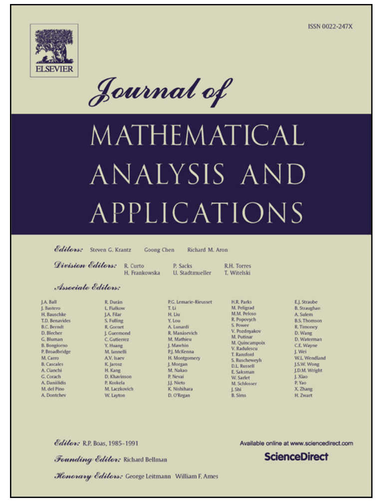

Strict and pointwise convergence of BV functions in metric spaces

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Strict and pointwise convergence of BV functions in metric spaces *

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Abstract

Consider a metric space X that is equipped with a doubling measure and supports a Poincaré inequality. We show that if $u_i \to u$ strictly in $\mathrm{BV}(X)$, i.e. if $u_i \to u$ in $L^1(X)$ and $\|Du_i\|(X) \to \|Du\|(X)$, then for a subsequence (not relabeled) we have $\widetilde{u}_i(x) \to \widetilde{u}(x)$ for $\mathcal{H}$-almost every $x \in X \setminus S_u$.

1 Introduction

During the past two decades, there has been active research on functions of bounded variation (BV functions) and more generally first-order analysis in the setting of metric measure spaces, see e.g. [1, 3, 4, 17]. Studying questions in such an abstract setting provides an opportunity to unify the theories developed in specific settings such as weighted Euclidean spaces, Riemannian manifolds, Carnot groups, etc. Moreover, metric spaces can be a natural setting for developing results that may be new even in Euclidean spaces.

Let (X, d, μ) be a metric space equipped with a doubling measure μ that supports a $(1,1)$-Poincaré inequality; see Section 2 for definitions. Take a sequence of BV functions $(u_i) \subset \mathrm{BV}(X)$. If $u_i \to u$ in $L^1(X)$, then of course for a subsequence (not relabeled) we have $u_i(x) \to u(x)$ for μ-almost every

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$x \in X$. If $X = \mathbb{R}^n$ and the functions u_i are defined as convolutions of $u \in \mathrm{BV}(\mathbb{R}^n)$ with a mollifier function at smaller and smaller scales, then $u_i(x) \to \widetilde{u}(x)$ for $\mathcal{H}$-almost every $x \in \mathbb{R}^n$ by [2, Corollary 3.80] (where $\mathcal{H}$ is the codimension one, or $n-1$-dimensional Hausdorff measure). See also [13, Proposition 4.1] for a slightly weaker analogous result in the metric setting.

In this paper we consider what kind of pointwise convergence can be obtained if we know that $u_i \to u$ *strictly* in $\mathrm{BV}(\Omega)$, that is, $u_i \to u$ in $L^1(\Omega)$ and $\|Du_i\|(\Omega) \to \|Du\|(\Omega)$, where $\Omega \subset X$ is an open set. The motivation for studying this question comes from the fact that strict convergence commonly arises for example in various minimization problems, as well as already in the definition of the total variation of a BV function. We show that in such a case, for a subsequence (not relabeled) we have $\widetilde{u}_i(x) \to \widetilde{u}(x)$ for $\mathcal{H}$-almost every $x \in \Omega \setminus S_u$, where S_u is the jump set of u. This is given in Corollary 3.3. We also show that in any compact subset of $\Omega \setminus S_u$, we can obtain uniform convergence outside sets of small 1-capacity. This is given in Corollary 3.11. Somewhat more general formulations of these results are given in Theorem 3.2 and Theorem 3.10.

Our results may be new even in the Euclidean setting. The main tool used in the proofs is a boxing inequality -type argument, which has been previously applied in e.g. [12]. At the end of the paper we give examples that demonstrate that the results appear to be optimal.

2 Preliminaries

In this section we introduce the notation, definitions, and assumptions used in this paper.

In this paper, (X, d, μ) is a complete metric space endowed with a Borel regular outer measure μ that satisfies a doubling property, meaning that there is a constant $C_d \geq 1$ such that

$$0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty$$

for every ball $B = B(x, r)$ with center $x \in X$ and radius $r > 0$. By iterating the doubling property, we obtain that for any $x \in X$ and $y \in B(x, R)$ with $0 < r \leq R < \infty$, we have

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \geq \frac{1}{C_d^2} \left(\frac{r}{R}\right)^Q, \tag{2.1}$$

where $Q > 0$ only depends on the doubling constant C_d.

A complete metric space endowed with a doubling measure is proper, that is, closed and bounded sets are compact, see e.g. [5, Proposition 3.1]. Given an open set $\Omega \subset X$, we write $\Omega' \Subset \Omega$ if Ω' is open and $\overline{\Omega'}$ is a compact subset of Ω. Since X is proper, we define $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ to be the space of functions that are in the Lipschitz class $\mathrm{Lip}(\Omega')$ for every $\Omega' \Subset \Omega$. Other local spaces of functions are defined analogously.

For any $0 < R < \infty$, the codimension one Hausdorff content of $A \subset X$ is defined by

$$\mathcal{H}_R(A) := \inf \left\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} :\ A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i),\ r_i \le R \right\}.$$

The codimension one Hausdorff measure of a set $A \subset X$ is then defined by

$$\mathcal{H}(A) := \lim_{R \to 0} \mathcal{H}_R(A).$$

By a curve γ we mean a rectifiable continuous mapping from a compact interval of the real line into X. A nonnegative Borel function g on X is an upper gradient of an extended real-valued function u on X if for all curves γ on X, we have

$$|u(x) - u(y)| \le \int_\gamma g\, ds,$$

where x and y are the end points of γ, and the curve integral is defined by using an arc-length parametrization, see [10, Section 2] where upper gradients were originally introduced. We interpret $|u(x) - u(y)| = \infty$ whenever at least one of $|u(x)|$, $|u(y)|$ is infinite.

We define the norm

$$\|u\|_{N^{1,1}(X)} := \|u\|_{L^1(X)} + \inf \|g\|_{L^1(X)},$$

where the infimum is taken over all upper gradients g of u. Then we define the Newton-Sobolev space

$$N^{1,1}(X) := \{u :\ \|u\|_{N^{1,1}(X)} < \infty\}.$$

The 1-capacity of a set $A \subset X$ is defined by

$$\mathrm{Cap}_1(A) := \inf \|u\|_{N^{1,1}(X)},$$

where the infimum is taken over all functions $u \in N^{1,1}(X)$ such that $u \geq 1$ in A. It is known that Cap_1 is an outer capacity, meaning that

$$\mathrm{Cap}_1(A) = \inf\{\mathrm{Cap}_1(U) : A \subset U, U \text{ is open}\}$$

for any $A \subset X$, see e.g. [5, Theorem 5.31]. For more on Newton-Sobolev spaces, see [18, 5, 11].

Let $E \subset X$ be an arbitrary set. The measure theoretic boundary $\partial^* E$ is defined as the set of points $x \in X$ where both E and its complement have positive upper density, i.e.

$$\limsup_{r \to 0} \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))} > 0 \quad \text{and} \quad \limsup_{r \to 0} \frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} > 0.$$

The measure theoretic interior and exterior of E are defined respectively by

$$I_E := \left\{ x \in X : \lim_{r \to 0} \frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} = 0 \right\}$$

and

$$O_E := \left\{ x \in X : \lim_{r \to 0} \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))} = 0 \right\}.$$

Note that the space is always partitioned into the disjoint sets $\partial^* E$, I_E, and O_E.

Next we recall the definition of functions of bounded variation, or BV functions, on metric spaces, following [17]. See also e.g. the monographs [2, 6, 7, 8, 19] for the classical theory in the Euclidean setting. Given $u \in L^1_{\mathrm{loc}}(X)$, the total variation of u in X is defined by

$$\|Du\|(X) := \inf \left\{ \liminf_{i \to \infty} \int_X g_{u_i} \, d\mu : u_i \in \mathrm{Lip}_{\mathrm{loc}}(X), \, u_i \to u \text{ in } L^1_{\mathrm{loc}}(X) \right\},$$

where each g_{u_i} is an upper gradient of u_i. A function $u \in L^1(X)$ is said to be of bounded variation, and we denote $u \in \mathrm{BV}(X)$, if $\|Du\|(X) < \infty$. By replacing X with an open set $\Omega \subset X$ in the definition of the total variation, we can define $\|Du\|(\Omega)$. For an arbitrary set $A \subset X$, we define

$$\|Du\|(A) = \inf\{\|Du\|(\Omega) : A \subset \Omega, \, \Omega \subset X \text{ is open}\}.$$

If $u \in \mathrm{BV}(\Omega)$, $\|Du\|(\cdot)$ is a finite Radon measure on Ω by [17, Theorem 3.4]. The BV norm is defined by

$$\|u\|_{\mathrm{BV}(\Omega)} := \|u\|_{L^1(\Omega)} + \|Du\|(\Omega).$$

A μ-measurable set $E \subset X$ is said to be of finite perimeter if $\|D\chi_E\|(X) < \infty$, where χ_E is the characteristic function of E. The perimeter of E in Ω is also denoted by

$$P(E, \Omega) := \|D\chi_E\|(\Omega).$$

Similarly as above, if $P(E, \Omega) < \infty$, then $P(E, \cdot)$ is a finite Radon measure on Ω. For any Borel sets $E_1, E_2 \subset X$ we have by [17, Proposition 4.7]

$$P(E_1 \cap E_2, X) + P(E_1 \cup E_2, X) \leq P(E_1, X) + P(E_2, X).$$

The proof works equally well for μ-measurable $E_1, E_2 \subset X$ and with X replaced by any open set $U \subset X$, so that

$$P(E_1 \cap E_2, U) + P(E_1 \cup E_2, U) \leq P(E_1, U) + P(E_2, U). \tag{2.2}$$

We have the following coarea formula from [17, Proposition 4.2]: if $\Omega \subset X$ is an open set and $v \in L^1_{\mathrm{loc}}(\Omega)$, then

$$\|Dv\|(\Omega) = \int_{-\infty}^{\infty} P(\{v > t\}, \Omega)\, dt. \tag{2.3}$$

In this paper we always assume that X supports a $(1, 1)$-Poincaré inequality, meaning that there exist constants $C_P \geq 1$ and $\lambda \geq 1$ such that for every ball $B(x, r)$, every $u \in L^1_{\mathrm{loc}}(X)$, and every upper gradient g of u, we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \leq C_P r \fint_{B(x,\lambda r)} g\, d\mu,$$

where

$$u_{B(x,r)} := \fint_{B(x,r)} u\, d\mu := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u\, d\mu.$$

If we apply the Poincaré inequality to sequences of approximating locally Lipschitz functions in the definition of the total variation, we get the following BV version of the Poincaré inequality. For every ball $B(x, r)$ and every $u \in L^1_{\mathrm{loc}}(X)$, we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \leq C_P r \frac{\|Du\|(B(x, \lambda r))}{\mu(B(x, \lambda r))}.$$

For a μ-measurable set $E \subset X$, the above implies the relative isoperimetric inequality

$$\min\{\mu(B(x, r) \cap E),\ \mu(B(x, r) \setminus E)\} \leq 2C_P r P(E, B(x, \lambda r)). \tag{2.4}$$

Given a set of finite perimeter $E \subset X$, for $\mathcal{H}$-almost every $x \in \partial^* E$ we have by [1, Theorem 5.4]

$$\gamma \le \liminf_{r\to 0} \frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))} \le \limsup_{r\to 0} \frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))} \le 1-\gamma, \tag{2.5}$$

where the number $\gamma \in (0, 1/2]$ only depends on the doubling constant and the constants in the Poincaré inequality. For an open set $\Omega \subset X$ and a μ-measurable set $E \subset X$ with $P(E,\Omega) < \infty$, we know that for any Borel set $A \subset \Omega$

$$P(E, A) = \int_{\partial^* E\cap A} \theta_E \, d\mathcal{H}, \tag{2.6}$$

where $\theta_E \colon X \to [\alpha, C_d]$ with $\alpha = \alpha(C_d, C_P, \lambda) > 0$, see [1, Theorem 5.3] and [3, Theorem 4.6].

The jump set of a function u on X is defined by

$$S_u := \{x \in X : u^\wedge(x) < u^\vee(x)\},$$

where $u^\wedge(x)$ and $u^\vee(x)$ are the lower and upper approximate limits of u defined by

$$u^\wedge(x) := \sup \left\{ t \in \mathbb{R} : \lim_{r\to 0} \frac{\mu(B(x,r)\cap \{u < t\})}{\mu(B(x,r))} = 0 \right\}$$

and

$$u^\vee(x) := \inf \left\{ t \in \mathbb{R} : \lim_{r\to 0} \frac{\mu(B(x,r)\cap \{u > t\})}{\mu(B(x,r))} = 0 \right\}.$$

Note that for $u = \chi_E$, we have $x \in I_E$ if and only if $u^\wedge(x) = u^\vee(x) = 1$, $x \in O_E$ if and only if $u^\wedge(x) = u^\vee(x) = 0$, and $x \in \partial^* E$ if and only if $u^\wedge(x) = 0$ and $u^\vee(x) = 1$.

We understand BV functions to be μ-equivalence classes. To consider pointwise properties, we need to consider the representatives $u^\wedge$ and $u^\vee$. We also define the representative

$$\widetilde{u} := (u^\wedge + u^\vee)/2.$$

We say that a set $A \subset X$ is 1-*quasiopen* if for every $\varepsilon > 0$ there is an open set $G \subset X$ with $\mathrm{Cap}_1(G) < \varepsilon$ such that $A \cup G$ is open.

3 The convergence results

In this section we give our main results: Theorem 3.2 on the pointwise convergence of BV functions, and Theorem 3.10, given at the end of the section, on uniform convergence.

The following fact about the Hausdorff content and measure is well known in the Euclidean setting, and proved in the metric setting in the below reference. See also [12, Lemma 7.9] for a previous similar result.

Lemma 3.1 ([15, Lemma 3.6]). *Let $R > 0$. If $A \subset X$ and $\mathcal{H}_R(A) = 0$, then $\mathcal{H}(A) = 0$.*

Theorem 3.2. *Let $\Omega \subset X$ be an open set, and let $u_i, u \in \mathrm{BV}(\Omega)$ such that $u_i \to u$ in $L^1(\Omega)$ and $\|Du_i\|(\Omega) \to \|Du\|(\Omega)$. Then there exists a subsequence (not relabeled) such that for $\mathcal{H}$-almost every $x \in \Omega$,*

$$u^\wedge(x) \le \liminf_{i\to\infty} u_i^\wedge(x) \le \limsup_{i\to\infty} u_i^\vee(x) \le u^\vee(x).$$

By the definitions of $u^\wedge$, $u^\vee$, and $\widetilde{u}$, we immediately get the following corollary.

Corollary 3.3. *Let $\Omega \subset X$ be an open set, and let $u_i, u \in \mathrm{BV}(\Omega)$ such that $u_i \to u$ in $L^1(\Omega)$ and $\|Du_i\|(\Omega) \to \|Du\|(\Omega)$. Then there exists a subsequence (not relabeled) such that $\widetilde{u}_i(x) \to \widetilde{u}(x)$ for $\mathcal{H}$-almost every $x \in \Omega \setminus S_u$.*

First we prove the theorem for sets of finite perimeter. In the proof below, the definition of the sets I_j and O_j is inspired by the proof of Federer's structure theorem, see [7, Section 4.5.11] or [6, p. 222].

Proposition 3.4. *Let $\Omega \subset X$ be an open set, and let $E_i, E \subset X$ be μ-measurable sets with $\chi_{E_i} \to \chi_E$ in $L^1_{\mathrm{loc}}(\Omega)$ and $P(E_i, \Omega) \to P(E, \Omega) < \infty$. Passing to a subsequence (not relabeled), for $\mathcal{H}$-almost every $x \in I_E \cap \Omega$ we have $x \in I_{E_i}$ for all sufficiently large $i \in \mathbb{N}$, and for $\mathcal{H}$-almost every $x \in O_E \cap \Omega$ we have $x \in O_{E_i}$ for all sufficiently large $i \in \mathbb{N}$.*

Proof. For each $j \in \mathbb{N}$, define (recall the number γ from (2.5))

$$I_j := \left\{ x \in X : \sup_{0<r<1/j} \frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} \le \gamma/2 \right\},$$

$$O_j := \left\{ x \in X : \sup_{0<r<1/j} \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))} \le \gamma/2 \right\}.$$

Note that these are increasing sequences of sets and

$$I_E \subset \bigcup_{j=1}^{\infty} I_j \quad \text{and} \quad O_E \subset \bigcup_{j=1}^{\infty} O_j. \tag{3.1}$$

Moreover, the sets I_j and O_j are closed, which can be seen as follows. For a fixed $j \in \mathbb{N}$, take a sequence of points $x_k \in I_j$ with $x_k \to x \in X$. Let $0 < r < 1/j$. Then by applying Lebesgue's dominated convergence theorem to both the numerator and the denominator, we obtain

$$\frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} = \lim_{k\to\infty} \frac{\mu(B(x_k, r - d(x_k,x)) \setminus E)}{\mu(B(x_k, r - d(x_k,x)))} \le \gamma/2,$$

so I_j is closed. The proof for the sets O_j is analogous. By (2.5) we also know that $\mathcal{H}(\partial^* E \cap (I_j \cup O_j) \cap \Omega) = 0$ for all $j \in \mathbb{N}$. Then by (2.6) we have

$$P\left(E, \bigcup_{j=1}^{\infty} (I_j \cup O_j) \cap \Omega\right) = 0. \tag{3.2}$$

We can find sets $\Omega'_j \Subset \Omega$ with $\Omega = \bigcup_{j=1}^{\infty} \Omega'_j$ such that for a fixed $x_0 \in X$, $\Omega'_j \subset B(x_0, j)$ for each $j \in \mathbb{N}$. Fix $j \in \mathbb{N}$. By the lower semicontinuity of perimeter with respect to L^1-convergence, we have for any open set $U \subset \Omega$

$$P(E,U) \le \liminf_{i\to\infty} P(E_i, U).$$

Since we also have $P(E_i, \Omega) \to P(E, \Omega)$, we get for any closed set $F \subset \Omega$

$$\limsup_{i\to\infty} P(E_i, F) \le P(E, F).$$

For any $A \subset X$ and $a > 0$, denote

$$A^a := \{x \in X : \operatorname{dist}(x, A) < a\}.$$

In particular, for any $0 < a < \operatorname{dist}(\Omega'_j, X \setminus \Omega)$ we have

$$\limsup_{i\to\infty} P(E_i, (I_j \cap \Omega'_j)^a) \le \limsup_{i\to\infty} P(E_i, \overline{(I_j \cap \Omega'_j)^a}) \le P(E, \overline{(I_j \cap \Omega'_j)^a}). \tag{3.3}$$

Using the fact that $P(E,\cdot)$ is a Radon measure on Ω in the second equality below, and then the fact that $\overline{I_j \cap \Omega'_j} \subset I_j \cap \overline{\Omega'_j}$ since I_j is closed,

$$\lim_{a\to 0} P(E, \overline{(I_j \cap \Omega'_j)^a}) = \lim_{a\to 0} P(E, (I_j \cap \Omega'_j)^a) = P\left(E, \bigcap_{a>0} (I_j \cap \Omega'_j)^a\right)$$
$$= P(E, \overline{I_j \cap \Omega'_j}) \le P(E, I_j \cap \overline{\Omega'_j}) = 0$$

by (3.2). Thus by choosing $0 < s_j < \min\{1/5, \operatorname{dist}(\Omega'_j, X \setminus \Omega)\}$ small enough, we get

$$P(E, \overline{(I_j \cap \Omega'_j)^{s_j}}) < 2^{-j}. \tag{3.4}$$

Noting that $\Omega'_j \subset B(x_0, j)$, by (2.1) we have

$$\inf_{y\in\Omega'_j} \mu(B(y, s_j/2\lambda)) \ge \frac{1}{C_d^2}\left(\frac{s_j}{2\lambda j}\right)^Q \mu(B(x_0,j)) > 0,$$

where $\lambda \ge 1$ is the constant from the relative isoperimetric inequality (2.4). By (3.3) and (3.4), we find $i_j \in \mathbb{N}$ such that

$$P(E_{i_j}, (I_j \cap \Omega'_j)^{s_j}) < 2^{-j} \tag{3.5}$$

and (by the fact that $\chi_{E_i} \to \chi_E$ in $L^1_{\mathrm{loc}}(\Omega)$)

$$2C_d\|\chi_{E_{i_j}} - \chi_E\|_{L^1((\Omega'_j)^{s_j})} \le \inf_{y\in\Omega'} \mu(B(y, s_j/2\lambda)). \tag{3.6}$$

Take $x \in I_j \cap O_{E_{i_j}} \cap \Omega'_j$. Since E has lower density at least $1-\gamma/2$ at x and E_{i_j} has density zero, for some $0 < s < s_j/2\lambda$ we have in particular

$$\fint_{B(x,s)} |\chi_{E_{i_j}} - \chi_E|\, d\mu > \frac{1}{2C_d}.$$

Next, double the radius s repeatedly and take the last number so obtained for which the above estimate holds, and call this number r. Note by (3.6) that $r < s_j/2\lambda$. Then necessarily

$$\frac{1}{2C_d} < \fint_{B(x,r)} |\chi_{E_{i_j}} - \chi_E|\, d\mu \le C_d \fint_{B(x,2r)} |\chi_{E_{i_j}} - \chi_E|\, d\mu \le \frac{C_d}{2C_d} = \frac{1}{2},$$

or

$$\frac{1}{2C_d} < \frac{\mu(B(x,r) \cap (E_{i_j} \Delta E))}{\mu(B(x,r))} \le \frac{1}{2}.$$

Then by the relative isoperimetric inequality (2.4) and (2.2)

$$\begin{aligned}\frac{\mu(B(x,r))}{2C_d} &\le \mu(B(x,r)\cap(E_{i_j}\Delta E))\\ &\le 2C_P r P(E_{i_j}\Delta E, B(x,\lambda r))\\ &\le 2C_P r\big(P(E_{i_j}\setminus E, B(x,\lambda r)) + P(E\setminus E_{i_j}, B(x,\lambda r))\big)\\ &\le 4C_P r\big(P(E_{i_j}, B(x,\lambda r)) + P(E, B(x,\lambda r))\big).\end{aligned} \tag{3.7}$$

Using these radii, we obtain a covering $\{B(x,\lambda r_x)\}_{x\in I_j\cap O_{E_{i_j}}\cap\Omega'_j}$ with $r_x < s_j/\lambda$. By the 5-covering theorem, we can extract a countable collection of pairwise disjoint balls $\{B(x_m,\lambda r_m)\}_{m\in\mathbb{N}}$ with $r_m < s_j/\lambda$ such that the balls $\{B(x_m, 5\lambda r_m)\}_{m\in\mathbb{N}}$ cover $I_j\cap O_{E_{i_j}}\cap\Omega'_j$. Denote by $\lceil a\rceil$ the smallest integer at least $a\in\mathbb{R}$. Then

$$\begin{aligned}\mathcal{H}_{5s_j}(I_j\cap O_{E_{i_j}}\cap\Omega'_j) &\le \sum_{m=1}^{\infty}\frac{\mu(B(x_m,5\lambda r_m))}{5\lambda r_m}\\ &\le C_d^{\lceil\log_2(5\lambda)\rceil}\sum_{m=1}^{\infty}\frac{\mu(B(x_m,r_m))}{r_m}\\ &\overset{(3.7)}{\le} 8C_d^{\lceil\log_2(5\lambda)\rceil+1}C_P\sum_{m=1}^{\infty}(P(E_{i_j},B(x_m,\lambda r_m)) + P(E,B(x_m,\lambda r_m)))\\ &\le 8C_d^{\lceil\log_2(5\lambda)\rceil+1}C_P(P(E_{i_j},(I_j\cap\Omega'_j)^{s_j}) + P(E,(I_j\cap\Omega'_j)^{s_j}))\\ &\le 8C_d^{\lceil\log_2(5\lambda)\rceil+1}C_P\times 2^{-j+1}\end{aligned}$$

by (3.4) and (3.5). In total, using also (2.6),

$$\begin{aligned}&\mathcal{H}_{5s_j}(I_j\cap(\partial^*E_{i_j}\cup O_{E_{i_j}})\cap\Omega'_j)\\ &\quad\le \mathcal{H}_{5s_j}(I_j\cap\partial^*E_{i_j}\cap\Omega'_j) + \mathcal{H}_{5s_j}(I_j\cap O_{E_{i_j}}\cap\Omega'_j)\\ &\quad\le \alpha^{-1}P(E_{i_j}, I_j\cap\Omega'_j) + 8C_d^{\lceil\log_2(5\lambda)\rceil+1}C_P\times 2^{-j+1}\\ &\quad\le 2^{-j}\alpha^{-1} + 2^{-j+4}C_d^{\lceil\log_2(5\lambda)\rceil+1}C_P\end{aligned}$$

by (3.5). Then for any $p \in \mathbb{N}$, since $I_p \subset I_j$ and $\Omega'_p \subset \Omega'_j$ as soon as $j \geq p$,

$$\begin{aligned}
\mathcal{H}_1\left(I_p \cap \Omega'_p \cap \bigcap_{l=1}^{\infty}\bigcup_{j=l}^{\infty}(\partial^* E_{i_j} \cup O_{E_{i_j}})\right) & \\
&\leq \lim_{l\to\infty} \mathcal{H}_1\left(I_p \cap \Omega'_p \cap \bigcup_{j=l}^{\infty}(\partial^* E_{i_j} \cup O_{E_{i_j}})\right) \\
&\leq \lim_{l\to\infty} \sum_{j=l}^{\infty} \mathcal{H}_1(I_p \cap \Omega'_p \cap (\partial^* E_{i_j} \cup O_{E_{i_j}})) \\
&\leq \lim_{l\to\infty} \sum_{j=l}^{\infty} \mathcal{H}_{5s_j}(I_j \cap \Omega'_j \cap (\partial^* E_{i_j} \cup O_{E_{i_j}})) \\
&\leq \lim_{l\to\infty} (2^{-l+1}\alpha^{-1} + 2^{-l+5}C_d^{\lceil \log_2(5\lambda)\rceil+1}C_P) \\
&= 0.
\end{aligned} \tag{3.8}$$

Thus by (3.1)

$$\begin{aligned}
\mathcal{H}_1\left(I_E \cap \Omega \cap \bigcap_{l=1}^{\infty}\bigcup_{j=l}^{\infty}(\partial^* E_{i_j} \cup O_{E_{i_j}})\right) & \\
&\leq \sum_{p=1}^{\infty} \mathcal{H}_1\left(I_p \cap \Omega'_p \cap \bigcap_{l=1}^{\infty}\bigcup_{j=l}^{\infty}(\partial^* E_{i_j} \cup O_{E_{i_j}})\right) = 0,
\end{aligned}$$

and so by Lemma 3.1,

$$\mathcal{H}\left(I_E \cap \Omega \cap \bigcap_{l=1}^{\infty}\bigcup_{j=l}^{\infty}(\partial^* E_{i_j} \cup O_{E_{i_j}})\right) = 0.$$

To conclude the proof, we note that for any $x \in I_E \cap \Omega \setminus \bigcap_{l=1}^{\infty}\bigcup_{j=l}^{\infty}(O_{E_{i_j}} \cup \partial^* E_{i_j})$ we have $x \in I_{E_{i_j}}$ starting from some index $j \in \mathbb{N}$. By an analogous argument, and by passing to a further subsequence (not relabeled), we obtain that for $\mathcal{H}$-almost every $x \in O_E \cap \Omega$ we have $x \in O_{E_{i_j}}$ starting from some index $j \in \mathbb{N}$. $\square$

We note the following relation between the 1-capacity and the Hausdorff content: for any $A \subset X$, we have

$$\mathrm{Cap}_1(A) \leq 2C_d\mathcal{H}_1(A), \tag{3.9}$$

see the proof of [12, Lemma 3.4].

Besides pointwise convergence, we wish to consider uniform convergence outside sets of small 1-capacity. For this, we need the following lemma, whose proof is again based on a boxing inequality -type argument.

Lemma 3.5. *Let $\Omega \subset X$ be an open set, let $E \subset X$ be a μ-measurable set with $P(E,\Omega) < \infty$, let $K \subset I_E \cap \Omega$ be a compact set, and let $\varepsilon > 0$. Then there exists an open set $V \subset X$ with $\mathrm{Cap}_1(V) < \varepsilon$ such that*

$$\frac{\mu(B(x,r)\setminus E)}{\mu(B(x,r))} \to 0 \quad \text{as } r \to 0$$

uniformly for $x \in K \setminus V$.

Proof. By (2.6) and the fact that $I_E \cap \partial^* E = \emptyset$, we have $P(E,K) = 0$. Recall the notation $K^a := \{x \in X : \mathrm{dist}(x,K) < a\}$. For each $j = 2,3,\ldots$, by the fact that $P(E,\cdot)$ is a Radon measure on Ω, we can choose

$$0 < s_j < \min\{1/5, \mathrm{dist}(K, X \setminus \Omega)\}$$

such that

$$P(E, K^{s_j}) < 2^{-j}. \tag{3.10}$$

Let

$$A_j := \left\{ x \in K : \frac{\mu(B(x,s)\setminus E)}{\mu(B(x,s))} > \frac{1}{j} \ \text{ for some } 0 < s < s_j/\lambda \right\},$$

where $\lambda \geq 1$ is the constant from the relative isoperimetric inequality (2.4). Fix $j \geq 2$. Take $x \in A_j$ and $0 < s < s_j/\lambda$ such that

$$\frac{\mu(B(x,s)\setminus E)}{\mu(B(x,s))} > \frac{1}{j}.$$

Halve the radius s repeatedly and take the last number so obtained for which the above estimate holds, and call this number r; we find such a number by the fact that $x \in I_E$. Then we have $\mu(B(x,r)\setminus E) > \mu(B(x,r))/j$ and $\mu(B(x,r/2)\setminus E) \leq \mu(B(x,r/2))/j$, and thus

$$\frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))} \geq \frac{1}{C_d}\frac{\mu(B(x,r/2)\cap E)}{\mu(B(x,r/2))} \geq \frac{1-1/j}{C_d} \geq \frac{1}{2C_d}.$$

Then

$$\min\{\mu(B(x,r)\cap E),\mu(B(x,r)\setminus E)\} \geq \min\left\{\frac{1}{j},\frac{1}{2C_d}\right\}\mu(B(x,r))$$
$$\geq \frac{\mu(B(x,r))}{2C_d j}.$$

Using these radii, we get a covering $\{B(x,\lambda r_x)\}_{x\in A_j}$ with $r_x < s_j/\lambda$. By the 5-covering theorem, we can extract a countable collection of disjoint balls $\{B(x_k,\lambda r_k)\}_{k=1}^{\infty}$ such that the balls $B(x_k, 5\lambda r_k)$ cover A_j. Thus

$$\begin{aligned}
\mathcal{H}_1(A_j) &\leq \sum_{k=1}^{\infty}\frac{\mu(B(x_k,5\lambda r_k))}{5\lambda r_k} \\
&\leq C_d^{\lceil 5\lambda\rceil}\sum_{k=1}^{\infty}\frac{\mu(B(x_k,r_k))}{r_k} \\
&\leq 2C_d^{\lceil 5\lambda\rceil+1} j\sum_{k=1}^{\infty}\frac{\min\{\mu(B(x,r_k)\cap E),\mu(B(x,r_k)\setminus E)\}}{r_k} \\
&\overset{(2.4)}{\leq} 4C_d^{\lceil 5\lambda\rceil+1}C_P j\sum_{k=1}^{\infty}P(E,B(x_k,\lambda r_k)) \\
&\leq 4C_d^{\lceil 5\lambda\rceil+1}C_P jP(E,K^{s_j}) \\
&\leq 4C_d^{\lceil 5\lambda\rceil+1}C_P j2^{-j}
\end{aligned}$$

by (3.10). Then

$$\mathcal{H}_1\left(\bigcup_{j=l}^{\infty}A_j\right) \leq \sum_{j=l}^{\infty}\mathcal{H}_1(A_j) \leq 4C_d^{\lceil 5\lambda\rceil+1}C_P\sum_{j=l}^{\infty}j2^{-j} < \frac{\varepsilon}{2C_d}$$

for sufficiently large $l\in\mathbb{N}$ and the $\varepsilon>0$ given in the statement of the lemma, and so by (3.9)

$$\mathrm{Cap}_1\left(\bigcup_{j=l}^{\infty}A_j\right) < \varepsilon.$$

By the fact that Cap_1 is an outer capacity, we can choose an open set $V\supset \bigcup_{j=l}^{\infty}A_j$ with $\mathrm{Cap}_1(V)<\varepsilon$. By the definition of the sets A_j, we obtain the desired uniform convergence in $K\setminus V$. □

Now we have the following uniform convergence result for sets of finite perimeter.

Proposition 3.6. *Let $\Omega \subset X$ be an open set, and let $E_i, E \subset X$ be μ-measurable sets with $\chi_{E_i} \to \chi_E$ in $L^1_{\mathrm{loc}}(\Omega)$ and $P(E_i, \Omega) \to P(E, \Omega) < \infty$. Then there exists a subsequence (not relabeled) such that whenever $K \subset I_E \cap \Omega$ is a compact set and $\varepsilon > 0$, there exists an open set $U \subset X$ with $\mathrm{Cap}_1(U) < \varepsilon$ and $l \in \mathbb{N}$ such that $K \setminus U \subset I_{E_i}$ for all $i \geq l$.*

Proof. Take the subsequence $\{i_j\}_{j=1}^{\infty}$ obtained in the proof of Proposition 3.4. Fix a compact set $K \subset I_E \cap \Omega$ and $\varepsilon > 0$. By Lemma 3.5 we find a set $V \subset X$ with $\mathrm{Cap}_1(V) < \varepsilon/2$ such that

$$\frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} \to 0 \quad \text{as } r \to 0$$

uniformly for $x \in K \setminus V$. This implies in particular that for some $p \in \mathbb{N}$ (recall the number γ from (2.5))

$$\frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} \leq \frac{\gamma}{2}$$

for all $x \in K \setminus V$ and $0 < r < 1/p$. Hence $K \setminus V \subset I_p$; recall the definition from the beginning of the proof of Proposition 3.4. Thus $\mathrm{Cap}_1(K \setminus I_p) \leq \mathrm{Cap}_1(V) < \varepsilon/2$.

Moreover, by choosing p even larger, if necessary, we have $K \subset \Omega'_p$, where the sets $\Omega'_k \Subset \Omega$ were also defined in the proof of Proposition 3.4. Now

$$\mathcal{H}_1\left(I_p \cap \Omega'_p \cap \bigcup_{j=l}^{\infty}(\partial^* E_{i_j} \cup O_{E_{i_j}})\right) < \frac{\varepsilon}{4C_d}$$

for a sufficiently large $l \in \mathbb{N}$, by the last five lines of (3.8). Thus by (3.9),

$$\mathrm{Cap}_1\left(I_p \cap \Omega'_p \cap \bigcup_{j=l}^{\infty}(\partial^* E_{i_j} \cup O_{E_{i_j}})\right) < \frac{\varepsilon}{2}.$$

Then since $\mathrm{Cap}_1(K \setminus (I_p \cap \Omega'_p)) < \varepsilon/2$, we conclude

$$\mathrm{Cap}_1\left(K \cap \bigcup_{j=l}^{\infty}(\partial^* E_{i_j} \cup O_{E_{i_j}})\right) < \varepsilon.$$

Since Cap_1 is an outer capacity, we can take an open set

$$U \supset K \cap \bigcup_{j=l}^{\infty} (\partial^* E_{i_j} \cup O_{E_{i_j}})$$

with $\mathrm{Cap}_1(U) < \varepsilon$. Now $K \setminus U \subset I_{E_{i_j}}$ for all $j \geq l$. □

The following lemma is essentially [2, Exercise 1.19].

Lemma 3.7. *Let $f_i, f \in L^1(\mathbb{R})$ be nonnegative functions and assume that $f(t) \leq \liminf_{i\to\infty} f_i(t)$ for almost every $t \in \mathbb{R}$, and that*

$$\limsup_{i\to\infty} \int_{\mathbb{R}} f_i \, dt \leq \int_{\mathbb{R}} f \, dt. \tag{3.11}$$

Then $f_i \to f$ in $L^1(\mathbb{R})$.

Proof. Define $g_i := \inf_{j\geq i} f_j$. Then for some function g, $g_i(t) \nearrow g(t) \geq f(t)$ for almost every $t \in \mathbb{R}$. By Lebesgue's monotone convergence theorem,

$$\int_{\mathbb{R}} g \, dt = \lim_{i\to\infty} \int_{\mathbb{R}} g_i \, dt \leq \limsup_{i\to\infty} \int_{\mathbb{R}} f_i \, dt \leq \int_{\mathbb{R}} f \, dt < \infty,$$

that is, $g \in L^1(\mathbb{R})$. Thus by Lebesgue's dominated convergence theorem, $g_i \to g$ in $L^1(\mathbb{R})$. But since $\lim_{i\to\infty} \int_{\mathbb{R}} g_i \, dt \leq \int_{\mathbb{R}} f \, dt$, we must have $g = f$ almost everywhere.

Thus $(f_i - f)_- = (f - f_i)_+ \leq (f - g_i)_+ \to 0$ in $L^1(X)$, and so

$$\begin{aligned}\limsup_{i\to\infty} \int_{\mathbb{R}} (f_i - f)_+ \, dt &= \limsup_{i\to\infty} \left(\int_{\mathbb{R}} (f_i - f) \, dt + \int_{\mathbb{R}} (f_i - f)_- \, dt \right) \\ &= \limsup_{i\to\infty} \int_{\mathbb{R}} (f_i - f) \, dt \leq 0\end{aligned}$$

by (3.11). □

Proof of Theorem 3.2. By passing to a subsequence (not relabeled), for almost every $t \in \mathbb{R}$ we have $\chi_{\{u_i>t\}} \to \chi_{\{u>t\}}$ in $L^1(\Omega)$, see e.g. [6, p. 188]. Hence by lower semicontinuity, for almost every $t \in \mathbb{R}$

$$P(\{u > t\}, \Omega) \leq \liminf_{i\to\infty} P(\{u_i > t\}, \Omega).$$

By the coarea formula (2.3) and the assumption of the theorem, we have also

$$\lim_{i\to\infty}\int_{-\infty}^{\infty} P(\{u_i>t\},\Omega)\,dt=\int_{-\infty}^{\infty} P(\{u>t\},\Omega)\,dt.$$

By Lemma 3.7 we conclude that $P(\{u_i>\cdot\},\Omega)\to P(\{u>\cdot\},\Omega)$ in $L^1(\mathbb{R})$. By passing to a subsequence (not relabeled), we have $P(\{u_i>t\},\Omega)\to P(\{u>t\},\Omega)<\infty$ for almost every $t\in\mathbb{R}$, and then in particular we can find a countable and dense set $T\subset\mathbb{R}$ such that

$$\chi_{\{u_i>t\}}\to\chi_{\{u>t\}}\quad\text{in } L^1(\Omega)\quad\text{and}\quad P(\{u_i>t\},\Omega)\to P(\{u>t\},\Omega)<\infty$$

for every $t\in T$. If $t\in T$, by Proposition 3.4 we can find a $\mathcal{H}$-negligible set $\widetilde{N}\subset X$ and a subsequence (not relabeled) such that if $x\in I_{\{u>t\}}\cap\Omega\setminus\widetilde{N}$, then $x\in I_{\{u_i>t\}}$ for sufficiently large $i\in\mathbb{N}$, and if $x\in O_{\{u>t\}}\cap\Omega\setminus\widetilde{N}$, then $x\in O_{\{u_i>t\}}$ for sufficiently large $i\in\mathbb{N}$.

By a diagonal argument, we find a $\mathcal{H}$-negligible set $N\subset X$ and a subsequence (not relabeled) such that if $t\in T$ and $x\in I_{\{u>t\}}\cap\Omega\setminus N$, then $x\in I_{\{u_i>t\}}$ for sufficiently large $i\in\mathbb{N}$, and if $x\in O_{\{u>t\}}\cap\Omega\setminus N$, then $x\in O_{\{u_i>t\}}$ for sufficiently large $i\in\mathbb{N}$.

By [13, Lemma 3.2], there exists a $\mathcal{H}$-negligible set $\widehat{N}\subset X$ such that for every $x\in\Omega\setminus\widehat{N}$, we have $-\infty<u^\wedge(x)\le u^\vee(x)<\infty$. Fix $x\in\Omega\setminus(N\cup\widehat{N})$. Also fix $\varepsilon>0$. There exists $t\in(u^\wedge(x)-\varepsilon,u^\wedge(x))\cap T$. Now

$$\lim_{r\to 0}\frac{\mu(B(x,r)\cap\{u\le t\})}{\mu(B(x,r))}=0,$$

so that $x\in I_{\{u>t\}}$. Thus for sufficiently large $i\in\mathbb{N}$, $x\in I_{\{u_i>t\}}$, and so $u_i^\wedge(x)\ge t\ge u^\wedge(x)-\varepsilon$. Hence

$$\liminf_{i\to\infty} u_i^\wedge(x)\ge u^\wedge(x)-\varepsilon.$$

Analogously, there exists $t\in(u^\vee(x),u^\vee(x)+\varepsilon)\cap T$, and then $x\in O_{\{u>t\}}$. For sufficiently large $i\in\mathbb{N}$, $x\in O_{\{u_i>t\}}$, and thus

$$\limsup_{i\to\infty} u_i^\vee(x)\le u^\vee(x)+\varepsilon.$$

Since $\varepsilon>0$ was arbitrary, we obtain the result. $\square$

In the remainder of this section, we give our main results on uniform convergence. Recall that a set $A \subset X$ is 1-*quasiopen* if for every $\varepsilon > 0$ there is an open set $G \subset X$ with $\mathrm{Cap}_1(G) < \varepsilon$ such that $A \cup G$ is open.

Proposition 3.8 ([14, Proposition 4.2]). *Let $\Omega \subset X$ be an open set and let $E \subset X$ be a μ-measurable set with $P(E, \Omega) < \infty$. Then the sets $I_E \cap \Omega$ and $O_E \cap \Omega$ are 1-quasiopen.*

Lemma 3.9 ([16, Eq. (4.1)]). *If $v \in \mathrm{BV}(X)$ and $t > 0$, then*

$$\mathrm{Cap}_1(\{v^\vee > t\}) \le C_1 \frac{\|v\|_{\mathrm{BV}(X)}}{t}$$

for some constant $C_1 = C_1(C_d, C_P, \lambda)$.

Theorem 3.10. *Let $\Omega \subset X$ be an open set, and let $u_i, u \in \mathrm{BV}(\Omega)$ such that $u_i \to u$ in $L^1(\Omega)$ and $\|Du_i\|(\Omega) \to \|Du\|(\Omega)$. Then there exists a subsequence (not relabeled) such that whenever $K \subset \Omega \setminus S_u$ is compact and $\varepsilon > 0$, there exists an open set $U \subset X$ with $\mathrm{Cap}_1(U) < \varepsilon$ and $l \in \mathbb{N}$ such that*

$$\widetilde{u}(x) - \varepsilon \le u_i^\wedge(x) \le u_i^\vee(x) \le \widetilde{u}(x) + \varepsilon.$$

for all $x \in K \setminus U$ and $i \ge l$.

Proof. Passing to a subsequence (not relabeled) just as in the proof of Theorem 3.2, we find a countable dense set $T \subset \mathbb{R}$ such that for all $t \in T$ we have $\chi_{\{u_i > t\}} \to \chi_{\{u > t\}}$ in $L^1(\Omega)$ and $P(\{u_i > t\}, \Omega) \to P(\{u > t\}, \Omega) < \infty$. By a diagonal argument, pick a subsequence (not relabeled) such that the conclusion of Proposition 3.6 is true with E replaced by any set $\{u > t\}$ or $\{u \le t\}$ with $t \in T$.

Fix a compact set $K \subset \Omega \setminus S_u$, and fix $\varepsilon > 0$. Choose $M \in \mathbb{N}$ as follows. Choose a function $\eta \in \mathrm{Lip}_c(X)$ with $0 \le \eta \le 1$, $\eta = 1$ in K, and $\eta = 0$ in $X \setminus \Omega$. Then $u\eta \in \mathrm{BV}(X)$ by e.g. [9, Lemma 3.2], and so by Lemma 3.9

$$\mathrm{Cap}_1(\{u^\vee > M\} \cap K) = \mathrm{Cap}_1(\{(u\eta)^\vee > M\} \cap K) \le C_1 \frac{\|u\eta\|_{\mathrm{BV}(X)}}{M}.$$

By similarly estimating the set $\{u^\wedge < -M\}$, we can fix a sufficiently large $M \in \mathbb{N}$ so that

$$\mathrm{Cap}_1(\{u^\wedge < -M\} \cap K) + \mathrm{Cap}_1(\{u^\vee > M\} \cap K) < \varepsilon/2. \tag{3.12}$$

Then take $L \in \mathbb{N}$ and a strictly increasing collection of numbers $S := \{t_j\}_{j=1,\ldots,L} \subset T$ such that $t_{j+1} - t_j < \varepsilon/2$ for all $j \in \{1, \ldots, L-1\}$, and $t_2 < -M$ and $t_{L-1} > M$.

Take open sets $U_j \supset \partial^*\{u > t_j\} \cap K$ with $\mathrm{Cap}_1(U_j \cap U_{j+1}) < \varepsilon/4L$ for all $j \in \{1, \ldots, L-1\}$; this can be done as follows. For each j, since $P(\{u > t_j\}, \Omega) < \infty$, by (2.6) we have $\mathcal{H}(\partial^*\{u > t_j\} \cap \Omega) < \infty$, and thus we can pick compact sets $F_j \subset \partial^*\{u > t_j\} \cap K$ such that $\mathcal{H}(\partial^*\{u > t_j\} \cap K \setminus F_j) < \varepsilon/16C_d L$. Then $\mathrm{Cap}_1(\partial^*\{u > t_j\} \cap K \setminus F_j) < \varepsilon/8L$ by (3.9), and so we find open sets $\widehat{U}_j \supset \partial^*\{u > t_j\} \cap K \setminus F_j$ with $\mathrm{Cap}_1(\widehat{U}_j) < \varepsilon/8L$. Moreover, since $S_u \cap K = \emptyset$, it follows that $\partial^*\{u > t_j\} \cap \partial^*\{u > t_k\} \cap K = \emptyset$ for any $j \neq k$; this follows in a straightforward manner from the definitions, see e.g. [3, Proposition 5.2]. Thus we can pick pairwise disjoint open sets $\widetilde{U}_j \supset F_j$. Then we can define $U_j := \widetilde{U}_j \cup \widehat{U}_j$, $j \in \{1, \ldots, L\}$.

By Proposition 3.8, we can also find open sets $V_j \subset X$ with $\mathrm{Cap}_1(V_j) < \varepsilon/8L$ such that each $(O_{\{u>t_j\}} \cap \Omega) \cup V_j$ is an open set. Thus each $((I_{\{u>t_j\}} \cup \partial^*\{u > t_j\}) \cup (X \setminus \Omega)) \setminus V_j$ is a closed set, and so each $(I_{\{u>t_j\}} \cup \partial^*\{u > t_j\}) \cap K \setminus V_j$ is a closed set. Then each

$$I_{\{u>t_j\}} \cap K \setminus (U_j \cup V_j) = (I_{\{u>t_j\}} \cup \partial^*\{u > t_j\}) \cap K \setminus (U_j \cup V_j)$$

is a compact subset of $I_{\{u>t_j\}} \cap \Omega$. By the conclusion of Proposition 3.6, there exist open sets $W_j \subset X$ with $\mathrm{Cap}_1(W_j) < \varepsilon/8L$ such that for some $l \in \mathbb{N}$ and all $j \in \{1, \ldots, L\}$,

$$I_{\{u>t_j\}} \cap K \setminus (U_j \cup V_j \cup W_j) \subset I_{\{u_i>t_j\}} \tag{3.13}$$

for all $i \geq l$.

Let

$$G := K \cap \left(\{u^\wedge < -M\} \cup \{u^\vee > M\} \cup \bigcup_{j=1}^{L} (V_j \cup W_j) \cup \bigcup_{j=1}^{L-1} (U_j \cap U_{j+1}) \right),$$

so that $\mathrm{Cap}_1(G) < \varepsilon$. Then take an open set $U \supset G$ with $\mathrm{Cap}_1(U) < \varepsilon$. Fix $x \in K \setminus U$. For some $j \in \{2, \ldots, L-1\}$ we have $t_j \in (\widetilde{u}(x) - \varepsilon/2, \widetilde{u}(x))$. Now

$$\lim_{r \to 0} \frac{\mu(B(x,r) \cap \{u \leq t_j\})}{\mu(B(x,r))} = 0,$$

so that $x \in I_{\{u>t_j\}}$. Then clearly also $x \in I_{\{u>t_{j-1}\}}$. Then by the definition of the set $G \not\ni x$, we conclude

$$x \in \big(I_{\{u>t_j\}} \setminus (U_j \cup V_j \cup W_j)\big) \cup \big(I_{\{u>t_{j-1}\}} \setminus (U_{j-1} \cup V_{j-1} \cup W_{j-1})\big).$$

By (3.13), $x \in I_{\{u_i>t_j\}} \cup I_{\{u_i>t_{j-1}\}} = I_{\{u_i>t_{j-1}\}}$ for all $i \geq l$, and so

$$u_i^\wedge(x) \geq t_{j-1} \geq t_j - \varepsilon/2 \geq \widetilde{u}(x) - \varepsilon.$$

By making l and U bigger, if necessary, we get analogously

$$u_i^\vee(x) \leq \widetilde{u}(x) + \varepsilon$$

for all $x \in K \setminus U$ and all $i \geq l$. □

Corollary 3.11. *Let $\Omega \subset X$ be an open set, and let $u_i, u \in \mathrm{BV}(\Omega)$ such that $u_i \to u$ in $L^1(\Omega)$ and $\|Du_i\|(\Omega) \to \|Du\|(\Omega)$. Then there exists a subsequence (not relabeled) such that whenever $K \subset \Omega \setminus S_u$ is compact and $\varepsilon > 0$, there exists an open set $U \subset X$ with $\mathrm{Cap}_1(U) < \varepsilon$ such that $\widetilde{u}_i \to \widetilde{u}$ uniformly in $K \setminus U$.*

4 Examples

In Theorem 3.2, there are three obvious ways in which the result could potentially be strengthened, presented in the following questions:

- Does the pointwise convergence hold for the original sequence, instead of a subsequence?
- Can we obtain $u_i^\wedge(x) \to u^\wedge(x)$ and $u_i^\vee(x) \to u^\vee(x)$ for $\mathcal{H}$-almost every $x \in S_u$?
- The sets where we do not obtain pointwise convergence are known to be $\mathcal{H}$-negligible; can we further restrict this family?

The following three examples show that the answer to each of these questions is no.

Example 4.1. Let $X = \mathbb{R}^2$ (unweighted) and for each $k \in \mathbb{N}$, let

$$E_i := B((2^{-k+1}(i - 2^{k-1}), 0), 2^{-k+1}), \qquad i = 2^{k-1}, \ldots, 2^k - 1.$$

Then we have $\|\chi_{E_i}\|_{\mathrm{BV}(\mathbb{R}^2)} \to 0$ as $i \to \infty$, but for all $x = (x_1, 0)$ with $0 \leq x_1 < 1$ there exists infinitely many $i \in \mathbb{N}$ such that $x \in E_i \subset I_{E_i}$. Clearly $\mathcal{H}(\{(x_1, x_2) : 0 \leq x_1 \leq 1, x_2 = 0\}) > 0$; note that the codimension one Hausdorff measure $\mathcal{H}$ is comparable to the usual 1-dimensional Hausdorff measure.

Thus we need to pass to a subsequence — for example $(u_{i_j})_{j=1}^\infty$ with $i_j = 2^{j-1}$ will do.

Example 4.2. Let $X = \mathbb{R}$ (unweighted). Let $u := \chi_{(0,1)}$ and

$$u_i(x) := \max\{0, \min\{1, 1/4 + ix\}\}\chi_{(-\infty,1]}(x), \quad i \in \mathbb{N}.$$

Clearly $u_i \to u$ in $L^1(\mathbb{R})$ and $\|Du_i\|(\mathbb{R}) = 2 = \|Du\|(\mathbb{R})$. However,

$$u_i^\wedge(0) = u_i^\vee(0) \equiv 1/4 \not\to 0 = u^\wedge(0).$$

Similarly, we do not have convergence to $u^\vee(0) = 1$ — or to $\widetilde{u}(0) = 1/2$. Yet $\mathcal{H}(0) = 2$ (note that the codimension one Hausdorff measure $\mathcal{H}$ is now exactly twice the usual 0-dimensional Hausdorff measure). Thus we cannot have pointwise convergence $\mathcal{H}$-almost everywhere in the jump set, for any subsequence.

If $\Omega \subset X$ is an open set and we even have $u_i \to u$ in $\mathrm{BV}(\Omega)$ (that is, in the BV norm), then for a subsequence (not relabeled) we have $u_i^\wedge(x) \to u^\wedge(x)$ and $u_i^\vee(x) \to u^\vee(x)$ for $\mathcal{H}$-almost every $x \in \Omega$; this follows e.g. from [16, Remark 4.1, Lemma 4.2]. Thus with such a stronger assumption, we do obtain pointwise convergence even in the jump set.

Example 4.3. Let $u \equiv 0$ and let $A \subset X$ be a $\mathcal{H}$-negligible set. For each $i \in \mathbb{N}$, take a covering $\{B(x_j^i, r_j^i)\}_{j\in\mathbb{N}}$ of A such that $r_j^i \le 1$ and

$$\sum_{j=1}^\infty \frac{\mu(B(x_j^i, r_j^i))}{r_j^i} < \frac{1}{i}.$$

By applying the coarea formula (2.3) with $v = d(\cdot, x_j^i)$ and $\Omega = B(x_j^i, 2r_j^i)$, for each $i, j \in \mathbb{N}$ we find $s_j^i \in [r_j^i, 2r_j^i]$ such that

$$P(B(x_j^i, s_j^i), X) \le 2C_d \frac{\mu(B(x_j^i, r_j^i))}{r_j^i}.$$

For each $i \in \mathbb{N}$, let $E_i := \bigcup_{j=1}^\infty B(x_j^i, s_j^i)$. Then $A \subset E_i \subset I_{E_i}$ for all $i \in \mathbb{N}$. However, by lower semicontinuity and (2.2),

$$P(E_i, X) \le \sum_{j=1}^\infty P(B(x_j^i, s_j^i), X) < \frac{2C_d}{i} \to 0 \qquad \text{as } i \to \infty.$$

Also

$$\|\chi_{E_i}\|_{L^1(X)} = \mu(E_i) \le \sum_{j=1}^\infty \mu(B(x_j^i, s_j^i)) \le C_d \sum_{j=1}^\infty \frac{\mu(B(x_j^i, r_j^i))}{r_j^i} < \frac{C_d}{i} \to 0.$$

Thus setting $u_i := \chi_{E_i}$, the assumptions of Theorem 3.2 are satisfied. In conclusion, given *any* $\mathcal{H}$-negligible set, pointwise convergence can fail at each point in the set, for all subsequences.

Concerning our results on uniform convergence, note first that according to Egorov's theorem, if $A \subset X$, ν is a positive Radon measure of finite mass on A, and v_i, v are ν-measurable functions on A such that $v_i(x) \to v(x)$ as $i \to \infty$ for ν-almost every $x \in A$, then for any $\varepsilon > 0$ there exists $D \subset A$ with $\nu(D) < \varepsilon$ such that $v_i \to v$ uniformly in $A \setminus D$.

However, if instead of a Radon measure we work with the 1-capacity, a problem arises from the fact that the 1-capacity is not a Borel measure. The following example demonstrates that a Egorov-type result fails even under very favorable conditions.

Example 4.4. Let $X = \mathbb{R}$ (unweighted) and let $\Omega = (0,1)$. Let $u \equiv 0$ and

$$u_i(x) := \begin{cases} ix & \text{for } 0 < x \le 1/i, \\ 2 - ix & \text{for } 1/i \le x \le 2/i, \\ 0 & \text{for } 2/i \le x < 1. \end{cases}$$

Then $u_i, u \in C(\Omega)$ and $u_i(x) \to u(x)$ for every $x \in \Omega$. Fix $\varepsilon \in (0,1)$. Since $\mathrm{Cap}_1(\{x\}) = 2$ for every $x \in \mathbb{R}$, the condition $\mathrm{Cap}_1(D) < \varepsilon$ implies $D = \emptyset$. However, we do not have that $u_i \to u$ uniformly in Ω.

Equally well we can consider the closed unit interval, so even with a compact set and continuous functions, things can go wrong. It is the condition $\|Du_i\|(\Omega) \to \|Du\|(\Omega)$ that allows us to obtain uniform convergence in Theorem 3.10 and Corollary 3.11.

Recall that according to Example 4.2, we cannot have even pointwise convergence in the jump set. If we consider the unit interval $(0,1) \subset X \setminus S_u$ in this example, it is clear that we do not have $\widetilde{u}_i \to \widetilde{u}$ uniformly in $(0,1)$, for any subsequence. Since again $\mathrm{Cap}_1(\{x\}) = 2$ for every $x \in \mathbb{R}$, we see that Corollary 3.11 (and Theorem 3.10) fail if we do not require the set K to be compact. Moreover, Example 4.3 demonstrates that we need, in general, to discard a further set of small capacity in order to obtain uniform convergence.

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