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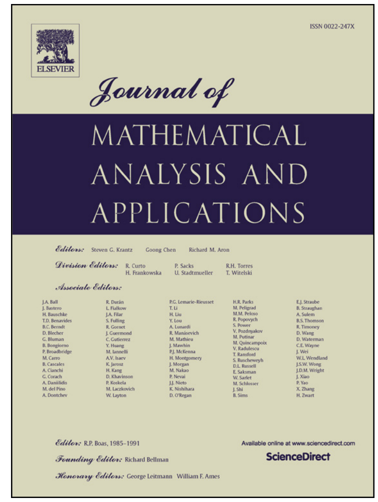

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Remark on upper bound for lifespan of solutions to semilinear evolution equations in a two-dimensional exterior domain

Masahiro Ikeda* and Motohiro Sobajima†

Abstract. In this paper we consider the following initial-boundary value problem with the power type nonlinearity $|u|^p$ with $1 < p \le 2$ in a two-dimensional exterior domain

$$\begin{cases} \tau\partial_t^2 u(x,t) - \Delta u(x,t) + e^{i\zeta}\partial_t u(x,t) = \lambda|u(x,t)|^p, & (x,t)\in\Omega\times(0,T),\\ u(x,t) = 0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0) = \varepsilon f(x), & x\in\Omega,\\ \partial_t u(x,0) = \varepsilon g(x), & x\in\Omega, \end{cases} \tag{0.1}$$

where Ω is given by $\Omega = \{x\in\mathbb{R}^2\ ;\ |x|>1\}$, $\zeta\in[-\frac{\pi}{2},\frac{\pi}{2}]$, $\lambda\in\mathbb{C}$ and $\tau\in\{0,1\}$ switches the parabolicity, dispersivity and hyperbolicity. Remark that $2 = 1 + 2/N$ is well-known as the Fujita exponent. If $p > 2$, then there exists a small global-in-time solution of (0.1) for some initial data small enough (see Ikehata [**13**]), and if $p < 2$, then global-in-time solutions cannot exist for any positive initial data (see Ogawa–Takeda [**24**] and Lai–Yin [**16**]). The result is that for given initial data $(f,\tau g)\in H_0^1(\Omega)\times L^2(\Omega)$ satisfying $(f+\tau g)\log|x|\in L^1(\Omega)$ with some requirement, the solution blows up at finite time, and moreover, the upper bound for lifespan of solutions to (0.1) is given as the following *double exponential type* when $p = 2$:

$$\mathrm{LifeSpan}(u) \le \exp[\exp(C\varepsilon^{-1})].$$

The crucial idea is to use test functions which approximates the harmonic function $\log|x|$ satisfying Dirichlet boundary condition and the technique modified from [**10**].

Mathematics Subject Classification (2010): 35K58, 35L71, 35Q55, 35Q56.

Key words and phrases: Evolution equations, exterior domain, two dimension, Small data blow-up, Upper bound of lifespan, Fujita exponent, double exponential type.

1 Introduction

In this paper we consider the following initial-boundary value problem with the power type nonlinearity $|u|^p$ with $1 < p \le 2$ in a two-dimensional exterior domain

$$\begin{cases} \tau\partial_t^2 u(x,t) - \Delta u(x,t) + e^{i\zeta}\partial_t u(x,t) = \lambda|u(x,t)|^p, & (x,t)\in\Omega\times(0,T),\\ u(x,t) = 0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0) = \varepsilon f(x), & x\in\Omega,\\ \partial_t u(x,0) = \varepsilon g(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is given by $\Omega = \{x\in\mathbb{R}^2\ ;\ |x|>1\}$, $\zeta\in[-\frac{\pi}{2},\frac{\pi}{2}]$, $\lambda\in\mathbb{C}$ and $\tau\in\{0,1\}$ switches the parabolicity, dispersivity and hyperbolicity.

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First we give some comment for the case of whole space $\mathbb{R}^N$. For the semilinear heat equation $\partial_t u - \Delta u = u^p$, Fujita [5] found blowing up solutions with small initial data when $p < p_F(N) = 1 + \frac{2}{N}$. The exponent $p_F(N)$ is well-known as the Fujita exponent. There are many subsequent papers about further detailed analysis (see e.g. Hayakawa [7], Kobayashi–Shirao–Tanaka [15], Mizoguchi–Yanagida [20, 21], Fujishima–Ishige [3, 4] and their references therein). Then Lee–Ni [18] gave a precise estimate for lifespan of solutions to $\partial_t u - \Delta u = u^p$ with $u(0) = \varepsilon f \geq 0$ as

$$\mathrm{LifeSpan}(u) \sim \begin{cases} C\varepsilon^{-(\frac{1}{p-1}-\frac{N}{2})^{-1}} & \text{if } 1 < p < p_F(N), \\ \exp(C\varepsilon^{-(p-1)}) & \text{if } p = p_F(N) \end{cases}$$

by using the structure of the heat kernel in the whole space.

For the Schrödinger equation without gauge invariance $i\partial_t u - \Delta u = \lambda |u|^p$, the study of blowup phenomena for L^1-initial data has been done in the literature (see Ikeda–Wakasugi [11], Fujiwara–Ozawa [6] and Oh–Okamoto–Pocovnicu [23]).

For the semilinear damped wave equation without gauge invariance $\partial_t^2 u - \Delta u + \partial_t u = \lambda |u|^p$, Li–Zhou [19] proved blowup phenomena with lifespan estimates for the solution with arbitrary small initial data when $N = 1, 2$ and $1 < p \leq p_F(N)$. Then the same question for $N = 3$ is solved by Nishihara [22]. For general dimensional case, Todorova–Yordanov [25] established existence of global solutions with sufficiently small initial data if $p > p_F(N)$. In the critical case $p = p_F(N)$, Zhang [27] proved blowup phenomena for the solution with arbitrary small initial data. Ikeda–Ogawa [9] gave an estimates of lifespan, but the gap between lower and upper bounds was not filled. Then Lai–Zhou [17] gives the precise lifespan estimate for blowup solution with small initial data for $p = p_F(N)$.

Later, our previous paper [10] deals with all of the above prototype of the evolution equations simultaneously and proves the upper estimates of lifespan of solutions to (1.1) when $1 < p \leq p_F(N)$ as the same upper bound as Lee–Ni [18]; note that in [10], a kind of space-dependent damping in a cone-like domain is also dealt with.

The problem for exterior domain also has been done for many mathematicians. Tsutsumi [26] proved the global existence of solutions to the Schrödinger equation with gauge invariance for $N \geq 3$ and $p \in 2\mathbb{N}$. Kobayashi–Misawa [14] deals with nonlinear heat equation $\partial_t u - \Delta u = (1 + |x| \log(B|x|))^{-1} |u|^{p-1} u$ with $N = 2$ via Hardy and BMO spaces and posed that the critical exponent for the nonlinearity is $p = \frac{3}{2}$.

If we focus our attention to the result for $N = 2$ with the nonlinearity $|u|^p$, Ikehata [13] constructed small global-in-time solutions of nonlinear damped wave equation when $p > 2 = p_F(2)$. If $p < 2$, then global-in-time solutions cannot exist for compactly supported initial data (see Ogawa–Takeda [24] and Lai–Yin [16]).

However, in the authors' knowledge there is no previous works dealing with the blowup phenomena with lifespan estimates for (1.1) in exterior domains via test function method which is applicable to the critical case and also to the Schrödinger equation. Moreover, the critical case $p = 2$ for two-dimensional exterior domain has been discussed (see e.g., [?]), but it is still open so far. The aim of this paper is the following. For given initial data $(f, \tau g) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfying $(f + \tau g) \log |x| \in L^1(\Omega)$ with some requirement, we shall prove that the solution blows up at finite time, and moreover, the upper bound for lifespan of solutions to (1.1) with $p = 2$ given as the following *double exponential type*:

$$\mathrm{LifeSpan}(u) \leq \exp[\exp(C\varepsilon^{-1})].$$

The crucial idea is to use test functions which approximates the harmonic function $\log |x|$ satisfying Dirichlet boundary condition and the technique for derivation of lifespan estimate in [10].

To state our main result, we give a definition of solutions to (1.1) as follows.

Definition 1.1. We say that u is a solution of (1.1) with $\tau = 0$ in $[0, T)$ if

$$u \in C([0,T); H_0^1(\Omega)) \cap L^p_{\mathrm{loc}}(\overline{\Omega} \times [0,T))$$

with $u(x,0) = \varepsilon f(x)$ and for every $\psi \in C^2(\Omega \times [0,T)$ with $\operatorname{supp}\psi \subset\subset \overline{\Omega} \times [0,T))$ and $\psi|_{\partial\Omega} = 0$

$$\begin{aligned}
&\varepsilon e^{i\zeta} \int_\Omega f(x)\psi(x,0)\,dx + \lambda \int_0^T \int_\Omega |u(x,t)|^p \psi(x,t)\,dx\,dt \\
&= \int_0^T \int_\Omega \Big(\nabla u(x,t) \cdot \nabla\psi(x,t) - e^{i\zeta} u(x,t)\partial_t\psi(x,t)\Big)\,dx\,dt.
\end{aligned}$$

Similarly, u is a solution of (1.1) with $\tau = 1$ in $[0, T)$ if

$$u \in C([0,T); H_0^1(\Omega)) \cap C^1([0,T); L^2(\Omega)) \cap L^p_{\mathrm{loc}}(\overline{\Omega} \times [0,T))$$

with $u(x,0) = \varepsilon f(x)$ and for every $\psi \in C^2(\Omega \times [0,T))$ with $\operatorname{supp}\psi \subset\subset \overline{\Omega} \times [0,T)$ and $\psi|_{\partial\Omega} = 0$

$$\begin{aligned}
&\varepsilon \int_\Omega g(x)\psi(x,0)\,dx + \int_0^T \int_\Omega |u(x,t)|^p \psi(x,t)\,dx\,dt \\
&= \int_0^T \int_\Omega \Big(\nabla u(x,t) \cdot \nabla\psi(x,t) - \partial_t u(x,t)\partial_t\psi(x,t) + a(x)\partial_t u(x,t)\psi(x,t)\Big)\,dx\,dt.
\end{aligned}$$

We remark that the existence and uniqueness of local-in-time solutions to (1.1) in this sense can be verified by the standard way via Dirichlet Laplacian $-\Delta$ endowed with the domain $H^2(\Omega) \cap H_0^1(\Omega)$ and the Gagliardo–Nirenberg inequality (see e.g., Cazenave–Haraux [**2**] and Cazenave [**1**]). Then we introduce the lifespan of the solution u.

Definition 1.2. We denote $\mathrm{LifeSpan}(u)$ as the maximal existence time of solutions to respective problem (1.1). Namely,

$$\mathrm{LifeSpan}(u) = \sup\{T > 0\ ;\ u \text{ is a unique weak solution of (1.1) in } [0,T)\}.$$

Now we are in a position to state the main result in this paper.

Theorem 1.1. *Let $\tau \in \{0,1\}$, $\zeta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\lambda \in \mathbb{C} \setminus \{0\}$, $1 < p < \infty$ and let u be a unique solution of* (1.1). *If $p \le 2$ and $f(x)\log|x|, \tau g(x)\log|x| \in L^1(\Omega)$ with*

$$\int_{C_\Sigma} (\tau g(x) + e^{i\zeta} f(x)) \log|x|\,dx \notin \{-\rho\lambda \in \mathbb{C}\ ;\ \rho \ge 0\},$$

then $\mathrm{LifeSpan}(u) < \infty$. *Moreover,* $\mathrm{LifeSpan}(u)$ *has the following upper bound:*

$$\mathrm{LifeSpan}(u) \le \begin{cases} C\varepsilon^{-\frac{p-1}{2-p}} (\log \varepsilon^{-1})^{p-1} & \text{if } 1 < p < 2, \\ \exp\exp\left(C\varepsilon^{-1}\right) & \text{if } p = 2. \end{cases}$$

Remark 1.1. If $N \ge 3$, then we can take $1 - |x|^{2-N}$ (instead of $\log|x|$) as a harmonic function satisfying Dirichlet boundary condition on $\partial\Omega$. Using this function as a test function, we can prove

$$\mathrm{LifeSpan}(u) \le \begin{cases} C\varepsilon^{-(\frac{1}{p-1} - \frac{N}{2})^{-1}} & \text{if } 1 < p < 1 + \frac{2}{N}, \\ \exp\left(C\varepsilon^{-(p-1)}\right) & \text{if } p = 1 + \frac{2}{N}. \end{cases}$$

This is exactly the same as the case of $\Omega = \mathbb{R}^N$. The exterior problem in two-dimension seems quite exceptional.

The present paper is organized as follows. In Section 2, we will prepare our cutoff functions and give a key property for deriving the upper bound of the lifespan of solutions to (1.1) (see Lemma 2.2 below); remark that the essence of the construction of family of cutoff functions are due to our previous paper [**10**]. Section 3 is devoted to prove Theorem 1.1 by applying Lemma 2.2.

2 Preliminaries

The choice of cutoff functions is a modified version of the one originated by [**10**]. We fix two kinds of functions $\eta \in C^2([0,\infty))$ and $\eta^* \in L^\infty((0,\infty))$ as follows, which will be used in the cut-off functions:

$$\eta(s)\begin{cases} = 1 & \text{if } s \in [0,1/2], \\ \text{is decreasing} & \text{if } s \in (1/2,1), \\ = 0 & \text{if } s \in [1,\infty), \end{cases} \qquad \eta^*(s) = \begin{cases} 0 & \text{if } s \in [0,1/2), \\ \eta(s) & \text{if } s \in [1/2,\infty). \end{cases}$$

Definition 2.1. For $p > 1$, we define for $R > 0$,

$$\psi_R(x,t) = [\eta(s_R(x,t))]^{2p'}, \quad (x,t) \in \Omega \times [0,\infty),$$
$$\psi_R^*(x,t) = [\eta^*(s_R(x,t))]^{2p'}, \quad (x,t) \in \Omega \times [0,\infty).$$

with

$$s_R(x,t) = \frac{(|x|-1)^2 + t}{R}.$$

We also set $P(R) = \left\{(x,t) \in \Omega \times [0,\infty)\ ;\ (|x|-1)^2 + t \le R\right\}$.

Remark 2.1. The choice $s_R = R^{-1}[(|x|-1)^2 + t]$ is crucial for our problem in an exterior domain in the sense of (2.1) (see below).

Then we have the following lemma.

Lemma 2.1. *Let ψ_R and ψ_R^* be as in Definition 2.1. Then ψ_R satisfies the following properties:*

(i) *If $(x,t) \in P(R/2)$, then $\psi_R(x,t) = 1$, and if $(x,t) \notin P(R)$, then $\psi_R(x,t) = 0$.*

(ii) *There exists a positive constant C_1 such that for every $(x,t) \in P(R)$,*

$$|\partial_t \psi_R(x,t)| \le C_1 R^{-1}[\psi_R^*(x,t)]^{\frac{1}{p}}.$$

(iii) *There exists a positive constant C_2 such that for every $(x,t) \in P(R)$,*

$$|\partial_t^2 \psi_R(x,t)| \le C_2 R^{-2}[\psi_R^*(x,t)]^{\frac{1}{p}}.$$

(iv) *There exists a positive constant C_3 such that for every $(x,t) \in P(R)$,*

$$|\nabla \psi_R(x,t)| \le C_3 R^{-1}|x|(\log|x|)[\psi_R^*(x,t)]^{\frac{1}{p}}.$$

(v) *There exists a positive constant C_4 such that for every $(x,t) \in P(R)$,*

$$|\Delta \psi_R(x,t)| \le C_4 R^{-1}[\psi_R^*(x,t)]^{\frac{1}{p}}.$$

Proof. All of the assertions follow from the direct calculation by noticing

$$\partial_t s_R = \frac{1}{R}, \quad \nabla s_R = \frac{2}{R}\left(1 - \frac{1}{|x|}\right)x, \quad \Delta s_R = \frac{2}{R}\left(2 - \frac{1}{|x|}\right)$$

and

$$1 - \frac{1}{|x|} \le \log|x| \tag{2.1}$$

for $x \in \Omega$. □

The following lemma is the key assetion of the present paper which is similar as [**10**, Lemma 2.10], but the situation with logarithmic function is included.

Lemma 2.2. *Let $\delta > 0$, $C_0 > 0$, $R_1 > 0$, $\theta \ge 0$, $\kappa \in \mathbb{R}$ and $0 \le w \in L^1_{\mathrm{loc}}([0,T); L^1(\Omega))$ for $T > R_1$. Assume that for every $R \in [R_1, T)$,*

$$\delta + \iint_{P(R)} w(x,t)\psi_R(x,t)\,dx\,dt \le C_0 R^{-\frac{\theta}{p'}}(\log R)^{\frac{\kappa}{p'}}\left(\iint_{P(R)} w(x,t)\psi_R^*(x,t)\,dx\,dt\right)^{\frac{1}{p}}. \tag{2.2}$$

Then T has to be bounded above as follows:

$$T \le \begin{cases} C\delta^{-\frac{1}{\theta}}(\log(\delta^{-1}))^{\frac{\kappa}{\theta}} & \text{if } \theta > 0,\ \kappa \in \mathbb{R}, \\ \exp\left(C\delta^{-\frac{p-1}{1-\kappa(p-1)}}\right) & \text{if } \theta = 0,\ \kappa < \frac{1}{p-1}, \\ \exp\exp\left(C\delta^{-(p-1)}\right) & \text{if } \theta = 0,\ \kappa = \frac{1}{p-1}. \end{cases}$$

Proof of Lemma 2.2. We set

$$y(r) := \iint_{P(r)} w(x,t)\psi_r^*(x,t)\,dx\,dt, \quad r \in (0,T),$$

Then as in [**10**, Lemma 2.10], we have

$$\begin{aligned} \int_0^R y(r)r^{-1}\,dr &= \iint_{P(R)} w(x,t)\left(\int_{((|x|-1)^2+t)/R}^{\infty} [\eta^*(s)]^{2p'} s^{-1}\,ds\right)dx\,dt \\ &\le (\log 2)\iint_{P(R)} w(x,t)\psi_R(x,t)\,dx\,dt. \end{aligned}$$

Taking

$$Y(R) = \int_0^R y(r)r^{-1}\,dr, \quad \rho \in (R_1, T),$$

we deduce from (2.2) that for $R \in (R_1, T)$,

$$\begin{aligned} \left(\delta + \frac{1}{\log 2}Y(R)\right)^p &\le C_0^p R^{-\theta(p-1)}(\log R)^{\kappa(p-1)}\iint_{P(R)} w(x,t)\psi_R^*(x,t)\,dx\,dt \\ &= C_0^p R^{1-\theta(p-1)}(\log R)^{\kappa(p-1)}Y'(R). \end{aligned}$$

Putting

$$Y(R) = Z\biggl(\int_{\log R_1}^{\log R} e^{\theta(p-1)s} s^{-\kappa(p-1)}\,ds\biggr), \quad 0 < \rho < \rho_T = \int_{\log R_1}^{\log T} e^{\theta(p-1)s} s^{-\kappa(p-1)}\,ds.$$

implies that

$$\frac{d}{d\rho}\Bigl((\log 2)\delta + Z(\rho)\Bigr)^{1-p} \le -(p-1)(\log 2)^{-p} C_1^{-p}, \quad \rho \in (0, \rho_T). \tag{2.3}$$

Integrating it over $[\rho_1, \rho_2] \subset (0, \rho_T)$, we obtain

$$\rho_2 < \rho_1 + (p-1)^{-1}(\log 2) C_1^p \delta^{-(p-1)}.$$

Letting $\rho_2 \uparrow \rho_T$ and $\rho_1 \downarrow 0$, we find

$$\rho_T = \int_{\log R_1}^{\log T} e^{\theta(p-1)s} s^{-\kappa(p-1)}\,ds \le (p-1)^{-1}(\log 2) C_1^p \delta^{-(p-1)}.$$

If the function $e^{\theta(p-1)s} s^{-\kappa(p-1)}$ is not integrable at infinity, then T has to be finite. More precisely, the asymptotics of ρ_T for large T is as follows, respectively. If $\theta > 0$, then $\rho_T \approx \frac{1}{(p-1)\theta} T^{\theta(p-1)}(\log T)^{-\kappa(p-1)}$. If $\theta = 0$ and $\kappa < \frac{1}{p-1}$, then $\rho_T \approx \frac{1}{1-\kappa(p-1)}(\log T)^{1-\kappa(p-1)}$. In the rest case $\theta = 0$ and $\kappa = \frac{1}{p-1}$, we have $\rho_T \approx \log\log T$. These imply the desired bounds of T for sufficiently small $\delta > 0$. □

3 Proof of Theorem 1.1

We show the assertion for $\tau = 0$ and $\tau = 1$ simultaneously. By the definition of weak solution to (1.1), we can verify that for every $\psi \in C^2(\overline{\Omega} \times [0, T))$ with $\operatorname{supp}\psi \subset\subset \overline{\Omega} \times [0, T)$ and $\psi|_{\partial\Omega} = 0$

$$\begin{aligned}
&\varepsilon \int_\Omega \tau g\psi(0) + f\Bigl(-\tau\partial_t\psi(0) + e^{i\zeta}\psi(0)\Bigr)\,dx + \lambda \int_0^T \int_\Omega |u(t)|^p \psi(t)\,dx\,dt \\
&= \int_0^T \int_\Omega u(t)\Bigl(-\Delta\psi(t) + \tau\partial_t^2\psi(t) - e^{i\zeta}\partial_t\psi(t)\Bigr)\,dx\,dt.
\end{aligned}$$

Noting that

$$\lim_{R\to\infty} \int_\Omega \tau g\Phi\psi_R(0) + f\Phi\Bigl(-\tau\partial_t\psi_R(0) + e^{i\zeta}\psi_R(0)\Bigr)\,dx = \int_\Omega \Bigl(\tau g + e^{i\zeta} f\Bigr)\Phi\,dx$$

with $\Phi(x) = \log|x|$, we see that there exist $R_0 > 0$, $\xi_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $c_0 > 0$ such that for every $R \ge R_0$,

$$\mathrm{Re}\left[\lambda^{-1} e^{i\xi} \int_\Omega \tau g\Phi\psi_R(0) + f\Phi\Bigl(-\tau\partial_t\psi_R(0) + e^{i\zeta}\psi_R(0)\Bigr)\,dx\right] \ge c_0.$$

Now we assume that LifeSpan$(u) > R_0$. Since Φ is independent of t, it follows from Lemma 2.1 that

$$\begin{aligned}
&\Bigl|-\Delta(\Phi\psi_R(t)) + \tau\partial_t^2(\Phi\psi_R(t)) - e^{i\zeta}\partial_t(\Phi\psi_R(t))\Bigr| \\
&= \Bigl|-2\nabla\Phi\cdot\nabla\psi_R(t) - \Phi\Delta\psi_R(t) + \Phi\partial_t^2\psi_R(t) - e^{i\zeta}\Phi\partial_t\psi_R(t)\Bigr| \\
&\le +\frac{2C_3}{R}\Phi[\psi_R^*(t)]^{\frac{1}{p}} + \frac{C_4}{R}\Phi[\psi_R^*(t)]^{\frac{1}{p}} + \frac{\tau C_2}{R^2}\Phi[\psi_R^*(t)]^{\frac{1}{p}} + \frac{C_1}{R}\Phi[\psi_R^*(t)]^{\frac{1}{p}} \\
&\le \frac{C_5}{R}\Phi[\psi_R^*(t)]^{\frac{1}{p}}
\end{aligned}$$

with $C_5 = 2C_3 + C_4 + \tau C_2 R_0^{-1} + C_1$. Therefore choosing the test function $\psi(x,t) = \Phi(x)\psi_R(x,t)$ which satisfies the Dirichlet boundary condition, we obtain

$$\begin{aligned}
&c_0\varepsilon + \cos\xi \iint_{P(R)} |u(t)|^p \Phi\psi_R(t)\,dx\,dt \\
&\le \mathrm{Re}\left[\lambda^{-1}e^{i\xi_0}\iint_{P(R)} u(t)\Big(\tau\partial_t^2(\Phi\psi_R(t)) - \Delta(\Phi\psi_R(t)) - e^{i\zeta}\partial_t(\Phi\psi_R(t))\Big)\,dx\,dt\right] \\
&\le \frac{C_5}{|\lambda|R}\iint_{P(R)} |u|\Phi[\psi_R^*(t)]^{\frac{1}{p}}\,dx\,dt \\
&\le \frac{C_5}{|\lambda|R}\left(\iint_{P(R)} \Phi\,dx\,dt\right)^{\frac{1}{p'}}\left(\iint_{P(R)} |u(t)|^p\Phi\psi_R^*(t)\,dx\,dt\right)^{\frac{1}{p}}.
\end{aligned}$$

Therefore noting that

$$\iint_{P(R)} \Phi\,dx\,dt \le \pi R(\sqrt{R}+1)^2\log(\sqrt{R}+1).$$

we deduce

$$c_0\varepsilon + \iint_{P(R)} |u(t)|^p\Phi\psi_R(t)\,dx\,dt \le C_6 R^{1-\frac{2}{p}}(\log R)^{\frac{1}{p'}}\left(\iint_{P(R)} |u(t)|^p\Phi\psi_R^*(t)\,dx\,dt\right)^{\frac{1}{p}}$$

for some positive constant $C_6 > 0$. Applying Lemma 2.2 with $w = |u|^p\Phi$, we have the desired estimate for LifeSpan(u). The proof is complete. □

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