcal{H}$.

$$\mathcal{N}_\omega^+ := \{\varphi \in \mathcal{H} : S_{\omega,V}(\varphi) < n_\omega, I_{\omega,V}(\varphi) \geq 0\},$$
$$\mathcal{N}_\omega^- := \{\varphi \in \mathcal{H} : S_{\omega,V}(\varphi) < n_\omega, I_{\omega,V}(\varphi) < 0\}.$$

These subsets satisfy the relations $\{\varphi \in \mathcal{H} : S_{\omega,V}(\varphi) < n_\omega\} = \mathcal{N}_\omega^+ \cup \mathcal{N}_\omega^-$ and $\mathcal{N}_\omega^+ \cap \mathcal{N}_\omega^- = \emptyset$. The following equivalency hold:

Proposition 1.7. *Under the same assumptions as in Proposition 1.6, let $\varphi \in \mathcal{H}$ and let $Q_1 \in H^1(\mathbb{R})$ be defined by (1.9) with $\omega = 1$ and n_ω be given by (1.23). Then the following two statements are equivalent:*

(1) *φ satisfies the estimate (1.10) with a replacement u_0 into φ.*
(2) *There exists $\omega = \omega(\varphi, p) > 0$ such that $S_{\omega,V}(\varphi) < n_\omega$.*

Moreover, the following equivalency also holds:

- *φ satisfies the estimates (1.10) and (1.11) with a replacement u_0 into φ, if and only if there exists $\omega > 0$ such that $\varphi \in \mathcal{N}_\omega^+$.*
- *φ satisfies the estimates (1.10) and (1.15) with a replacement u_0 into φ, if and only if there exists $\omega > 0$ such that $\varphi \in \mathcal{N}_\omega^-$.*

Especially, in order to prove the scattering result in Theorem 1.3, we use the above equivalency.

1.3. Strategy and difficulties for the proof of the theorems and idea to overcome them

The strategy of the proof of the dichotomy between an upper bound and a lower bound of solutions in Theorem 1.2 is based on combining the conservation laws of solutions to (NLS_V), the Sharp Gagliardo-Nirenberg inequality (2.1), the Pohozaev identity (2.2) and the continuity argument of functions. The similar arguments were used to prove Proposition 3.4 in [20] and Theorem 1.3 in [13] respectively. In order to prove the uniform bounds of the functionals (1.14) and (1.17), besides using them, the upper bound (1.12) and the lower bound (1.16) of the solution respectively and Sobolev's inequality are also employed. Virial Identity (Lemma 3.1) with the uniform bounds (1.14) and (1.17) is used to prove the scattering result in Theorem 1.3, especially rigidity theorem (Proposition 5.15) and the blow-up or glow-up result in Theorem 1.3 respectively. Different from our arguments, in the previous results [14,15], the uniform bounds of the functional P_V ($V = 0$ in [14], $V = \delta_0$ (Dirac's delta) in [15]) are derived (see Lemma 2.12 in [14] and Proposition 2.18 in [15] respectively) and utilized to prove the scattering result and the blow-up or glow-up result for their problems, where

$$P_V(\varphi) := \|\partial_x \varphi\|_{L^2}^2 - \frac{1}{2}\int_{-\infty}^{\infty} xV'(x)|\varphi(x)|^2 dx - \frac{p-1}{2(p+1)}\|\varphi\|_{L^{p+1}}^{p+1}.$$

However it is difficult to derive uniform bounds of the functional P_V in our case because of existence of the term xV' as the second term. To overcome this difficulty, we derive the uniform bounds of the functionals (1.14) and (1.17) and apply them to prove the scattering result and the blow-up or glow-up result in Theorem 1.3 respectively.

In order to prove the blow-up or glow-up result in Theorem 1.3, besides the uniform bound (1.17), we mainly follow the approach by Du–Wu–Zhang [6] with the assumption of the potential V, i.e. $xV'(x) +$

$2V(x) \geq 0$ for a.e. $x \in \mathbb{R}$, who proved the similar unboundedness result for the energy-critical or energy-supercritical Schrödinger equation without the potential by using the localized virial identity (Lemma 3.1) with an appropriate cut-off function similar to x^2 near the origin $x = 0$.

Our proof of the scattering result in Theorem 1.3 is based on the argument by Kenig-Merle [19], (1. dispersive estimates, 2. Strichartz estimates for non-admissible pairs, 3. small data scattering, 4. linear profile decomposition, 5. nonlinear profiles, 6. perturbation lemma, 7. critical element, 8. rigidity theorem). They proved the scattering result below the Talenti function to the energy-critical focusing Schrödinger equation in $H_r^1(\mathbb{R}^d)$ in $d = 3, 4, 5$. Their argument is utilized to prove scattering results for other nonlinear evolution equations including the cases of existence of potentials (see [3,20,13,15,21] for example). In order to apply their argument, we rewrite the conditions on the initial data in Theorem 1.2 into another equivalent form by using the frequency parameter ω (see Proposition 1.7). To do so, we investigate the minimizing problems for both (NLS_V) and (NLS) (see Proposition 1.7). We note that Proposition 1.7 is also utilized to construct a critical element (see Theorem 5.12), more precisely, appropriate nonlinear profiles (see Theorem 5.12), which is also quite different part from the defocusing case [21].

1.4. Construction of the paper

In Section 2, we give a proof of Theorem 1.2. In Section 3, we give a proof of the blow-up or glow-up result in Theorem 1.3. In Section 4, we give a proof of Proposition 1.7 and also prove several variational structures. In Section 5, we give a proof of the scattering result in Theorem 1.2.

2. Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2. The main ingredients for the proof are the Sharp Gagliardo-Nirenberg inequality (2.1), the conservation laws in Proposition 1.1, the Pohozaev identity (2.2), the continuity argument and the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$.

Proof of Theorem 1.2. First we note that the Sharp Gagliardo-Nirenberg inequality (see [31]) gives

$$\|f\|_{L^{p+1}}^{p+1} \leq C_{GN} \|f\|_{L^2}^{\frac{p+3}{2}} \|\partial_x f\|_{L^2}^{\frac{p-1}{2}}, \tag{2.1}$$

for $f \in H^1(\mathbb{R})$. Here $C_{GN} = C_{GN}(p, Q_1)$ is given by

$$C_{GN} := \frac{\|Q_1\|_{L^{p+1}}^{p+1}}{\|Q_1\|_{L^2}^{\frac{p+3}{2}} \|\partial_x Q_1\|_{L^2}^{\frac{p-1}{2}}} = \frac{2(p+1)}{p-1} \frac{1}{(\|Q_1\|_{L^2}^{\sigma} \|\partial_x Q_1\|_{L^2})^{\frac{p-5}{2}}},$$

where we have used the Pohozaev identity

$$\|Q_1\|_{L^{p+1}}^{p+1} = \frac{2(p+1)}{p-1} \|\partial_x Q_1\|_{L^2}^2. \tag{2.2}$$

We also note that since V is non-negative, the estimate

$$\|\partial_x f\|_{L^2} \leq \|H_V^{\frac{1}{2}} f\|_{L^2} \tag{2.3}$$

holds for $f \in \mathcal{H}$. Since $u \in C(I : \mathcal{H})$ is a solution to (NLS_V) on I, we can apply the estimates (2.1) and (2.3) and the L^2-conservation law to obtain

$$\|u(t)\|_{L^{p+1}}^{p+1} \leq \frac{2(p+1)}{p-1} \frac{1}{(\|Q_1\|_{L^2}^{\sigma} \|\partial_x Q_1\|_{L^2})^{\frac{p-5}{2}}} \|u_0\|_{L^2}^{\frac{p+3}{2}} \|H_V^{\frac{1}{2}} u(t)\|_{L^2}^{\frac{p-1}{2}} \tag{2.4}$$

for $t \in I$. Next for $\kappa > 0$, which will be chosen appropriately later, we introduce a function $h : [0, \infty) \mapsto \mathbb{R}$ defined by

$$h(x) := \frac{1}{2}x^2 - \frac{2}{(p-1)\kappa^{\frac{p-5}{2}}} x^{\frac{p-1}{2}}, \quad \text{for } x \in [0, \infty).$$

By differentiating the function h with respect to x, we have

$$h'(x) = x - \frac{1}{\kappa^{\frac{p-5}{2}}} x^{\frac{p-3}{2}} = \frac{x}{\kappa^{\frac{p-5}{2}}}(\kappa^{\frac{p-5}{2}} - x^{\frac{p-5}{2}}), \quad \text{for } x \geq 0.$$

Since $p > 5$, we find that $h' > 0$ on $(0, \kappa)$ and $h' < 0$ on (κ, ∞), which implies that f is increasing on $(0, \kappa)$ and f is decreasing on (κ, ∞). Therefore we have

$$h(x) < h(\kappa) = \max_{x>0} h(x), \quad \text{if } x \neq \kappa.$$

Moreover since u belongs to $C(I : \mathcal{H})$, we can define the following continuous function $g : I \mapsto [0, \infty)$ given by

$$g(t) := \|u_0\|_{L^2}^{\sigma} \|H_V^{\frac{1}{2}} u(t)\|_{L^2} \in C(I).$$

We note that the Pohozaev identity (2.2) again implies

$$E_0(Q_1) = \frac{p-5}{2(p-1)} \|\partial_x Q_1\|_{L^2}^2. \tag{2.5}$$

Here we choose $\kappa > 0$ such as

$$\kappa := \|Q_1\|_{L^2}^{\sigma} \|\partial_x Q_1\|_{L^2}.$$

Then we have

$$h(\kappa) = \frac{p-5}{2(p-1)} \kappa^2 = M(Q_1)^{\sigma} E_0(Q_1).$$

Noting that $2\sigma + \frac{p+3}{2} = \frac{(p-1)\sigma}{2}$, by the assumption (1.8), the energy-conservation law and the estimate (2.4), we have

$$\begin{aligned} h(\kappa) &= M(Q_1)^{\sigma} E_0(Q_1) > M(u_0)^{\sigma} E_V(u_0) = M(u_0)^{\sigma} E_V(u(t)) \\ &\geq \|u_0\|_{L^2}^{2\sigma} \left\{ \frac{1}{2} \|H_V^{\frac{1}{2}} u(t)\|_{L^2}^2 - \frac{2}{(p-1)\kappa^{\frac{p-5}{2}}} \|u_0\|_{L^2}^{\frac{p+3}{2}} \|H_V^{\frac{1}{2}} u(t)\|_{L^2}^{\frac{p-1}{2}} \right\} \\ &= h(g(t)), \quad \text{for } t \in I. \end{aligned}$$

From these facts, we see that for any $t \in I$,

$$\text{either } g(t) < \kappa \text{ or } g(t) > \kappa \tag{2.6}$$

is valid, which implies that $g(t) \neq \kappa$. Noting that the equivalency

$$g(0) = \kappa \Longleftrightarrow \|u_0\|_{L^2}^{\sigma} \|H_V^{\frac{1}{2}} u_0\|_{L^2} = \|Q_1\|_{L^2}^{\sigma} \|\partial_x Q_1\|_{L^2}$$

holds, we find that the right condition is impossible. If u_0 satisfies (1.15), then $g(0) > \kappa$. By the continuity of the function g and the relation (2.6), we have $g(t) > \kappa$ for $t \in I$, which implies that (1.16) holds for $t \in I$. On the other hand, if u_0 satisfies (1.11) holds, then $g(0) < \kappa$. By the continuity of the function g again, we have $g(t) < \kappa$ for any $t \in I$. Thus since V is non-negative, if $u_0 \neq 0$, then we have

$$\|\partial_x u(t)\|_{L^2} \leq \|H_V^{\frac{1}{2}} u(t)\|_{L^2} < \left(\frac{\|Q_1\|_{L^2}}{\|u_0\|_{L^2}}\right)^{\sigma} \|\partial_x Q_1\|_{L^2}, \quad \text{for } t \in I,$$

which implies that $T_\pm = \infty$ by the blow-up criterion in Proposition 1.1, and u satisfies the estimate (1.12). In the case of $u_0 = 0$, it is easy to see that $T_\pm = \infty$ and (1.12) holds. Next we prove the inequalities (1.13). The left inequality is easier to prove. Since we are considering the focusing case, the energy conservation law gives

$$E_V(u_0) = E_V(u(t)) = \frac{1}{2}\|H_V^{\frac{1}{2}} u(t)\|_{L^2}^2 - \frac{1}{p+1}\|u(t)\|_{L^{p+1}}^{p+1} \leq \frac{1}{2}\|H_V^{\frac{1}{2}} u(t)\|_{L^2}^2,$$

which implies that $2E(u_0) \leq \|H_V^{\frac{1}{2}} u(t)\|_{L^2}^2$ for any $t \in \mathbb{R}$. On the other hand, by the estimates (2.4) and (1.12), we have

$$\|u(t)\|_{L^{p+1}}^{p+1} \leq \frac{2(p+1)}{p-1} \left\{ \frac{\|u_0\|_{L^2}^{\sigma} \|H_V^{\frac{1}{2}} u(t)\|_{L^2}}{\|Q_1\|_{L^2}^{\sigma} \|\partial_x Q_1\|_{L^2}} \right\}^{\frac{p-5}{2}} \|H_V^{\frac{1}{2}} u(t)\|_{L^2}^2 \leq \frac{2(p+1)}{p-1} \|H_V^{\frac{1}{2}} u(t)\|_{L^2}^2 \tag{2.7}$$

for any $t \in \mathbb{R}$. By the energy conservation law and this estimate, we obtain

$$E_V(u_0) = E_V(u(t)) \geq \frac{1}{2}\|H_V^{\frac{1}{2}} u(t)\|_{L^2}^2 - \frac{2}{p-1}\|H_V^{\frac{1}{2}} u(t)\|_{L^2}^2 = \frac{p-5}{2(p-1)}\|H_V^{\frac{1}{2}} u(t)\|_{L^2}^2 \tag{2.8}$$

for any $t \in \mathbb{R}$. By this estimate and $p > 5$, we have the right inequality of (1.13).

Next we prove (1.14). By the sharp Gagliardo-Nirenberg inequality (2.1), the mass conservation law, the Pohozaev identity (2.5) and the right estimate of (1.13), we have

$$\|u(t)\|_{L^{p+1}}^{p+1} \leq \frac{2(p+1)}{p-1} \left\{ \frac{M(u_0)^{\sigma} E(u_0)}{M(Q_1)^{\sigma} E_0(Q_1)} \right\}^{\frac{p-5}{4}} \|\partial_x u(t)\|_{L^2}^2, \tag{2.9}$$

for any $t \in \mathbb{R}$. We note that since $u_0 \neq 0$, by the estimate (1.13), we have $E_V(u_0) > 0$. In the case of $V \in L^1(\mathbb{R})$, we can prove that there exists $\delta > 0$ depending only on $\|V\|_{L^1}, \|u_0\|_{L^1}$ and $E_V(u_0)$ such that the estimate $\|\partial_x u(t)\|_{L^2} \geq \delta$ holds for any $t \in \mathbb{R}$. Indeed, since $V \in L^1(\mathbb{R})$, by the estimate (1.13), the Sobolev inequality $\|f\|_{L^\infty}^2 \leq \|f\|_{L^2}\|\partial_x f\|_{L^2}$ due to $f \in H^1(\mathbb{R})$, and the mass conservation law, we have

$$\begin{aligned} 2E_V(u_0) &\leq \|\partial_x u(t)\|_{L^2}^2 + \int_{-\infty}^{\infty} V(x)dx \|u(t)\|_{L^\infty}^2 \leq \|\partial_x u(t)\|_{L^2}^2 + \|V\|_{L^1}\|u(t)\|_{L^2}\|\partial_x u(t)\|_{L^2} \\ &= \|\partial_x u(t)\|_{L^2}^2 + \|V\|_{L^1}\|u_0\|_{L^2}\|\partial_x u(t)\|_{L^2}, \end{aligned}$$

which implies that

$$\|\partial_x u(t)\|_{L^2} \geq \frac{-\|V\|_{L^1}\|u_0\|_{L^2} + \sqrt{(\|V\|_{L^1}\|u_0\|_{L^2})^2 + 8E_V(u_0)}}{2} =: \delta > 0, \tag{2.10}$$

for any $t \in \mathbb{R}$. By the estimates (2.9) and (2.10), we have

$$P_0(u(t)) \geq \left[1 - \left\{\frac{M(u_0)^\sigma E_V(u_0)}{M(Q_1)^\sigma E_0(Q_1)}\right\}^{\frac{p-5}{4}}\right] \|\partial_x u(t)\|_{L^2}^2 \geq \left[1 - \left\{\frac{M(u_0)^\sigma E(u_0)}{M(Q_1)^\sigma E_0(Q_1)}\right\}^{\frac{p-5}{4}}\right] \delta =: \delta_0,$$

for any $t \in \mathbb{R}$.

Finally we prove (1.17). By the definitions of the functionals P_0 and E_V, and the energy conservation law, the identities

$$\begin{aligned} &4\left\{P_0(u(t)) + \int_{-\infty}^{\infty} V(x)|u(t,x)|^2 dx\right\} \\ &= 2(p-1)E_V(u_0) - (p-5)\|\partial_x u(t)\|_{L^2}^2 - (p-5)\int_{-\infty}^{\infty} V(x)|u(t,x)|^2 dx \\ &= 2(p-1)E_V(u_0) - (p-5)\|H_V^{\frac{1}{2}} u(t)\|_{L^2}^2 \end{aligned} \tag{2.11}$$

hold for any $t \in I$. Here by the assumption (1.10) and $u_0 \neq 0$, we can choose $\varepsilon_1 = \varepsilon(u_0, Q_1, V, p) > 0$ such as

$$\varepsilon_1 := \frac{1}{2}\left\{\left(\frac{M(Q_1)}{M(u_0)}\right)^\sigma E_0(Q_1) - E_V(u_0)\right\}.$$

Then the estimate

$$E_V(u_0) < \left(\frac{M(Q_1)}{M(u_0)}\right)^\sigma E_0(Q_1) - \varepsilon_1 \tag{2.12}$$

holds. Moreover by the estimate (1.16) and $u_0 \neq 0$, the inequality

$$\|H_V^{\frac{1}{2}} u(t)\|_{L^2}^2 > \left(\frac{M(Q_1)}{M(u_0)}\right)^\sigma \|\partial_x Q_1\|_{L^2}^2 \tag{2.13}$$

holds for any $t \in I$. Here we choose $\delta_1 = \delta(u_0, Q_1, V, p) > 0$ such as

$$\delta_1 := \frac{p-1}{2}\varepsilon_1.$$

Then by combining the estimates (2.11), (2.12) and (2.13) and the Pohozaev identity (2.2), we have

$$\begin{aligned} (\text{the left hand side of (2.11)}) &< \left(\frac{M(Q_1)}{M(u_0)}\right)^\sigma \{2(p-1)E_0(Q_1) - (p-5)\|\partial_x Q_1\|_{L^2}^2\} \\ &\quad - 2(p-1)\varepsilon_1 \\ &= -2(p-1)\varepsilon_1 = -\delta_1, \end{aligned}$$

for any $t \in I$, which completes the proof of the proposition. □

3. Proof of the blow-up or glow-up result

In this section, we give a proof of the blow-up or glow-up result of Theorem 1.3 by mainly following the approach by Du-Wu-Zhang [6]. We note that one of the assumptions of the potential V, i.e. $xV'(x)+2V(x) \geq 0$ is used to deal with $\mathcal{R}_4$ in Lemma 3.3 and the uniform bound (1.17) of the functional is used to derive

(3.9). The approach is based on the following localized virial identity, which is also applied to prove the rigidity theorem (see Proposition 5.15) in the proof of the scattering result.

Lemma 3.1 *(Localized virial identity). Let $p \geq 1$, $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ be real-valued and satisfy $V' \in L^1(\mathbb{R})+L^\infty(\mathbb{R})$. Let $u_0 \in \mathcal{H}$ and $u \in C(I:\mathcal{H})\cap C^1(I:\mathcal{H}^{-1})$ be a solution to (NLS$_V$) on I. For $\phi \in W^{4,\infty}(\mathbb{R})$, set*

$$\mathcal{I}(t) = \mathcal{I}_\phi[u](t) := \int_{\mathbb{R}} \phi(x)|u(t,x)|^2 dx, \quad \text{for } t \in I.$$

Then $\mathcal{I} \in C^2(I)$ and the identities

$$\mathcal{I}'(t) = 2\mathrm{Im} \int_{\mathbb{R}} \phi'(x)\overline{u(t,x)}\partial_x u(t,x)dx, \tag{3.1}$$

$$\begin{aligned} \mathcal{I}''(t) = 4 \int_{\mathbb{R}} \phi''(x)|\partial_x u(t,x)|^2 dx - 2\int_{\mathbb{R}} \phi'(x)V'(x)|u(t,x)|^2 dx \\ - \frac{2(p-1)}{p+1}\int_{\mathbb{R}} \phi''(x)|u(t,x)|^{p+1}dx - \int_{\mathbb{R}} \phi^{(4)}(x)|u(t,x)|^2 dx \end{aligned} \tag{3.2}$$

hold for $t \in I$.

For the proof of this lemma, see [4] for example. For convenience of the readers, we give a formal proof of the lemma in Appendix.

In the following, we only consider the positive time direction, since the negative time direction can be treated in the same manner.

Lemma 3.2. *Let $p \geq 1$, $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ be non-negative, $u_0 \in \mathcal{H}\backslash\{0\}$, and u be the solution to (NLS$_V$) on $[0, T_+)$. We assume that $T_+ = \infty$ and*

$$C_0 := \sup_{t\in[0,\infty)} \|\partial_x u(t)\|_{L^2} < \infty.$$

Then there exists a constant $C_1 > 0$ such that for $\eta > 0$, $R > 0$ and $t \in \left[0, \frac{\eta R}{\|u_0\|_{L^2} C_0 C_1}\right]$, the estimate

$$\int_{|x|>R} |u(t,x)|^2 dx \leq o_R(1) + \eta \tag{3.3}$$

holds, where $o_R(1)$ denotes a function of R satisfying $o_R(1) \to 0$ as $R \to \infty$.

Proof. Let $R > 0$. We can construct a function $\phi^1 = \phi^1_R \in C^\infty(\mathbb{R})$ satisfying $0 \leq \phi^1(x) \leq 1$ for any $x \in \mathbb{R}$ and

$$\phi^1(x) = \begin{cases} 0, & 0 < |x| < \frac{R}{2}, \\ 1, & |x| \geq R, \end{cases} \qquad \left|\frac{d\phi^1}{dx}(x)\right| \leq \frac{C_1}{R},$$

where C_1 is a constant independent of x and R. Since $\phi^1 \in W^{4,\infty}(\mathbb{R})$, Lemma 3.2 implies that $\mathcal{I}(t)$ belongs to $C^1([0,\infty))$. By the fundamental formula, the identity (3.1), the Hölder inequality and the mass conservation law, we have

$$\begin{aligned}\mathcal{I}(t)=\mathcal{I}_{\phi^1}[u](t)=\mathcal{I}(0)+\int_0^t \mathcal{I}'(s)ds &\le \mathcal{I}(0)+\int_0^t |\mathcal{I}'(s)|ds\\ &\le \mathcal{I}(0)+t\,\|\phi'\|_{L^\infty}\sup_{t\in\mathbb{R}_+}\|u(t)\|_{L^2}\|\partial_x u(t)\|_{L^2}\\ &\le \mathcal{I}(0)+\frac{\|u_0\|_{L^2}C_0C_1t}{R},\quad t\in\mathbb{R}_+,\end{aligned}$$

where $\mathbb{R}_+ := [0,\infty)$. Since $u(0)=u_0\in\mathcal{H}\subset L^2(\mathbb{R})$, by a property of the function ϕ_1, we have

$$\mathcal{I}(0)=\int_{\mathbb{R}}\phi^1(x)|u_0(x)|^2dx=\int_{|x|>\frac{R}{2}}|u_0(x)|^2dx=o_R(1)\ \text{ as } R\to\infty.$$

Moreover we note that by the properties of the function ϕ^1, the estimate $\int_{|x|>R}|u(t,x)|^2dx\le\mathcal{I}(t)$ holds for any $t\in\mathbb{R}_+$. Therefore, by combining the above inequalities, we obtain (3.3), which completes the proof of the lemma. □

Next we introduce another function $\phi^2=\phi^2_R\in C_0^\infty(\mathbb{R})$ such that

$$0\le\phi^2(x)\le x^2,\quad \left|\frac{d\phi^2}{dx}(x)\right|\le C_2|x|,\quad \left|\frac{d^2\phi^2}{dx^2}(x)\right|\le 2,\quad \left|\frac{d^4\phi^2}{dx^4}(x)\right|\le\frac{4}{R^2},$$

and

$$\phi^2(x)=\begin{cases}x^2, & 0\le|x|\le R,\\ 0, & |x|\ge 2R.\end{cases}$$

Then we have the following lemma.

Lemma 3.3. *Besides the assumptions in Lemma 3.2, we assume that the potential V is differentiable in the distribution sense and satisfies $xV'\in L^1(\mathbb{R})+L^\infty(\mathbb{R})$ and the estimate*

$$xV'(x)+2V(x)\ge 0\quad \text{for a.e. } x\in\mathbb{R}.$$

Then for $q>p+1$, there exist constants $C_4=C_4(p,q,\|u_0\|_{L^2},C_0)>0$ and $\theta_q=\theta_q(p)>0$ such that for any $R>0$ and $t\in\mathbb{R}_+$, the estimate

$$\begin{aligned}\mathcal{I}''_{\phi^2}[u](t)\le&8\left\{P_0(u(t))+\int_{-\infty}^{\infty}V(x)|u(t,x)|^2dx\right\}+C_4\|u(t)\|^{(p+1)\theta_q}_{L^2(|x|>R)}+4R^{-2}\|u(t)\|^2_{L^2(|x|>R)}\\ &+2C_2\|xV'\|_{L^1+L^\infty}(\|u_0\|_{L^2}+C_0)\|u(t)\|_{L^2(|x|>R)}\end{aligned}$$

holds, where $\theta_q:=\frac{2\{q-(p+1)\}}{(p+1)(q-2)}\in\left(0,\frac{2}{p+1}\right]$.

Proof. Since u is the solution to (NLS_V) on $\mathbb{R}_+$, the localized virial identity (3.2) with $\phi=\phi^2\in W^{4,\infty}(\mathbb{R})$ can be applied to obtain

$$\mathcal{I}''(t)=\mathcal{I}''_{\phi^2}[u](t)=8\left\{P_0(u(t))+\int_{-\infty}^{\infty}V(x)|u(t,x)|^2dx\right\}\tag{3.4}$$

$$+\mathcal{R}_1+\mathcal{R}_2+\mathcal{R}_3+\mathcal{R}_4, \quad \text{for } t\in\mathbb{R}_+,$$

where $\mathcal{R}_k=\mathcal{R}_k(t)$ $(k=1,2,3,4)$ are defined by

$$\mathcal{R}_1 := 4\int_{\mathbb{R}}\left\{\frac{d^2\phi^2}{dx^2}(x)-2\right\}|\partial_x u(t,x)|^2dx,$$

$$\mathcal{R}_2 := -\frac{2(p-1)}{p+1}\int_{\mathbb{R}}\left\{\frac{d^2\phi^2}{dx^2}(x)-2\right\}|u(t,x)|^{p+1}dx,$$

$$\mathcal{R}_3 := -\int_{\mathbb{R}}\phi^{(4)}(x)|u(t,x)|^2dx,$$

$$\mathcal{R}_4 := -2\int_{\mathbb{R}}\frac{d\phi^2}{dx}(x)V'(x)|u(t,x)|^2dx-8\int_{-\infty}^{\infty}V(x)|u(t,x)|^2dx.$$

Due to $\left|\frac{d^2\phi^2}{dx^2}(x)\right|\le 2$, the estimate $\mathcal{R}_1\le 0$ holds for $t\in\mathbb{R}_+$. Next we estimate $\mathcal{R}_2$. We note that since $q\ge 2$, the Gagliardo-Nirenberg-Sobolev inequality can be applied to get

$$\|f\|_{L^q}\le C\|f\|_{L^2}^{\frac{q+2}{2q}}\|\partial_x f\|_{L^2}^{\frac{q-2}{2q}} \tag{3.5}$$

for any $f\in H^1(\mathbb{R})$, where C depends only on q. Thus due to $p\ge 1$, for any $q\in[p+1,\infty]$, by the mass conservation law $\|u(t)\|_{L^2}=\|u_0\|_{L^2}$ and the estimate (3.5), we have

$$\sup_{t\in\mathbb{R}_+}\|u(t)\|_{L^q}\le C\sup_{t\in\mathbb{R}_+}\|u(t)\|_{L^2}^{\frac{q+2}{2q}}\|\partial_x u(t)\|_{L^2}^{\frac{q-2}{2q}}\le C\|u_0\|_{L^2}^{\frac{q+2}{2q}}C_0^{\frac{q-2}{2q}}=:C_3, \tag{3.6}$$

where C_3 depends only on q, $\|u_0\|_{L^2}$ and C_0. We also note that due to $q>p+1$, by the Hölder inequality, the estimate

$$\|f\|_{L^{p+1}(|x|>R)}\le\|f\|_{L^2(|x|>R)}^{\theta_q}\|f\|_{L^q(|x|>R)}^{1-\theta_q}$$

is valid for any $f\in L^2(|x|>R)\cap L^q(|x|>R)$, where $\theta_q:=\frac{2\{q-(p+1)\}}{(q-2)(p+1)}\in(0,1]$. By these estimates, we have

$$\begin{aligned}\mathcal{R}_2&=-\frac{2(p-1)}{p+1}\int_{\mathbb{R}}\left\{\frac{d^2\phi^2}{dx^2}(x)-2\right\}|u(t,x)|^{p+1}dx\\
&\le\frac{8(p-1)}{p+1}\int_{|x|>R}|u(t,x)|^{p+1}dx\\
&\le\frac{8(p-1)}{p+1}\|u(t)\|_{L^q(|x|>R)}^{(p+1)(1-\theta_q)}\|u(t)\|_{L^2(|x|>R)}^{(p+1)\theta_q}\\
&\le\frac{8(p-1)C_3^{(p+1)(1-\theta_q)}}{p+1}\|u(t)\|_{L^2(|x|>R)}^{(1+p)\theta_q}, \quad \text{for } t\in\mathbb{R}_+.\end{aligned}$$

Moreover, we estimate $\mathcal{R}_3$. By the properties of ϕ^2, we have

$$\mathcal{R}_3=-\int_{\mathbb{R}}\frac{d^4\phi^2}{dx^4}(x)|u(t,x)|^2dx\le 4R^{-2}\int_{|x|>R}|u(t,x)|^2dx=4R^{-2}\|u(t)\|_{L^2(|x|>R)}^2.$$

Finally we estimate R_4. By the Gagliardo-Nirenberg-Sobolev inequality, the inequality

$$\|f\|^2_{L^\infty(|x|>R)} \leq \|f\|_{L^2(|x|>R)}\|\partial_x f\|_{L^2(|x|>R)}, \tag{3.7}$$

holds, for any $f \in H^1(|x|>R)$. Since the potential V is non-negative and satisfies

$$xV'(x) + 2V(x) \geq 0, \quad \text{for a.e. } x \in \mathbb{R},$$

by the properties of the function ϕ^2, the estimate (3.7), we have

$$\begin{aligned}
\mathcal{R}_4 &= -4 \int_{|x|<R} \{xV'(x) + 2V(x)\}|u(t,x)|^2 dx \\
&\quad - 8 \int_{|x|>R} V(x)|u(t,x)|^2 dx - 2 \int_{|x|>R} \frac{d\phi^2}{dx}(x)V'(x)|u(t,x)|^2 dx \\
&\leq 2C_2 \int_{|x|>R} |xV'(x)||u(t,x)|^2 dx \\
&\leq 2C_2 \|xV'\|_{L^1(|x|>R)+L^\infty(|x|>R)} \|u(t)\|_{L^2(|x|>R)} \left(\|u(t)\|_{L^2(|x|>R)} + \|\partial_x u(t)\|_{L^2(|x|>R)}\right) \\
&\leq 2C_2 \|xV'\|_{L^1+L^\infty} (\|u_0\|_{L^2} + C_0)\|u(t)\|_{L^2(|x|>R)},
\end{aligned}$$

which completes the proof of the lemma. □

We prove the blow-up or grow-up result (2) in Theorem 1.3 by using Lemmas 3.1, 3.2, 3.3.

Proof of (2) in Theorem 1.3. Let $u_0 \in \mathcal{H}$ and $u \in C(I : \mathcal{H}) \cap C^1(I : \mathcal{H}^{-1})$ be the unique solution to (NLS_V) on $I := (-T_-, T_+)$ obtained in Proposition 1.1. In the following, we only treat the positive time direction, since the negative time direction can be treated in the same manner. On the contrary, we assume that $T_+ = \infty$ and $\sup_{t\in\mathbb{R}_+} \|\partial_x u(t)\|_{L^2} < \infty$, where $\mathbb{R}_+ := [0,\infty)$. Then we can define $C_0 \in (0,\infty)$ such as

$$C_0 := \sup_{t\in\mathbb{R}_+} \|\partial_x u(t)\|_{L^2}.$$

Since u_0 satisfies the assumptions (2) in Theorem 1.2, there exists $\delta_1 > 0$ independent of t such that

$$P_0(u(t)) + \int_{-\infty}^{\infty} V(x)|u(t,x)|^2 dx < -\delta_1, \quad \text{for all } t \in \mathbb{R}_+. \tag{3.8}$$

By Lemma 3.3 with the estimate (3.8), we have

$$\begin{aligned}
\mathcal{I}''(t) := \mathcal{I}''_{\phi^2}[u](t) \leq & -8\delta_1 + C_4 \|u\|^{\theta_q}_{L^2(|x|>R)} + 4R^{-2}\|u\|^2_{L^2(|x|>R)} \\
& + 2C_2\|xV'\|_{L^1+L^\infty}(\|u_0\|_{L^2} + C_0)\|u(t)\|_{L^2(|x|>R)}, \quad \text{for any } t \in \mathbb{R}_+,
\end{aligned} \tag{3.9}$$

where C_2, $C_4 > 0$ and θ_q are given in Lemma 3.3. Here since $\theta_q > 0$, we can take $\eta_0 = \eta_0(\delta_1) > 0$ sufficiently small such as

$$C_4^{\frac{1}{2}} \eta_0^{\frac{\theta_q}{2}} + 2\eta_0 + \{2C_2\|xV'\|_{L^1+L^\infty}(\|u_0\|_{L^2} + C_0)\}^{\frac{1}{2}} \eta_0^{\frac{1}{2}} \leq \delta_1. \tag{3.10}$$

By Lemma 3.2 with $\eta = \eta_0$ and $R \geq 1$ and by the estimates (3.9)-(3.10) on $\mathbb{R}_+$, we have

$$\mathcal{I}''(t) \leq -7\delta_1 + o_R(1), \quad \text{for } t \in [0, T], \tag{3.11}$$

where $T = T(R)$ is defined by

$$T = T(R) := \frac{\eta_0 R}{\|u_0\|_{L^2} C_0 C_1},$$

due to $u_0 \neq 0$ and $C_1 > 0$ is given in Lemma 3.2. By integrating the inequality (3.11) twice with respect to time over $[0, T]$, we get

$$\mathcal{I}(T) \leq \mathcal{I}(0) + \mathcal{I}'(0)T + \frac{1}{2}(-7\delta_1 + o_R(1))T^2, \quad \text{for } R \geq 1.$$

Here we take sufficiently large $R = R(\delta_1) > 1$ satisfying

$$-7\delta_1 + o_R(1) < -6\delta_1.$$

Then we get

$$\mathcal{I}(T) \leq \mathcal{I}(0) + \frac{\eta_0}{\|u_0\|_{L^2} C_0 C_1} \mathcal{I}'(0) R - \alpha_0 R^2, \quad R \gg 1, \tag{3.12}$$

where α_0 is defined by

$$\alpha_0 := \frac{3\delta_1 \eta_0^2}{\|u_0\|_{L^2}^2 C_0^2 C_1^2} > 0,$$

and is especially independent of R. We can prove that $\mathcal{I}(0) = o_R(1)R^2$ and $\mathcal{I}'(0) = o_R(1)R$ as $R \to \infty$. Indeed, since $u_0 \in L^2(\mathbb{R})$, by the properties of ϕ^2, we have

$$\begin{aligned}
\mathcal{I}(0) &= \int\limits_{|x|<2R} \phi^2(x)|u_0(x)|^2 dx \leq \int\limits_{|x|<\sqrt{R}} x^2 |u_0(x)|^2 dx + \int\limits_{\sqrt{R}<|x|<2R} x^2 |u_0(x)|^2 dx \\
&\leq R\|u_0\|_{L^2}^2 + 4R^2 \int\limits_{\sqrt{R}<|x|} |u_0(x)|^2 dx = o_R(1)R^2, \quad \text{as } R \to \infty.
\end{aligned}$$

And since $u_0 \in \mathcal{H} \subset H^1(\mathbb{R})$, by the identity (3.1), the properties of ϕ^2 again and the Schwarz inequality, we have

$$\begin{aligned}
|\mathcal{I}'(0)| &\leq \int\limits_{|x|<\sqrt{R}} \left| \frac{d\phi^2}{dx}(x) \right| |u_0(x)||\partial_x u_0(x)| dx + \int\limits_{\sqrt{R}<|x|<2R} \left| \frac{d\phi^2}{dx}(x) \right| |u_0(x)||\partial_x u_0(x)| dx \\
&\leq C_3 \int\limits_{|x|<\sqrt{R}} |x||u_0(x)||\partial_x u_0(x)| dx + C_3 \int\limits_{\sqrt{R}<|x|<2R} |x||u_0(x)||\partial_x u_0(x)| dx \\
&\leq C_3 \|u_0\|_{L^2} \|\partial_x u_0\|_{L^2} \sqrt{R} + 2C_3 R \int\limits_{\sqrt{R}<|x|} |u_0(x)||\partial_x u_0(x)| dx \\
&= o_R(1)R.
\end{aligned}$$

Therefore by (3.12) and these estimates, we see that

$$\mathcal{I}(T) \leq o_R(1)R^2 - \alpha_0 R^2 = (o_R(1) - \alpha_0)R^2, \quad \text{as } R \to \infty.$$

We take $R \gg 1$ sufficiently large such that $o_R(1) - \alpha_0 < 0$. However, this contradicts $\mathcal{I}(T) = \int_{\mathbb{R}} \phi^2(x)|u(T,x)|^2 dx \geq 0$, which completes the proof of the blow-up or glow-up result in Theorem 1.3. □

4. Minimizing problems and variational structure

4.1. Minimizing problems

Let $\alpha, \beta \in \mathbb{R}$. For any function ϕ and $\lambda \in \mathbb{R}$, we define a scaling transformation

$$\phi_\lambda^{\alpha,\beta}(x) := e^{\alpha\lambda}\phi(e^{-\beta\lambda}x)$$

For any functional $S : H^1(\mathbb{R}) \to \mathbb{R}$ and $\lambda_0 \in \mathbb{R}$, the operator $\mathcal{L}_{\lambda_0}^{\alpha,\beta}$ is defined as

$$\mathcal{L}_{\lambda_0}^{\alpha,\beta} S(\phi) := \frac{d}{d\lambda} S(\phi_\lambda^{\alpha,\beta})|_{\lambda=\lambda_0}, \tag{4.1}$$

$$\mathcal{L}^{\alpha,\beta} S(\phi) := \mathcal{L}_0^{\alpha,\beta} S(\phi). \tag{4.2}$$

Let $\omega > 0$. We introduce the functionals $K_\omega^{\alpha,\beta} : H^1(\mathbb{R}) \to \mathbb{R}$ as follows:

$$\begin{aligned} K_\omega^{\alpha,\beta}(\phi) = K_{\omega,V}^{\alpha,\beta}(\phi) &:= \mathcal{L}^{\alpha,\beta} S_\omega(\phi) = \partial_\lambda S_\omega(e^{\alpha\lambda}\phi(e^{-\beta\lambda}\cdot))|_{\lambda=0} \\ &= \frac{2\alpha - \beta}{2} \|\partial_x \phi\|_{L^2}^2 + \frac{2\alpha + \beta}{2} \left\{ \omega \|\phi\|_{L^2}^2 + \int_{-\infty}^{\infty} V(x)|\phi(x)|^2 dx \right\} \\ &\quad + \frac{\beta}{2} \int_{-\infty}^{\infty} xV'(x)|\phi(x)|^2 dx - \frac{(p+1)\alpha + \beta}{p+1} \|\phi\|_{L^{p+1}}^{p+1}. \end{aligned} \tag{4.3}$$

We note that the third term and the fourth term in the right hand side of (4.3) are well defined if $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ and $xV' \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ respectively.

In the following, we always assume that (α, β) satisfies the following conditions:

$$\alpha > 0, \ \beta \leq 0, \ 2\alpha + \beta \geq 0. \tag{4.4}$$

We especially use the following two functionals $H^1(\mathbb{R}) \to \mathbb{R}$:

$$P(\phi) = P_V(\phi) := K_\omega^{1/2,-1}(\phi) = \|\partial_x \phi\|_{L^2}^2 - \frac{1}{2} \int_{-\infty}^{\infty} xV'(x)|\phi(x)|^2 dx - \frac{p-1}{2(p+1)} \|\phi\|_{L^{p+1}}^{p+1}, \tag{4.5}$$

$$I_\omega(\phi) = I_{\omega,V}(\phi) := K_\omega^{1,0}(\phi) = \|\partial_x \phi\|_{L^2}^2 + \int_{-\infty}^{\infty} V(x)|\phi(x)|^2 dx + \omega \|\phi\|_{L^2}^2 - \|\phi\|_{L^{p+1}}^{p+1}. \tag{4.6}$$

P is well defined if $xV' \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$, is related to the virial identity (Lemma 3.1), and is used to prove the blow-up result and the extinction of a so-called critical element (see Proposition 5.15) in the scattering result. I_ω is well defined if $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$, is called Nehari functional, and is used in the variational argument and the linear profile decomposition (Proposition 5.7).

For $\omega > 0$, $\alpha \in \mathbb{R}$, we also introduce the functional $J_\omega : L^{p+1}(\mathbb{R}) \mapsto [0, \infty)$ defined by

$$J_\omega(\phi) = J^{\alpha,0}_{\omega,V}(\phi) := S_\omega(\phi) - \frac{1}{2\alpha} K^{\alpha,0}_\omega(\phi) = \frac{p-1}{2(p+1)} \|\phi\|^{p+1}_{L^{p+1}}. \tag{4.7}$$

Remark 4.1. The reason why we restrict $\beta = 0$ in the definition of J_ω is that $J^{\alpha,\beta}_{\omega,V}(\phi)$ might be negative if $\beta < 0$ and $\|\phi\|_{H^1}$ is small, which is different from the case without a potential or with a Dirac's delta potential (see Section 2 in [15]).

Next, we see that $K^{\alpha,\beta}_\omega$ is positive near the origin in $H^1(\mathbb{R})$ under the assumption (4.4).

Lemma 4.1. *Let $\omega > 0$, $p > 5$ and (α, β) satisfy (4.4). Let $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ satisfy $V \geq 0$. If $\beta < 0$, then we further assume that $xV' \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ and $xV'(x) \leq 0$ for $x \in \mathbb{R}$. We assume that $\{\phi_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R}) \setminus \{0\}$ be bounded in $L^2(\mathbb{R})$ such that $\|\partial_x \phi_n\|_{L^2} \to 0$ as $n \to \infty$. Then $K^{\alpha,\beta}_\omega(\phi_n) > 0$ for sufficiently large n.*

This lemma is proved in the similar manner as the proof of Lemma 2.1 in [14] and Lemma 2.2 in [15].

Proof. Noting that $p \geq 1$, the Gagliardo-Nirenberg inequality gives

$$\|f\|^{p+1}_{L^{p+1}} \leq C_0 \|\partial_x f\|^{\frac{p-1}{2}}_{L^2} \|f\|^{\frac{p+3}{2}}_{L^2},$$

for $f \in H^1(\mathbb{R})$, where C_0 is dependent only on p. By this inequality and the assumptions of ω, p, (α, β) and V, we have

$$\begin{aligned} K^{\alpha,\beta}_\omega(\phi) &\geq \frac{2\alpha - \beta}{2} \|\partial_x \phi\|^2_{L^2} - \frac{(p+1)\alpha + \beta}{p+1} \|\phi\|^{p+1}_{L^{p+1}} \\ &\geq \frac{2\alpha - \beta}{2} \|\partial_x \phi\|^2_{L^2} - \frac{C_0\{(p+1)\alpha + \beta\}}{p+1} \|\partial_x \phi\|^{\frac{p-1}{2}}_{L^2} \|\phi\|^{\frac{p+3}{2}}_{L^2}, \end{aligned} \tag{4.8}$$

for any $\phi \in H^1(\mathbb{R})$. Since $\{\phi_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R})$, there exists a constant C_1 such that

$$C_1 = \sup_{n\in\mathbb{N}} \|\phi_n\|_{L^2} < \infty. \tag{4.9}$$

Since $\|\partial_x \phi_n\|_{L^2} \to 0$ as $n \to \infty$, there exists $N = N(\alpha, \beta, p) \in \mathbb{N}$ such that for $n \geq N$, the estimate

$$\|\partial_x \phi_n\|_{L^2} \leq \left\{ \frac{2\alpha - \beta}{4} \cdot \frac{p+1}{C_0 C_1^{\frac{p+3}{2}} \{(p+1)\alpha + \beta\}} \right\}^{\frac{2}{p-5}} \tag{4.10}$$

holds. Noting that $\partial_x \phi_n \neq 0$ for any $n \in \mathbb{N}$ and $p > 5$, for $n \geq N$, by combining the estimates (4.8)-(4.10), we have

$$\begin{aligned} K^{\alpha,\beta}_\omega(\phi_n) &\geq \frac{2\alpha - \beta}{2} \|\partial_x \phi_n\|^2_{L^2} - \frac{C_0\{(p+1)\alpha + \beta\}}{p+1} \|\partial_x \phi_n\|^{\frac{p-1}{2}}_{L^2} \|\phi_n\|^{\frac{p+3}{2}}_{L^2} \\ &= \|\partial_x \phi_n\|^2_{L^2} \left(\frac{2\alpha - \beta}{2} - \frac{C_0\{(p+1)\alpha + \beta\}}{p+1} \|\partial_x \phi_n\|^{\frac{p-5}{2}}_{L^2} \|\phi_n\|^{\frac{p+3}{2}}_{L^2} \right) \end{aligned}$$

$$\geq \|\partial_x \phi_n\|_{L^2}^2 \left(\frac{2\alpha - \beta}{2} - \frac{C_0 C_1^{\frac{p+3}{2}} (p+1)\alpha + \beta}{p+1} \|\partial_x \phi_n\|_{L^2}^{\frac{p-5}{2}} \right)$$
$$\geq \frac{2\alpha - \beta}{4} \|\partial_x \phi_n\|_{L^2}^2 > 0,$$

which completes the proof of the lemma. □

For $\omega > 0$ and $\alpha > 0$, we study the following minimizing problems:

$$n_\omega^\alpha = n_{\omega,V}^\alpha := \inf\{S_\omega(\phi) : \phi \in \mathcal{H} \setminus \{0\}, K_\omega^{\alpha,0}(\phi) = 0\}, \tag{4.11}$$
$$l_\omega^\alpha = n_{\omega,0}^\alpha := \inf\{S_{\omega,0}(\phi) : \phi \in H^1(\mathbb{R}) \setminus \{0\}, K_{\omega,0}^{\alpha,0}(\phi) = 0\}. \tag{4.12}$$

If $\alpha = 1$, then these are nothing but n_ω and l_ω respectively. We prove that these minimizing problems are independent of α and Proposition 1.6 in the following.

We prove that $n_\omega^\alpha = l_\omega^\alpha$ and $n_\omega^{\alpha,\beta}$ is not attained. To do so, we introduce

$$j_\omega^\alpha := \inf\{J_{\omega,0}^{\alpha,0}(\phi) : \phi \in \mathcal{H} \setminus \{0\}, K_{\omega,0}^{\alpha,0}(\phi) \leq 0\},$$

where $\omega > 0$ and $\alpha > 0$.

Lemma 4.2. *Let $\omega > 0$ and $\alpha > 0$. Let $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ be non-negative and $p > 5$. Then we have*

$$l_\omega^\alpha = j_\omega^\alpha.$$

Proof. First we prove $j_\omega^\alpha \leq l_\omega^\alpha$. By the definitions of j_ω^α and $J_{\omega,0}^\alpha$, we have

$$\begin{aligned} j_\omega^\alpha &\leq \inf\{J_{\omega,0}^{\alpha,0}(\phi) : \phi \in \mathcal{H} \setminus \{0\}, K_{\omega,0}^{\alpha,0}(\phi) = 0\} \\ &= \inf\{S_{\omega,0}(\phi) : \phi \in \mathcal{H} \setminus \{0\}, K_{\omega,0}^{\alpha,0}(\phi) = 0\} = l_\omega^\alpha. \end{aligned}$$

Next we prove $l_\omega^\alpha \leq j_\omega^\alpha$. In the proof, the assumptions are used. Let $\phi \in \mathcal{H} \setminus \{0\}$ such that $K_{\omega,0}^{\alpha,0}(\phi) \leq 0$. If $K_{\omega,0}^{\alpha,0}(\phi) = 0$, then by the definition of l_ω^α and $J_{\omega,0}^{\alpha,0}$, we have

$$l_\omega^\alpha \leq S_{\omega,0}(\phi) = J_{\omega,0}^{\alpha,0}(\phi).$$

If $K_{\omega,0}^{\alpha,0}(\phi) < 0$, then there exists $\lambda_* \in (0,1)$ such that $K_{\omega,0}^{\alpha,0}(\lambda_*\phi) = 0$. This follows from the continuity of the function $\lambda \mapsto K_{\omega,0}^{\alpha,0}(\lambda\phi)$ and the fact that $K_{\omega,0}^{\alpha,0}(\lambda\phi) > 0$ holds for small $\lambda \in (0,1)$ due to Lemma 4.1. Since the function $\lambda \mapsto J_{\omega,0}^{\alpha,0}(\lambda\phi)$ is monotone increasing on $[0,\infty)$ and $\lambda_* \in (0,1)$, the relations

$$l_\omega^\alpha \leq S_{\omega,0}(\lambda_*\phi) = J_{\omega,0}^{\alpha,0}(\lambda_*\phi) \leq J_{\omega,0}^{\alpha,0}(\phi)$$

hold. Namely, for any $\phi \in \mathcal{H} \setminus \{0\}$ satisfying $K_{\omega,0}^{\alpha,0}(\phi) \leq 0$, the estimate $l_\omega^\alpha \leq J_{\omega,0}^{\alpha,0}(\phi)$ holds, which implies $l_\omega^\alpha \leq j_\omega^\alpha$, which completes the proof of the lemma. □

For $y \in \mathbb{R}$, we introduce the translation operator τ_y defined by

$$(\tau_y\varphi)(x) := \varphi(x-y), \quad \text{for } x \in \mathbb{R}.$$

Proposition 4.3. *Addition to the same assumptions as in Lemma 4.2, we assume that $V \in L^1(\mathbb{R})$. Then the identity holds:*

$$n_\omega^\alpha = l_\omega^\alpha.$$

Proof. First we prove $n^\alpha_\omega \ge l^\alpha_\omega$. Take $\phi \in \mathcal{H} \setminus \{0\}$ arbitrarily such that $K^{\alpha,0}_\omega(\phi) = 0$. Since V is non-negative, the estimates $K^{\alpha,0}_{\omega,0}(\phi) \le K^{\alpha,0}_{\omega,V}(\phi) = 0$ hold. Thus Lemma 4.2 gives

$$l^\alpha_\omega = j^\alpha_\omega \le J^{\alpha,0}_{\omega,0}(\phi) = J^{\alpha,0}_{\omega,V}(\phi),$$

which implies that

$$\begin{aligned} l^\alpha_\omega &\le \inf\{J^{\alpha,0}_{\omega,V}(\phi) : \phi \in \mathcal{H} \setminus \{0\}, K^{\alpha,0}_\omega(\phi) = 0\} \\ &= \inf\{S_\omega(\phi) : \phi \in \mathcal{H} \setminus \{0\}, K^{\alpha,0}_\omega(\phi) = 0\} = n^\alpha_\omega. \end{aligned}$$

Next we prove $n^\alpha_\omega \le l^\alpha_\omega$. We note that the ground state Q_ω attains l^α_ω, i.e. $l^\alpha_\omega = S_{\omega,0}(Q_\omega)$. Since the identity $\lim_{n\to\infty} Q_\omega(x-n) = 0$ holds for any $x \in \mathbb{R}$, the estimate $0 < Q_\omega(x-n) \le Q_\omega(0)$ holds for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$ and $V \in L^1(\mathbb{R})$ is non-negative, Lebesgue's convergence theorem gives

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} V(x)|Q_\omega(x-n)|^2 dx = 0, \tag{4.13}$$

which implies that $\lim_{n\to\infty} S_{\omega,V}(Q_\omega(\cdot - n)) = S_{\omega,0}(Q_\omega) = l^\alpha_\omega$. Since V is non-negative, $K^{\alpha,0}_{\omega,V}(Q_\omega(\cdot - n)) \ge K^{\alpha,0}_{\omega,0}(Q_\omega(\cdot - n)) = K^{\alpha,0}_{\omega,0}(Q_\omega) = 0$ holds for all $n \in \mathbb{N}$. We only consider the case $V \ne 0$, since in the case $V \equiv 0$, $l^\alpha_\omega = n^\alpha_\omega$. In this case, $K^{\alpha,0}_{\omega,V}(Q_\omega(\cdot - n)) > 0$ for any $n \in \mathbb{N}$. On the other hand, for any $n \in \mathbb{N}$, $K^{\alpha,0}_{\omega,V}(\lambda Q_\omega(\cdot - n)) < 0$ for large $\lambda > 1$ due to $p > 1$. Thus by combining these facts and the continuity of the function $[1,\infty) \ni \lambda \mapsto K^{\alpha,0}_{\omega,V}(\lambda Q_\omega(\cdot - n)) \in \mathbb{R}$, there exists $\lambda_n > 1$ such that $K^{\alpha,0}_{\omega,V}(\lambda_n Q_\omega(\cdot - n)) = 0$. For this sequence $\{\lambda_n\}_{n\in\mathbb{N}}$, we can prove $\lambda_n \searrow 1$ as $n \to \infty$. Indeed, since the identity $K^{\alpha,0}_{\omega,V}(\lambda_n Q_\omega(\cdot - n)) = 0$ and the estimate $\lambda_n > 1$ hold for any $n \in \mathbb{N}$ and $\alpha > 0$, we have for any $n \in \mathbb{N}$,

$$0 = \|\partial_x Q_\omega\|^2_{L^2} + \omega\|Q_\omega\|^2_{L^2} - \lambda_n^{p-1}\|Q_\omega\|^{p+1}_{L^{p+1}} + \int_{-\infty}^{\infty} V(x)|Q_\omega(x-n)|^2 dx. \tag{4.14}$$

Moreover since $K^{1,0}_{\omega,0}(Q_\omega) = 0$, we have

$$\|\partial_x Q_\omega\|^2_{L^2} + \omega\|Q_\omega\|^2_{L^2} = \|Q_\omega\|^{p+1}_{L^{p+1}}. \tag{4.15}$$

Thus by combining the identities (4.14) and (4.15), we have

$$(\lambda_n^{p-1} - 1)\|Q_\omega\|^{p+1}_{L^{p+1}} = \int_{-\infty}^{\infty} V(x)|Q_\omega(x-n)|^2 dx.$$

By combining this and (4.13), we have $\lambda_n \searrow 1$ as $n \to \infty$, which implies that

$$\lim_{n\to\infty} |S_{\omega,V}(\lambda_n Q_\omega(\cdot - n)) - S_{\omega,V}(Q_\omega(\cdot - n))| = 0.$$

Finally, we obtain, $S_{\omega,V}(\lambda_n Q_\omega(\cdot - n)) \to S_{\omega,0}(Q_\omega) = l^\alpha_\omega$ as $n \to \infty$. Recalling the identity $K^{\alpha,0}_{\omega,V}(\lambda_n \tau_n Q_\omega) = 0$ holds for all $n \in \mathbb{N}$, we have $n^\alpha_\omega \le l^\alpha_\omega$, which completes the proof of the proposition. □

Proposition 4.4. *Addition to the assumptions of Proposition 4.3, we assume that*

$$\mu(\{x \in \mathbb{R} : V(x) \ne 0\}) > 0, \tag{4.16}$$

where μ denotes the Lebesgue measure. Then we see that n_ω^α is not attained, namely, there does not exist $\varphi \in \mathcal{H}\setminus\{0\}$ such that $K_{\omega,V}^\alpha(\varphi) = 0$ and $S_{\omega,V}(\varphi) = n_\omega^\alpha$.

Proof. On the contrary, we assume that φ attains n_ω^α, i.e. there exists $\varphi \in \mathcal{H}\setminus\{0\}$ such that $S_{\omega,V}(\varphi) = n_\omega^\alpha$ and $K_{\omega,V}^\alpha(\varphi) = 0$. If $V(x)|\varphi(x)|^2 = 0$ a.e. $x \in \mathbb{R}$, then $S_{\omega,0}(\varphi) = S_\omega(\varphi) = n_\omega^\alpha = l_\omega^\alpha$ and $K_{\omega,0}^\alpha(\varphi) = K_{\omega,V}^\alpha(\varphi) = 0$ holds, that is, φ also attains $l_\omega^{\alpha,\beta}$. By the uniqueness of the ground state for l_ω^α, we have $\varphi = Q_\omega$. Thus we obtain $\varphi(x) = Q_\omega(x) > 0$ for any $x \in \mathbb{R}$. Therefore we get $V(x) = 0$ a.e. $x \in \mathbb{R}$, which contradicts (4.16). Therefore, we have $V(x)|\varphi(x)|^2 > 0$ a.e. $x \in \mathbb{R}$. Since $\lim_{|y|\to\infty} |\varphi(x-y)|^2 = 0$ for a.e. $x \in \mathbb{R}$ due to $\varphi \in \mathcal{H} \subset H^1(\mathbb{R})$ and $V \in L^1(\mathbb{R})$, Lebesgue's convergence theorem gives

$$\lim_{|y|\to\infty} \int_{-\infty}^{\infty} V(x)|\varphi(x-y)|^2 dx = 0.$$

Thus there exists $Y > 0$ such that for any $y \in \mathbb{R}$ satisfying $|y| \geq Y$

$$0 < \int_{-\infty}^{\infty} V(x)|\varphi(x-y)|^2 dx < \int_{-\infty}^{\infty} V(x)|\varphi(x)|^2 dx,$$

which implies that for $y \in \mathbb{R}$ satisfying $|y| > Y$, we have

$$K_\omega^\alpha(\tau_y\varphi) < K_\omega^\alpha(\varphi) = 0.$$

Moreover we see that $K_\omega^\alpha(\lambda_0\tau_y\varphi) > 0$ for small $\lambda_0 \in (0,1)$ by Lemma 4.1. Therefore there exists $\lambda_* \in (0,1)$ such that $K_\omega^{\alpha,\beta}(\lambda_*\tau_y\varphi) = 0$ by the continuity of the function $\lambda \in [\lambda_0, 1] \mapsto K_\omega^\alpha(\lambda\tau_y\varphi)$. By the definitions of n_ω^α and J_ω^α, Proposition 4.3, Lemma 4.2, $|\lambda_*| < 1$, we have

$$n_\omega^\alpha \leq J_\omega^{\alpha,\beta}(\lambda_*\tau_y\varphi) < J_\omega^\alpha(\tau_y\varphi) = J_\omega^\alpha(\varphi) = S_\omega^\alpha(\varphi) = n_\omega^\alpha.$$

This is a contradiction, which completes the proof of the proposition. □

4.2. Rewriting the conditions in Theorem 1.2 into another form dependent on the frequency

In this subsection, we give a proof of Proposition 1.7.

Proof of Proposition 1.7. Let $\varphi \in \mathcal{H}$. First we consider the case $\varphi \neq 0$. We introduce the function $f : (0,\infty) \mapsto \mathbb{R}$ defined by

$$f(\omega) := l_\omega - S_{\omega,V}(\varphi), \quad \text{for } \omega > 0$$

We note that from Proposition 4.3, we see that the identity $n_\omega = l_\omega$ holds for any $\omega > 0$. Thus we see that (1) is valid if and only if $\sup_{\omega>0} f(\omega) > 0$. Since $l_\omega = S_{\omega,0}(Q_\omega)$, by using changing variables, the identity $l_\omega = \omega^{\frac{p+3}{2(p-1)}} S_{1,0}(Q_1) > 0$ holds for any $\omega > 0$. Thus f can be written as

$$f(\omega) = \omega^{\frac{p+3}{2(p-1)}} S_{1,0}(Q_1) - E_V(\varphi) - \frac{\omega}{2} M(\varphi).$$

By differentiating f with respect to the frequency ω, we have

$$f'(\omega) = \frac{p+3}{2(p-1)} \omega^{\frac{5-p}{2(p-1)}} S_{1,0}(Q_1) - \frac{1}{2} M(\varphi).$$

Noting that $p > 5$, since $\varphi \neq 0$, we can define

$$\omega_0 = \omega_0(\varphi, p) := \left\{ \frac{M(\varphi)}{\frac{p+3}{p-1} S_{1,0}(Q_1)} \right\}^{-\frac{2(p-1)}{p-5}} > 0.$$

Then $f'(\omega) = 0$ if and only if $\omega = \omega_0$. Since f' is monotone decreasing on $(0, \infty)$, due to $p > 5$, f is maximum at $\omega = \omega_0$, i.e.

$$f(\omega_0) = \max_{\omega > 0} f(\omega)$$

Therefore we see that the statement (1) holds if and only if $f(\omega_0) > 0$. A direct computation gives

$$f(\omega_0) = \frac{p-5}{2(p+3)} \frac{\left\{ \frac{p+3}{p-1} S_{1,0}(Q_1) \right\}^{\frac{2(p-1)}{p-5}}}{M(\varphi)^{\frac{p+3}{p-5}}} - E_V(\varphi).$$

Thus $f(\omega_0) > 0$ if and only if

$$\frac{p-5}{2(p+3)} \left\{ \frac{p+3}{p-1} S_{1,0}(Q_1) \right\}^{\frac{2(p-1)}{p-5}} > E_V(\varphi) M(\varphi)^{\sigma}$$

holds. Since the ground state Q_1 satisfies the energy identity and the Pohozaev identity respectively:

$$\|Q_1'\|_{L^2}^2 + \|Q_1\|_{L^2}^2 = \|Q_1\|_{L^{p+1}}^{p+1} \quad \text{and} \quad -\|Q_1'\|_{L^2}^2 + \|Q_1\|_{L^2}^2 = \frac{2}{p+1} \|Q\|_{L^{p+1}}^{p+1},$$

the identities

$$\|Q_1\|_{L^2}^2 = \frac{p+3}{p-1} \|Q_1'\|_{L^2}^2 = \frac{p+3}{2(p+1)} \|Q_1\|_{L^{p+1}}^{p+1}$$

hold. By using these identities, we have

$$\frac{p-5}{2(p+3)} \left\{ \frac{p+3}{p-1} S_{1,0}(Q_1) \right\}^{\frac{2(p-1)}{p-5}} = E_0(Q_1) M(Q_1)^{\sigma}. \tag{4.17}$$

Next we consider the case $\varphi \equiv 0$. Then $S_{\omega,V}(\varphi) = E_V(\varphi) M(\varphi)^{\sigma} = 0$. Since $l_\omega > 0$, (1) holds for any $\omega > 0$. On the other hand, since $S_{1,0}(Q_1) > 0$, by the identity (4.17), (2) also holds.

Next we prove that if φ satisfies (1.10) and (1.11) with $u_0 = \varphi$, then $I_{\omega_0,V}(\varphi) \geq 0$. When $\varphi = 0$, $I_{\omega_0,V}(\varphi) = 0$. When $\varphi \neq 0$, since

$$\omega_0 = \left\{ \frac{M(\varphi)}{\frac{p+3}{p-1} S_{1,0}(Q_1)} \right\}^{-\frac{2(p-1)}{p-5}} = \frac{2(p+3)}{p-5} E_0(Q_1) M(Q_1)^{\sigma} M(\varphi)^{-\frac{2(p-1)}{p-5}},$$

the identity

$$I_{\omega_0,V}(\varphi) = \|H_V^{\frac{1}{2}} \varphi\|_{L^2}^2 + \frac{2(p+3)}{p-5} E_0(Q_1) \left\{ \frac{M(Q_1)}{M(\varphi)} \right\}^{\sigma} - \|\varphi\|_{L^{p+1}}^{p+1} \tag{4.18}$$

holds. Since φ satisfies (1.11), we can apply the estimate (2.7) with $u(t) = \varphi$ to get

$$I_{\omega_0,V}(\varphi) > \frac{2(p+3)}{p-5}E_0(Q_1)\left\{\frac{M(Q_1)}{M(\varphi)}\right\}^{\sigma} - \frac{p+3}{p-1}\|H_V^{\frac{1}{2}}\varphi\|_{L^2}^2.$$

By the estimate (2.8), we obtain

$$I_{\omega_0,V}(\varphi) > \frac{2(p+3)}{p-5}M(\varphi)^{-\sigma}\{E_0(Q_1)M(Q_1)^{\sigma} - E_V(\varphi)M(\varphi)^{\sigma}\} > 0,$$

where we have used the assumption (1.10).

Finally, we prove that if φ satisfies (1.10) and (1.12) with $u_0 = \varphi$, then $I_{\omega_0,V}(\varphi) < 0$. Noting that the identity

$$\|\varphi\|_{L^{p+1}}^{p+1} = \frac{p+1}{2}\|H_V^{\frac{1}{2}}\varphi\|_{L^2}^2 - (p+1)E_V(\varphi)$$

holds, by the identity (4.18), we have

$$I_{\omega_0,V}(\varphi) = -\frac{p-1}{2}\|H_V^{\frac{1}{2}}\varphi\|_{L^2}^2 + \frac{2(p+3)}{p-5}E_0(Q_1)M(Q_1)^{\sigma}M(\varphi)^{-\sigma} + (p+1)E_V(\varphi).$$

Since φ satisfies (1.12), we can apply the estimate (2.13) to get

$$\begin{aligned} I_{\omega_0,V}(\varphi) &> \left[-\frac{p-1}{2}\|\partial_x Q_1\|_{L^2}^2 + \left\{\frac{2(p+3)}{p-5} + (p+1)\right\}E_0(Q_1)\right]\left(\frac{M(Q_1)}{M(\varphi)}\right)^{\sigma} \\ &= 0 \end{aligned}$$

due to the assumption (1.10) and the identity (2.5), which completes the proof of the proposition. □

4.3. Variational structure

For $\omega > 0$ and $V \in L^1(\mathbb{R}) + L^{\infty}(\mathbb{R})$ satisfying $V(x) \geq 0$ for a.e. $x \in \mathbb{R}$, we introduce a norm $\|\cdot\|_{H^{\frac{1}{2}}_{\omega,V}}$

$$\|\varphi\|_{H^{\frac{1}{2}}_{\omega,V}}^2 := \frac{1}{2}\|\partial_x\varphi\|_{L^2}^2 + \frac{\omega}{2}\|\varphi\|_{L^2}^2 + \frac{1}{2}\int_{-\infty}^{\infty} V(x)|\varphi(x)|^2 dx,$$

for $\varphi \in \mathcal{H}$.

Lemma 4.5 *(Equivalency of the norms and action). Let $\omega > 0$, $V \in L^1(\mathbb{R}) + L^{\infty}(\mathbb{R})$, $p > 1$ and $\varphi \in H^1(\mathbb{R})$. We assume that $I_{\omega,V}(\varphi) \geq 0$. Then the inequality*

$$S_{\omega,V}(\varphi) \leq \|\varphi\|_{H^{\frac{1}{2}}_{\omega,V}}^2 \leq \frac{p+1}{p-1}S_{\omega,V}(\varphi), \tag{4.19}$$

holds. Moreover the estimate

$$c\|\varphi\|_{H^1} \leq \|\varphi\|_{H^{\frac{1}{2}}_{\omega,V}} \leq C\|\varphi\|_{H^1} \tag{4.20}$$

also holds, where c is a positive constant dependent only on ω and C is a constant dependent only on ω and $\|V\|_{L^1+L^{\infty}}$. The estimates (4.19) and (4.20) imply that the norms $\|\cdot\|_{H^{\frac{1}{2}}_{\omega,V}}^2$, $\|\cdot\|_{H^1}^2$ and $\|\cdot\|_{\mathcal{H}}^2$ are equivalent to $S_{\omega,V}(\cdot)$.

Proof. The left inequality of (4.19) is trivial, since we are considering the focusing nonlinearity. Next we prove the right inequality of (4.19). By the assumption of this lemma, we have

$$\begin{aligned}0 &\le I_{\omega,V}(\varphi)\\ &= \|\partial_x\varphi\|_{L^2}^2 + \int_{-\infty}^{\infty} V(x)|\varphi(x)|^2 dx + \omega\|\varphi\|_{L^2}^2 - \|\varphi\|_{L^{p+1}}^{p+1}\\ &= (1-p)\|\varphi\|^2_{H^{\frac{1}{2}}_{\omega,V}} + (p+1)S_\omega(\varphi),\end{aligned}$$

which implies the right inequality of (4.19). It is easy to see that the left inequality of (4.20) holds due to $V(x) \ge 0$ a.e. $x \in \mathbb{R}$. Indeed,

$$\|\varphi\|_{H^1}^2 = \|\partial_x\varphi\|_{L^2}^2 + \|\varphi\|_{L^2}^2 \le \max\left(2, \frac{2}{\omega}\right)\left(\frac{1}{2}\|\partial_x\varphi\|_{L^2}^2 + \frac{\omega}{2}\|\varphi\|_{L^2}^2\right) \le \max\left(2, \frac{2}{\omega}\right)\|\varphi\|^2_{H^{\frac{1}{2}}_{\omega,V}}.$$

By the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$, the estimate

$$\int_{-\infty}^{\infty} V(x)|\varphi(x)|^2 dx \le C\|V\|_{L^1+L^\infty}\|\varphi\|_{H^1}^2$$

holds where C is a constant. By using this estimate, we have

$$\|\varphi\|^2_{H^{\frac{1}{2}}_{\omega,V}} \le \max\left(\frac{1}{2}, \frac{\omega}{2}, C\|V\|_{L^1+L^\infty}\right)\|\varphi\|_{H^1}^2,$$

which completes the proof of the lemma. □

Lemma 4.6 *(Invariant sets). Let $p > 1$, $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ be non-negative, $u_0 \in \mathcal{H}$ and let $u \in C(I, \mathcal{H})$ be a solution to (NLS$_V$) with $u|_{t=0} = u_0 \in \mathcal{H}$ on $I = (-T_-, T_+)$, $t_0 \in \mathbb{R}$ and $\omega > 0$. If $u(t_0) \in \mathcal{N}_\omega^+$, $u(t) \in \mathcal{N}_\omega^+$ for any $t \in I$. On the other hand, if $u(t_0) \in \mathcal{N}_\omega^-$, then $u \in \mathcal{N}_\omega^-$ for any $t \in I$.*

Proof. Since u is the solution to (NLS$_V$) on I, the energy and the mass conservation laws give $u(t) \in \mathcal{N}_\omega^+ \cup \mathcal{N}_\omega^-$ for any $t \in I$ due to $S_{\omega,V}(u(t_0)) < n_\omega$. First we consider the case $u(t_0) \in \mathcal{N}_\omega^+$. We only consider the case of $t \ge t_0$, since the case $t < t_0$ can be treated in the same manner. On the contrary, we assume that there exists $t_* \in (t_0, T_+)$ such that $u(t_*) \in \mathcal{N}_\omega^-$. By the continuity of the function $t \in I \mapsto I_{\omega,V}(u(t)) \in \mathbb{R}$, there exists $t_{**} \in [t_0, t_*)$ such that $I_{\omega,V}(u(t_{**})) = 0$. By the definition of n_ω and the conservation laws again, we have

$$n_\omega > S_{\omega,V}(u(t_0)) = S_{\omega,V}(u(t_{**})) \ge n_\omega,$$

which leads to a contradiction. Thus for any $t \in (t_0, T_+)$, $u(t) \in \mathcal{N}_\omega^+$.

In the same manner as above, the second statement can be proved, which completes the proof of the lemma. □

Corollary 4.7 *(Global existence of solution in $\mathcal{N}_\omega^+$). Let $\omega > 0$, $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ be non-negative, $p > 1$ and $t_0 \in \mathbb{R}$. Let u be the solution to (NLS$_V$) with the initial condition $u|_{t=t_0} = u(t_0) \in \mathcal{H}$ on $(-T_-, T_+)$, where $T_\pm$ denote the maximal existence times of the solution u. We further assume that $u(t_0) \in \mathcal{N}_\omega^+$. Then $T_+ = T_- = \infty$.*

Proof. We only prove that $T_+ = \infty$, since $T_- = \infty$ can be proved in the similar manner. On the contrary, we assume that $T_+ < \infty$. Then by the blow-up criterion in Proposition 1.1 with $u_0 = u(t_0) \in \mathcal{H}$, we have $\lim_{t\to T_+-0} \|\partial_x u(t)\|_{L^2} = \infty$. On the other hand, since $u(t_0) \in \mathcal{N}_\omega^+$, Lemma 4.6 implies that $I_{\omega,V}(u(t)) \geq 0$ for any $t \in I$. Thus noting that $\omega > 0$ and $p > 1$, we can apply Lemma 4.5 with the conservation laws and $V \geq 0$ to get

$$\|\partial_x u(t)\|_{L^2}^2 \leq 2\|u(t)\|_{H^1}^2 \leq CS_{\omega,V}(u(t)) = CS_{\omega,V}(u(t_0)) < \infty, \text{ for } t \in I,$$

where C is a constant independent of $t \in I$. This contradicts $\lim_{t\to T_+-0} \|\partial_x u(t)\|_{L^2} = \infty$. Therefore we find $T_+ = \infty$, which completes the proof of the corollary. $\square$

Lemma 4.8. *Let $\omega > 0$, $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ be non-negative, $p > 5$, $\varepsilon > 0$, $\delta > 0$ satisfy $2\varepsilon < \delta$, and k be a nonnegative integer. Let $\{\varphi_l\}_{l=0}^k \subset H^1(\mathbb{R})$ be a sequence satisfying*

$$\begin{aligned}
&S_{\omega,V}\left(\textstyle\sum_{l=0}^k \varphi_l\right) \leq n_\omega - \delta, \quad &&S_{\omega,V}\left(\textstyle\sum_{l=0}^k \varphi_l\right) \geq \textstyle\sum_{l=0}^k S_{\omega,V}(\varphi_l) - \varepsilon,\\
&I_{\omega,V}\left(\textstyle\sum_{l=0}^k \varphi_l\right) \geq -\varepsilon, &&I_{\omega,V}\left(\textstyle\sum_{l=0}^k \varphi_l\right) \leq \textstyle\sum_{l=0}^k I_{\omega,V}(\varphi_l) + \varepsilon.
\end{aligned}$$

Then we have $0 \leq S_{\omega,V}(\varphi_l) < n_\omega$ and $I_{\omega,V}(\varphi_l) \geq 0$ for all $l \in \{0, 1, 2, \cdots, k\}$, which implies that $\varphi_l \in \mathcal{N}_\omega^+$, for all $l \in \{0, 1, 2, \cdots, k\}$.

This lemma can be proved in the similar manner as the proof of Lemma 6.4 in [14] or Lemma 3.6 in [15].

Proof. On the contrary, we assume that there exists an $l \in \{0, 1, 2, \cdots, k\}$ such that $I_{\omega,V}(\varphi_l) < 0$. First we prove that $J_{\omega,V}^{1,0}(\varphi_l) \geq n_\omega$, where the functional $J_{\omega,V}^{1,0}$ is defined by (4.7). We note that $\varphi_l \neq 0$ due to $I_{\omega,V}(\varphi_l) < 0$, and $\lim_{\lambda\to 0} \|\partial_x(\lambda\varphi_l)\|_{L^2} = \|\partial_x\varphi_l\|_{L^2} \lim_{\lambda\to 0} \lambda = 0$. These facts and $p > 5$ allow us to apply Lemma 4.1 with $\alpha = 1$ and $\beta = 0$ to obtain $I_{\omega,V}(\lambda_0\varphi_l) > 0$ for small $\lambda_0 \in (0, 1)$. By $I_{\omega,V}(\lambda_0\varphi_l) > 0$ and $I_\omega(\varphi_l) < 0$ and the continuity of the function $[0, \infty) \ni \lambda \mapsto I_{\omega,V}(\lambda\varphi_l) \in \mathbb{R}$, there exists $\lambda_* \in (\lambda_0, 1)$ such that $I_{\omega,V}(\lambda_*\varphi_l) = 0$. Noting that the function $[0, \infty) \ni \lambda \mapsto J_{\omega,V}^{1,0}(\lambda\varphi_l)$ is monotone increasing, by the definitions of n_ω and $J_{\omega,V}^{1,0}$ and $\lambda_* < 1$, we obtain

$$n_\omega \leq S_{\omega,V}(\lambda_*\varphi_l) = J_{\omega,V}^{1,0}(\lambda_*\varphi_l) < J_{\omega,V}^{1,0}(\varphi_l).$$

By the positivity and the definition of the functional $J_{\omega,V}^{1,0}$ and the assumptions of this lemma, we obtain

$$\begin{aligned}
n_\omega < J_{\omega,V}^{1,0}(\varphi_l) &\leq \sum_{l=0}^k J_{\omega,V}^{1,0}(\varphi_l) = \sum_{l=0}^k \left\{ S_{\omega,V}(\varphi_l) - \frac{1}{2} I_{\omega,V}(\varphi_l) \right\}\\
&= \sum_{l=0}^k S_{\omega,V}(\varphi_l) - \frac{1}{2} \sum_{l=0}^k I_{\omega,V}(\varphi_l)\\
&\leq S_{\omega,V}\left(\sum_{l=0}^k \varphi_l\right) + \varepsilon - \frac{1}{2}\left\{ I_{\omega,V}\left(\sum_{l=0}^k \varphi_l\right) - \varepsilon \right\}\\
&\leq n_\omega - \delta + \varepsilon + \varepsilon < n_\omega,
\end{aligned}$$

which leads to a contradiction. Thus, $I_{\omega,V}(\varphi_l) \geq 0$ for all $l \in \{0, 1, 2, \cdots, k\}$. Moreover, by combining this estimate and the positivity of the functional $J_{\omega,V}^{1,0}$ again, for any $l \in \{0, 1, 2, \cdots, k\}$, we have

$$S_{\omega,V}(\varphi_l) = J^{1,0}_{\omega,V}(\varphi_l) + \frac{1}{2} I_{\omega,V}(\varphi_l) \geq 0.$$

By combining this estimate and the assumptions of this lemma, we have for any $l \in \{0, 1, 2, \cdots, k\}$,

$$S_{\omega,V}(\varphi_l) \leq \sum_{l=0}^{k} S_{\omega,V}(\varphi_l) \leq S_{\omega,V}\left(\sum_{l=0}^{k} \varphi_l\right) + \varepsilon \leq n_\omega - \delta + \varepsilon < n_\omega - \frac{1}{2}\delta < n_\omega.$$

Therefore, we get $\varphi_l \in \mathcal{N}^+_\omega$ for all $l \in \{0, 1, 2, \cdots, k\}$, which completes the proof of the proposition. □

5. Proof of the scattering part

5.1. Dispersive estimate, Strichartz estimates and small data scattering

In this subsection, we recall the dispersive estimate and the Strichartz estimates for the linear Schrodinger evolution group $\{e^{itH_V}\}_{t\in\mathbb{R}}$ and a small data scattering result for (NLS$_V$) in the mass-critical or supercritical case $p \geq 5$.

Lemma 5.1 *(Dispersive estimate). Let $V \in L^1_1(\mathbb{R})$ be non-negative and $\phi \in L^1(\mathbb{R})$. Then there exists a constant $C = C(V) > 0$ such that the estimate*

$$\|e^{-itH_V}\phi\|_{L^\infty} \leq \frac{C}{|t|^{\frac{1}{2}}} \|\phi\|_{L^1}, \tag{5.1}$$

holds for $t \in \mathbb{R}\backslash\{0\}$. Moreover, let $a \in [2, \infty]$ and $\phi \in L^{a'}(\mathbb{R})$. Then there exists a constant $C = C(a, V) > 0$ such that the estimate

$$\|e^{-itH_V}\phi\|_{L^a(\mathbb{R})} \leq C|t|^{-\frac{1}{2}+\frac{1}{a}} \|\phi\|_{L^{a'}} \tag{5.2}$$

holds for $t \in \mathbb{R}\backslash\{0\}$, where a' is the Hölder conjugate of a: $\frac{1}{a} + \frac{1}{a'} = 1$.

For the proof of the estimate (5.1), see [10]. The estimate (5.2) can be proved by Riesz-Thorin's theorem with the estimate (5.1) and the L^2-conservation law.

Remark 5.1. The reason why $V \in L^1_1(\mathbb{R})$ is assumed in the scattering part of Theorem 1.3 is due to the use of Lemma 5.1 for the proof.

Next we state the Strichartz estimate for $\{e^{-itH_V}\}_{t\in\mathbb{R}}$ for L^2-admissible pairs. We say that (q, r) is an L^2-admissible pair, if and only if (q, r) satisfies

$$2 \leq q \leq \infty \quad \text{and} \quad \frac{2}{q} = \frac{1}{2} - \frac{1}{r}.$$

Lemma 5.2 *(Strichartz estimates for L^2-admissible pairs). Let $V \in L^1_1(\mathbb{R})$ be non-negative, and let (q, r) be an L^2-admissible pair and $f \in L^2(\mathbb{R})$. Then for any time interval I, there exists C depending only on q, such that the estimate*

$$\|e^{-itH_V} f\|_{L^q_t(I:L^r_x)} \leq C\|f\|_{L^2},$$

holds. Moreover let (q_j, r_j) $(j = 1, 2)$ be L^2-admissible pairs. Then for any time interval I, there exists a constant C depending only on q_1, q_2 such that the estimate

$$\left\| \int_0^t e^{-(t-s)H_V} F(s) ds \right\|_{L_t^{q_1}(I:L_x^{r_1})} \leq C \|F\|_{L_t^{q_2'}\left(I:L_x^{r_2'}\right)}$$

holds for $F \in L_t^{q_2'}\left(I : L_x^{r_2'}\right)$, *where* q_2' *and* r_2' *are Hölder conjugate of* q_2 *and* r_2 *respectively.*

This lemma can be proved by the dispersive estimate (Lemma 5.1) and so-called TT^*-argument (see [18] for example).

We need Strichartz estimates for non-admissible pairs in order to treat scattering results in the mass-supercritical case $p > 5$.

Lemma 5.3 *(Strichartz estimates for non-admissible pairs). Let* $V \in L_1^1(\mathbb{R})$ *be non-negative and let* $p \geq 5$ *and the exponents* r, a, b *and* γ *be defined by*

$$r := p+1, \quad a := \frac{2(p-1)(p+1)}{p+3}, \quad b := \frac{2(p-1)(p+1)}{(p-1)^2-(p-1)-4}, \quad \gamma := \frac{2(p-1)}{p-3}. \tag{5.3}$$

Then for any time interval I, *the estimates*

$$\begin{aligned}
\left\|e^{-itH_V}\varphi\right\|_{L_t^a(I:L_x^r)} &\leq C\|\varphi\|_{H^1}, \\
\left\|e^{-itH_V}\varphi\right\|_{L_t^{p-1}(I:L_x^\infty)} &\leq C\|\varphi\|_{H^1}, \\
\left\| \int_0^t e^{-i(t-s)H_V} F(s) ds \right\|_{L_t^a(I:L_x^r)} &\leq C\|F\|_{L_t^{b'}(I:L_x^{r'})}, \\
\left\| \int_0^t e^{-i(t-s)H_V} F(s) ds \right\|_{L_t^{p-1}(I:L_x^\infty)} &\leq C\|F\|_{L_t^{b'}(I:L_x^{r'})}, \\
\left\| \int_0^t e^{-i(t-s)H_V} G(s) ds \right\|_{L_t^a(I:L_x^r)} &\leq C\|G\|_{L_t^{\gamma'}(I:L_x^1)},
\end{aligned}$$

hold, where $\varphi \in \mathcal{H}$, $F \in L_t^{b'}(I : L_x^{r'}(\mathbb{R}))$ *and* $G \in L_t^{\gamma'}(I : L_x^1)$ *and* b', r' *and* γ' *denote the Hölder conjugate of* b, r *and* γ *respectively, i.e.* $\frac{1}{b'} + \frac{1}{b} = 1$, $\frac{1}{r'} + \frac{1}{r} = 1$ *and* $\frac{1}{\gamma'} + \frac{1}{\gamma} = 1$, *and* C *depends on* p *and* V.

For the proof of this lemma, see [3, Section 3.1 and 3.2] and Proposition 2 in [21] (see also [9]).

Next we state a sufficient condition on Strichartz spaces to obtain the scattering result in $\mathcal{H}$.

Proposition 5.4. *Let* $p \geq 5$, $V \in L_1^1(\mathbb{R})$ *be non-negative and* a, r *be defined by (5.3),* $u_0 \in \mathcal{H}$ *and* $u \in C(\mathbb{R} : \mathcal{H})$ *be a solution to (NLS$_V$) on* $\mathbb{R}$. *Then if* u *belongs to* $L_t^a(\mathbb{R} : L_x^r)$, *then the solution* u *also belongs to* $L_t^{p-1}(\mathbb{R} : L_x^\infty)$ *and scatters in* $\mathcal{H}$ *as* $t \to \pm\infty$.

Proof of Proposition 5.4. Since u is the solution to (NLS$_V$) on $\mathbb{R}$, the identity

$$u(t) = (e^{-itH_V}u_0) + i\int_0^t e^{-i(t-s)H_V}\left\{|u(s)|^{p-1}u(s)\right\} ds, \quad t \in \mathbb{R} \tag{5.4}$$

holds in $\mathcal{H}$-sense. The Strichartz estimates for non-admissible pairs (Lemma 5.2) give

$$\|u\|_{L_t^{p-1}(\mathbb{R}:L_x^\infty)} \le C\|u_0\|_{H^1} + C\||u|^{p-1}u\|_{L_t^{b'}(\mathbb{R}:L_x^{r'})} \le C\|u_0\|_{H^1} + C\|u\|^p_{L_t^a(\mathbb{R}:L_x^r)} < \infty \tag{5.5}$$

due to $u_0 \in \mathcal{H} \subset H^1(\mathbb{R})$ and $u \in L_t^a(\mathbb{R} : L_x^r)$, which implies $u \in L_t^{p-1}(\mathbb{R} : L_x^\infty)$.

We only consider the positive time direction, since the negative time direction can be treated in the same manner. Take t_1, t_2 such as $0 < t_1 < t_2$. Since L_V commutes e^{-itH_V}, e^{itH_V} is unitary on $L^2(\mathbb{R})$ and the norm $\|\cdot\|_{\mathcal{H}}$ is equivalent to the norm $\|\cdot\|_{H^1}$, we have

$$\begin{aligned}
\|e^{it_2H_V}u(t_2) - e^{it_1H_V}u(t_1)\|_{\mathcal{H}} &= \left\| \int_{t_1}^{t_2} e^{-isH_V}\{|u(s)|^{p-1}u(s)\}ds \right\|_{\mathcal{H}} \\
&= \left\| \int_{t_1}^{t_2} e^{-isH_V} L_V \left\{|u(s)|^{p-1}u(s)\right\} ds \right\|_{L^2} \\
&\le C\int_{t_1}^{t_2} \||u(s)|^{p-1}u(s)\|_{H^1} ds \le C\int_{t_1}^{t_2} \|u(s)\|^{p-1}_{L^\infty}\|u(s)\|_{H^1} ds \\
&\le C_*\|u\|^{p-1}_{L_t^{p-1}(t_1,t_2:L_x^\infty)}\|u\|_{L_t^\infty(\mathbb{R}:\mathcal{H})} \to 0, \quad \text{as } t_2 > t_1 \to \infty,
\end{aligned} \tag{5.6}$$

due to $u \in L_t^\infty(\mathbb{R} : \mathcal{H}) \cap L_t^{p-1}(\mathbb{R} : L_x^r)$, where C_* is dependent only on p, V. Thus since $\mathcal{H}$ is Hilbert space and e^{-itH_V} is symmetric on $\mathcal{H}$, we can see that there exists $u_+ \in \mathcal{H}$ such that the identities hold:

$$\lim_{t\to\infty} \|u(t) - e^{-itH_V}u_+\|_{\mathcal{H}} = \lim_{t\to\infty} \|e^{itH_V}u(t) - u_+\|_{\mathcal{H}} = 0,$$

which completes the proof of the proposition. □

Next we state a small data scattering result to (NLS$_V$) in the energy space $\mathcal{H}$ in the mass-critical or mass-supercritical case ($p \ge 5$):

Proposition 5.5 *(Small data scattering result in the energy space $\mathcal{H}$). Let $p \ge 5$, $V \in L_1^1(\mathbb{R})$ be non-negative, (a, r) be defined by (5.3). Then there exists $\varepsilon = \varepsilon(p, V) > 0$ such that for any $\phi \in \mathcal{H}$ satisfying $\|e^{-itH_V}\phi\|_{L_t^a(\mathbb{R}:L_x^r)} \le \varepsilon$, there exists a unique solution $u \in L_t^a(\mathbb{R} : L_x^r)$ to (NLS$_V$) such that*

$$\|u\|_{L_t^a(\mathbb{R}:L_x^r)} \le 2\varepsilon.$$

Moreover we have

$$u \in L_t^{p-1}(\mathbb{R} : L_x^r) \cap L_t^\infty(\mathbb{R} : \mathcal{H}).$$

The proof is standard and can be done by a fixed point argument via the non-admissible Strichartz estimates (Lemma 5.3).

Proof of Proposition 5.5. Take $\varepsilon > 0$ sufficiently small, which will be determined later. Let $\phi \in \mathcal{H}$ such that $\|e^{-itH_V}\phi\|_{L_t^a(\mathbb{R}:L_x^r)} \le \varepsilon$ (the existence of ϕ can be proved via the non-admissible Strichartz estimate (Lemma 5.3)). We introduce the closed ball $X(\varepsilon)$ in $L_t^a(\mathbb{R} : L_x^r)$ as

$$X(\varepsilon) := \left\{u \in L_t^a(\mathbb{R} : L_x^r) : \ \|u\|_{L_t^a(\mathbb{R}:L_x^r)} \le 2\varepsilon\right\}$$

with the metric

$$d(u,v) := \|u-v\|_{L^a_t(\mathbb{R}:L^r_x)}$$

for $u, v \in L^a_t(\mathbb{R} : L^r_x)$. We prove that the nonlinear mapping $J : X(\varepsilon) \mapsto X(\varepsilon)$

$$J[u](t) := e^{-itH_V}\phi + i\int_0^t e^{-i(t-s)H_V}|u(s)|^{p-1}u(s)ds, \quad t \in \mathbb{R},$$

is contractive on $X(\varepsilon)$. The Strichartz estimates (Lemma 5.3) give

$$\begin{aligned}\|J[u]\|_{L^a_t(\mathbb{R}:L^r_x)} &\le \|e^{-itH_V}\phi\|_{L^a_t(\mathbb{R}:L^r_x)} + C_1\||u|^{p-1}u\|_{L^{b'}_t(\mathbb{R}:L^{r'}_x)} \le \varepsilon + C_1\|u\|^p_{L^a_t(\mathbb{R}:L^r_x)} \\ &\le \varepsilon + C_1 2^p\varepsilon^p,\end{aligned} \tag{5.7}$$

for $u \in X(\varepsilon)$, where C_1 is dependent only on p and V. Here we choose $\varepsilon = \varepsilon(p,V) > 0$ such as $\varepsilon < \frac{1}{(2^pC_1)^{\frac{1}{p-1}}}$. Then we have $\|J[u]\|_{L^a_t(\mathbb{R}:L^r_x)} \le 2\varepsilon$, which implies that the mapping J is well defined from $X(\varepsilon)$ into itself. We note that by the fundamental formula, the estimate

$$\begin{aligned}||a|^{p-1}a - |b|^{p-1}b| &= \left|\int_0^1 \frac{d}{d\theta}|\theta a + (1-\theta)b|^{p-1}(\theta a + (1-\theta)b)d\theta\right| \\ &\le p2^{p-1}(|a|^{p-1} + |b|^{p-1})|a-b|\end{aligned} \tag{5.8}$$

holds for $a, b \in \mathbb{C}$. In the same manner as the proof of (5.7), we obtain

$$d(J[u], J[v]) \le C_1 p 2^{p-1}(2\varepsilon)^{p-1}d(u,v) \tag{5.9}$$

for $u, v \in X(\varepsilon)$. Here we take $\varepsilon > 0$ sufficiently small such as

$$\varepsilon \le \frac{1}{4(2C_1p)^{\frac{1}{p-1}}}.$$

Then (5.9) gives

$$d(J[u], J[v]) \le \frac{1}{2}d(u,v),$$

which implies that the mapping J is contractive on $X(\varepsilon)$. Thus by the contraction mapping principle, we see that there exists a unique $u \in X(\varepsilon)$ such that $J[u](t) = u(t)$ on $t \in \mathbb{R}$. Therefore we have $\|u\|_{L^a_t(\mathbb{R}:L^r_x)} \le 2\varepsilon$. Moreover, in the similar manner as the proof of Proposition 5.4, we have $u \in L^{p-1}_t(\mathbb{R} : L^r_x)$. Next we prove $u \in L^\infty_t(\mathbb{R} : \mathcal{H})$. By the local well-posedness result in $\mathcal{H}$ (Proposition 1.1), we have $u \in L^\infty_t(-t, t : \mathcal{H})$ for small $t > 0$. Set

$$T_m := \sup\{T \in (0,\infty] : \|u\|_{L^\infty_t(0,T:\mathcal{H})} < \infty\}.$$

We assume that $T_m < \infty$. Then $\|u\|_{L^\infty_t(0,T_m:\mathcal{H})} = \infty$ by the local-wellposedness result again. Since $u \in L^{p-1}_t(0, T_m : L^\infty_x)$, there exists $t_1 \in (0, T_m)$ such that $\|u\|_{L^{p-1}_t(t_1,T_m:L^\infty_x)} \le \frac{1}{(2C_*)^{\frac{1}{p-1}}}$, where C_* appears in the estimate (5.6). Since $t_1 \in (0, T_m)$, by the definition of T_m, we have $\|u\|_{L^\infty_t(0,t_1:\mathcal{H})} < \infty$ and $\|u\|_{L^\infty_t(t_1,T_m;\mathcal{H})} = \infty$. In the same manner as the proof of the estimate (5.6), we have

$$\|u\|_{L^\infty_t(t_1,T_m:\mathcal{H})} \le \|u(t_1)\|_{\mathcal{H}} + C_*\|u\|^{p-1}_{L^{p-1}_t(t_1,T_m:\mathcal{H})}\|u\|_{L^\infty_t(t_1,T_m:\mathcal{H})},$$

which implies

$$\frac{1}{2}\|u\|_{L_t^\infty(t_1,T_m:\mathcal{H})} \leq (1 - C_*\|u\|^{p-1}_{L_t^{p-1}(t_1,T_m:L_x^\infty)})\|u\|_{L_t^\infty(t_1,T_m:\mathcal{H})} \leq \|u(t_1)\|_{\mathcal{H}} < \infty.$$

This contradicts $\|u\|_{L_t^\infty(t_1,T_m;\mathcal{H})} = \infty$. Thus we have $T_m = \infty$, which gives $\|u\|_{L_t^\infty(0,\infty;\mathcal{H})} < \infty$. In the same manner, we can prove $\|u\|_{L_t^\infty(-\infty,0:\mathcal{H})} < \infty$. Therefore we have $u \in L_t^\infty(\mathbb{R} : \mathcal{H})$, which completes the proof of the lemma. $\square$

Corollary 5.6. *Under the same assumptions as in Proposition 5.5, then there exists $\varepsilon_1 = \varepsilon_1(p, V) > 0$ such that for any $\phi \in \mathcal{H}$ satisfying $\|\phi\|_{\mathcal{H}} \leq \varepsilon_1$, the same conclusion as in Proposition 5.5 holds*

The similar statements as Proposition 5.4 and Corollary 5.6 also hold for the focusing semilinear Schrödinger equation without the potential with a replacement $\mathcal{H}$ into $H^1(\mathbb{R})$:

$$\begin{cases} i\partial_t u + \partial_x^2 u + |u|^{p-1}u = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u|_{t=0} = u_0 \in H^1(\mathbb{R}), \end{cases} \tag{NLS}$$

5.2. Linear profile decomposition

In order to prove the scattering result, more precisely, to construct a critical element (see Theorem 5.12 for the definition), we use a linear profile decomposition, which is proved in [21]. The abstract version was obtained in Theorem 2.1 in [3] (see [2] for the energy-critical wave equation and [8] for the mass-supercritical and energy-subcritical Schrödinger equation).

Proposition 5.7 *(Linear profile decomposition). Let $V \in L_1^1(\mathbb{R})$ be non-negative and satisfy $V' \in L_1^1(\mathbb{R})$. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{H}$. Then, up to subsequence, we can write*

$$\varphi_n = \sum_{j=1}^{J} e^{it_n^j H_V}\tau_{x_n^j}\psi^j + W_n^J, \quad \forall J \in \mathbb{N},$$

where $t_n^j \in \mathbb{R}$, $x_n^j \in \mathbb{R}$, $\psi^j \in H^1(\mathbb{R})$, and the following statements hold:

- *for any fixed $j \in \{1, 2, \cdots, J\}$, we have:*

$$\text{either } t_n^j = 0 \text{ for any } n \in \mathbb{N}, \text{ or } t_n^j \to \pm\infty \text{ as } n \to \infty,$$
$$\text{either } x_n^j = 0 \text{ for any } n \in \mathbb{N}, \text{ or } x_n^j \to \pm\infty \text{ as } n \to \infty.$$

- *orthogonality of the parameters:*

$$|t_n^j - t_n^k| + |x_n^j - x_n^k| \to \infty \text{ as } n \to \infty, \quad \forall j \neq k. \tag{5.10}$$

- *smallness of the reminder:*

$$\forall\varepsilon > 0, \exists J = J(\varepsilon) \in \mathbb{N} \text{ such that } \limsup_{n\to\infty} \left\|e^{-itH_V}W_n^J\right\|_{L_t^\infty(\mathbb{R}:L_x^\infty(\mathbb{R}))} < \varepsilon.$$

- *orthogonality in norms: for any $J \in \mathbb{N}$,*

$$\|\varphi_n\|_{L^2}^2 = \sum_{j=1}^{J}\left\|\psi^j\right\|_{L^2}^2 + \left\|W_n^J\right\|_{L^2}^2 + o_n(1),$$

$$\|\varphi_n\|_{H_V^{\frac{1}{2}}}^2 = \sum_{j=1}^{J} \left\|\tau_{x_n^j}\psi^j\right\|_{H_V^{\frac{1}{2}}}^2 + \left\|W_n^J\right\|_{H_V^{\frac{1}{2}}}^2 + o_n(1),$$

as $n \to \infty$, where the norm $\|\cdot\|_{H_V^{\frac{1}{2}}}$ is defined by (1.4). Moreover, we have

$$\|\varphi_n\|_{L^q}^q = \sum_{j=1}^{J} \left\|e^{it_n^j H_V}\tau_{x_n^j}\psi^j\right\|_{L^q}^q + \|W_n^J\|_{L^q}^q + o_n(1), \quad q \in (2,\infty), \quad \forall J \in \mathbb{N},$$

as $n \to \infty$ and in particular, for any $J \in \mathbb{N}$,

$$S_{\omega,V}(\varphi_n) = \sum_{j=1}^{J} S_{\omega,V}(e^{it_n^j H_V}\tau_{x_n^j}\psi^j) + S_{\omega,V}(W_n^J) + o_n(1),$$

$$I_{\omega,V}(\varphi_n) = \sum_{j=1}^{J} I_{\omega,V}(e^{it_n^j H_V}\tau_{x_n^j}\psi^j) + I_{\omega,V}(W_n^J) + o_n(1),$$

as $n \to \infty$.

For the proof of this proposition, see Theorem 2.1 in [3], and Proposition 6 in [21]. More precisely, the linear profile decomposition for more general Schrödinger operator was given by Theorem 2.1 in [3], and Proposition 6 in [21] says that if the potential V satisfies the assumptions in Proposition 5.7, then the Schrödinger operator H_V satisfies the assumptions of Theorem 2.1 in [3].

5.3. Perturbation lemma

In order to prove the scattering result in Theorem 1.3, especially to construct a critical element (see Theorem 5.12), we also use so-called perturbation lemma.

Lemma 5.8 *(Perturbation lemma). Let $p \geq 5$, $V \in L_1^1(\mathbb{R})$ be non-negative and (a,r) be defined by (5.3). For any $M > 0$, there exist $\varepsilon = \varepsilon(M) > 0$ and $C = C(M) > 0$ such that the following occurs: Let $v \in C(\mathbb{R} : \mathcal{H}) \cap L_t^a(\mathbb{R} : L_x^r)$ satisfying $\|v\|_{L_t^a(\mathbb{R}:L_x^r)} \leq M$ be a solution of the integral equation with a source term $e \in L_t^a(\mathbb{R} : L_x^r)$ with $\|e\|_{L_t^a(\mathbb{R}:L_x^r)} \leq \varepsilon$:*

$$v(t) = e^{-itH_V}\varphi + i\int_0^t e^{-i(t-s)H_V}\{|v(s)|^{p-1}v(s)\}ds + e(t). \tag{5.11}$$

Moreover we assume that $\varphi_0 \in \mathcal{H}$ satisfies $\left\|e^{-itH_V}\varphi_0\right\|_{L_t^a(\mathbb{R}:L_x^r)} \leq \varepsilon$. Then the solution $u \in C(\mathbb{R} : \mathcal{H})$ to (NLS$_V$) with the initial data $\varphi + \varphi_0$, i.e.

$$u(t) = e^{-itH_V}(\varphi + \varphi_0) + i\int_0^t e^{-i(t-s)H_V}\{|u(s)|^{p-1}u(s)\}ds, \tag{5.12}$$

satisfies the estimate $\|u - v\|_{L_t^a(\mathbb{R}:L_x^r)} < C\varepsilon$, which implies $u \in L_t^a(\mathbb{R} : L_x^r)$.

This lemma can be proved in the similar manner as the proof of Proposition 4.7 in [8]. For the completeness of the paper, we give a proof of this lemma in Appendix C.

5.4. Nonlinear profile decomposition

In order to construct a critical element, we construct so-called nonlinear profiles. By the following Lemma 5.9, 5.10, and 5.11, we can construct nonlinear profiles to (NLS_V) and can apply the perturbation lemma (Lemma 5.8) in order to prove existence of a critical element.

Lemma 5.9 *(Cauchy problem for (NLS) and nonlinear profile). Let $V \in L^1_1(\mathbb{R})$ be non-negative and $p \geq 5$, $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $|x_n| \to \infty$ as $n \to \infty$, $u_0 \in H^1(\mathbb{R})$ and $U \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L^a_t(\mathbb{R} : L^r_x(\mathbb{R}))$ be a solution to (NLS) with the initial data u_0, where (a, r) is defined by (5.3). Set $U_n(t, x) := U(t, x - x_n)$ and we write*

$$U_n(t,x) = \{e^{-itH_V}(\tau_{x_n}u_0)\}(x) + i\int_0^t e^{-i(t-s)H_V}\left\{|U_n(s,x)|^{p-1}U_n(s,x)\right\}ds + g_n(t,x).$$

Then we have

$$\lim_{n\to\infty} \|g_n\|_{L^a_t(\mathbb{R}:L^r_x)} = 0.$$

This lemma can be proved by combining Proposition 7 and Proposition 8 in [21]. For the completeness of the paper, we give a proof of the lemma in Appendix D.

Lemma 5.10 *(Final state problem for (NLS_V) and nonlinear profile). Let $\omega > 0$, $p \geq 5$, $V \in L^1_1(\mathbb{R})$ be non-negative, $\varphi \in \mathcal{H}$ satisfying*

$$\frac{1}{2}\|\varphi\|^2_{H_V^{\frac{1}{2}}} + \frac{\omega}{2}M(\varphi) < n_\omega, \tag{5.13}$$

and the exponent (a, r) be defined by (5.3). Then the following statements hold, where the following double-sign corresponds:

- *(Existence and Uniqueness) There exist only two solutions*

$$W_\pm \in C(\mathbb{R} : \mathcal{H}) \cap L^a_t(\mathbb{R}_\pm : L^r_x)$$

  to the final state problems

$$W_\pm(t) = e^{-itH_V}\varphi \pm \int_t^{\pm\infty} e^{-i(t-s)H_V}\{|W_\pm(s)|^{p-1}W_\pm(s)\}ds \tag{5.14}$$

  such that

$$\lim_{t\to\pm\infty} \left\|W_\pm(t) - e^{-itH_V}\varphi\right\|_{\mathcal{H}} = 0,$$

  where $\mathbb{R}_- := (-\infty, 0]$.
- *(Conservation laws) For any $t \in \mathbb{R}$, the identities*

$$M(W_\pm(t)) = M(\varphi), \quad E_V(W_\pm(t)) = \frac{1}{2}\|\varphi\|^2_{H_V^{\frac{1}{2}}}$$

hold, which implies that for any $t \in \mathbb{R}$, the identities

$$S_{\omega,V}(W_\pm(t)) = \frac{1}{2}\|\varphi\|^2_{H_V^{\frac{1}{2}}} + \frac{\omega}{2}M(\varphi) < n_\omega, \quad I_{\omega,V}(W_\pm(t)) = \|\varphi\|^2_{H_V^{\frac{1}{2}}} + \omega M(\varphi) \geq 0$$

hold, namely, for any $t \in \mathbb{R}$, $W_\pm(t) \in \mathcal{N}_\omega^+$.

- *(Nonlinear profiles) Let $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence such that $\lim_{n\to\infty} t_n = \mp\infty$, $\varphi_n := e^{it_nH_V}\varphi$ for any $n \in \mathbb{N}$ and $W_\pm$ be obtained in the above, $W_{\pm,n}(t,x) := W_\pm(t-t_n,x)$ for any $n \in \mathbb{N}$. We assume that $W_\pm \in L_t^a(\mathbb{R} : L_x^r)$. Then if we write*

$$W_{\pm,n}(t) = e^{-itH_V}\varphi_n + i\int_0^t e^{-i(t-s)H_V}\{|W_{\pm,n}(s)|^{p-1}W_{\pm,n}(s)\}ds + f_{\pm,n}(t), \tag{5.15}$$

then

$$\lim_{n\to\infty} \|f_{\pm,n}\|_{L_t^a(\mathbb{R}:L_x^r)} = 0. \tag{5.16}$$

Proof of Lemma 5.10. (Existence and Uniqueness) The proof of the existence and the uniqueness of solutions to the final state problem for (NLS_V) is based on the contraction mapping principle with the non-admissible Strichartz estimates (Lemma 5.3). We only consider the positive time direction, since the negative time direction can be treated in the similar manner.

For $T \geq 1$ and $\Theta > 0$, which will be determined later, we introduce the closed ball $X(T,\Theta)$ in $L_t^\infty(T,\infty : \mathcal{H}) \cap L_t^a(T,\infty : L_x^r)$ as

$$\begin{aligned} X(T,\Theta) : = \{u \in L_t^\infty(T,\infty : \mathcal{H}) \cap L_t^a(T,\infty : L_x^r) : \\ \|u - e^{-i(\cdot)H_V}\varphi\|_{L_t^\infty(T,\infty:\mathcal{H})} + \|u\|_{L_t^a(T,\infty:L_x^r)} \leq \Theta\Big\} \end{aligned}$$

with the metric

$$d_T(u,v) := \|u-v\|_{L_t^\infty(T,\infty:L_x^2)} + \|u-v\|_{L_t^a(T,\infty:L_x^r)}$$

for $u, v \in X(T,\Theta)$. We prove that the nonlinear mapping $J : X(T,\Theta) \mapsto X(T,\Theta)$

$$J[u](t) := e^{-itH_V}\varphi - i\int_t^\infty e^{-i(t-s)H_V}\{|u(s)|^{p-1}u(s)\}ds, \quad t \in [T,\infty),$$

is contractive on $X(T,\Theta)$, if T is sufficiently large and Θ is sufficiently small.

Let $u \in X(T,\Theta)$. The non-admissible Strichartz estimates (Lemma 5.3) give

$$\begin{aligned} \|J[u]\|_{L_t^a(T,\infty:L_x^r)} &\leq \|e^{-itH_V}\varphi\|_{L_t^a(T,\infty:L_x^r)} + C_0\||u|^{p-1}u\|_{L_t^{b'}(T,\infty:L_x^{r'})} \\ &= \|e^{-itH_V}\varphi\|_{L_t^a(T,\infty:L_x^r)} + C_0\|u\|^p_{L_t^a(T,\infty:L_x^r)} \\ &\leq \|e^{-itH_V}\varphi\|_{L_t^a(T,\infty:L_x^r)} + C_0\Theta^p, \end{aligned} \tag{5.17}$$

where C_0 is a positive constant depending only on p and V. Since $\varphi \in \mathcal{H} \subset H^1(\mathbb{R})$ and $a < \infty$, we can choose $\Theta = \Theta(p,V)$ and $T = T(\varphi,\Theta)$ such as

$$\Theta \le \left(\frac{1}{4C_0}\right)^{\frac{1}{p-1}} \quad \text{and} \quad \|e^{-itH_V}\varphi\|_{L_t^a(T,\infty:L_x^r)} \le \frac{1}{4}\Theta.$$

Then we have

$$\|J[u]\|_{L_t^a(T,\infty:L_x^r)} \le \frac{1}{2}\Theta. \tag{5.18}$$

In the same manner as the proof of the estimate (5.5), we have

$$\begin{aligned}\|u\|_{L_t^{p-1}(T,\infty:L_x^\infty)} &\le \|e^{-itH_V}\varphi\|_{L_t^{p-1}(T,\infty:L_x^\infty)} + C_1\|u\|^p_{L_t^a(T,\infty:L_x^r)} \\ &\le \|e^{-itH_V}\varphi\|_{L_t^{p-1}(T,\infty:L_x^\infty)} + C_1\Theta^p,\end{aligned}$$

where C_1 is a positive constant depending only on p and V. Thus since the estimate $(a+b)^{p-1} \le 2^{p-2}(a^{p-1} + b^{p-1})$ holds for any $a, b \ge 0$ due to $p \ge 5$, we have

$$\|u\|^{p-1}_{L_t^{p-1}(T,\infty:L_x^\infty)} \le C_2\|e^{-itH_V}\varphi\|^{p-1}_{L_t^{p-1}(T,\infty:L_x^\infty)} + C_3\Theta^{p(p-1)}, \tag{5.19}$$

where C_2 and C_3 are positive constants depending only on p and V. Here $\Theta = \Theta(p, V)$ and $T = T(\varphi, \Theta)$ are chosen sufficiently small such as

$$\Theta \le \left(\frac{1}{4C_3C_*}\right)^{\frac{1}{p(p-1)}} \quad \text{and} \quad \|e^{-itH_V}\varphi\|_{L_t^{p-1}(T,\infty:L_x^\infty)} \le \left(\frac{1}{4C_2C_*}\right)^{\frac{1}{p-1}},$$

where C_* is defined by (5.6). Then we have

$$\|u\|^{p-1}_{L_t^{p-1}(T,\infty:L_x^\infty)} \le \frac{1}{2C_*}. \tag{5.20}$$

By this estimate and in the same manner as the proof of the estimate (5.6), we obtain

$$\|J[u] - e^{-itH_V}\varphi\|_{L_t^\infty(T,\infty:\mathcal{H})} \le C_*\|u\|^{p-1}_{L_t^{p-1}(T,\infty:L_x^\infty)}\|u\|_{L_t^\infty(T,\infty:\mathcal{H})} \le \frac{1}{2}\Theta. \tag{5.21}$$

By combining the estimates (5.18) and (5.21), we obtain

$$\|J[u] - e^{-itH_V}\varphi\|_{L_t^\infty(T,\infty:\mathcal{H})} + \|J[u]\|_{L_t^a(T,\infty:L_x^r)} \le \Theta,$$

which implies that the mapping J is well defined on $X(T, \Theta)$. Next let $u, v \in X(T, \Theta)$. By using the estimate (5.8) and in the same manner as the proof of the estimate (5.17), we have

$$\begin{aligned}\|J[u] - J[v]\|_{L_t^a(T,\infty:L_x^r)} &\le C_0\||u|^{p-1}u + |v|^{p-1}v\|_{L_t^{b'}(T,\infty:L_x^{r'})} \\ &\le C_0 p 2^{p-1}\|(|u|^{p-1} + |v|^{p-1})|u - v|\|_{L_t^{b'}(T,\infty:L_x^{r'})} \\ &\le C_0 p 2^p \Theta^{p-1}\|u - v\|_{L_t^a(T,\infty:L_x^r)}.\end{aligned}$$

Here we choose $\Theta = \Theta(p, V)$ sufficiently small such as

$$\Theta \le \frac{1}{(4C_0 p 2^p)^{\frac{1}{p-1}}}.$$

Then we have

$$\|J[u] - J[v]\|_{L_t^a(T,\infty:L_x^r)} \le \frac{1}{4}\|u - v\|_{L_t^a(T,\infty:L_x^r)}. \tag{5.22}$$

Let $t \in [T, \infty)$. Since e^{-itH_V} is a unitary operator on $L^2(\mathbb{R})$, by the estimate (5.8), we have

$$\begin{aligned}
\|J[u](t) - J[v](t)\|_{L^2} &= \left\| \int_t^\infty e^{-i(t-s)H_V} \left\{ |u(s)|^{p-1}u(s) - |v(s)|^{p-1}v(s) \right\} ds \right\|_{L^2} \\
&\le \int_t^\infty \||u(s)|^{p-1}u(s) - |v(s)|^{p-1}v(s)\|_{L^2} ds \\
&\le p2^{p-1} \int_t^\infty \|(|u(s)|^{p-1} + |v(s)|^{p-1})|u(s) - v(s)|\|_{L^2} ds \\
&\le p2^{p-1} \int_t^\infty (\|u(s)\|_{L^\infty}^{p-1} + \|v(s)\|_{L^\infty}^{p-1})\|u(s) - v(s)\|_{L^2} ds \\
&\le p2^{p-1}(\|u\|^{p-1}_{L_t^{p-1}(t,\infty:L_x^\infty)} + \|v\|^{p-1}_{L_t^{p-1}(t,\infty:L_x^\infty)})\|u - v\|_{L_t^\infty(t,\infty:L_x^2)},
\end{aligned} \tag{5.23}$$

for $t \ge T$. Here we take $\Theta = \Theta(p, V)$ sufficiently small and $T = T(\varphi, \Theta)$ sufficiently large such as

$$\Theta \le \left(\frac{1}{C_3 p 2^{p+3}}\right)^{\frac{1}{p-1}} \quad \text{and } \|e^{-itH_V}\varphi\|_{L_t^{p-1}(T,\infty:L_x^\infty)} \le \left(\frac{1}{C_2 p 2^{p+3}}\right)^{\frac{1}{p-1}}.$$

Then by the estimate (5.19), we have

$$p2^{p-1}(\|u\|^{p-1}_{L_t^{p-1}(t,\infty:L_x^\infty)} + \|v\|^{p-1}_{L_t^{p-1}(t,\infty:L_x^\infty)}) \le p2^p C_2 \|e^{-itH_V}\varphi\|^{p-1}_{L_t^{p-1}(T,\infty:L_x^\infty} + p2^p C_3 \Theta^{p(p-1)} \le \frac{1}{4}.$$

By this estimate and (5.23), we obtain

$$\|J[u] - J[v]\|_{L_t^\infty(T,\infty:L^2)} \le \frac{1}{4}\|u - v\|_{L_t^\infty(T,\infty:L_x^2)}. \tag{5.24}$$

By combining the estimates (5.22) and (5.24), we have

$$d_T(J[u], J[v]) \le \frac{1}{2} d_T(u, v),$$

which implies that the mapping J is contractive. Thus by the contraction mapping principle, we see that there exists a unique solution $W_+ \in X(T, \Theta)$ such that $J[W_+](t) = W_+(t)$ on $[T, \infty)$. Moreover, since the operator e^{-itH_V} commutes with the Schrödinger operator $L_V^{\frac{1}{2}}$ and is unitary on $L^2(\mathbb{R})$ for any $t \in \mathbb{R}$, we have

$$\|W_+\|_{L_t^\infty(T,\infty:\mathcal{H})} \le \|W_+ - e^{-i(\cdot)H_V}\varphi\|_{L_t^\infty(T,\infty:\mathcal{H})} + \|e^{-i(\cdot)H_V}\varphi\|_{L_t^\infty(T,\infty\mathcal{H})} \le \Theta + \|\varphi\|_{\mathcal{H}},$$

which implies $W_+ \in L_t^\infty(T, \infty : \mathcal{H})$. We can also prove that $W_+ \in C([T, \infty) : \mathcal{H})$ in the standard argument, so we omit the detail. In the same manner as the proof of (5.6), we find that

$$\lim_{t\to\infty} \|W_+(t) - e^{-itH_V}\varphi\|_{\mathcal{H}} = 0. \tag{5.25}$$

Noting that $\mathcal{H} \subset L^2(\mathbb{R})$ and e^{-itH_V} is unitary on $L^2(\mathbb{R})$ again, the above relation implies that

$$\lim_{t\to\infty} \|W_+(t)\|_{L^2} = \|\varphi\|_{L^2}. \tag{5.26}$$

Let $t, t_1 \in [T, \infty)$. By the mass conservation law in Proposition 1.1, we have $M(W_+(t)) = M(W_+(t_1))$. Letting $t_1 \to \infty$ with (5.26), we obtain

$$M(W_+(t)) = M(\varphi). \tag{5.27}$$

Next we prove that

$$E_V(W_+(t)) = \frac{1}{2}\|\varphi\|^2_{H_V^{\frac{1}{2}}}, \quad \text{for any } t \in [T, \infty). \tag{5.28}$$

In order to prove this, we first show

$$\lim_{t\to\infty} \|W_+(t)\|_{L^{p+1}} = 0. \tag{5.29}$$

Indeed, take $\varepsilon > 0$ arbitrarily. Since $C_0^\infty(\mathbb{R})$ is dense in $\mathcal{H}$, there exists $\varphi_\varepsilon \in C_0^\infty(\mathbb{R})$ such that $\|\varphi - \varphi_\varepsilon\|_{\mathcal{H}} \le \frac{1}{3}\varepsilon$. Since $\varphi_\varepsilon \in L^{1+\frac{1}{p}}(\mathbb{R})$, we can apply the dispersive estimate for $\{e^{-itH_V}\}_{t\in\mathbb{R}}$ (Lemma 5.1), to get $\|e^{-itH_V}\varphi_\varepsilon\|_{L^{p+1}} \le C_4|t|^{-\frac{p-1}{2(p+1)}}\|\varphi_\varepsilon\|_{L^{1+\frac{1}{p}}}$ for any $t \neq 0$, where C_4 is a constant depending only on p and V. Here we choose $T_1 > 0$ such as

$$T_1 = T_1(\varepsilon, \varphi_\varepsilon) := \left(\frac{\varepsilon}{3C_4\|\varphi_\varepsilon\|_{L^{1+\frac{1}{p}}}}\right)^{-\frac{2(p+1)}{p-1}}$$

Then the above estimate gives that for any $t \ge T_1$, the estimate $\|e^{-itH_V}\varphi_\varepsilon\|_{L^{p+1}} \le \frac{1}{3}\varepsilon$ holds. By (5.25), there exists $T_2 = T_2(\varepsilon) \ge T$, such that for any $t \ge T_2$, the estimate $\|W_+(t) - e^{-itH_V}\varphi\|_{\mathcal{H}} \le \frac{1}{3}\varepsilon$ holds. Thus for any $t \ge \max(T_1, T_2)$, by combining these estimates, we have $\|W_+(t)\|_{L^{p+1}} \le \varepsilon$, which implies (5.29). Since the operator e^{-itH_V} commutes with the Schrödinger operator $H_V^{\frac{1}{2}}$ and is unitary on $L^2(\mathbb{R})$, by using the relations (5.25) and (5.29), we have

$$\lim_{t\to\infty}\left|E_V(W_+(t)) - \frac{1}{2}\|\varphi\|^2_{H_V^{\frac{1}{2}}}\right| \le \frac{1}{2}\lim_{t\to\infty}\|W_+(t) - e^{-itH_V}\varphi\|^2_{H_V^{\frac{1}{2}}} + \frac{1}{p+1}\lim_{t\to\infty}\|W_+(t)\|_{L^{p+1}} = 0, \tag{5.30}$$

which implies (5.28). Take $t, t_1 \in [T, \infty)$ arbitrarily. The energy conservation law in Proposition 1.1 gives $E_V(W_+(t)) = E_V(W_+(t_1))$. Thus letting $t_1 \to \infty$ with (5.30), we obtain (5.28). By the definition of $S_{\omega,V}$, the identities (5.27) and (5.28) and the assumption (5.13), we have

$$S_{\omega,V}(W_+(t)) = \frac{1}{2}\|\varphi\|^2_{H_V^{\frac{1}{2}}} + \frac{\omega}{2}\|\varphi\|^2_{L^2} < n_\omega \tag{5.31}$$

for $t \ge T$. In the same manner as the proof of the estimates (5.30) with the assumption (5.13), we can prove

$$I_\omega(W_+(t)) = \|\varphi\|^2_{H_V^{\frac{1}{2}}} + \omega\|\varphi\|^2_{L^2} \ge 0, \tag{5.32}$$

for $t \ge T$. The estimates (5.31) and (5.32) imply that $W_+(t) \in \mathcal{N}_\omega^+$ for any $t \in [T, \infty)$. Thus by Corollary 4.7, we find that W_+ can be extended globally and belongs to $C(\mathbb{R} : \mathcal{H})$. Next we prove that W_+ belongs to $L_t^a(0, \infty : L_x^r)$. Since $W_+ \in L_t^a(T, \infty : L_x^r)$, it suffices to prove that $W_+ \in L_t^a(0, T : L_x^r)$. Since W_+ satisfies the integral equation

$$W_+(t) = e^{-itH_V} W_+(0) + i \int_0^t e^{-i(t-s)H_V} \{|W_+(s)|^{p-1} W_+(s)\} ds$$

in $\mathcal{H}$-sense, by the non-admissible Strichartz estimates (Lemma 5.3) and the Sobolev embedding $\mathcal{H} \subset H^1(\mathbb{R}) \subset L^p(\mathbb{R})$, we have

$$\begin{aligned}
&\|W_+\|_{L_t^a(0,T:L_x^r)} \\
&\leq \|e^{-itH_V} W_+(0)\|_{L_t^a(0,T:L_x^r)} + \left\| \int_0^t e^{-i(t-s)H_V} \{|W_+(s)|^{p-1} W(s)\} ds \right\|_{L_t^a(0,T:L_x^r)} \\
&\leq C\|W_+(0)\|_{\mathcal{H}} + C\||W_+|^{p-1} W_+\|_{L_t^{\gamma'}(0,T:L_x^1)} = C\|W_+(0)\|_{\mathcal{H}} + C\|\|W_+(t)\|_{L_x^p}\|_{L_t^{\gamma'}(0,T)} \\
&\leq C\|W_+(0)\|_{\mathcal{H}} + C\|\|W_+(t)\|_{\mathcal{H}}\|_{L_t^{\gamma'}(0,T)} \leq C\|W_+(0)\|_{\mathcal{H}} + CT^{\frac{1}{\gamma'}} \|W_+\|_{L_t^\infty(0,T:\mathcal{H})} < \infty,
\end{aligned}$$

which implies that $W_+ \in L_t^a(0, T : L_x^r)$.
(Nonlinear profile) We prove (5.16) only for the positive time direction, since the negative time direction can be proved in the similar manner. Since W_+ satisfies the integral equation (5.14) on $[0, \infty)$, by the definition (5.15) of $f_{+,n}$ and a simple calculation, $f_{+,n}$ satisfies

$$f_{+,n}(t, x) = e^{-itH_V} \{W_+(-t_n) - \varphi_n\}(x) = e^{-itH_V} \{W_+(-t_n) - e^{it_n H_V} \varphi\}(x),$$

for $(t, x) \in \mathbb{R} \times \mathbb{R}$. Thus by using the non-admissible Strichartz estimate (Lemma 5.3), the identity (5.25) and $\lim_{n\to\infty} t_n = -\infty$, we have (5.16), which completes the proof of the proposition. □

Lemma 5.11 *(Final state problem for (NLS) and nonlinear profile). Let $\omega > 0$, $V \in L_1^1(\mathbb{R})$ be non-negative, $p \geq 5$, $\varphi \in H^1(\mathbb{R})$ satisfying*

$$\frac{1}{2}\|\partial_x \varphi\|_{L^2}^2 + \frac{\omega}{2} M(\varphi) < n_\omega.$$

Then the following statements hold, where double-sign corresponds:

- *(Existence and Uniqueness) Then there exist only two solutions*

  $$Y_\pm \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R}_\pm : L_x^r(\mathbb{R}))$$

  to (NLS) such that the identity

  $$\lim_{t\to\pm\infty} \left\|Y_\pm(t) - e^{-itH_0}\varphi\right\|_{H^1},$$

  holds.
- *(Conservation laws) For any $t \in \mathbb{R}$, the identities*

  $$M(Y_\pm(t)) = M(\varphi), \quad E_0(Y_\pm(t)) = \frac{1}{2}\|\partial_x \varphi\|_{L^2}^2$$

  hold, which implies that for any $\omega > 0$, $t \in \mathbb{R}$, the identities

  $$S_{\omega,0}(Y_\pm(t)) = \frac{1}{2}\|\partial_x \varphi\|_{L^2}^2 + \frac{\omega}{2}\|\varphi\|_{L^2}^2 \quad \textit{and} \quad I_{\omega,0}(Y_\pm(t)) = \|\partial_x \varphi\|_{L^2}^2 + \omega\|\varphi\|_{L^2}^2$$

  hold.

- *(Nonlinear profile) Let $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be sequences of real numbers such that*

$$\lim_{n\to\infty} t_n = \mp\infty \text{ and } \lim_{n\to\infty} |x_n| = \infty.$$

We write

$$Y_{\pm,n}(t,x) = e^{-itH_V}\varphi_n + i\int_0^t e^{-i(t-s)H_V}\{|Y_{\pm,n}(s)|^{p-1}Y_{\pm,n}(s)\}ds + e_{\pm,n}(t,x),$$

for $t \in \mathbb{R}$ and $x \in \mathbb{R}$, where $\varphi_n = e^{it_nH_V}\tau_{x_n}\varphi$, $Y_{\pm,n}(t,x) = Y_\pm(t-t_n, x-x_n)$. Then

$$\|e_{\pm,n}\|_{L_t^a L_x^r} \to 0$$

as $n \to \infty$.

Proof of Lemma 5.11. Noting that $n_\omega = l_\omega$ due to $V \in L^1(\mathbb{R})$, Existence and Uniqueness part and Conservation laws part can be proved in the same manner as the proof of Lemma 5.10. Nonlinear profile part is proved in the similar manner as the proof of Proposition 9 in [21]. □

5.5. Construction of a critical element

In this subsection, we construct a so-called critical element, the definition of which is given in Theorem 5.12, under the assumption that the scattering result in Theorem 1.3 does not hold. For $\omega > 0$, we define the critical action level S_ω^c as follows:

$$S_\omega^c := \sup\{S : S_{\omega,V}(\varphi) < S \text{ for any } \varphi \in \mathcal{N}_\omega^+ \text{ implies } u \in L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))\},$$

where u is a unique global solution to (NLS$_V$) with $u|_{t=0} = \varphi$ and the exponents a, r are defined by (5.3). The fact that the solution u to (NLS$_V$) can be extended globally follows from Corollary 4.7. By the small data scattering result (Corollary 5.6) and the equivalency between $\|\cdot\|_{\mathcal{H}}$ and $S_{\omega,V}(\cdot)$ (Lemma 4.5) due to $I_{\omega,V}(\cdot) \geq 0$, we see that $S_\omega^c > 0$. By the contradiction argument, we will prove that $S_\omega^c \geq n_\omega$ in the following, which completes the proof of the scattering part in Theorem 1.3. We assume that $S_\omega^c < n_\omega$. Then by the definition of S_ω^c, we can take a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\omega^+$ such that $S_{\omega,V}(\varphi_n) \searrow S_\omega^c$ as $n \to \infty$, and $\|u_n\|_{L_t^a(\mathbb{R}:L_x^r(\mathbb{R}))} = \infty$ for all $n \in \mathbb{N}$, where u_n is a unique global solution to (NLS$_V$) with the initial data $\varphi_n \in \mathcal{N}_\omega^+$. In the following, we prove that up to subsequence, the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ converges strongly in $\mathcal{H}$ as $n \to \infty$.

Theorem 5.12 *(Existence of a critical element). Let $\omega > 0$, $p > 5$ and $V \in L_1^1(\mathbb{R})$ be non-negative satisfy $V' \in L_1^1(\mathbb{R})$. We assume that $S_\omega^c < n_\omega$. Then there exists a global solution $u^c \in C(\mathbb{R} : \mathcal{H})$ to (NLS$_V$) such that $u^c(t) \in \mathcal{N}_\omega^+$ for any $t \in \mathbb{R}$ and the identities holds:*

$$S_\omega(u^c(t)) = S_\omega^c, \text{ for } t \in \mathbb{R}, \quad \|u^c\|_{L_t^a(\mathbb{R}:L_x^r(\mathbb{R}))} = \infty.$$

This solution u^c is called critical element.

Proof of Theorem 5.12. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be same as in the above. Since $\varphi_n \in \mathcal{N}_\omega^+$ for any $n \in \mathbb{N}$ and $V \in L_1^1(\mathbb{R})$ is non-negative, from Lemma 4.5, we see that there exists a constant $C > 0$ depending only on ω, p, V such that

$$\|\varphi_n\|_{\mathcal{H}}^2 \le CS_{\omega,V}(\varphi_n) < Cn_\omega$$

for all $n \in \mathbb{N}$, which implies that $\{\varphi_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$. This allows us to apply the linear profile decomposition (Proposition 5.7), to obtain

$$\varphi_n = \sum_{j=1}^{J} e^{it_n^j H_V} \tau_{x_n^j} \psi^j + W_n^J, \quad \forall J \in \mathbb{N}, \tag{5.33}$$

up to subsequence, where $\{t_n^j\}_{n=1}^\infty \subset \mathbb{R}$, $\{x_n^j\}_{n=1}^\infty \subset \mathbb{R}$ and $\psi^j \in H^1(\mathbb{R})$ are given in Proposition 5.7. Set $\delta = \delta(\omega) := \frac{1}{2}(n_\omega - S_\omega^c)$. Then by the assumption $S_\omega^c < n_\omega$, the estimates $\delta > 0$ and $S_\omega^c + \delta < n_\omega$ hold. Moreover, since $S_{\omega,V}(\varphi_n) \searrow S_\omega^c$ as $n \to \infty$, for sufficiently large $n \in \mathbb{N}$, $S_{\omega,V}(\varphi_n) + \delta \le n_\omega$. By the orthogonality of the action $S_{\omega,V}$ and the Nehari functional $I_{\omega,V}$ in Proposition 5.7, we have

$$S_{\omega,V}(\varphi_n) = \sum_{j=1}^{J} S_{\omega,V}\left(e^{it_n^j H_V} \tau_{x_n^j} \psi^j\right) + S_{\omega,V}(W_n^J) + o_n(1), \tag{5.34}$$

$$I_{\omega,V}(\varphi_n) = \sum_{j=1}^{J} I_{\omega,V}\left(e^{it_n^j H_V} \tau_{x_n^j} \psi^j\right) + I_{\omega,V}(W_n^J) + o_n(1), \tag{5.35}$$

as $n \to \infty$, where $o_n(1) \to 0$ as $n \to \infty$. Here we set $\varepsilon = \varepsilon(\delta) := \frac{1}{4}\delta > 0$. Then we have $\varepsilon < \frac{1}{2}\delta$. Then by the identities (5.34) and (5.35) and $\varphi_n \in \mathcal{N}_\omega^+$ for any $n \in \mathbb{N}$, we have

$$\begin{aligned}
&S_{\omega,V}(\varphi_n) \le n_\omega - \delta,\\
&S_{\omega,V}(\varphi_n) \ge \sum_{j=1}^{J} S_{\omega,V}\left(e^{it_n^j H_V} \tau_{x_n^j} \psi^j\right) + S_{\omega,V}(W_n^J) - \varepsilon,\\
&I_{\omega,V}(\varphi_n) \ge -\varepsilon,\\
&I_{\omega,V}(\varphi_n) \le \sum_{j=1}^{J} I_{\omega,V}\left(e^{it_n^j H_V} \tau_{x_n^j} \psi^j\right) + I_{\omega,V}(W_n^J) + \varepsilon,
\end{aligned}$$

for sufficiently large n. Therefore, we can apply Lemma 4.8, to obtain for $j \in \{0, \cdots, J\}$,

$$e^{it_n^j H_V} \tau_{x_n^j} \psi^j \in \mathcal{N}_\omega^+ \text{ and } W_n^J \in \mathcal{N}_\omega^+, \text{ for sufficiently large } n,$$

and

$$S_{\omega,V}(e^{it_n^j H_V} \tau_{x_n^j} \psi^j) \ge 0 \text{ and } S_{\omega,V}(W_n^J) \ge 0, \text{ for sufficiently large } n. \tag{5.36}$$

Thus by combining (5.34) and (5.36), we have

$$S_\omega^c = \limsup_{n\to\infty} S_{\omega,V}(\varphi_n) \ge \limsup_{n\to\infty} \sum_{j=1}^{J} S_{\omega,V}\left(e^{it_n^j H_V} \tau_{x_n^j} \psi^j\right), \tag{5.37}$$

for any $J \in \mathbb{N}$. In the following, we will prove that

$$S_\omega^c = \limsup_{n\to\infty} S_{\omega,V}\left(e^{it_n^j H_V} \tau_{x_n^j} \psi^j\right) \text{ for some } j \in \{1, \cdots, J\}. \tag{5.38}$$

We may assume that $j = 1$ by reordering. If this is proved, then we can prove that

$$J = 1 \quad \text{and} \quad \lim_{n\to\infty} \|W_n^J\|_{H^1} = 0. \tag{5.39}$$

Indeed, if $J \geq 2$, then by (5.37) and (5.38), we have

$$S_\omega^c \geq \limsup_{n\to\infty} \left\{ S_{\omega,V}\left(e^{it_n^1 H_V}\tau_{x_n^1}\psi^1\right) + \sum_{j=2}^{J} S_{\omega,V}\left(e^{it_n^j H_V}\tau_{x_n^j}\psi^j\right) \right\} > S_\omega^c,$$

which leads to a contradiction. Thus we have $J = 1$. Moreover (5.34), (5.37) and (5.38) give

$$\limsup_{n\to\infty} S_{\omega,V}(W_n^1) = 0. \tag{5.40}$$

Since W_n^1 belongs to $\mathcal{N}_\omega^+$ for sufficiently large $n \in \mathbb{N}$, by combining Lemma 4.5 and (5.40), we obtain

$$\limsup_{n\to\infty} \left\|W_n^1\right\|_{H^1} = 0.$$

Now we give a proof of (5.38). On the contrary, we assume that for any $j \in \{1, \cdots, J\}$,

$$S_\omega^c \neq \limsup_{n\to\infty} S_{\omega,V}(e^{it_n^j H_V}\tau_{x_n^j}\psi^j).$$

Then for any $j \in \{1, \cdots, J\}$, by the estimate (5.37), there exists $\delta_j > 0$ such that

$$\limsup_{n\to\infty} S_{\omega,V}(e^{it_n^j H_V}\tau_{x_n^j}\psi^j) < S_\omega^c - \delta_j. \tag{5.41}$$

By reordering, we can choose $0 \leq J_1 \leq J_2 \leq J_3 \leq J_4 \leq J_5 \leq J$ such that

$$\begin{array}{rccc}
1 \leq j \leq J_1 : & t_n^j = 0,\ \forall n & \text{and} & x_n^j = 0,\ \forall n, \\
J_1 + 1 \leq j \leq J_2 : & t_n^j = 0,\ \forall n & \text{and} & \lim_{n\to\infty} |x_n^j| = \infty, \\
J_2 + 1 \leq j \leq J_3 : & \lim_{n\to\infty} t_n^j = +\infty, & \text{and} & x_n^j = 0,\ \forall n, \\
J_3 + 1 \leq j \leq J_4 : & \lim_{n\to\infty} t_n^j = -\infty, & \text{and} & x_n^j = 0,\ \forall n, \\
J_4 + 1 \leq j \leq J_5 : & \lim_{n\to\infty} t_n^j = +\infty, & \text{and} & \lim_{n\to\infty} |x_n^j| = \infty, \\
J_5 + 1 \leq j \leq J : & \lim_{n\to\infty} t_n^j = -\infty, & \text{and} & \lim_{n\to\infty} |x_n^j| = \infty.
\end{array}$$

In the above cases, we assume that if $a > b$, then there is no j such that $a \leq j \leq b$. We see that $J_1 \in \{0, 1\}$ from the orthogonality of the parameters (5.10). In the following, we only treat the case $J_1 = 1$, since the case of $J_1 = 0$ is easier to treat. Then by (5.41), we have $0 < S_{\omega,V}(\psi^1) < S_\omega^c - \delta_1$, since $(t_n^1, x_n^1) = (0, 0)$ for all $n \in \mathbb{N}$. Hence, by the definition of the critical action level S_ω^c, we can construct a solution $N \in C(\mathbb{R} : \mathcal{H}) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$ to (NLS$_V$) with $N|_{t=0} = \psi^1$, i.e.

$$N(t,x) = (e^{-itH_V}\psi^1)(x) + i\int_0^t e^{-i(t-s)H_V}\{|N(s,x)|^{p-1}N(s,x)\}ds.$$

For every j such that $J_1 + 1 \leq j \leq J_2$, let U^j be the solution of (NLS) with the initial data $\psi^j \in H^1(\mathbb{R})$. Since V is non-negative, by the estimate (5.41) and Proposition 4.3, we see that for sufficiently large n,

$$S_{\omega,0}(\psi^j) = S_{\omega,0}(\tau_{x_n^j}\psi^j) \leq S_{\omega,V}(\tau_{x_n^j}\psi^j) \leq S_\omega^c - \delta_j < n_\omega = l_\omega.$$

Since the identities

$$I_{\omega,0}(\psi^j) = I_{\omega,0}(\tau_{x_n^j}\psi^j) = I_{\omega,V}(\tau_{x_n^j}\psi^j) - \int_{-\infty}^{\infty} V(x)|\psi^j(x-x_n^j)|^2 dx$$

hold, and $\tau_{x_n^j}\psi^j$ belongs to $\mathcal{N}_\omega^+$ for sufficiently large $n \in \mathbb{N}$ and $V \in L^1(\mathbb{R})$ and $\psi^j \in H^1(\mathbb{R})$, we have

$$\begin{aligned} I_{\omega,0}(\psi^j) &\geq \liminf_{n\to\infty} I_{\omega,V}(\tau_{x_n^j}\psi^j) - \limsup_{n\to\infty} \int_{-\infty}^{\infty} V(x)|\psi^j(x-x_n^j)|^2 dx \\ &\geq 0 - 0 = 0, \end{aligned}$$

where we have used the Lebesgue convergence theorem. Therefore, we see that the solution U^j to (NLS) belongs to $\in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$ from the scattering result obtained in [8] or [1]. For $j \in \mathbb{N}$ satisfying $J_1 + 1 \leq j \leq J_2$ and $n \in \mathbb{N}$, we set

$$U_n^j(t,x) := U^j(t, x - x_n^j), \quad \text{for } t \in \mathbb{R}, x \in \mathbb{R},$$

and we write U_n^j as

$$U_n^j(t,x) = e^{-itH_V}(\tau_{x_n^j}\psi^j)(x) + i\int_0^t e^{-i(t-s)H_V}\{|U_n^j(s,x)|^{p-1}U_n^j(s,x)\}ds + g_n^j(t,x).$$

For every j such that $J_2 + 1 \leq j \leq J_3$, since $\psi^j \in H^1(\mathbb{R}) \subset \mathcal{H}$, by Lemma 5.10, we can construct a solution $W_-^j \in C((-\infty, T) : \mathcal{H}) \cap L_t^a(-\infty, T : L_x^r(\mathbb{R}))$ to the final state problem of

$$W_-^j(t,x) = (e^{-itH_V}\psi^j)(x) + i\int_{-\infty}^t e^{-i(t-s)H_V}|W_-^j(s,x)|^{p-1}W_-^j(s,x)ds$$

on $(-\infty, T)$, where T denotes the maximal existence time of the function W_-. We prove that $T = \infty$ and

$$W_-^j \in C(\mathbb{R} : \mathcal{H}) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R})).$$

Let $t_0 \in (-\infty, T)$. We note that the identity

$$\lim_{t\to-\infty} \|W_-^j(t) - e^{-itH_V}\varphi^j\|_{\mathcal{H}} = 0 \tag{5.42}$$

holds by Lemma 5.10. Since $t_n^j \to \infty$ as $n \to \infty$ in this case, by the Sobolev embedding $\mathcal{H} \subset L^{p+1}(\mathbb{R})$, the estimates (5.42) and (5.41), the conservation laws and the assumption $S_\omega^c < n_\omega$, we have

$$S_{\omega,V}(W_-^j(t_0)) = \lim_{n\to\infty} S_{\omega,V}(e^{it_n^j H_V}\psi^j) < S_\omega^c - \delta_j < S_\omega^c < n_\omega. \tag{5.43}$$

Next we prove that

$$I_{\omega,V}(W_-^j(t_0)) \geq 0. \tag{5.44}$$

On the contrary, we assume that $I_{\omega,V}(W_-^j(t_0)) < 0$. In the same manner as the proof of (5.43), we have

$$\lim_{n\to\infty}|I_{\omega,V}(W_-^j(-t_n^j))-I_{\omega,V}(e^{itH_V}\psi^j)|=0.$$

Since $e^{itH_V}\psi^j\in\mathcal{N}_\omega^+$ and $\psi^j\neq 0$, we also have $I_{\omega,V}(e^{itH_V}\psi^j)>0$. Thus there exists $n_0\in\mathbb{N}$ such that $I_{\omega,V}(W_-^j(-t_{n_0}^j))>0$. By the continuity of the function $t\in\mathbb{R}\mapsto I_{\omega,V}(W_-^j(t))\in\mathbb{R}$, there exists $n_1\in\mathbb{N}$ such that $I_{\omega,V}(W_-^j(-t_{n_1}^j))=0$. By the definition of n_ω, the conservation laws and the estimate (5.43), we have

$$n_\omega\le S_{\omega,V}(W_-^j(-t_{n_1}^j))=S_{\omega,V}(W_-^j(t_0))<n_\omega,$$

which leads to a contradiction. Therefore we obtain (5.44). Thus by the estimates (5.43) and (5.44), we find that $W_-^j(t_0)\in\mathcal{N}_\omega^+$. Thus we see that $T=\infty$ and $W_-^j\in C(\mathbb{R}:H^1(\mathbb{R}))$ from Corollary 4.7. Moreover, by the estimate (5.43), the conservation laws and the definition of the critical action level S_ω^c, we see that $W_-^j\in L_t^a(\mathbb{R}:L_x^r(\mathbb{R}))$. Then for $j\in\mathbb{N}$ satisfying $J_2+1\le j\le J_3$, we define

$$W_{-,n}^j(t,x):=W_-^j(t-t_n^j,x),\quad\text{for } t\in\mathbb{R},x\in\mathbb{R},$$

and we write $W_{-,n}^j$ as

$$W_{-,n}^j(t,x)=\{e^{-itH_V}(e^{it_n^jH_V}\varphi^j)\}(x)+i\int_0^t e^{-i(t-s)H_V}\{|W_{-,n}^j(s,x)|^{p-1}W_{-,n}^j(s,x)\}ds+f_{-,n}^j(t,x).$$

For every j such that $J_3+1\le j\le J_4$, in the same manner as in the case $J_2+1\le j\le J_3$, by the function $\psi^j\in H^1(\mathbb{R})\subset\mathcal{H}$, we can construct a solution $W_+^j\in C(\mathbb{R}_+:H^1(\mathbb{R}))\cap L_t^a(\mathbb{R}_+:L_x^r(\mathbb{R}))$ to the final state problem of

$$W_+^j(t,x)=\left(e^{-itH_V}\psi^j\right)(x)-i\int_t^\infty e^{-i(t-s)H_V}|W_+^j(s,x)|^{p-1}W_+^j(s,x)ds$$

on $\mathbb{R}^2$. For $j\in\mathbb{N}$ satisfying $J_3+1\le j\le J_4$, we define

$$W_{+,n}^j(t,x):=W_+^j(t-t_n^j,x),\quad\text{for } t\in\mathbb{R},x\in\mathbb{R},$$

and write $W_{+,n}^j$ as

$$W_{+,n}^j(t,x)=\{e^{-itH_V}(e^{it_n^jH_V}\varphi^j)\}(x)+i\int_0^t e^{-i(t-s)H_V}\{|W_{+,n}^j(s,x)|^{p-1}W_{+,n}^j(s,x)\}ds+f_{+,n}^j(t,x).$$

For every j such that $J_4+1\le j\le J_5$, by using $\psi^j\in H^1(\mathbb{R})$, we can construct a solution $Y_-^j\in C((-\infty,T):H^1(\mathbb{R}))\cap L_t^a(-\infty,T:L_x^r(\mathbb{R}))$ to the final state problem of

$$Y_-^j(t,x)=\left(e^{-itH_0}\psi^j\right)(x)+i\int_{-\infty}^t e^{-i(t-s)H_0}|Y_-^j(s,x)|^{p-1}Y_-^j(s,x)ds,$$

on $(t,x)\in(-\infty,T)\times\mathbb{R}$, where T denotes the maximal existence time of the function Y_-^j, by Lemma 5.11. We prove that $T=\infty$ and $Y_-^j\in C(\mathbb{R}:H^1(\mathbb{R}))\cap L_t^a(\mathbb{R}:L_x^r(\mathbb{R}))$. Let $t_0\in(-\infty,T)$. Since $p+1>2$, $\psi^j\in H^1(\mathbb{R})$ and $t_n^j\to\infty$ as $n\to\infty$ in this case, in the same manner as the proof of (3.3) in [21], we can prove

$$\lim_{n\to\infty} \|e^{it_n^j H_V}\tau_{x_n^j}\psi^j\|_{L^{p+1}} = 0. \tag{5.45}$$

Since the solution operator e^{-itH_V} commutes with the fractional operator $H_V^{\frac{1}{2}}$ and is unitary on $L^2(\mathbb{R})$ for any $t \in \mathbb{R}$ and V is non-negative, by the conservation law in Lemma 5.11 and the translation invariance in L^2-norm, we obtain for any $n \in \mathbb{N}$,

$$\begin{aligned} S_{\omega,0}(Y_-^j(t_0)) &= \frac{1}{2}\|\partial_x\psi^j\|_{L^2}^2 + \frac{\omega}{2}\|\psi^j\|_{L^2}^2 = \frac{1}{2}\|\partial_x\tau_{x_n^j}\psi^j\|_{L^2}^2 + \frac{\omega}{2}\|\tau_{x_n^j}\psi^j\|_{L^2}^2 \\ &\leq \frac{1}{2}\|H_V^{\frac{1}{2}}\tau_{x_n^j}\psi^j\|_{L^2}^2 + \frac{\omega}{2}\|\tau_{x_n^j}\psi\|_{L^2}^2 \\ &= S_{\omega,V}\left(e^{it_n^j H_V}\tau_{x_n^j}\psi^j\right) + \frac{1}{p+1}\|e^{it_n^j H_V}\tau_{x_n^j}\psi^j\|_{L^{p+1}}^{p+1}. \end{aligned}$$

By this estimate, (5.45), the relation (5.41) and Proposition 4.3, we obtain

$$\begin{aligned} S_{\omega,0}(Y_-^j(t_0)) &\leq \limsup_{n\to\infty} S_{\omega,V}\left(e^{it_n^j H_V}\tau_{x_n^j}\psi^j\right) + \frac{1}{p+1}\limsup_{n\to\infty}\|e^{it_n^j H_V}\tau_{x_n^j}\psi^j\|_{L^{p+1}}^{p+1} \\ &\leq S_\omega^c - \delta_j < n_\omega = l_\omega. \end{aligned}$$

In the same manner as above, we can prove $I_{\omega,0}(Y_-^j(t_0)) \geq 0$. Thus we can apply the scattering result for (NLS) obtained in [8] or [1] to find that $T = \infty$ and $Y_-^j \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$. For $j \in \mathbb{N}$ such that $J_4 + 1 \leq j \leq J_5$, set

$$Y_{-,n}^j(t,x) := Y_-^j(t - t_n^j, x - x_n^j), \quad \text{for } t \in \mathbb{R}, x \in \mathbb{R}$$

and write $Y_{-,n}^j$ as

$$Y_{-,n}^j(t,x) = \{e^{-itH_V}(e^{it_n^j H_V}\psi^j)\}(x) + i\int_0^t e^{-i(t-s)H_V}\{Y_{-,n}^j(s,x)|^{p-1}Y_{-,n}^j(s,x)\}ds + e_{-,n}^j(t,x).$$

For every j such that $J_5 + 1 \leq j \leq J$, in the same manner as in the case $J_4 + 1 \leq j \leq J_5$, by using $\psi^j \in H^1(\mathbb{R})$, we can construct a solution $Y_+^j \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$ to the final state problem

$$Y_+^j(t,x) = \left(e^{-itH_0}\psi^j\right)(x) - i\int_t^\infty e^{-i(t-s)H_0}\{|Y_+^j(s,x)|^{p-1}Y_+^j(s,x)\}ds,$$

on $\mathbb{R}^2$. In the same manner as the proof of above, we can prove $Y_+^j \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$. For $j \in \mathbb{N}$ such that $J_5 + 1 \leq j \leq J$, set

$$Y_{+,n}^j(t,x) := Y_+^j(t - t_n^j, x - x_n^j), \quad \text{for } t \in \mathbb{R},\ x \in \mathbb{R},$$

and write $Y_{+,n}^j$ as

$$Y_{+,n}^j(t,x) = \{e^{-itH_V}(e^{it_n^j H_V}\psi^j)\}(x) + i\int_0^t e^{-i(t-s)H_V}\{|Y_{+,n}^j(s,x)|^{p-1}Y_{+,n}^j(s,x)\}ds + e_{+,n}^j(t,x).$$

For $J \in \mathbb{N}$ and $n \in \mathbb{N}$, we introduce an approximate solution Z_n^J to (NLS$_V$), which is called nonlinear profile, a function z_n^J and a remainder term r_n^J as

$$Z_n^J(t,x) := N(t,x) + \sum_{j=J_1+1}^{J_2} U_n^j(t,x) + \sum_{j=J_2+1}^{J_3} W_{-,n}^j(t,x) + \sum_{j=J_3+1}^{J_4} W_{+,n}^j(t,x) \tag{5.46}$$

$$+ \sum_{j=J_4+1}^{J_5} Y_{-,n}^j(t,x) + \sum_{j=J_5+1}^{J_6} Y_{+,n}^j(t,x),$$

$$z_n^J(t,x) := \int_0^t e^{-i(t-s)H_V} \left\{ |N(s,x)|^{p-1}N(s,x) + \sum_{j=J_1+1}^{J_2} |U_n^j(s,x)|^{p-1}U_n^j(s,x) \right.$$

$$+ \sum_{j=J_2+1}^{J_3} |W_{-,n}^j(s,x)|^{p-1}W_{-,n}^j(s,x) + \sum_{j=J_3+1}^{J_4} |W_{+,n}^j(s,x)|^{p-1}W_{+,n}^j(s,x)$$

$$\left. + \sum_{j=J_4+1}^{J_5} |Y_{-,n}^j(s,x)|^{p-1}Y_{-,n}^j(s,x) + \sum_{j=J_5+1}^{J} |Y_{+,n}^j(s,x)|^{p-1}Y_{+,n}^j(s,x) \right\} ds,$$

$$r_n^J(t,x) := \sum_{j=J_1+1}^{J_2} g_n^j(t,x) + \sum_{j=J_2+1}^{J_3} f_{+,n}^j(t,x) + \sum_{j=J_3+1}^{J_4} f_{-,n}^j(t,x) + \sum_{j=J_4+1}^{J_5} e_{-,n}^j + \sum_{j=J_5+1}^{J_6} e_{+,n}^j,$$

respectively, for $(t,x) \in \mathbb{R} \times \mathbb{R}$. By the identity (5.33), the function Z_n^J can be written as

$$Z_n^J(t,x) = \{e^{-itH_V}(\varphi_n - W_n^J)\}(x) + iz_n^J(t,x) + r_n^J(t,x).$$

For any $J \in \mathbb{N}$, Lemmas 5.9, 5.10, 5.11 give

$$\lim_{n\to\infty} \|r_n^J\|_{L_t^a(\mathbb{R}:L_x^r)} = 0. \tag{5.47}$$

Since each term in the right-hand side of (5.46) belongs to $C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$ and we have the orthogonality of the parameters (5.10), we can apply Corollary 5.1 with $N = J$ in [3] with the non-admissible Strichartz estimate (Lemma 5.3), to obtain

$$\lim_{n\to\infty} \left\| z_n^J - \int_0^t e^{-i(t-s)H_V} \left\{ |Z_n^J(s)|^{p-1} Z_n^J(s) \right\} ds \right\|_{L_t^a(\mathbb{R}:L_x^r)} = 0. \tag{5.48}$$

Therefore, if we write Z_n^J as

$$Z_n^J(t,x) = \{e^{-itH_V}(\varphi_n - W_n^J)\}(x) + i\int_0^t e^{-i(t-s)H_V}\{|Z_n^J(s,x)|^{p-1}Z_n^J(s,x)\}ds + s_n^J(t,x),$$

then by the identities (5.47) and (5.48), we have

$$\lim_{n\to\infty} \|s_n^J\|_{L_t^a(\mathbb{R}:L_x^r)} = 0.$$

In order to apply the perturbation lemma (Lemma 5.8 with $v = Z_n^J$, $\varphi = \varphi_n - W_n^J$, $\varphi_0 = -W_n^J$ and $e = s_n^J$), we have to show a bound on $\sup_{J\in\mathbb{N}} \left\{ \limsup_{n\to\infty} \|Z_n^J\|_{L_t^a(\mathbb{R}:L_x^r)} \right\}$. We can apply Corollary 5.2 in [3] in this case to obtain

$$\begin{aligned}\limsup_{n\to\infty}\left\|Z_n^J\right\|^p_{L^a_t(\mathbb{R}:L^r_x)} &\le 2\left\|N\right\|^p_{L^a_t(\mathbb{R}:L^r_x)} + 2\sum_{j=J_1+1}^{J_2}\left\|U^j\right\|^p_{L^a_t(\mathbb{R}:L^r_x)}\\ &+2\sum_{j=J_2+1}^{J_3}\left\|W^j_-\right\|^p_{L^a_t(\mathbb{R}:L^r_x)} + 2\sum_{j=J_3+1}^{J_4}\left\|W^j_+\right\|^p_{L^a_t(\mathbb{R}:L^r_x)}\\ &+2\sum_{j=J_4+1}^{J_5}\left\|V^j_-\right\|^p_{L^a_t(\mathbb{R}:L^r_x)} + 2\sum_{j=J_5+1}^{J}\left\|V^j_+\right\|^p_{L^a_t(\mathbb{R}:L^r_x)},\\ &=:\sum_{j=1}^{J}a^j,\end{aligned}$$

where we have used the translation invariance of the norm $L^a_t(\mathbb{R}:L^r_x)$ with respect to time and space and we set $a^j := 2\left\|N\right\|^p_{L^a_t(\mathbb{R}L^r_x)}$ if $1\le j\le J_1$, $2\left\|U^j\right\|^p_{L^a_t(\mathbb{R}L^r_x)}$ if $J_1+1\le j\le J_2$, and so on.

Since $\{\varphi_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$ and V is non-negative, by the orthogonality of the L^2-norm and $H_V^{\frac{1}{2}}$-norm in the linear profile decomposition (Proposition 5.7), we find that there exists a finite set $\mathcal{J}\subset\{1,2,\cdots,J\}$ such that $\left\|\psi^j\right\|_{H^1}\le\|\psi^j\|_{\mathcal{H}}\le\varepsilon_0$ for any $j\notin\mathcal{J}$, where ε_0 is a constant given in the small data scattering result (Proposition 5.5). Thus by Proposition 5.5, we have

$$\begin{aligned}\limsup_{n\to\infty}\left\|Z_n^J\right\|^p_{L^a_t(\mathbb{R}:L^r_x)} &\le \sum_{j\in\mathcal{J}}a^j+\sum_{j\notin\mathcal{J}}a^j \le \sum_{j\in\mathcal{J}}a^j + C\sum_{j\notin\mathcal{J}}\|\psi^j\|^p_{H_V^{\frac{1}{2}}}\\ &\le \sum_{j\in\mathcal{J}}a^j + C\lim_{n\to\infty}\sum_{j\notin\mathcal{J}}\|\tau_{x_n^j}\psi^j\|^p_{H_V^{\frac{1}{2}}}\\ &\le \sum_{j\in\mathcal{J}}a^j + C\lim_{n\to\infty}\sum_{j\notin\mathcal{J}}\|e^{it_j^nH_V}\tau_{x_n^j}\psi^j\|^p_{H_V^{\frac{1}{2}}}\\ &\le \sum_{j\in\mathcal{J}}a^j + Cn_\omega =: M,\end{aligned}$$

where M is a positive constant independent of J.

Set $\varepsilon_M=\varepsilon(M)>0$, which is given in Lemma 5.8. Then by Proposition 5.7, we find that there exists $J_0=J_0(\varepsilon_M)\in\mathbb{N}$ such that for any $J\in\mathbb{N}$ satisfying $J\ge J_0$, the estimate $\limsup_{n\to\infty}\left\|e^{-itH_V}W_n^J\right\|_{L^a_t(\mathbb{R}:L^r_x)}<\varepsilon_M$ holds due to $a,r<\infty$ and $W_n^J\in\mathcal{H}$ for any $J\in\mathbb{N}$ and sufficiently large $n\in\mathbb{N}$. Then by Lemma 5.8, we find that $u_n\in L^a_t(\mathbb{R}:L^r_x)$ for sufficiently large n, which contradicts $\|u_n\|_{L^a_t(\mathbb{R}:L^r_x)}=\infty$ for any $n\in\mathbb{N}$. Therefore, we obtain $J=1$.

Then by (5.33), (5.38), (5.39), we have

$$\varphi_n=e^{it_n^1H_V}\tau_{x_n^1}\psi^1+W_n^1,\quad S_\omega^c=\limsup_{n\to\infty}S_{\omega,V}(e^{it_n^1H_V}\tau_{x_n^1}\psi^1),\quad \lim_{n\to\infty}\|W_n^1\|_{H^1}=0.$$

In the same argument as the proof of Lemma 6.3 in [8], we can prove that $\{t_n^1\}_{n\in\mathbb{N}}$ is bounded. Thus we may assume that $t_n^1=0$ for any $n\in\mathbb{N}$ up to subsequence. In the same argument as the proof of Proposition 4.1 in [3], we get $x_n^1=0$ for any $n\in\mathbb{N}$. Let u^c be the solution to (NLS$_V$) with the initial data $\psi^1\in\mathcal{N}_\omega^+$. Then by the conservation law, we have $S_\omega^c=S_{\omega,V}(u^c)$. The global solution u^c does not scatter by a contradiction argument and the perturbation lemma (see the proof of Proposition 6.1 in [8] for more detail). □

5.6. Extinction of the critical element

In this subsection, we study properties of the critical element to (NLS$_V$) obtained in Theorem 5.12. First we prove precompactness of the flow of the critical element in the energy space $\mathcal{H}$. Second we prove a so-called rigidity lemma by using the repulsiveness of the potential, which concludes the proof of the scattering part of Theorem 1.3.

Lemma 5.13 *(Precompactness of the critical element). Let $\omega > 0$, $p > 5$, $V \in L^1_1(\mathbb{R})$ satisfy $V' \in L^1_1(\mathbb{R})$, u be a critical element, i.e. $u \in C(\mathbb{R} : \mathcal{H})$ is a solution to (NLS$_V$) on $\mathbb{R}$ which satisfies $u(t) \in \mathcal{N}^+_\omega$ for any $t \in \mathbb{R}$ and*

$$S_{\omega,V}(u(t)) = S^c_\omega, \quad \|u\|_{L^a_t(\mathbb{R}:L^r_x(\mathbb{R}))} = \infty.$$

Then $\mathbf{H} := \{u(t) \in \mathcal{H} : t \in \mathbb{R}\} \subset \mathcal{H}$, which is the orbit of the solution u, is precompact in $\mathcal{H}$, i.e. $\overline{\mathbf{H}}$ is compact in $\mathcal{H}$.

Proof of Lemma 5.13. Let $\{t_n\} \subset \mathbb{R}$ be a sequence. It suffices to prove that there exist a subsequence of $\{u(t_n)\}_{n\in\mathbb{N}}$ and the function $\psi \in \mathcal{H}$ such that $\lim_{n\to\infty} u(t_n) = \psi$ in $\mathcal{H}$.

For any $t \in \mathbb{R}$, since $u(t) \in \mathcal{N}^+_\omega$, the estimate $I_{\omega,V}(u(t)) \geq 0$ holds. Since $S_{\omega,V}(u(t)) = S^c_{\omega,V}$ for any $t \in \mathbb{R}$ by the assumption, Lemma 4.5 gives

$$\|u(t)\|_{\mathcal{H}} \leq C\sqrt{S_\omega(u(t))} = C\sqrt{S^c_\omega} < \infty$$

for any $t \in \mathbb{R}$, where C is dependent only on ω and p, which implies that $\{u(t_n)\}_{n\in\mathbb{R}}$ is bounded in $\mathcal{H}$. We note that the identity $S_\omega(u(t_n)) = S^c_{\omega,V}$ holds for any $n \in \mathbb{N}$. Set $\varphi_n := u(t_n) \in \mathcal{H}$ for $n \in \mathbb{N}$ and let u_n be the unique solution to (NLS$_V$) with $u_n(0) = \varphi_n$ (see Proposition 1.1). Then we can prove that for any $n \in \mathbb{N}$, u_n is global, belongs to $C(\mathbb{R} : \mathcal{H})$ and $\|u_n\|_{L^a_t(\mathbb{R}:L^r_x)} = \infty$. Indeed, since $S_{\omega,V}(\varphi_n) = S_{\omega,V}(u(t_n)) < n_\omega$ and $I_{\omega,V}(\varphi_n) = I_{\omega,V}(u(t_n)) \geq 0$, we see that u_n is global and belongs to $C(\mathbb{R} : \mathcal{H})$ from Corollary 4.7 with $t_0 = t_n$. And on the contrary, we assume that there exists $n_0 \in \mathbb{N}$ such that $\|u_{n_0}\|_{L^a_t(\mathbb{R}:L^r_x)} < \infty$. Then by the perturbation lemma (Lemma 5.8), we can find $\tilde{\psi} \in \mathcal{H}$ such that $S^c_\omega = S_{\omega,V}(u(t_{n_0})) = S_{\omega,V}(\varphi_{n_0}) < S_\omega(\tilde{\psi}) < n_\omega$ and if $\tilde{u}$ is defined as the solution to (NLS$_V$) with $\tilde{u}(0) = \tilde{\psi}$ on $\mathbb{R}$, then $\|\tilde{u}\|_{L^a_t(\mathbb{R}:L^r_x)} < \infty$, which contradicts the definition of S^c_ω. The existence of such $\tilde{\psi}$ is proved as follows more precisely. We define the function $f : [-1, \infty) \mapsto \mathbb{R}$ as

$$\begin{aligned} f(\lambda) &:= S_{\omega,V}((1+\lambda)\varphi_{n_0}) = \|\varphi_{n_0}\|^2_{H^{\frac{1}{2}}_{\omega,V}}(1+\lambda)^2 - \frac{1}{p+1}\|\varphi_{n_0}\|^{p+1}_{L^{p+1}}(1+\lambda)^{p+1} \\ &=: A(1+\lambda)^2 - B(1+\lambda)^{p+1}, \end{aligned}$$

where $A := \|\varphi_{n_0}\|^2_{H^{\frac{1}{2}}_{\omega,V}}$, $B := \frac{1}{p+1}\|\varphi_{n_0}\|^{p+1}_{L^{p+1}}$. Noting that

$$f'(\lambda) = (1+\lambda)\left\{2A - B(p+1)(1+\lambda)^{p-1}\right\},$$

we see that if λ_0 is defined by $\lambda_0 := -1 + \left\{\frac{2A}{B(p+1)}\right\}^{\frac{1}{p-1}}$, then f is increasing on $(-1, \lambda_0)$ and f is decreasing on (λ_0, ∞). Therefore, in the case of $\lambda_0 > 0$, by letting λ such as $\lambda \to +0$ with the perturbation lemma (Lemma 5.8), we can find such $\tilde{\psi}$, on the other hand, in the case of $\lambda_0 < 0$, by letting λ such as $\lambda \to -0$ with the perturbation lemma, we can find such $\tilde{\psi}$.

Thus we can repeat the same argument as in the proof of Theorem 5.12 with $\varphi_n = u(t_n)$ and we can find a subsequence of $\{u(t_n)\}_{n\in\mathbb{N}}$ and the function $\psi \in \mathcal{H}$ such that $\lim_{n\to\infty} u(t_n) = \psi$ in $\mathcal{H}$, which completes the proof of the lemma. □

Lemma 5.14 *(Precompactness of the flow implies uniform localization). Let $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ be non-negative, $u \in C(\mathbb{R} : \mathcal{H})$ and $p \geq 1$ and let $\mathbf{H} \subset \mathcal{H}$ be the set of the flow defined by*

$$\mathbf{H} := \{u(t) \in \mathcal{H} : t \in \mathbb{R}\}.$$

Then if $\mathbf{H}$ *is precompact in* $\mathcal{H}$*, for any* $\varepsilon > 0$*, there exists* $R = R(\varepsilon) > 0$ *such that*

$$\int_{|x|>R} |\partial_x u(t,x)|^2 dx + \int_{|x|>R} |u(t,x)|^2 dx + \int_{|x|>R} |u(t,x)|^{p+1} dx < \varepsilon, \text{ for any } t \in \mathbb{R}.$$

This lemma can be proved in the same manner as the proof of Lemma 5.6 in [11] and Corollary 3.3 in [7]. For the convenience of readers, we give a proof of the lemma in Appendix E.

Next we give a proof of the so-called rigidity theorem under the precompactness of the orbit of the flow, and the repulsiveness of the potential ($xV' \leq 0$).

Proposition 5.15 *(Rigidity theorem). Besides the assumptions (1) in Theorem 1.2, we assume that* V *belongs to* $L^1(\mathbb{R})$*,* xV' *belongs to* $L^1(\mathbb{R})+L^\infty(\mathbb{R})$ *and satisfies* $xV'(x) \leq 0$ *for a.e.* $x \in \mathbb{R}$*, i.e.* V *is repulsive. Let* $u \in C(\mathbb{R} : \mathcal{H})$ *be the unique solution to (*NLS_V*) with* $u(0) = u_0$*. If the orbit of the flow* $\mathbf{H} := \{u(t) \in \mathcal{H} : t \in \mathbb{R}\}$ *to (*NLS_V*) is precompact in* $\mathcal{H}$*. Then* $u = 0$ *for any* $t \in \mathbb{R}$*.*

The repulsiveness of the potential V in the scattering part is used only in the proof of the proposition (see also Proposition 13 in [21]). The proof of the proposition is based on the localized virial identity (Lemma 3.1).

Proof of Proposition 5.15. On the contrary, we assume that there exists $t_0 \in \mathbb{R}$ such that $u(t_0) \neq 0$. Let $R > 0$ be a parameter, which will be determined later. We can take $\phi = \phi(x) \in C_0^\infty(\mathbb{R})$ such that

$$0 \leq \phi(x) \leq x^2, \quad |\phi'(x)| \leq C_1|x|, \quad |\phi''(x)| \leq 2, \quad |\phi^{(4)}(x)| \leq \frac{4}{R^2},$$

for $x \in \mathbb{R}$, where C_1 is a constant independent of x, and

$$\phi(x) = \begin{cases} x^2, & 0 \leq |x| \leq R, \\ 0, & |x| \geq 2R. \end{cases}$$

Since u is the solution to (NLS_V) on $\mathbb{R}$, we can apply the localized virial identity (Lemma 3.1) to obtain

$$\mathcal{I}''(t) = 8P_0(u(t)) + \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + \mathcal{R}_4, \quad \text{for } t \in \mathbb{R}, \tag{5.49}$$

where $\mathcal{R}_j$ ($j = 1, 2, 3, 4$) are defined by

$$\mathcal{R}_1 := 4\int_{\mathbb{R}} \{\phi''(x) - 2\}|\partial_x u(t,x)|^2 dx,$$

$$\mathcal{R}_2 := -\frac{2(p-1)}{p+1}\int_{\mathbb{R}} \{\phi''(x) - 2\}|u(t,x)|^{p+1} dx,$$

$$\mathcal{R}_3 := -\int_{\mathbb{R}} \phi^{(4)}(x)|u(t,x)|^2 dx,$$

$$\mathcal{R}_4 := -2\int_{\mathbb{R}} \phi'(x)V'(x)|u(t,x)|^2 dx.$$

By the properties of the function ϕ, we have

$$|\mathcal{R}_1| = 4\left|\int_{\mathbb{R}} \{\phi''(x) - 2\}|\partial_x u(t,x)|^2 dx\right| \leq 16 \int_{|x|>R} |\partial_x u(t,x)|^2 dx,$$

$$|\mathcal{R}_2| = \left|\frac{2(p-1)}{p+1}\int_{\mathbb{R}} \{\phi''(x) - 2\}|u(t,x)|^{p+1} dx\right| \leq \frac{8(p-1)}{p+1} \int_{|x|>R} |u(t,x)|^{p+1} dx,$$

$$|\mathcal{R}_3| = \left|\int_{\mathbb{R}} \phi^{(4)}(x)|u(t,x)|^2 dx\right| \leq \frac{4}{R^2} \int_{|x|>R} |u(t,x)|^2 dx.$$

Since V satisfies $xV'(x) \leq 0$ for a.e. $x \in \mathbb{R}$, by the properties of ϕ, we have

$$\begin{aligned}
\mathcal{R}_4 &= -4 \int_{|x|\leq R} xV'(x)|u(t,x)|^2 dx - 2 \int_{|x|>R} \phi'(x)V'(x)|u(t,x)|^2 dx \\
&\geq -2 \int_{|x|>R} \phi'(x)V'(x)|u(t,x)|^2 dx \\
&\geq -2 \int_{|x|>R} |\phi'(x)||V'(x)||u(t,x)|^2 dx \\
&\geq -2C_1 \int_{|x|>R} |xV'(x)||u(t,x)|^2 dx,
\end{aligned}$$

for any $t \in \mathbb{R}$ and $R > 0$. Since $xV' \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$, there exist $V_1 \in L^1(\mathbb{R})$ and $V_2 \in L^\infty(\mathbb{R})$ such that the identity $xV' = V_1 + V_2$ holds. By the estimate (1.13) in Theorem 1.2 and the energy conservation law, we have

$$-\|\partial_x u(t)\|_{L^2} \geq -\|H_V^{\frac{1}{2}} u(t)\|_{L^2} > -\sqrt{\frac{2(p-1)}{p-5} E_V(u_0)} = -\sqrt{\frac{2(p-1)}{p-5} E_V(u(t_0))}$$

for any $t \in \mathbb{R}$. Thus by these facts, the Gagliardo-Nirenberg-Sobolev inequality $\|f\|_{L^\infty}^2 \leq \|f\|_{L^2}\|\partial_x f\|_{L^2}$ for $f \in H^1(\mathbb{R})$ and the mass conservation law, we have

$$\begin{aligned}
\mathcal{R}_4 &\geq -2C_1\|u(t)\|_{L^\infty}^2 \int_{|x|>R} |V_1(x)| dx - 2C_1\|V_2\|_{L^\infty} \int_{|x|>R} |u(t,x)|^2 dx \\
&\geq -2C_1\|u(t)\|_{L^2}\|\partial_x u(t)\|_{L^2} \int_{|x|>R} |V_1(x)| dx - 2C_1\|V_2\|_{L^\infty} \int_{|x|>R} |u(t,x)|^2 dx \\
&\geq -2C_1\|u(t_0)\|_{L^2}\sqrt{\frac{2(p-1)}{p-5} E_V(u(t_0))} \int_{|x|>R} |V_1(x)| dx - 2C_1\|V_2\|_{L^\infty} \int_{|x|>R} |u(t,x)|^2 dx =: \tilde{\mathcal{R}}_4,
\end{aligned}$$

for $t \in \mathbb{R}$. Therefore, by combining the above estimates, for any $R \geq 1$ and $t \in \mathbb{R}$, the estimate

$$\begin{aligned}\mathcal{I}''(t) \geq & 8P_0(u(t)) - (|\mathcal{R}_1| + |\mathcal{R}_2| + |\mathcal{R}_3|) + \mathcal{R}_4 \\ \geq & 8P_0(u(t)) \\ & - 16 \left\{ \int_{|x|>R} |\partial_x u(t,x)|^2 dx + \int_{|x|>R} |u(t,x)|^2 dx + \int_{|x|>R} |u(t,x)|^{p+1} dx \right\} + \tilde{\mathcal{R}_4} \end{aligned} \tag{5.50}$$

holds. By $u(t_0) \neq 0$ and the mass conservation law, we have $u_0 \neq 0$, which allows us to apply Theorem 1.2 to find that there exists $\delta_0 > 0$ independent of t such that $P_0(u(t)) > \delta_0$. Since $\mathbf{H}$ is precompact in $\mathcal{H}$, by Lemma 5.13, there exists $R_0 = R_0(\delta_0)$ such that for $R \geq R_0$, the estimate holds

$$\int_{|x|>R} |\partial_x u(t,x)|^2 dx + \int_{|x|>R} |u(t,x)|^2 dx + \int_{|x|>R} |u(t,x)|^{p+1} dx \leq \frac{\delta_0}{4}, \tag{5.51}$$

for any $t \in \mathbb{R}$. Since $V_1 \in L^1(\mathbb{R})$, by the precompactness of $\mathbf{H}$ again, there exists $R_1 = R_1(\delta_0, V) > 0$ independent of t such that for $R \geq R_1$, the estimate

$$\tilde{R_4} < \delta_0 \tag{5.52}$$

holds for any $t \in \mathbb{R}$. Fix $R \geq \max(R_0, R_1)$ arbitrarily. By combining the estimates (5.50)-(5.52), the estimate

$$\mathcal{I}''(t) \geq \delta_0$$

holds for any $t \in \mathbb{R}$. For any $t > 0$, by integrating the above estimate twice with respect to time over $[0, t]$, we have

$$\mathcal{I}(t) > \frac{\delta_0}{2} t^2 + \mathcal{I}'(0) t + \mathcal{I}(0),$$

which implies that

$$\lim_{t \to \infty} \mathcal{I}(t) = \infty. \tag{5.53}$$

On the other hand, by the definition of the function $\mathcal{I}$, the property of ϕ and the mass conservation law, the estimate

$$\mathcal{I}(t) = \int_{|x| \leq 2R} \phi(x) |u(t,x)|^2 dx \leq 4R^2 \|u(t)\|_{L^2}^2 = 4R^2 \|u(0)\|_{L^2} =: C_2,$$

holds for any $t \in \mathbb{R}$, where $C_2 = C_2(R, \|u(0)\|_{L^2})$ is independent of t, which contradicts (5.53). Therefore we see that for any $t \in \mathbb{R}$, $u(t) = 0$, which completes the proof of the proposition. □

Finally, we complete the proof of the scattering part of Theorem 1.3 by combining Theorem 5.12, Lemma 5.14 and Proposition 5.15.

Proof of the scattering part of Theorem 1.3. By the definition of S_ω^c, it suffices to prove $n_\omega \leq S_\omega^c$. On the contrary, we assume that $n_\omega > S_\omega^c$. Then from Theorem 5.12, we can find a critical element $u^c \in C(\mathbb{R} : \mathcal{H})$ to (NLS$_V$). Lemma 5.13 implies that the orbit of the flow $\{u^c(t) \in \mathcal{H} : t \in \mathbb{R}\}$ is precompact in $\mathcal{H}$. Thus the rigidity theorem (Proposition 5.15) implies that $u^c(t) = 0$ for any $t \in \mathbb{R}$, which gives $S_{\omega,V}(u^c(t)) = 0$ for any $t \in \mathbb{R}$. However, this contradicts $S_{\omega,V}(u^c(t)) = S_\omega^c > 0$ for any $t \in \mathbb{R}$. Therefore we see that $n_\omega \leq S_\omega^c$, which completes the proof of the scattering part of the theorem. □

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Appendix A. Proof of the local well-posedness in the energy space $\mathcal{H}$

In this appendix, we give a proof of the local well-posedness result for (NLS$_V$) in the energy space $\mathcal{H}$ (Proposition 1.1) via the standard argument (see also Theorem 3.7.1 in [4] for more general operator H_V).

Proof of Proposition 1.1. (Existence) Let $\varrho > 0$ and $u_0 \in \mathcal{H}$ such that $\|u_0\|_{\mathcal{H}} \leq \varrho$. For $T > 0$ and $\Theta > 0$, which will be determined later (see (A.4)), we set $I_T := (-T, T)$ and we introduce a closed ball $X(T, \Theta)$ in $L_t^\infty(I_T : \mathcal{H})$ as

$$X(T,\Theta) := \left\{u \in L_t^\infty(I_T : \mathcal{H}) : \ \|u\|_{L_t^\infty(I_T:\mathcal{H})} \leq \Theta\right\}$$

with the metric $d_T(u, v) := \|u - v\|_{L_t^\infty(I_T:L^2)}$, for $u, v \in L_t^\infty(I_T : \mathcal{H})$. We prove that the nonlinear mapping $J : X(T, \Theta) \mapsto X(T, \Theta)$

$$J[u](t) := e^{-itH_V} u_0 + i\int_0^t e^{-i(t-s)H_V}|u(s)|^{p-1}u(s)ds, \quad t \in I_T$$

is well defined and is contractive on $X(T, \Theta)$, if $\Theta = \Theta(\rho)$ is sufficiently large and $T = T(\Theta)$ is sufficiently small.

Since V is non-negative and belongs to $L^1(\mathbb{R}) + L^\infty(\mathbb{R})$, by the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$, the estimate

$$\|f\|_{H^1}^2 \leq \|f\|_{\mathcal{H}}^2 = \|f\|_{H^1}^2 + \int_{-\infty}^{\infty} V(x)|f(x)|^2 dx \leq (1 + C\|V\|_{L^1+L^\infty})\|f\|_{H^1}^2 \tag{A.1}$$

holds for any $f \in \mathcal{H}$, where C is a positive constant independent of f and V. Since $L_V^{\frac{1}{2}}$ and e^{-itH_V} are commute with each other for any $t \in \mathbb{R}$ and $\{e^{-itH_V}\}_{t\in\mathbb{R}}$ is unitary on $L^2(\mathbb{R})$, by the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$ again and the above equivalency of the norms, we have

$$\begin{aligned}
\|J[u](t)\|_{\mathcal{H}} &\leq \|u_0\|_{\mathcal{H}} + \left|\int_0^t \|L_V^{\frac{1}{2}}(|u(s)|^{p-1}u(s))\|_{L^2} ds\right| \\
&\leq \|u_0\|_{\mathcal{H}} + C\left|\int_0^t \||u(s)|^{p-1}u(s)\|_{H^1} ds\right| \\
&\leq \varrho + C_1 T\Theta^p, \quad \text{for } t \in I_T,
\end{aligned} \tag{A.2}$$

where C_1 is a positive constant dependent only on p and $\|V\|_{L^1+L^\infty}$. Here we choose $\Theta = \Theta(\varrho)$ and $T = T(\Theta)$ such as

$$\Theta \geq 2\varrho \quad \text{and} \quad 0 < T \leq \frac{1}{2C_1\Theta^{p-1}}. \tag{A.3}$$

Then the estimate

$$\|J[u]\|_{L^\infty(I_T:\mathcal{H})} \leq \varrho,$$

holds, which implies that the mapping J is well defined from $X(T,\Theta)$ into itself.

By the fundamental formula of calculus, the estimate

$$\begin{aligned} ||a|^{p-1}a - |b|^{p-1}b| &= \left| \int_0^1 \frac{d}{d\theta} |\theta a + (1-\theta)b|^{p-1}(\theta a + (1-\theta)b) d\theta \right| \\ &\leq p2^{p-1}(|a|^{p-1} + |b|^{p-1})|a-b| \end{aligned}$$

holds for $a, b \in \mathbb{C}$. By using this estimate and in the same manner as the proof of (A.2), we obtain

$$d_T(J[u], J[v]) \leq C_2 T \Theta^{p-1} d_T(u,v)$$

for $u, v \in X(T,\Theta)$, where C_2 is dependent only on p and $\|V\|_{L^1+L^\infty}$. Here we take $T = T(\Theta)$ such that

$$0 < T \leq \frac{1}{2C_2\Theta^{p-1}}. \tag{A.4}$$

Then the estimate

$$d_T(J[u], J[v]) \leq \frac{1}{2} d_T(u,v)$$

holds, which implies that the mapping J is contractive on $X(T,\Theta)$. Thus by the contraction mapping principle, we see that there exists a unique $u \in X(T,\Theta)$ such that $J[u](t) = u(t)$ on $t \in I_T$.

(Uniqueness) On the contrary, we assume that there exists $t \in (0, T_1)$ such that $u(t) \neq v(t)$ in L^2-sense. Then we can define $t_0 := \inf\{t \in [0,T_1), u(t) \neq v(t)\} > 0$. Since both u and v belong to $C([0,T_1) : L^2)$, we have $u(t_0) = v(t_0)$ by the continuity. Since (NLS_V) is invariant with respect to the time translation, we may assume $t_0 = 0$. In the same manner as the proof of the Existence part, for small $\tau \in (0, T_1)$, we have

$$d_\tau(u,v) \leq C_2 \tau (\|u\|^{p-1}_{L_t(I_{T_1}:\mathcal{H})} + \|v\|^{p-1}_{L_t(I_{T_1}:\mathcal{H})}) d_\tau(u,v) \leq \frac{1}{2} d_\tau(u,v),$$

which implies $u(t) = v(t)$ on $[0,\tau)$. This contradicts the definition of t_0. Therefore $u(t) = v(t)$ on $[0,T_1)$. We can also prove $u(t) = v(t)$ on $(-T_1, 0]$ in the same manner.

(Continuity of the flow map) Let $u_0 \in \mathcal{H}$ and $v_0 \in \mathcal{H}$ such that $\|u_0\|_{\mathcal{H}} \leq \varrho$ and $\|v_0\|_{\mathcal{H}} \leq \varrho$ respectively. Let $u \in X(T,\Theta)$ and $v \in X(T,\Theta)$ be solutions to (NLS_V) with $u(0) = u_0$ and $v(0) = v_0$ respectively. In the same manner as the proof of the Existence part, we have

$$d_T(u,v) \leq \|u_0 - v_0\|_{L^2} + C_2 T \Theta^{p-1} d_T(u,v) \leq \|u_0 - v_0\|_{L^2} + \frac{1}{2} d_T(u,v),$$

which implies that the flow map $\Xi : \{f \in \mathcal{H} : \|f\|_{\mathcal{H}} \leq \varrho\} \mapsto X(T,\Theta)$ is Lipschitz continuous with the Lipschitz constant 2.

(Conservation laws) The proof of the conservation laws is standard. So we omit the detail.

(Blow-up criterion) We only consider the positive time direction, since the negative time direction can be treated in the same manner. We assume that

$$\liminf_{t \to T_+ - 0} \|\partial_x u(t)\|_{L^2} < \infty.$$

Then we can define

$$C_3 := \liminf_{t\to T_+-0} \|\partial_x u(t)\|_{L^2}.$$

We can find a sequence $\{t_k\}_{k\in\mathbb{N}} \subset [0, T_+)$ such that

$$\lim_{k\to\infty} t_k = T_+ \tag{A.5}$$

$$\sup_{k\in\mathbb{N}} \|\partial_x u(t_k)\|_{L^2} \leq C_3 + 1 \tag{A.6}$$

The identity (A.1), the L^2-conservation law and (A.6) give

$$\begin{aligned}\sup_{k\in\mathbb{N}} \|u(t_k)\|_{\mathcal{H}}^2 &\leq (1 + C\|V\|_{L^1+L^\infty})(\|u_0\|_{L^2}^2 + \sup_{k\in\mathbb{N}} \|\partial_x u(t_k)\|_{L^2}^2) \\ &\leq (1 + C\|V\|_{L^1+L^\infty})(\|u_0\|_{L^2}^2 + 1 + C_3) =: \rho_1\end{aligned}$$

Thus by the result of the Existence part, there exists a positive time $T = T(\rho_1)$ independent of k such that for any $k \in \mathbb{N}$, there exists a unique solution $u \in C([t_k, t_k + T(r_1) : \mathcal{H}) \cap C^1([t_k, t_k + T(\rho_1) : \mathcal{H}^{-1})$ to (NLS$_V$). However, by (A.5), we can take t_k such as $t_k + T(\rho_1) > T_+$, which contradicts the maximality of T_+. Therefore we obtain

$$\liminf_{t\to T_+-0} \|\partial_x u(t)\|_{L^2} = \infty,$$

which completes the proof of the proposition. □

Appendix B. Proof of the localized virial identity (Lemma 3.1)

In this appendix, we give a proof of the localized virial identity (Lemma 3.1) only for smooth rapidly decaying solutions. For the proof of H^1-solutions, see [17].

Proof of Lemma 3.1. Since V is real-valued and u is a smooth rapidly decaying solution to (NLS$_V$) on I, by the definition of the function $\mathcal{I}$ and the integration by parts, we have

$$\begin{aligned}\mathcal{I}'(t) &= 2\int_{\mathbb{R}} \phi(x)\mathrm{Re}\{\overline{u(t,x)}\partial_t u(t,x)\}dx = -2\mathrm{Im}\int_{\mathbb{R}} \phi(x)\overline{u(t,x)}\partial_x^2 u(t,x)dx \\ &= 2\mathrm{Im}\int_{\mathbb{R}} \phi'(x)\overline{u(t,x)}\partial_x u(t,x)dx,\end{aligned}$$

for $t \in I$. By the integration by parts again, we get

$$\begin{aligned}\mathcal{I}''(t) &= 2\mathrm{Im}\int_{\mathbb{R}} \phi'(x)\overline{\partial_t u(t,x)}\partial_x u(t,x)dx + 2\mathrm{Im}\int_{\mathbb{R}} \phi'(x)\overline{u(t,x)}\partial_x\partial_t u(t,x)dx \\ &= 4\mathrm{Im}\int_{\mathbb{R}} \phi'(x)\overline{\partial_t u(t,x)}\partial_x u(t,x)dx - 2\mathrm{Im}\int_{\mathbb{R}} \phi''(x)\overline{u(t,x)}\partial_t u(t,x)dx \\ &=: A_1 + A_2.\end{aligned}$$

We note that the following identities hold for $(t, x) \in I \times \mathbb{R}$:

$$i\partial_t u(t,x) = -\partial_x^2 u(t,x) + V(x)u(t,x) - |u(t,x)|^{p-1}u(t,x), \tag{B.1}$$

$$\begin{aligned}
&\partial_x(|\partial_x u(t,x)|^2) = 2\mathrm{Re}\{\overline{\partial_x^2 u(t,x)}\partial_x u(t,x)\},\\
&\partial_x(|u(t,x)|^2) = 2\mathrm{Re}\{\overline{u(t,x)}\partial_x u(t,x)\},\\
&\partial_x(|u(t,x)|^{p+1}) = (p+1)|u(t,x)|^{p-1}\mathrm{Re}\{\overline{u(t,x)}\partial_x u(t,x)\}.
\end{aligned} \tag{B.2}$$

By using the identities and the integration by parts again, we have

$$\begin{aligned}
A_1 = &-4\mathrm{Re}\int_{\mathbb{R}} \phi'(x)\overline{\partial_x^2 u(t,x)}\partial_x u + 4\mathrm{Re}\int_{\mathbb{R}} \phi'(x)V(x)\overline{u(t,x)}\partial_x u(t,x)dx\\
&- 4\mathrm{Re}\int_{\mathbb{R}} \phi'(x)|u(t,x)|^{p-1}\overline{u(t,x)}\partial_x u(t,x)dx\\
=& \, 2\int_{\mathbb{R}} \phi''(x)|\partial_x u(t,x)|^2dx - 2\int_{\mathbb{R}} \phi''(x)V(x)|u(t,x)|^2dx\\
&- 2\int_{\mathbb{R}} \phi'(x)V'(x)|u(t,x)|^2dx + \frac{4}{p+1}\int_{\mathbb{R}} |u(t,x)|^{p+1}dx.
\end{aligned} \tag{B.3}$$

By the identities (B.1) and (B.2) and the integration by parts, we obtain

$$\begin{aligned}
A_2 = &-2\mathrm{Re}\int_{\mathbb{R}} \phi''(x)\partial_x^2 u(t,x)\overline{u(t,x)}dx\\
&+ 2\int_{\mathbb{R}} \phi''(x)V(x)|u(t,x)|^2dx - 2\int_{\mathbb{R}} \phi''(x)|u(t,x)|^{p+1}dx\\
=& -\int_{\mathbb{R}} \phi^{(4)}(x)|u(t,x)|^2dx + 2\int_{\mathbb{R}} \phi''(x)|\partial_x u(t,x)|^2dx\\
&+ 2\int_{\mathbb{R}} \phi''(x)V(x)|u(t,x)|^2dx - 2\int_{\mathbb{R}} \phi''(x)|u(t,x)|^{p+1}dx.
\end{aligned} \tag{B.4}$$

By adding the identities (B.3) and (B.4), we obtain (3.2), which completes the proof of the lemma. □

Appendix C. Proof of the perturbation lemma (Lemma 5.8)

In this appendix, we give a proof of the perturbation lemma (Lemma 5.8). In order to prove the lemma, we use the following Gronwall-type inequality.

Lemma C.1. *Let $1 \le \mu < \nu \le \infty$, $q = q(\mu,\nu) := \frac{\mu\nu}{\nu-\mu} \in [1,\infty)$, $0 < T \le \infty$, $f \in L_t^q(-T,T)$, $g \in L_{loc}^\nu(-T,T)$ and $\eta > 0$. Then the function $fg : (-T,T) \mapsto \mathbb{R}$ belongs to $L_t^\mu(-T,T)$. Moreover, if the estimate*

$$\|g\|_{L_t^\nu(-t,t)} \le \eta + \|fg\|_{L_t^\mu(-t,t)}$$

holds for any $t \in (0,T]$, then the estimate

$$\|g\|_{L_t^\nu(-t,t)} \le \eta\Psi(\|f\|_{L_t^q(-t,t)})$$

holds for any $t \in (0,T]$, where $\Psi(s) := 2\Gamma(3+2s)$ for $s > 0$ and Γ is the Gamma function defined by

$$\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}dt, \quad \textit{for } s > 0.$$

For the proof of this lemma, see Lemma 8.1 in [8]. Now we give a proof of Lemma 5.8.

Proof of Lemma 5.8. Set $w := u - v \in C(\mathbb{R} : \mathcal{H})$. Then since u and v satisfies (5.12) and (5.11) respectively, w satisfies

$$w(t) = e^{-itH_V}\varphi_0 + i\int_0^t e^{-i(t-s)H_V}[|w(s)+v(s)|^{p-1}\{w(s)+v(s)\} - |v(s)|^{p-1}v(s)]ds - e(t), \tag{C.1}$$

on $t \in \mathbb{R}$. Here we fix $\varepsilon > 0$ and $T > 0$ arbitrarily, which will be determined later. We note that since $p \geq 5$, there exists a positive constant C_p depending only on p, such that for any $a, b \in \mathbb{C}$, the estimate

$$||a+b|^{p-1}(a+b) - |b|^{p-1}b| \leq C_p(|a|^p + |b|^{p-1}|a|) \tag{C.2}$$

holds. Let $t \in (0, T)$. By taking $L_t^a(-t, t : L_x^r)$-norm of the equation (C.1), using the assumptions $\|e^{-itH_V}\varphi\|_{L_t^a(\mathbb{R}:L_x^r)} \leq \varepsilon$ and $\|e\|_{L_t^a(\mathbb{R}:L_x^r)} \leq \varepsilon$, the non-admissible Strichartz estimate (Lemma 5.3) due to $V \in L_1^1(\mathbb{R})$ and $p \geq 5$, the Hölder inequality and the identity $pb' = a$, we have

$$\begin{aligned}
\|w\|_{L_t^a(-t,t:L_x^r)} &\leq \varepsilon + \||w+v|^{p-1}(w+v) - |v|^{p-1}v\|_{L_t^{b'}(-t,t:L_x^{r'})} + \varepsilon \\
&\leq 2\varepsilon + C_p\||v|^{p-1}|w| + |w|^p\|_{L_t^{b'}(-t,t:L_x^{r'})} \\
&\leq 2\varepsilon + C_p\|\|v(t)\|_{L_x^r}^{p-1}\|w(t)\|_{L_x^r}\|_{L_t^{b'}(-t,t)} + C_p\|\|w(t)\|_{L_x^r}^p\|_{L_t^{b'}(-t,t)} \\
&=: 2\varepsilon + C_p\|fg\|_{L_t^{b'}(-t,t)} + C_p\|g\|_{L_t^a(-t,t)}^p \\
\|g\|_{L_t^a(-t,t)} &\leq A\varepsilon + A\|fg\|_{L_t^{b'}(-t,t)} + A\|g\|_{L_t^a(-t,t)}^p,
\end{aligned} \tag{C.3}$$

where $A := \max(2, C_p)$, the functions $f : \mathbb{R} \mapsto [0, \infty)$ and $g : \mathbb{R} \mapsto [0, \infty)$ are defined by

$$f(t) := \|v(t)\|_{L_x^r}^{p-1} \in L_t^\omega(\mathbb{R}), \quad \text{and } g(t) := \|w(t)\|_{L_x^r} \in L_t^a(-T, T)$$

respectively. Here ω is defined by $\omega = \omega(p) := \frac{2(p+3)}{p+1}$ and satisfies $\frac{1}{\omega} = \frac{1}{b'} - \frac{1}{a}$, and we have used the fact that $v \in L_t^a(\mathbb{R} : L_x^r)$, $u \in L_t^\infty(\mathbb{R} : \mathcal{H})$ and the Sobolev embedding $\mathcal{H} \subset H^1(\mathbb{R}) \subset L_x^r(\mathbb{R})$. Moreover, since the estimate $\|v\|_{L_t^a(\mathbb{R}:L_x^r)} \leq M$ holds, the inequality

$$\|f\|_{L_t^\omega(\mathbb{R})} = \|v\|_{L_t^a(\mathbb{R}:L_x^r)}^{p-1} \leq M^{p-1} \tag{C.4}$$

is valid. Here we choose $\varepsilon = \varepsilon(M) = \varepsilon(M, A) > 0$ satisfying

$$\varepsilon \leq 2^{-(p-1)}\{2A\Psi(M^{p-1})\}^{-\frac{p}{p-1}}, \tag{C.5}$$

where Ψ is same as in Lemma C.1. Next we will prove that $\|g\|_{L_t^a(\mathbb{R})} < \infty$. Since $g \in L_t^\infty(\mathbb{R})$ and $a < \infty$, there exists a positive $T_0 = T_0(\varepsilon)$ such that $\|g\|_{L_t^a(-T_0,T_0)}^p < \varepsilon$. In fact, it suffices to take $0 < T_0 < \frac{1}{2}\left(\frac{\varepsilon}{\|g\|_{L_t^\infty(\mathbb{R})}}\right)^{\frac{a}{p}}$. Thus we can define the value

$$T_* := \sup\{T \in (0, \infty] : \|g\|_{L_t^a(-T,T)}^p < \varepsilon\} > 0.$$

We claim that $T_* = \infty$. Indeed, on the contrary, we assume that $T_* < \infty$. Then by the continuity, we have $\|g\|^p_{L^a_t(-T_*,T_*)} = \varepsilon$. Then by the estimate (C.3), we have

$$\|g\|_{L^a_t(-t,t)} \le 2A\varepsilon + A\|fg\|_{L^{b'}_t(-t,t)}$$

for any $t \in (0, T_*]$. We can apply the Gronwall-type estimate (Lemma C.1) with $\mu = b', \nu = a$, $q = \omega$ and $\eta = 2A\varepsilon$, to obtain

$$\|g\|_{L^a_t(-t,t)} \le 2A\varepsilon\Psi(\|f\|_{L^\omega_t(-t,t)}) \le 2A\varepsilon\Psi(M^{p-1}),$$

for any $t \in (0, T_*]$, where we have used the estimate (C.4). By this estimate and the inequality (C.5), we obtain $\|g\|^{p-1}_{L^a_t(-t,t)} \le \frac{\varepsilon}{2}$, for any $t \in (0, T_*]$, which leads to a contradiction. Thus we see that $T_* = \infty$. Here we set

$$C(M) := 2A\Psi(M^{p-1}).$$

By the repeating above argument, we find that

$$\|w\|_{L^a_t(\mathbb{R}:L^r_x)} = \|g\|_{L^a_t(\mathbb{R})} \le 2A\varepsilon\Psi(M^{p-1}) = C(M)\varepsilon,$$

which completes the proof of the lemma. □

Appendix D. Proof of Lemma 5.9

Proof of Lemma 5.9. Up to subsequence, we may assume that $x_n \to \infty$ or $x_n \to -\infty$ as $n \to \infty$. We only consider the case $x_n \to \infty$ as $n \to \infty$, since the case $x_n \to -\infty$ can be treated in the same manner.

First we claim that for $\psi \in H^1(\mathbb{R})$, the identity

$$\lim_{n\to\infty} \left\| e^{it\partial_x^2}(\tau_{x_n}\psi) - e^{-itH_V}(\tau_{x_n}\psi) \right\|_{L^a_t(\mathbb{R}:L^r_x)} = 0 \tag{D.1}$$

holds. To prove this, we will show that

$$\lim_{T\to\infty} \sup_{n\in\mathbb{N}} \|e^{-itH_V}(\tau_{x_n}\psi)\|_{L^a_t((|t|>T):L^r_x)} = 0. \tag{D.2}$$

Let $\epsilon > 0$. Since $C_0^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R})$, there exists $\tilde{\psi} \in C_0^\infty(\mathbb{R})$ depending on $\epsilon > 0$ such that the estimate holds:

$$\|\psi - \tilde{\psi}\|_{H^1} \le \epsilon.$$

By this estimate and the non-admissible Strichartz estimate (Lemma 5.3) due to $p \ge 5$ and $V \in L^1_1(\mathbb{R})$, we have

$$\|e^{-itH_V}(\tau_{x_n}\tilde{\psi} - \tau_{x_n}\psi)\|_{L^a_t(\mathbb{R}:L^r_x)} \le C_0\|\tau_{x_n}\tilde{\psi} - \tau_{x_n}\psi\|_{H^1} = C_0\|\tilde{\psi} - \psi\|_{H^1} \le C_0\epsilon, \tag{D.3}$$

where we have used the translation invariance of the norm $\|\cdot\|_{H^1}$ and C_0 depends only on p and V. For any $n \in \mathbb{N}$, since $\tau_{x_n}\tilde{\psi} \in L^{r'}$, we can apply the dispersive estimate (5.2) to get

$$\|e^{-itH_V}(\tau_{x_n}\tilde{\psi})\|_{L^r} \le C_1|t|^{-\frac{1}{2}\left(\frac{1}{r'}-\frac{1}{r}\right)}\|\tau_{x_n}\tilde{\psi}\|_{L^{r'}} = C_1|t|^{-\frac{1}{2}(1-\frac{2}{r})}\|\tilde{\psi}\|_{L^{r'}}$$

for $t \neq 0$, where we have also used the translation invariance of the $L^{r'}$-norm and C_1 depends only on p and V. We note that since the estimates

$$\frac{a}{2}\left(1-\frac{2}{r}\right) = \frac{(p-1)^2}{p+3} > 1$$

hold due to $p \geq 5 > \frac{3+\sqrt{17}}{2}$, the function $|t|^{\frac{1}{2}(1-\frac{2}{r})}$ belongs to $L_t^a(|t| \geq 1, \infty)$. From this integrability, we see that there exists large $T_0 \gg 1$ such that for any $T \geq T_0$, the estimate

$$\sup_{n\in\mathbb{N}} \|e^{-itH_V}(\tau_{x_n}\tilde{\psi})\|_{L_t^a((|t|>T):L_x^r)} \leq \epsilon, \tag{D.4}$$

holds. We note that the identity $\tau_{x_n}\psi = \tau_{x_n}\tilde{\psi} + (\tau_{x_n}\psi - \tau_{x_n}\tilde{\psi})$ holds for any $n \in \mathbb{N}$. Thus combining the estimates (D.3) and (D.4) gives that for $T \geq T_0$, the estimate

$$\sup_{n\in\mathbb{N}} \|e^{-itH_V}(\tau_{x_n}\psi)\|_{L_t^a((|t|>T):L_x^r)} \leq (1+C_0)\epsilon,$$

holds, which completes the proof of the identity (D.2). Next in order to prove (D.1), we will show that for any $T > 0$, the identity

$$\lim_{n\to\infty} \left\|e^{it\partial_x^2}(\tau_{x_n}\psi) - e^{-itH_V}(\tau_{x_n}\psi)\right\|_{L_t^a(-T,T:L_x^r)} = 0 \tag{D.5}$$

holds. Indeed, for any $n \in \mathbb{N}$, set $u_n(t,x) := e^{it\partial_x^2}(\tau_{x_n}\psi)(x) - e^{-itH_V}(\tau_{x_n}\psi)(x)$ on $(t,x) \in \mathbb{R}\times\mathbb{R}$. We note that for any $n \in \mathbb{N}$, since $\tau_{x_n}\psi \in \mathcal{H}$, we see that the function u_n belongs to $C(\mathbb{R}:\mathcal{H}) \cap L_t^a(\mathbb{R}:L_x^r)$ and is the unique solution to the Schrödinger equation with the potential and the inhomogeneous term $-Ve^{it\partial_x^2}\tau_{x_n}\psi$:

$$i\partial_t u_n - H_V u_n = -Ve^{it\partial_x^2}\tau_{x_n}\psi, \quad (t,x) \in \mathbb{R}\times\mathbb{R}.$$

Thus the inhomogeneous non-admissible Strichartz estimate (Lemma 5.3) due to $p \geq 5$ and $V \in L_1^1(\mathbb{R})$ gives

$$\begin{aligned} \|u_n\|_{L_t^a(-T,T:L_x^r)} &\leq C_2\|Ve^{it\partial_x^2}\tau_{x_n}\psi\|_{L_t^{\gamma'}(-T,T:L_x^1)} \leq C_2(2T)^{\frac{1}{\gamma'}}\|Ve^{it\partial_x^2}\tau_{x_n}\psi\|_{L_t^\infty(-T,T:L_x^1)} \\ &= C_2(2T)^{\frac{1}{\gamma'}}\|V\tau_{x_n}e^{it\partial_x^2}\psi\|_{L_t^\infty(-T,T:L_x^1)} = C_2(2T)^{\frac{1}{\gamma'}}\|(\tau_{-x_n}V)e^{it\partial_x^2}\psi\|_{L_t^\infty(-T,T:L_x^1)} \end{aligned} \tag{D.6}$$

for any $n \in \mathbb{N}$, where we have used the fact that the translation operator τ_y commutes with the free Schrödinger solution operator $e^{it\partial_x^2}$ for any $y \in \mathbb{R}$ and $t \in \mathbb{R}$. In the same manner as the proof of (3.17) in [21], we can prove

$$\lim_{n\to\infty} \|(\tau_{-x_n}V)e^{it\partial_x^2}\psi\|_{L_t^\infty(-T,T:L_x^1)} = 0. \tag{D.7}$$

Indeed, let $\varepsilon > 0$. Since $e^{it\partial_x^2}\psi \in C([-T,T]:H^1)$ due to $\psi \in H^1(\mathbb{R})$ and $[-T,T]$ is closed, $e^{it\partial_x^2}\psi$ is uniformly continuous in $H^1(\mathbb{R})$ on $[-T,T]$, that is, there exists $\delta = \delta(\varepsilon) > 0$ such that for any $t_0, t_1 \in [-T,T]$ satisfying $|t_0 - t_1| < \delta$,

$$\|e^{it_0\partial_x^2}\psi - e^{it_1\partial_x^2}\psi\|_{H^1} \leq \varepsilon. \tag{D.8}$$

Take $t \in [-T,T]$ and $t_0 = t_0(t,\delta) \in [-T,T]$ with $|t-t_0| < \delta$ arbitrarily. Since $e^{it_0\partial_x^2}\psi \in H^1(\mathbb{R})$, there exists $M_1 = M_1(\varepsilon, t_0) > 0$ such that for any $x \in \mathbb{R}$ satisfying $|x| > M_1$, the estimate

$$|e^{it_0\partial_x^2}\psi(x)| \le \varepsilon \tag{D.9}$$

holds. For $x \in \mathbb{R}$ satisfying $|x| > M_1$, the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$ gives

$$\begin{aligned}|e^{it\partial_x^2}\psi(x)| &\le |e^{it\partial_x^2}\psi(x) - e^{it_0\partial_x^2}\psi(x)| + |e^{it_0\partial_x^2}\psi(x)| \\ &\le C_3\|e^{it\partial_x^2}\psi - e^{it_0\partial_x^2}\psi\|_{H^1} + \varepsilon \le (C_3+1)\varepsilon,\end{aligned}$$

where C_3 is a positive constant, which implies that for any $t \in [-T, T]$, there exists $M_1 = M_1(t, \varepsilon)$ such that the estimate holds:

$$\|e^{it\partial_x^2}\psi\|_{L_x^\infty(|x|>M_1)} \le (C_3+1)\varepsilon.$$

On the other hand, since $V \in L^1(\mathbb{R})$, there exists $M_2 = M_2(V, \varepsilon) > 0$ such that the estimate holds:

$$\int_{|x|>M_2} |V(x)|dx \le \varepsilon. \tag{D.10}$$

Set $M = M(t, V, \varepsilon) := \max(M_1, M_2)$. Since we are considering the case $\lim_{n\to\infty} x_n = \infty$, there exists $n_0 = n_0(M) \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ satisfying $n \ge n_0$, $x_n \ge 2M$. Then for $n \in \mathbb{N}$ with $n \ge n_0$, the relation holds:

$$x \in \mathbb{R} \ \text{ with } \ |x + x_n| \le M \Longrightarrow |x| \ge M. \tag{D.11}$$

For $n \in \mathbb{N}$ satisfying $n \ge n_0$ and $t \in [-T, T]$, the relations (D.10)-(D.11) and the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$ give

$$\begin{aligned}\|(\tau_{-x_n}V)e^{it\partial_x^2}\psi\|_{L^1} &= \int_{|x+x_n|\ge M} |V(x+x_n)||(e^{it\partial_x^2}\psi)(x)|dx + \int_{|x+x_n|<M} |V(x+x_n)||(e^{it\partial_x^2}\psi)(x)|dx \\ &\le \|e^{it\partial_x^2}\psi\|_{L_x^\infty} \int_{|x+x_n|\ge M} |V(x+x_n)|dx + \int_{|x|\ge M} |V(x+x_n)||(e^{it\partial_x^2}\psi)(x)|dx \\ &\le C_3\|\psi\|_{H^1} \int_{|x|\ge M} |V(x)|dx + \|e^{it\partial_x^2}\psi\|_{L_x^\infty(|x|\ge M)} \int_{|x|>M} |V(x+x_n)|dx \\ &\le \{C_3\|\psi\|_{H^1} + (C_3+1)\|V\|_{L^1}\}\varepsilon =: C_4\varepsilon,\end{aligned}$$

where C_4 is a positive constant depending on $\|V\|_{L^1}$ and $\|\psi\|_{H^1}$. This gives that for any $n \in \mathbb{N}$ satisfying $n \ge n_0$, the estimate

$$\|(\tau_{-x_n})e^{it\partial_x^2}\psi\|_{L_t^\infty(-T,T:L_x^1)} \le C_4\varepsilon$$

holds, which completes the proof of (D.7). By combining (D.6) and (D.7), we have (D.5). Moreover by combining (D.2) and (D.5), we can prove (D.1). In the same manner as the proof of Proposition 8 in [21], we can prove

$$\lim_{n\to\infty} \left\| \int_0^t e^{i(t-s)\partial_x^2}\left\{|U_n(s)|^{p-1}U_n(s)\right\} ds - \int_0^t e^{-i(t-s)H_V}\{|U_n(s)|^{p-1}U_n(s)\}ds \right\|_{L_t^a(\mathbb{R}:L_x^r)} = 0.$$

So we omit the detail. $\square$

Appendix E. Proof of the uniform localization via precompactness

In this appendix, we give a proof of Lemma 5.14, which can be proved in the similar manner as the proof of Lemma 5.6 in [11] and Corollary 3.3 in [7].

Proof of Lemma 5.14. On the contrary, there exists $\varepsilon > 0$ such that for any $n \in \mathbb{N}$, there exists $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\int_{|x|>n} |\partial_x u(t_n, x)|^2 dx + \int_{|x|>n} |u(t_n, x)|^2 dx + \int_{|x|>n} |u(t_n, x)|^{p+1} dx \geq \varepsilon. \tag{E.1}$$

Since $\mathbf{H}$ is precompact in $\mathcal{H}$, there exists $\phi \in \mathcal{H}$ such that, passing to a subsequence of $\{t_n\}_{n\in\mathbb{N}}$, the identity $\lim_{n\to\infty} u(t_n) = \phi$ in $\mathcal{H}$ holds. By combining this fact, the equivalency of the norms $\|\cdot\|_{H^1}$ and $\|\cdot\|_{\mathcal{H}}$, and the Sobolev embedding $H^1(\mathbb{R}) \subset L^{p+1}(\mathbb{R})$, there exists $N_1 = N_1(\varepsilon, p) \in \mathbb{N}$ such that for $n \geq N_1$, the estimate holds:

$$\int_{\mathbb{R}} |\partial_x u(t_n, x) - \partial_x \phi(x)|^2 dx + \int_{\mathbb{R}} |u(t_n, x) - \phi(x)|^2 dx + \int_{\mathbb{R}} |u(t_n, x) - \phi(x)|^{p+1} dx < \frac{\varepsilon}{2(p+1)}. \tag{E.2}$$

Since $\phi \in \mathcal{H} \subset H^1(\mathbb{R}) \subset L^{p+1}(\mathbb{R})$, there exists $N_2 = N_2(\varepsilon, p) \in \mathbb{N}$ such that for $n \geq N_2$, the estimate

$$\int_{|x|>n} |\partial_x \phi(x)|^2 dx + \int_{|x|>n} |\phi(x)|^2 dx + \int_{|x|>n} |\phi(x)|^{p+1} dx < \frac{\varepsilon}{2(p+1)}, \tag{E.3}$$

holds. Thus for any $n \in \mathbb{N}$ satisfying $n \geq \max(N_1, N_2)$, by combining the estimates (E.2) and (E.3), we have

$$\int_{|x|>n} |\partial_x u(t, x)|^2 dx + \int_{|x|>n} |u(t, x)|^2 dx + \int_{|x|>n} |u(t, x)|^{p+1} dx < \varepsilon.$$

This contradicts (E.1), which completes the proof of the lemma. □

References

[1] T. Akahori, H. Nawa, Blowup and scattering problems for the nonlinear Schrödinger equations, Kyoto J. Math. 53 (3) (2013) 629–672.
[2] H. Bahouri, P. Gérard, High frequency approximation of solutions to critical nonlinear wave equations, Am. J. Math. 121 (1) (1999) 131–175.
[3] V. Banica, N. Visciglia, Scattering for NLS with a delta potential, J. Differ. Equ. 260 (5) (2016) 4410–4439.
[4] T. Cazenave, Semilinear Schrödinger Equations, Courant Lecture Notes in Mathematics, vol. 10, American Mathematical Society, Courant Institute of Mathematical Sciences, 2003.
[5] R.Y. Chiao, E. Garmine, C.H. Townes, Self-trapping of optical beams, Phys. Rev. Lett. 13 (1964) 479–482.
[6] D. Du, Y. Wu, K. Zhang, On blow-up criterion for the nonlinear Schrödinger equation, Discrete Contin. Dyn. Syst. 36 (7) (2016) 3639–3650.
[7] T. Duyckaerts, J. Holmer, S. Roudenko, Scattering for the non-radial 3D cubic nonlinear Schrödinger equation, Math. Res. Lett. 15 (6) (2008) 1233–1250.
[8] D.Y. Fang, J. Xie, T. Cazenave, Scattering for the focusing energy-subcritical nonlinear Schrödinger equation, Sci. China Math. 54 (10) (2011) 2037–2062.
[9] D. Foschi, Inhomogeneous Strichartz estimates, J. Hyperbolic Differ. Equ. 2 (1) (2005) 1–24.
[10] M. Goldberg, W. Schlag, Dispersive estimates for Schrödinger operators in dimension one and three, Commun. Math. Phys. 251 (1) (2004) 157–178.
[11] J. Holmer, S. Roudenko, A sharp condition for scattering of the radial 3D cubic nonlinear Schrödinger equation, Commun. Math. Phys. 282 (2) (2008) 435–467.
[12] J. Holmer, S. Roudenko, Divergence of infinite-variance nonradial solutions to the 3D NLS equation, Commun. Partial Differ. Equ. 35 (5) (2010) 878–905.

[13] Y. Hong, Scattering for a nonlinear Schrödinger equation with a potential, Commun. Pure Appl. Anal. 15 (2016) 1571–1601.
[14] S. Ibrahim, N. Masmoudi, K. Nakanishi, Scattering threshold for the focusing nonlinear Klein-Gordon equation, Anal. PDE 4 (3) (2011) 405–460.
[15] M. Ikeda, T. Inui, Global dynamics below the standing waves for the focusing semilinear Schrödinger equation with a repulsive Dirac delta potential, Anal. PDE 10 (2) (2017) 481–512.
[16] T. Inui, Global dynamics of solutions with group invariance for the nonlinear Schrödinger equation, Commun. Pure Appl. Anal. 16 (2017) 557–590.
[17] O. Kavian, A remark on the blowing-up of solutions to the Cauchy problem for nonlinear Schrödinger equations, Trans. Am. Math. Soc. 299 (1) (1987) 193–203.
[18] M. Keel, T. Tao, Endpoint Strichartz estimates, Am. J. Math. 120 (5) (1998) 955–980.
[19] C.E. Kenig, F. Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case, Invent. Math. 166 (3) (2006) 645–675.
[20] R. Killip, J. Murphy, M. Visan, J. Zheng, The focusing cubic NLS with inverse-square potential in three space dimensions, Differ. Integral Equ. 30 (2017) 161–206.
[21] D. Lafontaine, Scattering for NLS with a potential on the line, Asymptot. Anal. 100 (2016) 21–39.
[22] J. Lu, C. Miao, J. Murphy, Scattering in H^1 for the intercritical NLS with an inverse-square potential, preprint, arXiv:1702.04064v1.
[23] K. Nakanishi, Global dynamics below excited solitons for the nonlinear Schrödinger equation with a potential, J. Math. Soc. Japan 69 (4) (2017) 1353–1401.
[24] K. Nakanishi, Global dynamics above the first excited energy for the nonlinear Schrödinger equation with a potential, preprint, arXiv:1603.02361.
[25] T. Ogawa, T. Tsutsumi, Blow-up of H^1 solution for the nonlinear Schrödinger equation, J. Differ. Equ. 92 (1991) 317–330.
[26] T. Ogawa, Y. Tsutsumi, Blow-up of H^1 solutions for the one-dimensional nonlinear Schrödinger equation with critical power nonlinearity, Proc. Am. Math. Soc. 111 (1991) 487–496.
[27] M. Reed, B. Simon, Methods of Modern Mathematical Physics. I. Functional Analysis, second edition, Academic Press, Inc, New York, 1980, xv+400 pp.
[28] H.A. Rose, M.I. Weinstein, On the bound states of the nonlinear Schrödinger equation with a linear potential, Physica D 30 (1988) 207–218.
[29] C. Sulem, P.-L. Sulem, The Nonlinear Schrödinger Equation. Self-Focusing and Wave Collapse, Applied Mathematical Sciences, vol. 139, Springer-Verlag, New York, 1999.
[30] R. Weder, L^p-$L^{p'}$ estimates for the Schrödinger equation on the line and inverse scattering for the nonlinear Schrödinger equation with a potential, J. Funct. Anal. 170 (2000) 37–68.
[31] M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Commun. Math. Phys. 84 (1983) 567–576.
[32] V.E. Zakharov, Collapse of Langmuir waves, Sov. Phys. JETP 35 (1972) 908–912.