J. Math. Anal. Appl. 464 (2018) 456–472

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Boundary blow-up solutions to the k-Hessian equation with a weakly superlinear nonlinearity

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ARTICLE INFO

Article history:
Received 28 September 2017
Submitted by V. Radulescu

Keywords:
k-Hessian equation
Boundary blow up
Sub-supersolution method
k-convex solution
Singular weight function

ABSTRACT

By constructing new sub- and super-solutions, we are concerned with determining values of β, for which there exist k-convex solutions to the boundary blow-up k-Hessian problem

$$S_k(D^2u(x)) = H(x)[u(x)]^k[\ln u(x)]^\beta > 0 \text{ for } x \in \Omega,\ u(x) \to +\infty$$
$$\text{as } \operatorname{dist}(x, \partial\Omega) \to 0.$$

Here $k \in \{1, 2, \cdots, N\}$, $S_k(D^2u)$ is the k-Hessian operator, $\beta > 0$ and Ω is a smooth, bounded, strictly convex domain in $\mathbb{R}^N$ $(N \geq 2)$. We suppose that the nonlinearity behaves like $u^k \ln^\beta u$ as $u \to \infty$, which is more complex and difficult to deal with than the nonlinearity grows like u^p with $p > k$ or faster at infinity. Further, several new results of nonexistence, global estimates and estimates near the boundary for the solutions are also given.

1. Introduction

We consider the boundary blow-up problem for the k-Hessian equation

$$S_k(D^2u(x)) = H(x)[u(x)]^k[\ln u(x)]^\beta > 0 \text{ in } \Omega,\ u = +\infty \text{ on } \partial\Omega, \tag{1.1}$$

where $\beta > 0, \Omega$ is a smooth, bounded, strictly convex domain in $\mathbb{R}^N (N \geq 2)$, and $H(x)$ is smooth positive function on Ω. The boundary blow-up condition $u = +\infty$ on $\partial\Omega$ means

$$u(x) \to +\infty \text{ as } \operatorname{dist}(x, \partial\Omega) \to 0.$$

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https://doi.org/10.1016/j.jmaa.2018.04.014
0022-247X/

$S_k(D^2u)(k \in \{1, 2, \ldots, N\})$ denotes the kth elementary symmetric function of the eigenvalues of D^2u, the Hessian of u, i.e.

$$S_k(D^2u) = S_k(\lambda_1, \lambda_2 \ldots, \lambda_N) = \sum_{1 \leq i_1 < \cdots < i_k \leq N} \lambda_{i_1} \ldots \lambda_{i_k},$$

where $\lambda_1, \lambda_2 \ldots, \lambda_N$ are the eigenvalues of D^2u. For $k = 1$, i.e.

$$S_1(D^2u) = \sum_{i=1}^{N} \lambda_i = \Delta u$$

is the well known classical Laplace operator; for $K = N$, i.e.

$$S_N(D^2u) = \prod_{i=1}^{N} \lambda_i = det(D^2u)$$

is the Monge–Ampère operator. A great interest has been shown by many authors in the subject of Laplace problems and Monge–Ampère problems, and many excellent results for Laplace problems and Monge–Ampère problems have been obtained, for instance, see ([3,30,31,37,21,32–34,50,49,52,35,38,42]) and the references cited therein. In contrast to numerous results on the case $k = 1$ or $k = N$ less is known about the situation $k \in \{2, \ldots, N - 1\}$. Only in recent years there is a good number of investigations k-Hessian equations (see by instance [51,8,15,27,39,20,14,41,28,29,53,16,40,47,45,46]).

For $k \in \{1, 2, \ldots, N\}$, let Γ_k be the component of $\{\lambda \in R^N : S_k(\lambda) > 0\} \subset R^N$ containing the positive cone

$$\Gamma^+ = \{\lambda \in R^N : \lambda_i > 0, i = 1, 2, \ldots, N\}.$$

It follows from [8] that

$$\Gamma^+ = \Gamma_N \subset \cdots \subset \Gamma_{k+1} \subset \Gamma_k \subset \cdots \Gamma_1.$$

Definition 1.1. (See [39]) Let $k \in \{1, 2, \ldots, N\}$, and let Ω be an open bounded subset of R^N; a function $u \in C^2(\Omega)$ is k-convex if $(\lambda_1, \lambda_2, \ldots, \lambda_N) \in \bar{\Gamma}_k$ for every $x \in \Omega$, where $\lambda_1, \lambda_2, \ldots, \lambda_N$ are the eigenvalues of D^2u. Equivalently, we can say that u is k-convex if $S_i(D^2u) \geq 0$ in Ω for $i = 1, \ldots, k$.

Definition 1.2. (See [39]) Let $\Omega \subset R^N$ be an open set with boundary of class C^2 and let $k \in \{1, \ldots, N - 1\}$; we say that Ω is strictly convex if $S_i(\kappa_1(x), \ldots, \kappa_{N-1}(x)) > 0$, for $i = 1, \ldots, N - 1$ and for every $x \in \partial\Omega$, where $\kappa_i(x)$, $i = 1, \ldots, N - 1$, are the principal curvatures of $\partial\Omega$ at x.

It is generally accepted that the concept of k-convexity arises naturally from the involved operator S_k. In fact, when Ω is strictly $(k - 1)$-convex and $H \in C^\infty(\bar{\Omega})$, $S_k(D^2u)$ changes into elliptic in the class of k-convex functions and the related Dirichlet problem

$$\begin{cases} S_k(D^2u) = H(x) > 0 \ \text{ in } \ \Omega, \\ u|_{\partial\Omega} = \phi \in C(\partial\Omega) \end{cases} \tag{1.2}$$

admits a unique k-convex solution. Moreover the $(k-1)$-convexity of the domain is necessary if ϕ is constant (see [8]).

At the same time, we notice that the subject of blow-up solutions has received much attention starting with the pioneering work of Bieberbach [4]. It was about the following model involving the classical Laplace operator

$$\begin{cases} \Delta u = H(x)f(u) \text{ in } \Omega, \\ u = +\infty \text{ on } \partial\Omega \end{cases} \tag{1.3}$$

with $H(x) \equiv 1$ in Ω, $f(u) = e^u$ and $N = 2$. If $f(u) = u^p$, $H(x)$ is growing like a negative power of $d(x)$ near $\partial\Omega$, the radial case was completely discussed in [12], the general case was discussed when $p > 1$ in [11], the authors obtained existence, nonexistence, uniqueness, multiplicity and estimates for all positive solutions. When f may exhibit non monotone behaviour, in [17], Dumont, Dupaigne, Goubet, and Radulescu studied the existence, asymptotic behaviour, uniqueness and numerical approximation of solutions of (1.3). An overview of the asymptotic behaviour of solutions of elliptic problems can be found in Ghergu and Radulescu [19].

Recently, in [48], Yang and Chang considered the boundary blow up solutions of Monge–Ampère equations

$$M[u] = K(x)u^p \text{ in } \Omega, \ u = +\infty \text{ on } \partial\Omega \tag{1.4}$$

with $K(x)$ growing like a negative power of $d(x)$ near $\partial\Omega$. They showed some results on the uniqueness, nonexistence, and the exact boundary blow up rate of the strictly convex solutions of (1.4). The latest results of blow-up solutions of partial differential equations can be found in ([1,36]).

However, even for problem (1.3) there is few work on the case $f(u) = u[\ln u]^\beta$. It is worth mention that, in [13], the authors studied the asymptotic behaviour near the boundary for large solutions of the semilinear equation

$$\begin{cases} \Delta u + au = b(x)f(u) \text{ in } \Omega, \\ u = +\infty \text{ on } \partial\Omega \end{cases} \tag{1.5}$$

with $f(u)$ behaves like $u[\ln u]^\beta$ for $\beta > 2$, and showed that this case is more difficult to handle than those where $f(u)$ grows like $u^p(p > 1)$ or faster at infinity. In [10], the authors extended some results of Laplacian case in [13] to the context of the p-Laplacian and improve the description of the asymptotic behaviour of the large solutions of problem (1.5).

Moreover, for $k \geq 2$ we know that the k-Hessian operator is a fully nonlinear partial differential operator, and we notice that some fully nonlinear degenerate elliptic operators have attracted the attention of Harvey and Lawson ([22–25]), Caffarelli, Li and Nirenberg ([5,6]), Amendola, Galise and Vitolo [2], Galise and Vitolo [18], Capuzzo–Dolcetta, Leoni and Vitolo [9], and Vitolo [43]. However, in literature there aren't articles on boundary blow-up solutions to the k-Hessian equation with a weakly superlinear nonlinearity. More precisely, the study of $f(u) = u[\ln u]^\beta$ for $\beta > 2$ is still open for the k-Hessian problem.

Motivated by the above works, we would like do some research on problem (1.1). We'll launch our research according to three different cases on H. Here we point out that our problem is new in the way of k-Hessian equation with a weakly superlinear nonlinearity introduced here. To the best of our knowledge, when the weight function H is singular, the k-convex solutions to the boundary blow-up k-Hessian problem has not yet to be studied, especially when the nonlinearity is weakly superlinear. In consequence, our main results of the present work will make new contribution to the existing literatures on the topic of k-Hessian equation. The existence, nonexistence and global estimates for the given problem are new, though they are proved by applying the method of sub- and super-solutions.

The rest of the paper is organized as follows. In Section 2 we collect some known results to be used in the subsequent sections. In Sections 3–5, we give the proof of the main results. In appendix, we list some relevant definitions.

2. Some preliminary results

In this section, we collect some results for the convenience of later use and reference.

Lemma 2.1. *(See [29], Lemma 2.1) Suppose that $\Omega \subset R^N$ is a bounded domain, and $u, v \in C^2(\Omega)$ are k-convex. If*

(1) $\psi(x,z,p) \geq \phi(x,z,p),\ \forall\ (x,z,p) \in (\Omega \times R \times R^N)$;
(2) $S_k(D^2u) \geq \psi(x,u,Du)$ and $S_k(D^2v) \leq \phi(x,v,Dv)$ in Ω;
(3) $u \leq v$ on $\partial\Omega$;
(4) $\psi_z(x,z,p) > 0$ or $\phi_z(x,z,p) > 0$,

then $u \leq v$ in Ω.

Remark 2.2. It is not difficult to find that "$\psi_z(x,z,p) > 0$ or $\phi_z(x,z,p) > 0$" in Lemma 2.1 can be relaxed to "$\psi_z(x,z,p) \geq 0$ or $\phi_z(x,z,p) \geq 0$" provided that one of the inequalities in (2) is replaced by a strict inequality. This observation will be used later in the paper.

In the following lemma we'll use some definitions from matrix analysis, for convenience of the reader, we list them in Appendix.

Lemma 2.3. *Let $u \in C^2(\Omega)$ be such that all of the principal submatrix of $(u_{x_i x_j})$ are invertible for $x \in \Omega$, and let g be a C^2 function defined on an interval containing the range of u. Then*

$$S_k(D^2 g(u)) = S_k(D^2u)[g'(u)]^k + [g'(u)]^{k-1} g''(u) \sum_{i=1}^{C_N^k} det(u_{x_{is}x_{ij}})(\nabla u_i)^T B(u_i)\nabla u_i, \tag{2.1}$$

where A^T denotes the transpose of the matrix A, $B(u_i)$ denotes the inverse of the i-th principal submatrix $(u_{x_{is}x_{ij}})$, $det(u_{x_{is}x_{ij}})$ is the determinant of $(u_{x_{is}x_{ij}})$ and

$$\nabla u_i = (u_{x_{i1}}, u_{x_{i2}}, \cdots, u_{x_{ik}})^T, i = 1, 2, \ldots, C_N^k,$$

here $C_N^k = \frac{N!}{(N-k)!k!}$.

Proof. As is well known, the matrix $(D^2g(u))$ has C_N^k distinct principal minors, and $S_k(D^2g(u))$ equals to the sum of all the principal minor of size k of matrix $(D^2g(u))$. For convenience, we only compute the i-th principal minor of size k. Denote i-th principal minor of size k by D_i, then

$$D_i = \begin{vmatrix} g''(u)u_{x_{i1}}u_{x_{i1}} + g'(u)u_{x_{i1}x_{i1}} & \cdots & g''(u)u_{x_{i1}}u_{x_{ik}} + g'(u)u_{x_{i1}x_{ik}} \\ g''(u)u_{x_{i2}}u_{x_{i1}} + g'(u)u_{x_{i2}x_{i1}} & \cdots & g''(u)u_{x_{i2}}u_{x_{ik}} + g'(u)u_{x_{i2}x_{ik}} \\ \vdots & \vdots\ \vdots\ \vdots & \vdots \\ g''(u)u_{x_{ik}}u_{x_{i1}} + g'(u)u_{x_{ik}x_{i1}} & \cdots & g''(u)u_{x_{ik}}u_{x_{ik}} + g'(u)u_{x_{ik}x_{ik}} \end{vmatrix}.$$

The jth column of D_i is the sum of two columns. The first of these two has entries

$$g'(u)u_{x_{is}x_{ij}} \quad (s = 1, 2, \ldots, k)$$

and the second has entries

$$g''(u)u_{x_{is}}u_{x_{ij}} \quad (s = 1, 2, \ldots, k).$$

Since the determinant of a matrix is linear in each of its columns, D_i can be expressed as a sum of 2^k determinants where each summand has as its jth column one of the two types given above. Since for $j \neq t$ the two columns

$$col(u_{x_{i1}}u_{x_{ij}}, u_{x_{i2}}u_{x_{ij}}, \ldots, u_{x_{ik}}u_{x_{ij}})$$

and

$$col(u_{x_{i1}}u_{x_{it}}, u_{x_{i2}}u_{x_{it}}, \ldots, u_{x_{ik}}u_{x_{it}})$$

are proportional, any of the 2^k summands which have two different columns of the second type are zero. Therefore

$$D_i = [g'(u)]^k \bar{D}_i + [g'(u)]^{k-1} g''(u) \sum_{j=1}^{k} \bar{D}_{ij},$$

where $\bar{D}_i$ is the determinant whose (s, j) th entry is

$$u_{x_{is}x_{ij}}$$

and $\bar{D}_{ij}(j = 1, 2 \ldots, k)$ is the determinant obtained from $\bar{D}_i$ by replacement of the jth column of $\bar{D}_i$ by the column with entries

$$u_{x_{is}}u_{x_{ij}} \quad (s = 1, 2, \ldots, k).$$

Therefore, if we denote the cofactor of the (s, j) th entry of $\bar{D}_{ij}$ by $C_{sj}(u_i)$, then

$$D_i = [g'(u)]^k det(u_{x_{is}x_{ij}}) + [g'(u)]^{k-1} g''(u) \sum_{j=1}^{k} \sum_{s=1}^{k} C_{sj}(u_i) u_{x_{is}} u_{x_{ij}},$$

where $det(u_{x_{is}x_{ij}})$ is the i-th principal minor of size k of matrix (D^2u).

Since the matrix $(u_{x_{is}x_{ij}})$ is symmetric, if we write

$$\nabla u_i = col(u_{x_{i1}}, u_{x_{i2}}, \cdots, u_{x_{ik}}),$$

then, by the formula for the inverse of a matrix,

$$\sum_{j=1}^{k} \sum_{s=1}^{k} C_{sj}(u_i) u_{x_{is}} u_{x_{ij}} = det(u_{x_{is}x_{ij}})(\nabla u_i)^T B(u_i) \nabla u_i,$$

where $B(u_i)$ is the inverse of the i-th principal submatrix $(u_{x_{is}x_{ij}})$.

Thus

$$\begin{aligned}
S_k(D^2g(u)) &= \sum_{i=1}^{C_N^k} D_i \\
&= \sum_{i=1}^{C_N^k} \left\{ [g'(u)]^k det(u_{x_{is}x_{ij}}) + [g'(u)]^{k-1} g''(u) det(u_{x_{is}x_{ij}})(\nabla u_i)^T B(u_i) \nabla u_i \right\} \\
&= S_k(D^2u)[g'(u)]^k + [g'(u)]^{k-1} g''(u) \sum_{i=1}^{C_N^k} det(u_{x_{is}x_{ij}})(\nabla u_i)^T B(u_i) \nabla u_i. \qquad \square
\end{aligned}$$

Let $\Gamma_\mu = \{x \in \bar{\Omega} : d(x, \partial\Omega) < \mu\}$ for $\mu > 0$.

Lemma 2.4. *(Corollary 2.3 of [27]) Let Ω be bounded with $\partial\Omega \in C^l$ for $l \geq 2$. Assume that $\mu > 0$ is small such that $d \in C^2(\Gamma_\mu)$ and g is a C^2-function on $(0, \mu)$. Let $x_0 \in \Gamma_\mu \setminus \partial\Omega$ and $y_0 \in \partial\Omega$ be such that $d(x_0) = |x_0 - y_0|$. Then we have*

$$\begin{aligned} S_k(D^2 g(d(x_0))) \quad &= [-g'(d(x_0)]^k S_k(\varepsilon_1, \ldots, \varepsilon_{N-1}) \\ &+ [-g'(d(x_0)]^{k-1} g''(d(x_0)) S_{k-1}(\varepsilon_1, \ldots, \varepsilon_{N-1}), \end{aligned} \tag{2.2}$$

where

$$\varepsilon_i = \frac{\kappa_i(y_0)}{1 - \kappa_i(y_0) d(x_0)}, i = 1, 2, \ldots, N-1,$$

and $\kappa_1(y_0), \ldots, \kappa_{N-1}(y_0)$ are the principal curvatures of $\partial\Omega$ at y_0.

The following interior estimate for derivatives of smooth solutions is a simple variant of Lemma 2.2 in [32], which can be proved by following the idea of Theorem 3.1 and Remark 3.1 of [44].

Lemma 2.5. *Let Ω be a bounded strictly $(k-1)$-convex domain in $\mathbb{R}^N, N \geq 2$, with $\partial\Omega \in C^\infty$. Let $\eta \in [-\infty, +\infty)$ and $f \in C^\infty(\overline{\Omega} \times (\eta, \infty))$ with $f(x, u) > 0$ for $(x, u) \in \overline{\Omega} \times (\eta, \infty)$. Let $u \in C^\infty(\overline{\Omega})$ be a solution of the Dirichlet problem*

$$\begin{cases} S_k(D^2 u) = f(x, u), & x \in \Omega, \\ u(x) = c = \mathit{constant}, & x \in \partial\Omega \end{cases} \tag{2.3}$$

with $\eta < u(x) < c$ in Ω. Let Ω' be a subdomain of Ω with $\overline{\Omega'} \subset \Omega$ and assume that $\eta < a \leq u(x) \leq b$ for $x \in \overline{\Omega'}$ and let $\tau \geq 1$ be an integer. Then there exists a constant C which depends only on τ, a, b, bounds for the derivatives of $f(x, u)$ for $(x, u) \in \overline{\Omega'} \times [a, b]$, and $\operatorname{dist}(\Omega', \partial\Omega)$ such that

$$||u||_{C^\tau(\overline{\Omega'})} \leq C.$$

The existence result below is a special case of Theorem 1.1 in [20].

Lemma 2.6. *Let Ω be an open domain in R^N with boundary of class C^∞ and let $g(x, t)$ be a C^∞ function such that $g > 0$ and $g_t \geq 0$ in $\Omega \times R$. The problem*

$$\begin{cases} S_k(D^2 u) = g(x, u), & x \in \Omega, \\ u|_{\partial\Omega} = \varphi \in C(\partial\Omega) \end{cases}$$

has a unique k-convex solution if there exists a k-convex strict subsolution v, i.e. a k-convex function v such that $v|_{\partial\Omega} = \varphi$ and $S_k(v) \geq g(x, v) + \delta$ in Ω, for some $\delta > 0$.

For the proofs in Section 3–5, we introduce a function $z(x)$.

Let Ω be a smooth, bounded, strictly convex domain in $\mathbb{R}^N$, by Theorem 1.1 of [7], there exists $u_0 \in C^\infty(\overline{\Omega})$, which is the unique strictly convex solution to

$$M[u_0] = 1 \text{ in } \Omega, \ u_0 = 1 \text{ on } \partial\Omega,$$

where $M[u_0] = det(u_{0ij})$ is the Monge–Ampère operator. Set $z(x) := 1 - u_0(x)$. Then $z(x) > 0$ in Ω and it is the unique strictly concave solution to

$$(-1)^N M[z] = 1 \text{ in } \Omega,\ z = 0 \text{ on } \partial\Omega. \tag{2.4}$$

Since $(z_{x_i x_j})$ is negative definite on $\overline{\Omega}$, its trace is negative, that is $\Delta z < 0$, and hence one can apply the Hopf boundary lemma to conclude that $|\nabla z| > 0$ for $x \in \partial\Omega$. It follows that there exist positive constants b_1 and b_2 such that

$$b_1 d(x) \le z(x) \le b_2 d(x) \text{ for } x \in \Omega. \tag{2.5}$$

Remark 2.7. Since $(z_{x_i x_j})$ is negative definite, the eigenvalues are negative, then, for all principal minor Δ_k of order k to $det(D^2 z)$, we have $(-1)^k \Delta_k > a$ for some positive constant a, and then $(-1)^k S_k(D^2 z) > aC_N^k$.

3. $H(x) \in C^\infty(\bar{\Omega})$ is positive on $\bar{\Omega}$

Theorem 3.1. *Let Ω be a smooth, bounded, strictly convex domain in $R^N, N \ge 2$. Suppose $H \in C^\infty(\bar{\Omega})$ is positive. If $\beta > k+1$, then problem (1.1) has a k-convex solution $u \in C^\infty(\Omega)$ verifying*

$$e^{c_1 d(x)^{-\alpha}} \le u(x) \le e^{c_2 d(x)^{-\alpha}},\ x \in \Omega \tag{3.1}$$

for some positive numbers $c_1 < c_2$, where

$$\alpha = \frac{k+1}{\beta - (k+1)}. \tag{3.2}$$

Proof. By the fact that $H \in C^\infty(\bar{\Omega})$ is positive, there exist $h_1 > 0, h_2 > 0$ such that $h_1 \le H(x) \le h_2$. We divide the proof into several steps.

Step1. Subsolution and supersolution.

Let $v(x) = e^{c[z(x)]^{-\alpha}}$, where z is defined by (2.4), α is defined by (3.2), c is a constant to be determined. By (2.1) we get

$$\begin{aligned} &S_k(D^2 v) \\ &= c^k \alpha^k z^{-(k+1)(\alpha+1)} e^{kcz^{-\alpha}} \\ &\times \{(-1)^k S_k(D^2 z) z^{\alpha+1} - [c\alpha + (\alpha+1) z^\alpha] \sum_{i=1}^{C_N^k} (-1)^k det(z_{x_{i_s} x_{i_j}}) (\nabla z_i)^T B(z_i) \nabla z_i\}. \end{aligned} \tag{3.3}$$

Let

$$\Delta = (-1)^k S_k(D^2 z) z^{\alpha+1} - [c\alpha + (\alpha+1) z^\alpha] \sum_{i=1}^{C_N^k} (-1)^k det(z_{x_{i_s} x_{i_j}}) (\nabla z_i)^T B(z_i) \nabla z_i.$$

By the definition of z and Remark 2.7 we have $(z_{x_i x_j})$ is negative definite, then all of its principal submatrices of order k are negative definite. It follows that there exist $e_1, e_2 > 0$ such that

$$-e_1 ||\nabla z_i||^2 \le (\nabla z_i)^T B(z_i) \nabla z_i \le -e_2 ||\nabla z_i||^2,$$

and trace $(z_{x_{i_s} x_{i_j}}) = z_{x_{i1} x_{i1}} + \cdots + z_{x_{ik} x_{ik}} = \Delta z_i < 0$. Therefore, since $\Delta(-z_i) > 0$ on Ω and $-z_i$ attains its maximum on $\bar{\Omega}$ at each point of $\partial\Omega$, it follows from the maximum principle that there exists an open set U containing $\partial\Omega$ such that

$$\|\nabla z_i\| \geq e > 0$$

for some i.

On the other hand, it is easy to see that z is bounded below by a positive constant on $\Omega - \mathrm{U}$. Combining this with the fact that

$$(-1)^k det(z_{x_{is}x_{ij}}) > a, \ (-1)^k S_k(D^2 z) > aC_N^k,$$

we get Δ is positive on Ω. Considering (3.2) and (3.3) we have

$$S_k(D^2 v) = c^{k-\beta}\alpha^k v^k (\ln v)^\beta \Delta.$$

By $\beta > k+1$, $h_1 \leq H(x) \leq h_2$ we can see there exists $c_2 > 0$ large enough such that

$$S_k(D^2 v)(x) \leq H(x)[v(x)]^k[\ln v(x)]^\beta, \ x \in \Omega \tag{3.4}$$

and there exists $c_1 > 0$ small enough such that

$$S_k(D^2 v)(x) \geq H(x)[v(x)]^k[\ln v(x)]^\beta, \ x \in \Omega. \tag{3.5}$$

Let $v_1(x) = e^{c_1[z(x)]^{-\alpha}}, v_2(x) = e^{c_2[z(x)]^{-\alpha}}$. Then $v_1(x)$ is a subsolution, and $v_2(x)$ is a supersolution.

Step2. The existence of a k-convex solution $u(x) \in C^\infty(\Omega)$.

Let $\{\sigma_n\}_1^\infty$ be a strictly increasing sequence of positive numbers such that $\sigma_n \to \infty$ as $n \to \infty$, and let $\Omega_n = \{x \in \Omega | v_1(x) < \sigma_n\}$. Since any level surface of v_1 is a level surface of z and z is strictly convex, for each $n \geq 1, \partial\Omega_n$ is a strictly convex C^∞-submanifold of R^N of dimension $N-1$.

By Lemma 2.6 there exists $u_n \in C^\infty(\bar{\Omega}_n)$ for $n \geq 1$ such that

$$\begin{cases} S_k(D^2 u_n) = H(x)u_n^k[\ln u_n]^\beta, \ x \in \Omega_n, \\ u_n|_{\partial\Omega_n} = \sigma_n = v_1|_{\partial\Omega_n} \end{cases} \tag{3.6}$$

and $u_n(x)$ is k-convex on Ω_n and satisfies

$$u_n(x) \geq v_1(x), \ x \in \bar{\Omega}_n. \tag{3.7}$$

From the fact that

$$u_n(x) = v_1(x) \leq v_2(x), \ x \in \partial\Omega_n, \tag{3.8}$$

(3.4) and Lemma 2.1, we see that

$$v_2(x) \geq u_n(x), \ x \in \bar{\Omega}_n. \tag{3.9}$$

Clearly, for $n > 1$

$$\bar{\Omega}_n \subset \Omega_{n+1}$$

and

$$\Omega = \bigcup_{n=1}^\infty \Omega_n.$$

We claim that

$$u_n(x) \le u_{n+1}(x),\ x \in \Omega_n. \tag{3.10}$$

Indeed, since u_n and u_{n+1} are both positive solutions of (3.6) on $\bar{\Omega}$, u_n is k-convex in Ω_n and for $x \in \partial\Omega_n \subset \Omega_{n+1}$,

$$u_{n+1}(x) \ge v_1(x) = u_n(x),$$

the inequality (3.10) is a consequence of Lemma 2.1.

Let $x_0 \in \Omega$ be fixed. If m is so large that $x_0 \in \Omega_m$, then for all $n \ge m$, we have

$$v_1(x_0) \le u_n(x_0) \le u_{n+1}(x_0) \le v_2(x_0).$$

Therefore, $\lim\limits_{n\to\infty} u_n(x_0)$ exists. Since x_0 is arbitrary, we see that for $x \in \Omega$,

$$\lim_{n\to\infty} u_n(x) = u(x)$$

exists and

$$v_1(x) \le u(x) \le v_2(x), \quad x \in \Omega.$$

Next we prove $u \in C^\infty(\Omega)$ satisfies (1.1).

Fix an integer m. For $n > m$

$$\bar{\Omega}_m \subset \Omega_n.$$

Suppose u_n is a k-convex solution of (3.6), then for $x \in \bar{\Omega}_m$, $a \le u(x) \le b$, where a is the minimum of $v_1(x)$ on $\bar{\Omega}_m$ and b is the maximum of $v_2(x)$ on $\bar{\Omega}_m$. Moreover, for $n > m$,

$$0 < dist(\bar{\Omega}_m, \partial\Omega_{m+1}) \le dist(\bar{\Omega}_m, \partial\Omega_n) < dist(\Omega_m, \partial\Omega).$$

Let $j \ge 3$ be an integer. Since u_n is k-convex on $\bar{\Omega}_n$, it follows from Lemma 2.5 that there exists a constant C^* such that if $n > m$, then

$$|D^\gamma u_n(x)| \le C^*, \quad \forall x \in \bar{\Omega}_m,$$

where $D^\gamma u_n$ is any partial derivative of u_n of order $\le j$. It follows from Ascoli's lemma that there exists a subsequence $\{u_{n_j}(x)\}_1^\infty$ of $\{u_n(x)\}_1^\infty$ such that if D^η is any partial derivative operator of order $\le j-1$, then the sequence $\{D^\eta u_{n_j}(x)\}_1^\infty$ converges uniformly on $\bar{\Omega}_m$. Hence $u \in C^{j-1}(\bar{\Omega})$ and for $x \in \bar{\Omega}_m$,

$$\begin{aligned}
&S_k(D^2u)(x) = \lim_{j\to\infty} S_k(D^2u_{n_j})(x)\\
&= \lim_{j\to\infty} H(x)[u_{n_j}(x)]^k[\ln u_{n_j}(x)]^\beta\\
&= H(x)[u(x)]^k[\ln u(x)]^\beta.
\end{aligned}$$

Since $j \ge 3$ was arbitrary and $m \ge 1$ is arbitrary, this argument proves that $u \in C^\infty(\Omega)$ satisfies (1.1).

Step3. Establish the estimates (3.1).

By (2.5), the form of v_1 and v_2 given in Step1 and the fact that $v_1(x) \le u(x) \le v_2(x)$ for all $x \in \Omega$, we infer the existence of constants $c_1 > 0$ and c_2 such that (3.1) is true. $\square$

The following Theorem deals with the nonexistence result.

Theorem 3.2. *Let Ω be a smooth, bounded, strictly convex domain in $R^N, N \geq 2$. Suppose $H \in C^\infty(\bar{\Omega})$ is positive. If $\beta \leq k+1$, then problem (1.1) has no k-convex solution on Ω.*

Proof. Let $a = (a_1, a_2, \cdots, a_N) \in R^N - \bar{\Omega}$ and

$$v(x) = exp\sum_{j=1}^{N} t(x_j - a_j)^2, \quad x \in \Omega,$$

where $t \geq 1$ is a constant to be determined. Then we have for $1 \leq i, j \leq N$,

$$v_{x_i x_j}(x) = 4t^2(x_i - a_i)(x_j - a_j)v(x), \quad i \neq j,$$

and for $1 \leq i \leq N$,

$$v_{x_i x_i}(x) = (2t + 4t^2(x_i - a_i)^2)v(x).$$

Then by the property of determinant and a direct calculation we obtain

$$S_k(D^2v)(x) = \sum_{i=1}^{C_N^k}\{(2tv(x))^{k-1}v(x)[2t + 4t^2\sum_{s=1}^{k}(x_{is} - a_{is})^2]\}.$$

Therefore, for $x \in R^N$,

$$det(v_{x_{is}x_{ij}}) \geq 2^k t^k[v(x)]^k, \ S_k(D^2v)(x) \geq C_N^k 2^k t^k [v(x)]^k. \tag{3.11}$$

Assume, contrary to the assertion of the theorem, that there exists a k-convex solution u of (1.1). Let $w(x) = e^{cv(x)}$. We have for $x \in R^N$,

$$S_k(D^2w) = c^k e^{kcv}[S_k(D^2v) + c\sum_{i=1}^{C_N^k} det(v_{x_{is}x_{ij}})(\nabla v_i)^T B(v_i)\nabla v_i]. \tag{3.12}$$

Since the matrix $(v_{x_i x_j})$ is positive definite, $B(v_i)$ is positive definite and there exist constants $0 < l_1 < l_2$ such that

$$l_1||\nabla v_i||^2 \leq (\nabla v_i)^T B(v_i)\nabla v_i \leq l_2||\nabla v_i||^2. \tag{3.13}$$

By (3.11), (3.12), (3.13) and the fact that $\|\nabla v_i\|^2 = 4t^2v^2(x)\sum\limits_{s=1}^{k}(x_{is} - a_{is})^2 > 0$ we get

$$\begin{aligned} S_k(D^2w) &\geq c^{k+1}l_1(2t)^{2+k}v^{2+k}e^{kcv}\sum_{i=1}^{C_N^k}\sum_{s=1}^{k}(x_{is} - a_{is})^2 \\ &= c^{k+1-\beta}l_1(2t)^{2+k}w^k[\ln w]^\beta v^{2+k-\beta}\sum_{i=1}^{C_N^k}\sum_{s=1}^{k}(x_{is} - a_{is})^2. \end{aligned} \tag{3.14}$$

If $\beta < k+1$, it follows from (3.14) that there exists c large enough such that

$$S_k(D^2w) \geq H(x)w^k[\ln w]^\beta.$$

Fix $x_1 \in \Omega$ and by further enlarging c if necessary we may assume that

$$w(x_1) > u(x_1) \text{ and } S_k(D^2 w) > H(x) w^k [\ln w]^\beta \text{ in } \Omega.$$

Since $u(x) \to \infty$ as $d(x) \to 0$, while $w(x)$ is continuous on $\overline{\Omega}$, there exists an open connected set D such that

$$x_1 \in D,\ \overline{D} \subset \Omega,\ u(x) < w(x) \text{ in } D \text{ and } u(x) = w(x) \text{ on } \partial D.$$

On the other hand, since

$$S_k(D^2 u) = H(x) u^k [\ln u]^\beta \text{ in } D \text{ and } u = w \text{ on } \partial D,$$

and the matrix $(w_{x_i x_j})$ is positive definite on $\overline{D}$, we can apply Lemma 2.1 to conclude that $w(x) \le u(x)$ in D. It is a contradiction.

If $\beta = k+1$, from (3.14) we have that

$$S_k(D^2 w) \ge H(x) w^k [\ln w]^\beta$$

for large t. Similar to the proof above, we can get contradiction. This contradiction completes our proof. □

4. $H(x) \in C^\infty(\Omega)$ is unbounded on $\partial\Omega$

In this case we assume that $H(x) \in C^\infty(\Omega)$ satisfy

(H) $C_1 d(x)^{-\gamma} \le H(x) \le C_2 d(x)^{-\gamma}, x \in \Omega$ for some positive constants C_1, C_2, γ;

or the stronger condition

(H′) $\lim\limits_{d(x)\to 0} H(x) d(x)^\gamma = C_0$ for some positive constants γ, C_0.

Theorem 4.1. *Let Ω be a smooth, bounded, strictly convex domain in $R^N, N \ge 2$. Suppose that $H(x) \in C^\infty(\Omega)$ satisfies* **(H)** *and $1 < \gamma < k+1$. If $\beta > k+1$, then problem (1.1) has a k-convex solution $u \in C^\infty(\Omega)$ verifying*

$$e^{d_1 d(x)^{-\alpha_1}} \le u(x) \le e^{d_2 d(x)^{-\alpha_1}}, x \in \Omega \tag{4.1}$$

for some positive numbers $d_1 < d_2$, where

$$\alpha_1 = \frac{k+1-\gamma}{\beta-(k+1)}. \tag{4.2}$$

If $\beta \le k$, then problem (1.1) has no k-convex solution.

Proof. If $\beta > k+1$, then let $v(x) = e^{d[z(x)]^{-\alpha_1}}$. Similar to the proof of Theorem 3.1, we can prove the existence of the solution and (4.1). So we omit it.

Suppose $\beta \le k$.

We first prove that if $1 < \gamma < k+1$, then

$$\begin{cases} S_k(D^2u) = H(x), \text{ in } \Omega, \\ u = 0, \quad \text{ on } \partial\Omega \end{cases} \tag{4.3}$$

has a solution.

Let $w = -c(z(x))^{\frac{k+1-\gamma}{k}}$, where z is defined by (2.4). By (2.1) we obtain

$$S_k(D^2w) = c^k(\frac{k+1-\gamma}{k})^k z^{-\gamma}[(-1)^k S_k(D^2z)z + \frac{1-\gamma}{k}\sum_{i=1}^{C_N^k}(-1)^k det(z_{x_{i_s}x_{i_j}})(\nabla z_i)^T B(z_i)\nabla z_i].$$

Because of $\gamma > 1$, similar to the analysis in Theorem 3.1 there exists $M_1 > 0$ such that

$$(-1)^k S_k(D^2z)z + \frac{1-\gamma}{k}\sum_{i=1}^{C_N^k}(-1)^k det(z_{x_{i_s}x_{i_j}})(\nabla z_i)^T B(z_i)\nabla z_i \geq M_1.$$

By (2.5) and (H) we get

$$S_k(D^2w) \geq H(x)$$

for large c. It follows from Lemma 2.6 that (4.3) has a k-convex solution $w_1(x)$.

Let $w_2(x) = w_1(x) - \min_{x\in\bar\Omega} w_1(x) + 1$. Then $S_k(D^2w_2) = H(x), x \in \Omega$, and $w_2 \geq 1$.

We suppose contrary that problem (1.1) has a solution $u(x)$. If $\beta < k$, let $v(x) = e^{c[w_2(x)]^{\frac{k+1}{k-\beta}}}, x \in \Omega$, where c is a positive constant to be determined. By (2.1) we have

$$\begin{aligned}
&S_k(D^2v) \\
&= c^k(\tfrac{k+1}{k-\beta})^k e^{kc[w_2(x)]^{\frac{k+1}{k-\beta}}} w_2^{k(\frac{\beta+1}{k-\beta})-1}[S_k(D^2w_2)w_2 \\
&+(c\tfrac{k+1}{k-\beta}w_2^{\frac{k+1}{k-\beta}} + \tfrac{\beta+1}{k-\beta})\sum_{i=1}^{C_N^k} det(w_{2x_{i_s}x_{i_j}})(\nabla w_{2i})^T B(w_{2i})\nabla w_{2i}] \\
&\geq c^{k-\beta}(\tfrac{k+1}{k-\beta})^k v^k[\ln v]^\beta S_k(D^2w_2)w_2 \\
&\geq H(x)v^k[\ln v]^\beta, x \in \Omega
\end{aligned}$$

for large c.

Fix $x_1 \in \Omega$ and by further enlarging c if necessary we may assume that

$$v(x_1) > u(x_1) \text{ and } S_k(D^2v) > H(x)v^k[\ln v]^\beta \text{ in } \Omega.$$

Since $u(x) \to \infty$ as $d(x) \to 0$, while $v(x)$ is continuous on $\overline{\Omega}$, there exists an open connected set D such that

$$x_1 \in D,\ \overline{D} \subset \Omega,\ u(x) < v(x) \text{ in } D \text{ and } u(x) = v(x) \text{ on } \partial D.$$

On the other hand, since

$$S_k(D^2u) = H(x)u^k[\ln u]^\beta \text{ in } D \text{ and } u = v \text{ on } \partial D,$$

and the matrix $(v_{x_ix_j})$ is positive definite on $\overline{D}$, we can apply Lemma 2.1 to conclude that $v(x) \leq u(x)$ in D. This contradiction completes our proof.

If $\beta = k$, then let $v(x) = e^{e^{cw_2(x)}}$, similar to the proof above we can get the contradiction. □

Theorem 4.2. *Let Ω be a smooth, bounded, strictly convex domain in $R^N, N \geq 2$. Suppose $H(x) \in C^\infty(\Omega)$ satisfies* $(\mathbf{H}')$ *and $1 < \gamma < k+1$. If $\beta > k+1$, then for any k-convex solution $u(x)$ of (1.1), it holds*

$$\xi^- \leq \liminf_{d(x)\to 0} d(x)^{\alpha_1} \ln u(x) \text{ and } \limsup_{d(x)\to 0} d(x)^{\alpha_1} \ln u(x) \leq \xi^+,$$

where α_1 is defined by (4.2) and

$$\xi^- = [C_0^{-1} C_{N-1}^{k-1} (M^-)^{k-1} \alpha_1^{k+1}]^{\frac{1}{\beta-(k+1)}},$$
$$\xi^+ = [C_0^{-1} C_{N-1}^{k-1} (M^+)^{k-1} \alpha_1^{k+1}]^{\frac{1}{\beta-(k+1)}},$$

here

$$M^- = \min_{y\in\partial\Omega} \{\kappa_1(y), \kappa_2(y), \ldots, \kappa_{N-1}(y)\},$$
$$M^+ = \max_{y\in\partial\Omega} \{\kappa_1(y), \kappa_2(y), \ldots, \kappa_{N-1}(y)\}.$$

If further suppose Ω is a ball of radius $R > 0$, then

$$\lim_{d(x)\to 0} d(x)^{\alpha_1} \ln u(x) = \left(\frac{C_{N-1}^{k-1} \alpha_1^{k+1}}{C_0 R^{k-1}} \right)^{\frac{1}{\beta-(k+1)}}.$$

Proof. For $\delta > 0$, we set $\Omega_\delta = \{x \in \Omega : 0 < d(x) < \delta\},\ \Gamma_\delta = \{x \in \Omega : d(x) = \delta\}$.

Fix $\varepsilon \in (0, \frac{1}{2})$ and choose $\delta > 0$ small enough such that

$$C_0(1-\varepsilon)d(x)^{-\gamma} \leq H(x) \leq C_0(1+\varepsilon)d(x)^{-\gamma}$$

and

$$(1 - M^+ d(x))^k \geq 1 - \varepsilon$$

for $x \in \Omega_\delta$.

Suppose u is a k-convex solution of (1.1).

Let $v^+ = e^{\eta^+(d(x))^{-\alpha_1-\mu} + \max_{\Gamma_\delta} u(x)}, x \in \Omega_\delta$ with $\mu > 0$,

$$\eta^+ = [\frac{(\alpha_1+\mu)^{k+1}\Delta_1}{C_0(1-\varepsilon)^2}]^{\frac{1}{\beta-(k+1)}},\ \eta^- = [\frac{(\alpha_1-\mu)^{k+1}\Delta_2}{C_0(1-\varepsilon)}]^{\frac{1}{\beta-(k+1)}},$$

where

$$\Delta_1 = \frac{(d(x))^{\alpha_1+\mu+1}}{\eta^+(\alpha_1+\mu)} (M^+)^k C_{N-1}^k + [1 + \frac{\alpha_1+\mu+1}{\eta^+(\alpha_1+\mu)} (d(x))^{\alpha_1+\mu}] (M^+)^{k-1} C_{N-1}^{k-1},$$
$$\Delta_2 = (M^-)^{k-1} C_{N-1}^{k-1}.$$

Since $|Dd(x)| = 1$ in Ω_δ, we can choose a coordinate system such that

$$Dd(x) = (0, 0, \ldots, 0, 1),$$
$$D^2 d(x) = diag(d_{11}(x), \ldots, d_{N-1,N-1}(x), 0),$$

where $d_{ii}(x) = -\kappa_i(y)/(1 - \kappa_i(y)d(x))$, and $y \in \partial\Omega$ is such that $|x - y| = d(x)$ as in Lemma 2.4.

Hence

$$\begin{aligned}
&S_k(D^2v^+)\\
&= (v^+)^k(\eta^+)^{k+1}(\alpha_1+\mu)^{k+1}(d(x))^{-(k+1)(\alpha_1+\mu+1)}\{[1+\tfrac{\alpha_1+\mu+1}{\eta^+(\alpha_1+\mu)}(d(x))^{\alpha_1+\mu}]S_{k-1}(D^2(-d(x)))\\
&+\tfrac{(d(x))^{\alpha_1+\mu+1}}{\eta^+(\alpha_1+\mu)}S_k(D^2(-d(x)))\}\\
&\le \eta^{+(k+1-\beta)}C_0^{-1}(1-\varepsilon)^{-2}(\alpha_1+\mu)^{k+1}(d(x))^{[\beta-(k+1)]\mu}H(x)(v^+)^k[\ln v^+]^\beta\Delta_1\\
&\le H(x)(v^+)^k[\ln v^+]^\beta,\ x\in\Omega_\delta,
\end{aligned}$$

i.e. v^+ is a supersolution to (1.1) in Ω_δ. By (4.1), for any $\mu>0$,

$$\lim_{d(x)\to 0}(v^+-u(x))=+\infty.$$

Thus

$$u(x)\le v^+(x),\ x\in\Omega_\delta.$$

Letting $\mu\to 0,\varepsilon\to 0$, we get

$$\limsup_{d(x)\to 0} d(x)^{\alpha_1}\ln u(x)\le [C_0^{-1}C_{N-1}^{k-1}(M^+)^{k-1}\alpha^{k+1}]^{\frac{1}{\beta-(k+1)}}.$$

Let $v^-=e^{\eta^-[(d(x))^{-\alpha+\mu}-\delta^{-\alpha+\mu}]},\ x\in\Omega_\delta$ with $\mu>0$. Similarly we obtain

$$S_k(D^2v^-)\ge H(x)(v^-)^k[\ln v^-]^\beta,\ x\in\Omega_\delta,$$

which shows that v^- is a subsolution to (1.1) in Ω_δ. By (4.1), for any $\mu>0$,

$$\lim_{d(x)\to 0}(u(x)-v^-)=+\infty.$$

So we get

$$\liminf_{d(x)\to 0} d(x)^{\alpha_1}\ln u(x)\ge [C_0^{-1}C_{N-1}^{k-1}(M^-)^{k-1}\alpha^{k+1}]^{\frac{1}{\beta-(k+1)}}.$$

If Ω is a ball of radius $R>0$, then $M^-=M^+=\frac{1}{R}$. It follows that

$$\lim_{d(x)\to 0} d(x)^{\alpha_1}\ln u(x)=\left(\frac{C_{N-1}^{k-1}\alpha^{k+1}}{C_0R^{k-1}}\right)^{\frac{1}{\beta-(k+1)}}. \qquad \square$$

5. $H(x)\in C^\infty(\bar\Omega)$ can vanish on $\partial\Omega$

In this case we assume that $H(x)\in C^\infty(\Omega)$ satisfies

$(\mathbf{H_1})$ $D_1d(x)^\theta\le H(x)\le D_2d(x)^\theta, x\in\Omega$ for some positive constants D_1,D_2,θ;

or the stronger condition

$(\mathbf{H_1'})$ $\lim_{d(x)\to 0} H(x)d(x)^{-\theta}=D_0$ for some positive constants θ, D_0.

Theorem 5.1. *Let Ω be a smooth, bounded, strictly convex domain in $R^N, N \geq 2$. Suppose that $H(x) \in C^\infty(\bar{\Omega})$ satisfies $(\mathbf{H_1})$. If $\beta > k+1$, then problem (1.1) has a k-convex solution $u \in C^\infty(\Omega)$ verifying*

$$e^{d_1' d(x)^{-\alpha_2}} \leq u(x) \leq e^{d_2' d(x)^{-\alpha_2}}, x \in \Omega \tag{5.1}$$

for some positive numbers $d_1' < d_2'$, where

$$\alpha_2 = \frac{k+1+\theta}{\beta-(k+1)}. \tag{5.2}$$

If $\beta \leq k+1$, then problem (1.1) has no k-convex solution.

Proof. The proof is similar to that of Theorem 3.1 and Theorem 3.2, so we omit it. □

Theorem 5.2. *Let Ω be a smooth, bounded, strictly convex domain in $R^N, N \geq 2$. Suppose $H(x) \in C^\infty(\bar{\Omega})$ satisfies $(\mathbf{H}_1')$. If $\beta > k+1$, then for any k-convex solution $u(x)$ of (1.1), it holds*

$$\xi^- \leq \liminf_{d(x)\to 0} d(x)^{\alpha_2} \ln u(x) \text{ and } \limsup_{d(x)\to 0} d(x)^{\alpha_2} \ln u(x) \leq \xi^+,$$

where α_2 is defined by (5.2) and

$$\xi^- = [D_0^{-1} C_{N-1}^{k-1} (M^-)^{k-1} \alpha_2^{k+1}]^{\frac{1}{\beta-(k+1)}},$$
$$\xi^+ = [D_0^{-1} C_{N-1}^{k-1} (M^+)^{k-1} \alpha_2^{k+1}]^{\frac{1}{\beta-(k+1)}},$$

here

$$M^- = \min_{y\in\partial\Omega}\{\kappa_1(y), \kappa_2(y), \ldots, \kappa_{N-1}(y)\},$$
$$M^+ = \max_{y\in\partial\Omega}\{\kappa_1(y), \kappa_2(y), \ldots, \kappa_{N-1}(y)\}.$$

If further suppose Ω is a ball of radius $R > 0$, then

$$\lim_{d(x)\to 0} d(x)^{\alpha_1} \ln u(x) = \left(\frac{C_{N-1}^{k-1}\alpha_2^{k+1}}{D_0 R^{k-1}}\right)^{\frac{1}{\beta-(k+1)}}.$$

Proof. The proof is similar to that of Theorem 4.2, so we omit it. □

Acknowledgments

This work is sponsored by the National Natural Science Foundation of China (11301178), the Beijing Natural Science Foundation (1163007) and the Scientific Research Project of Construction for Scientific and Technological Innovation Service Capacity (KM201611232017). The authors are grateful to anonymous referees for their constructive comments and suggestions, which has greatly improved this paper.

Appendix

The following definitions and propositions are from [26].

Let A be a square matrix and

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

For index sets $\alpha \subseteq \{1, 2, \ldots, n\}, \beta \subseteq \{1, 2, \ldots, n\}$, we denote by $A[\alpha, \beta]$ the submatrix of entries that lie in the rows of A indexed by α and the columns indexed by β. If $\alpha = \beta$, the submatrix $A[\alpha] = A[\alpha, \beta]$ is a principal submatrix of A. For $k \in \{1, 2, \ldots, n\}$, $A[\{1, 2, \ldots, k\}]$ is a leading principal submatrix. If the k-by-k submatrix is a principal submatrix, then its determinant is a principal minor (of size k); if the submatrix is a leading principal matrix, then its determinant is a leading principal minor. It is clear that an n-by-n matrix has C_n^k distinct principal submatrices of size k, but has only one leading principal submatrix of size k.

There are some conclusions on positive (negative) definite matrix. We list them here without proofs.

Proposition 1. *Suppose A is a real symmetric matrix, then A is positive definite if and only if all of its principal minors are positive.*

Proposition 2. *Suppose A is a real symmetric matrix, then A is negative definite if and only if for all of its principal minors Δ_k, we have $(-1)^k \Delta_k > 0$.*

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