



# On eigenvalues of the linearization of a free boundary problem modeling two-phase tumor growth <sup>☆</sup>



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## ABSTRACT

In this paper we study eigenvalues of the linearization of a free boundary problem modeling the growth of a tumor containing two species of cells: proliferating cells and quiescent cells. Such eigenvalues are potential bifurcation points from which nonradial solutions of the free boundary problem might bifurcate from the radial solution. A special feature of this problem is that it contains a singular ordinary differential equation which causes the main difficulty of this problem. By using the spherical harmonic expansion method combined with some techniques for solving singular differential integral equations developed in some previous literature, eigenvalues of the linearized problem are completely determined. Invertibility of some linear operators related to the linearized problem in suitable function spaces is also studied which might be useful in the analysis of the original free boundary problem.

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## 1. Introduction

It has long been observed that under a constant circumstance, a solid tumor will finally evolve into a dormant or stationary state. In a dormant state, the tumor's macrostructure such as size, shape and etc. does not vary in time, while cells inside the tumor are alive and keep undergoing the process of proliferation and movement before they die. In 1972 Greenspan established the first mathematical model in the form of a free boundary problem of a system of partial differential equations to illustrate this phenomenon [23,24]. Since then an increasing number of tumor models in similar forms have appeared in the literature; see the reviewing articles [1,14,16–18,26] and references cited therein. Rigorous mathematical analysis of such models has drawn great attention during the past thirty years, and many interesting results have been obtained, cf., [2–12], [14–21], [25], [30], [31] and references cited therein.

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This paper is concerned with the following free boundary problem modeling the dormant state of a solid tumor with two species of cells – proliferating cells and quiescent cells (see [27]):

$$\Delta\sigma = F(\sigma) \quad \text{for } x \in \Omega, \quad (1.1)$$

$$\sigma = 1 \quad \text{for } x \in \partial\Omega, \quad (1.2)$$

$$\nabla \cdot (\vec{v}p) = [K_B(\sigma) - K_Q(\sigma)]p + K_P(\sigma)q \quad \text{for } x \in \Omega, \quad (1.3)$$

$$\nabla \cdot (\vec{v}q) = K_Q(\sigma)p - [K_P(\sigma) + K_D(\sigma)]q \quad \text{for } x \in \Omega, \quad (1.4)$$

$$p + q = 1 \quad \text{for } x \in \Omega, \quad (1.5)$$

$$\vec{v} = -\nabla\varpi \quad \text{for } x \in \Omega, \quad (1.6)$$

$$\varpi = \gamma\kappa \quad \text{for } x \in \partial\Omega, \quad (1.7)$$

$$V_n \equiv \vec{v} \cdot \vec{n} = 0 \quad \text{for } x \in \partial\Omega. \quad (1.8)$$

Here  $\Omega$  is the domain occupied by the dormant tumor,  $\sigma = \sigma(x)$ ,  $p = p(x)$  and  $q = q(x)$  are the concentration of nutrient, the density of proliferating cells and the density of quiescent cells, respectively,  $\vec{v} = \vec{v}(x)$  is the velocity of tumor cell movement,  $\varpi = \varpi(x)$  is the pressure distribution in the tumor,  $\kappa$  is the mean curvature of the tumor surface whose sign is designated by the convention that  $\kappa \geq 0$  at points where  $\partial\Omega$  is convex,  $\vec{n}$  is the unit outward normal vector of  $\partial\Omega$ , and  $V_n$  is the normal velocity of the tumor surface. Besides,  $F(\sigma)$  is the consumption rate of nutrient by tumor cells,  $K_B(\sigma)$  is the birth rate of tumor cells,  $K_P(\sigma)$  and  $K_Q(\sigma)$  are respectively the transferring rates of tumor cells from quiescent state to proliferating state and from proliferating state to quiescent state, and  $K_D(\sigma)$  is the death rate of quiescent cells. Finally,  $\gamma$  is a positive constant and is referred as surface tension coefficient. For illustration of biological implications of each equation in the above model, we refer the reader to see [16–18,27] and references therein.

A main feature of the above model compared with various other models describing the growth of tumors consisting of only one species of cells, or one-phase tumor model in short, is that it contains balance equations, i.e., the equations (1.3) and (1.4). This determines that the above model is much more difficult to make analysis than one-phase tumor models. Indeed, for one-phase tumor models of the stationary form, we know that they contain only elliptic equations (cf. [7–9,11,12,15,19–21,25,30,31]). But in the above two-phase model, the system contains both elliptic equations and hyperbolic equations. Since hyperbolic equations have quite different and much worse properties compared with elliptic equations, such a system is much harder to tackle. For instance, as far as radial stationary solution is concerned, existence and uniqueness is not very hard to prove for the one-phase tumor model (cf. [20]); but for the above two-phase model the same topic needs a lot of work (cf. [2,10]). The same situation occurs in the analysis of asymptotic stability of the radial stationary solution (cf. [20] and [2,4,5]). This is perhaps the main reason that Friedman called on researcher's attention many times to rigorous mathematical analysis of the above tumor model and its various extensions, and expressed the difficulty of such analysis as “challenging”; see the reviewing articles [16–18], for instance. Indeed, up to now these reviewing articles have been published for over ten years; but little progress has been made on the open problems proposed in them except those made in [2–6,10].

In [10] and [2] it was proved that the above model has a unique radial (i.e. spherically symmetric) solution under the following assumptions:

$$F, K_B, K_D, K_P \text{ and } K_Q \text{ are } C^\infty\text{-functions}; \quad (1.9)$$

$$F(0) = 0 \text{ and } F'(c) > 0 \quad \text{for } 0 \leq c \leq 1; \quad (1.10)$$

$$\begin{cases} K'_B(c) > 0 \text{ and } K'_D(c) < 0 \text{ for } 0 \leq c \leq 1, K_B(0) = 0 \text{ and } K_D(1) = 0; \\ K_P \text{ and } K_Q \text{ satisfy the same conditions as } K_B \text{ and } K_D, \text{ respectively;} \\ K'_B(c) + K'_D(c) > 0 \text{ for } 0 \leq c \leq 1. \end{cases} \quad (1.11)$$

Naturally, we may ask: Does this model have any non-radial solutions? This is a very difficult question to answer. As a first step, in this paper we make a systematic study to the linearized problem of the above model around its radial stationary solution.

Let  $(\sigma_s, p_s, q_s, \varpi_s, v_s, \Omega_s)$ , where  $\Omega_s = \{x \in \mathbb{R}^n : r < R_s\}$ , be the unique radial stationary solution of the system (1.1)–(1.8) ensured by [10] and [2]. After simplification, the linearized system of (1.1)–(1.8) at  $(\sigma_s, p_s, q_s, \varpi_s, v_s, \Omega_s)$  is as follows (see the next section):

$$\Delta \chi = F'(\sigma_s(r))\chi, \quad x \in \Omega_s, \quad (1.12)$$

$$\chi|_{r=R_s} = -\sigma'_s(R_s)\eta(\omega), \quad \omega \in \mathbb{S}^{n-1}, \quad (1.13)$$

$$v_s(r)\varphi_r = p'_s(r)\psi_r + f_\sigma^*(r)\chi + f_p^*(r)\varphi, \quad x \in \Omega_s, \quad (1.14)$$

$$\vec{w} = -\nabla \psi, \quad x \in \Omega_s, \quad (1.15)$$

$$-\Delta \psi = g_\sigma^*(r)\chi + g_p^*(r)\varphi, \quad x \in \Omega_s, \quad (1.16)$$

$$\psi|_{r=R_s} = -\frac{\gamma}{R_s^2}[\eta(\omega) + \frac{1}{n-1}\Delta_\omega \eta(\omega)], \quad \omega \in \mathbb{S}^{n-1}, \quad (1.17)$$

$$\psi_r|_{r=R_s} = g(1, 1)\eta(\omega), \quad \omega \in \mathbb{S}^{n-1}. \quad (1.18)$$

Here  $\chi = \chi(r, \omega)$ ,  $\varphi = \varphi(r, \omega)$ ,  $\psi = \psi(r, \omega)$ ,  $\vec{w} = \vec{w}(r, \omega)$  and  $\eta = \eta(\omega)$ , where  $r = |x|$  and  $\omega = x/|x|$ , are new unknown functions, the subscript  $r$  denotes the derivative in radial direction (e.g.,  $\varphi_r = \frac{\partial \varphi}{\partial r} = \frac{x}{r} \cdot \nabla \varphi$  etc.),  $\Delta_\omega$  denotes the Laplace–Beltrami operator on the unit sphere  $\mathbb{S}^{n-1}$ , and

$$\begin{aligned} f_\sigma^*(r) &= f_\sigma(\sigma_s(r), p_s(r)), & f_p^*(r) &= f_p(\sigma_s(r), p_s(r)), \\ g_\sigma^*(r) &= g_\sigma(\sigma_s(r), p_s(r)), & g_p^*(r) &= g_p(\sigma_s(r), p_s(r)), \end{aligned}$$

where

$$\begin{cases} f(\sigma, p) = K_P(\sigma) + [K_M(\sigma) - K_N(\sigma)]p - K_M(\sigma)p^2, \\ g(\sigma, p) = K_M(\sigma)p - K_D(\sigma), \end{cases}$$

where

$$K_M(\sigma) = K_B(\sigma) + K_D(\sigma), \quad K_N(\sigma) = K_P(\sigma) + K_Q(\sigma).$$

Note that for all  $0 \leq r \leq 1$  (see Lemma 3.1 of [5]),

$$f_p^*(r) < 0, \quad f_\sigma^*(r) > 0, \quad g_p^*(r) > 0 \quad \text{and} \quad g_\sigma^*(r) > 0. \quad (1.19)$$

We note that in the system (1.12)–(1.18), the unknown functions  $\chi = \chi(r, \omega)$ ,  $\varphi = \varphi(r, \omega)$ ,  $\psi = \psi(r, \omega)$  and  $\eta = \eta(\omega)$  can be decoupled with  $\vec{w} = \vec{w}(r, \omega)$ , so that it can be regarded as a system of equations in the unknowns  $\chi = \chi(r, \omega)$ ,  $\varphi = \varphi(r, \omega)$ ,  $\psi = \psi(r, \omega)$  and  $\eta = \eta(\omega)$  only. The equation for  $\chi$  is the elliptic equation (1.12) subject to the Dirichlet boundary condition (1.13) which contains the unknown  $\eta$ . The equation for  $\psi$  is the elliptic equation (1.16) subject to the Dirichlet boundary condition (1.18) which contains the unknowns  $\chi$ ,  $\eta$  and  $\varphi$ . The equation for  $\eta$  is (1.17), which is an elliptic equation on the compact manifold  $\mathbb{S}^{n-1}$  and this equation contains the unknown  $\psi$ . The main difficulty is caused by the equation (1.14) for the unknown  $\varphi$ , which is a first-order singular ordinary differential equation (in the variable  $r$ , with  $\omega$  regarded as a parameter) because  $v_s(0) = v_s(R_s) = 0$ . Indeed, from the analysis made in the references

[2,10] we see that dynamics of singular ordinary differential equations are usually very complex and very hard to analyze.

For any  $\gamma \in \mathbb{R}$ , the system (1.12)–(1.18) has the following family of nontrivial solutions:

$$\begin{cases} \chi(r, \omega) = \sigma'_s(r)z \cdot \omega, & \varphi(r, \omega) = p'_s(r)z \cdot \omega, & \psi(r, \omega) = -v_s(r)z \cdot \omega, \\ \vec{w}(r, \omega) = \frac{v_s(r)}{r}[z - (z \cdot \omega)\omega] + v'_s(r)(z \cdot \omega)\omega, & \eta(\omega) = -z \cdot \omega, \end{cases} \quad (1.20)$$

where  $z$  is an arbitrary nonzero vector in  $\mathbb{R}^n$ . This is actually a reflection to the system (1.12)–(1.18) of the property of translation invariance of the system (1.1)–(1.8). Indeed, since  $(\sigma_s, p_s, v_s, \varpi_s, \Omega_s)$  is a solution of an equivalent system of (1.1)–(1.8) (see (2.2)–(2.8) in the next section), translation invariance implies that for any  $z \in \mathbb{R}^n$  and any  $\varepsilon \in \mathbb{R}$  with  $|\varepsilon|$  sufficiently small,  $(\sigma_\varepsilon, p_\varepsilon, \vec{v}_\varepsilon, \varpi_\varepsilon, \Omega_s - \varepsilon z)$  is also a solution of that system, where

$$\sigma_\varepsilon(x) = \sigma_s(|x + \varepsilon z|), \quad p_\varepsilon(x) = p_s(|x + \varepsilon z|), \quad \varpi_\varepsilon(x) = \varpi_s(|x + \varepsilon z|)$$

and  $\vec{v}_\varepsilon(x) = v_s(|x + \varepsilon z|)(x + \varepsilon z)/|x + \varepsilon z|$ . Differentiating  $(\sigma_\varepsilon, p_\varepsilon, \vec{v}_\varepsilon, \varpi_\varepsilon, \Omega_s - \varepsilon z)$  in  $\varepsilon$  at  $\varepsilon = 0$ , we obtain the above nontrivial solutions of the system (1.12)–(1.18). The purpose of this paper is to investigate for what values of  $\gamma$ , the system (1.12)–(1.18) has nontrivial solutions different from (1.20), and study invertibility and ranges of some linear operators related to the system (1.12)–(1.18) in certain function spaces.

To state the main result of this paper, we first recall some basic notion of analysis in the unit sphere  $\mathbb{S}^{n-1}$ . For every  $k \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ , let  $\lambda_k$  be the  $k+1$ -th eigenvalue of the operator  $-\Delta_\omega$  and  $d_k$  be the dimension of the space  $\mathcal{H}_k$  of all spherical harmonics of degree  $k$ , i.e. (cf. [28,29])

$$\lambda_k = (n+k-2)k \quad \text{and} \quad d_k = \dim \mathcal{H}_k, \quad k = 0, 1, 2, \dots,$$

where

$$\mathcal{H}_k = \{\phi \in C^\infty(\mathbb{S}^{n-1}) : \Delta_\omega \phi = -\lambda_k \phi\}, \quad k = 0, 1, 2, \dots$$

Recall that (cf. [28])

$$d_0 = 1, \quad d_1 = n \quad \text{and} \quad d_k = \binom{n+k-1}{k} - \binom{n+k-3}{k-2} \quad \text{for } k \geq 2.$$

For every  $k \in \mathbb{Z}_+$ , let  $Y_{kl}(\omega)$ ,  $l = 1, 2, \dots, d_k$ , be a normalized orthogonal basis of the space  $\mathcal{H}_k$ , i.e.

$$\begin{aligned} \Delta_\omega Y_{kl}(\omega) &= -\lambda_k Y_{kl}(\omega), \\ \int_{\mathbb{S}^{n-1}} Y_{kl}(\omega) Y_{kl'}(\omega) d\omega &= 0 \quad (l \neq l'), \quad \int_{\mathbb{S}^{n-1}} Y_{kl}^2(\omega) d\omega = 1, \end{aligned}$$

where  $d\omega$  is the induced element on  $\mathbb{S}^{n-1}$  of the Lebesgue measure  $dx$  in  $\mathbb{R}^n$ . Note that in particular,

$$Y_{01}(\omega) = \frac{1}{\sqrt{\sigma_n}} \quad \text{and} \quad Y_{1l}(\omega) = \frac{\sqrt{n}\omega_l}{\sqrt{\sigma_n}}, \quad l = 1, 2, \dots, n, \quad (1.21)$$

where  $\sigma_n$  denotes the surface area of  $\mathbb{S}^{n-1}$ , i.e.  $\sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ , and  $\omega_l$  denotes the  $l$ -th component of  $\omega \in \mathbb{S}^{n-1}$  regarded as a vector in  $\mathbb{R}^n$ . We note that the  $\eta$ -component of the nontrivial solution given by (1.20) ranges over all nonzero functions in  $\mathcal{H}_1$ .

The main result of this paper is as follows:

**Theorem 1.1.** *There exists a null sequence  $\{\gamma_k\}_{k=2}^\infty$ , which is strictly monotone decreasing for sufficiently large  $k$  and satisfies the property  $\gamma_k \sim ck^{-3}$  as  $k \rightarrow \infty$ , where  $c$  is a positive constant independent of  $k$ , such that if  $\gamma = \gamma_k$  for some  $k \geq 2$  then the system (1.12)–(1.18) has a family of nontrivial solutions with the  $\eta$ -component ranging over all nonzero functions in  $\bigoplus_{\gamma_{k'}=\gamma_k} \mathcal{H}_{k'}$ , so that they are different from (1.20). If  $\gamma \neq \gamma_k$  for any  $k \geq 2$  then (1.12)–(1.18) does not have other nontrivial solutions than (1.20).*

The exact expression of  $\gamma_k$  ( $k = 2, 3, \dots$ ) will be given in Section 3; see (3.14). The idea for the proof of the above result is as follows: By solving (1.12)–(1.13) and (1.16)–(1.17) in terms of  $\eta$  and  $\varphi$ , we get  $\chi$  and  $\psi$  as functionals of  $\eta$  and  $\varphi$ . It follows that the system (1.12)–(1.18) reduces into a 2-system containing only the unknown functions  $\eta$  and  $\varphi$ . In such a reduced system, the equation obtained from (1.14) is a non-local singular differential-integral equation: Singularity comes from the fact that  $v_s(0) = v_s(R_s) = 0$  (see (2.14) and (2.17) in the next section), and non-localness is caused by the term  $\psi_r$  in (1.14) because  $\psi$  is the solution of an elliptic boundary value problem containing  $\varphi$ . This is the main difficulty encountered in the proof of the above theorem. We shall appeal to Fourier expansions of functions in  $\mathbb{S}^{n-1}$  via the sequence of spherical harmonics  $\{Y_{kl}(\omega) : k = 0, 1, 2, \dots; l = 1, 2, \dots, d_k\}$  and some techniques for solving singular differential equations developed in [2,3,10] to overcome this difficulty; see Sections 4 and 5 for details.

In addition to the above result, we shall also study invertibility and ranges of some linear operators related to the system (1.12)–(1.18) in certain function spaces. This has potential applications in the study of non-radial solutions of the original system (1.1)–(1.8). Since the exact statements of such results require a big number of new notations, we leave them for later presentation; see Theorems 6.2 and 6.3 in the last section.

The structure of the rest part is as follows. In the next section we compute the linearization of the system of (1.1)–(1.8) around its radial solution  $(\sigma_s, p_s, q_s, \varpi_s, v_s, \Omega_s)$  and reduce the linearized system into a 2-system. In Section 3 we use Fourier expansions of functions in  $\mathbb{S}^{n-1}$  via spherical harmonics to further reduce the PDE 2-system into a sequence of ODE systems, and use them to derive the eigenvalues  $\gamma_k$ ,  $k = 2, 3, \dots$ , by assuming existence and uniqueness of a solution to a nonlocal singular differential-integral equation. In Section 4 we give the proof of the assertion stated in the last sentence. Section 5 aims at studying properties of the eigenvalues  $\gamma_k$ . In the last section we study invertibility and ranges of some linear operators related to the system (1.12)–(1.18) in certain function spaces.

## 2. Linearization

In this section we derive the system (1.12)–(1.18) and make some basic reduction to it.

We first make a basic simplification to the system (1.1)–(1.8). Firstly, by summing up (1.3) and (1.4) and using (1.5), we get

$$\nabla \cdot \vec{v} = K_M(\sigma)p - K_D(\sigma) \quad \text{for } x \in \Omega. \quad (2.1)$$

Substituting this relation into (1.3) and using (1.5) we get

$$\vec{v} \cdot \nabla p = f(\sigma, p) \quad \text{for } x \in \Omega.$$

Moreover, substituting (1.6) into (1.12) and (1.8) we respectively get

$$\begin{aligned} -\Delta \varpi &= g(\sigma, p) \quad \text{for } x \in \Omega, \\ \frac{\partial \varpi}{\partial \vec{n}} &= 0 \quad \text{for } x \in \partial\Omega. \end{aligned}$$

Hence, the system (1.1)–(1.8) reduces into the following system of equations:

$$\Delta\sigma = F(\sigma) \quad \text{for } x \in \Omega, \quad (2.2)$$

$$\sigma = 1 \quad \text{for } x \in \partial\Omega, \quad (2.3)$$

$$\vec{v} \cdot \nabla p = f(\sigma, p) \quad \text{for } x \in \Omega, \quad (2.4)$$

$$\vec{v} = -\nabla\varpi \quad \text{for } x \in \Omega, \quad (2.5)$$

$$-\Delta\varpi = g(\sigma, p) \quad \text{for } x \in \Omega, \quad (2.6)$$

$$\varpi = \gamma\kappa \quad \text{for } x \in \partial\Omega, \quad (2.7)$$

$$\frac{\partial\varpi}{\partial\vec{n}} = 0 \quad \text{for } x \in \partial\Omega. \quad (2.8)$$

Let  $(\sigma_s, p_s, \varpi_s, v_s, \Omega_s)$ , where  $\Omega_s = \{x \in \mathbb{R}^n : r < R_s\}$ , be the unique radial stationary solution of (2.2)–(2.8), i.e.,  $(\sigma_s, p_s, \varpi_s, v_s, R_s)$  is the unique solution of the following system of equations:

$$\sigma_s''(r) + \frac{n-1}{r}\sigma_s'(r) = F(\sigma_s(r)), \quad 0 < r < R_s, \quad (2.9)$$

$$\sigma_s'(0) = 0, \quad \sigma_s(R_s) = 1, \quad (2.10)$$

$$v_s(r)p_s'(r) = f(\sigma_s(r), p_s(r)), \quad 0 < r < R_s, \quad (2.11)$$

$$v_s'(r) + \frac{n-1}{r}v_s(r) = g(\sigma_s(r), p_s(r)), \quad 0 < r < R_s, \quad (2.12)$$

$$v_s(r) = -\varpi_s'(r), \quad 0 < r < R_s, \quad (2.13)$$

$$v_s(0) = 0, \quad v_s(R_s) = 0. \quad (2.14)$$

Later on we shall also use the following simplified notations:

$$f^*(r) = f(\sigma_s(r), p_s(r)), \quad g^*(r) = g(\sigma_s(r), p_s(r)).$$

As we mentioned before, existence and uniqueness of the above system has been proved in [10,2] in the 3-dimension case. Moreover, this solution satisfies the following properties (cf. [10]):

$$0 < \sigma_s(r) < 1 \quad \text{for } 0 \leq r < R_s, \quad \sigma_s'(r) > 0 \quad \text{for } 0 < r \leq R_s, \quad (2.15)$$

$$0 < p_s(r) < 1 \quad \text{for } 0 \leq r < R_s, \quad p_s'(r) > 0 \quad \text{for } 0 < r \leq R_s, \quad (2.16)$$

and there exist positive constants  $c_1, c_2$  such that

$$-c_1r(R_s - r) \leq v_s(r) \leq -c_2r(R_s - r) \quad \text{for } 0 \leq r \leq R_s. \quad (2.17)$$

For the general  $n$ -dimension case ( $n \geq 2$ ), the argument is quite similar so that we omit it here. Note that the above properties are also valid in the general  $n$ -dimension case.

Consider a perturbation of  $(\sigma_s, p_s, v_s, \varpi_s, \Omega_s)$  of the following form:

$$\begin{cases} \sigma(x) = \sigma_s(r) + \varepsilon\chi(r, \omega), & p(x) = p_s(r) + \varepsilon\varphi(r, \omega), \\ \varpi(x) = \varpi_s(r) + \varepsilon\psi(r, \omega), & \vec{v}(x) = v_s(r)\omega + \varepsilon\vec{w}(r, \omega), \\ \Omega = \{x \in \mathbb{R}^n : r < R_s + \varepsilon\eta(\omega)\}, \end{cases}$$

where  $r = |x|$ ,  $\omega = x/|x|$ ,  $\varepsilon$  is a small parameter and  $\chi$ ,  $\varphi$ ,  $\psi$ ,  $\vec{w}$ ,  $\eta$  are new unknown functions. Substituting these expressions into (2.2)–(2.8), making the first-order Taylor expansions to all nonlinear functions containing  $\varepsilon$ , subtracting the corresponding equations in (2.9)–(2.14), then dividing both sides of all equations with  $\varepsilon$  and finally letting  $\varepsilon \rightarrow 0$ , we obtain the system (1.12)–(1.18).

Indeed, deductions of the equations (1.12), (1.13), (1.15), (1.16) and (1.18) are quite standard, see [7,8] for instance. To get (1.17) we need to use the following asymptotic formula for the mean curvature  $\kappa$  of the hypersurface  $r = R_s + \varepsilon\eta(\omega)$  (cf. [22]):

$$\kappa = \frac{1}{R_s} - \frac{\varepsilon}{R_s}[\eta(\omega) + \frac{1}{n-1}\Delta_\omega\eta(\omega)] + o(\varepsilon).$$

Here we only give the deduction of the equation (1.14). Substituting the relations  $\sigma(x) = \sigma_s(r) + \varepsilon\chi(x)$ ,  $p(x) = p_s(r) + \varepsilon\varphi(x)$  and  $\vec{v}(x) = v_s(r)\omega + \varepsilon\vec{w}(x)$  into the third equation in (2.4), we get

$$[v_s(r)\omega + \varepsilon\vec{w}] \cdot [\nabla p_s(r) + \varepsilon\nabla\varphi] = f(\sigma_s(r) + \varepsilon\chi, p_s(r) + \varepsilon\varphi). \quad (2.18)$$

By (2.10) we have

$$v_s(r)\omega \cdot \nabla p_s(r) = v_s(r)p'_s(r) = f(\sigma_s(r), p_s(r)). \quad (2.19)$$

Subtracting both sides of (2.18) with the left and the right terms in (2.19), respectively, next dividing both sides with  $\varepsilon$ , using the first-order Taylor expansion of the function  $f$  at the point  $(\sigma_s(r), p_s(r))$  and finally letting  $\varepsilon \rightarrow 0$ , we get

$$v_s(r)\omega \cdot \nabla\varphi + \vec{w} \cdot \nabla p_s(r) = f_\sigma(\sigma_s(r), p_s(r))\chi + f_p(\sigma_s(r), p_s(r))\varphi. \quad (2.20)$$

Note that  $\omega \cdot \nabla\varphi = \varphi_r$  and, by virtue of (1.15),

$$\vec{w} \cdot \nabla p_s(r) = -\nabla\psi \cdot p'_s(r)\omega = -p'_s(r)\psi_r.$$

Substituting these expressions into (2.20), we see that (1.14) follows.

Since all the rest equations in (1.12)–(1.18) can be decoupled from (1.15), in what follows we neglect (1.15). This system can be reduced into a 2-system of linear equations in the unknowns  $\varphi$  and  $\eta$  only. To see this we denote by  $\mathcal{J}$ ,  $\mathcal{J}_0$  and  $\mathcal{G}$  respectively the following operators: Given  $\eta \in C^2(\mathbb{S}^{n-1})$ , we let  $u = \mathcal{J}(\eta) \in C^{2*}(\overline{\Omega}_s)$  and  $v = \mathcal{J}_0(\eta) \in C^{2*}(\overline{\Omega}_s)$ , where  $C^{2*}(\overline{\Omega}_s)$  denotes the second-order Zygmund space on  $\overline{\Omega}_s$ , be respectively solutions of the following elliptic boundary value problems:

$$\begin{cases} \Delta u = F'(\sigma_s(r))u, & x \in \Omega_s, \\ u|_{x=R_s\omega} = \eta(\omega), & \omega \in \mathbb{S}^{n-1}; \\ \Delta v = 0, & x \in \Omega_s, \\ v|_{x=R_s\omega} = \eta(\omega), & \omega \in \mathbb{S}^{n-1}. \end{cases}$$

Next, given  $h \in C(\overline{\Omega}_s)$ , we let  $w = \mathcal{G}(h) \in C^{2*}(\overline{\Omega}_s)$  be the solution of the following elliptic boundary value problem:

$$\begin{cases} \Delta w = h, & x \in \Omega_s, \\ w = 0, & x \in \partial\Omega_s. \end{cases}$$

Then from (1.12), (1.13), (1.16) and (1.17) we have

$$\chi = -\sigma'_s(R_s) \mathcal{J}(\eta), \quad \psi = \Phi + \Upsilon + \Psi,$$

where

$$\begin{cases} \Phi = -\mathcal{G}[g_p^*(r)\varphi], \\ \Upsilon = -\mathcal{G}[g_\sigma^*(r)\chi] = \sigma'_s(R_s)\mathcal{G}[g_\sigma^*(r)\mathcal{J}(\eta)], \\ \Psi = -\frac{\gamma}{R_s^2}\mathcal{J}_0(\eta + \frac{1}{n-1}\Delta_\omega\eta). \end{cases}$$

Substituting these expressions into (1.14) and (1.18), we see that the system (1.12)–(1.18) reduces into the following 2-system:

$$\begin{cases} \mathcal{A}_\gamma(\varphi, \eta) = 0, \\ \mathcal{B}_\gamma(\varphi, \eta) = 0, \end{cases} \quad (2.21)$$

where

$$\begin{aligned} \mathcal{A}_\gamma(\varphi, \eta) &= -v_s(r)\partial_r\varphi + f_p^*(r)\varphi + p'_s(r)\partial_r\Phi + p'_s(r)\partial_r\Upsilon + p'_s(r)\partial_r\Psi + f_\sigma^*(r)\chi \\ &= -v_s(r)\partial_r\varphi + f_p^*(r)\varphi - p'_s(r)\partial_r\mathcal{G}[g_p^*(r)\varphi] + \sigma'_s(R_s)p'_s(r)\partial_r\mathcal{G}[g_\sigma^*(r)\mathcal{J}(\eta)] \\ &\quad - \frac{\gamma}{R_s^2}p'_s(r)\partial_r\mathcal{J}_0(\eta + \frac{1}{n-1}\Delta_\omega\eta) - \sigma'_s(R_s)f_\sigma^*(r)\mathcal{J}(\eta), \\ \mathcal{B}_\gamma(\varphi, \eta) &= -\partial_r\Phi|_{r=R_s} - \partial_r\Upsilon|_{r=R_s} - \partial_r\Psi|_{r=R_s} + g(1,1)\eta \\ &= \partial_r\mathcal{G}[g_p^*(r)\varphi]|_{r=R_s} - \sigma'_s(R_s)\partial_r\mathcal{G}[g_\sigma^*(r)\mathcal{J}(\eta)]|_{r=R_s} \\ &\quad + \frac{\gamma}{R_s^2}\partial_r\mathcal{J}_0(\eta + \frac{1}{n-1}\Delta_\omega\eta)|_{r=R_s} + g(1,1)\eta. \end{aligned}$$

Hence, to get nontrivial solutions of the system (1.12)–(1.18) we only need to find nontrivial solutions of the system (2.21). This is the task of the next two sections.

We note that the operator  $\varphi \mapsto \mathcal{A}_\gamma(\varphi, \eta)$  (for fixed  $\eta$ ) is a first-order nonlocal singular differential-integral operator. Since the Dirichlet–Neumann operator  $\eta \mapsto \partial_r\mathcal{G}(\eta)|_{r=R_s}$  is a first-order elliptic pseudo-differential operator in  $\mathbb{S}^{n-1}$  (cf. [13]), and  $\Delta_\omega$  is a second-order elliptic partial differential operator in  $\mathbb{S}^{n-1}$ , we see that the operator  $\eta \mapsto \mathcal{B}_\gamma(\varphi, \eta)$  (for fixed  $\varphi$ ) is a third-order elliptic pseudo-differential operator in the unit sphere  $\mathbb{S}^{n-1}$ . Main difficulty for solving the system (2.21) comes from the singularity and non-localness of the operator  $\mathcal{A}_\gamma$ .

### 3. Expansion via spherical harmonics

Recall that in the polar coordinate  $(r, \omega)$  the Laplacian  $\Delta$  on  $R^n$  has the following expression (cf. [28,29]):

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_\omega. \quad (3.1)$$

Let  $Y_{kl}$ ,  $k = 0, 1, 2, \dots$ ,  $l = 1, 2, \dots, d_k$ , be the basis of spherical harmonics introduced in Section 1. We expand  $\varphi$  and  $\eta$  in (2.21) via  $Y_{kl}$ 's:

$$\varphi(r, \omega) = \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} \varphi_{kl}(r) Y_{kl}(\omega), \quad \eta(\omega) = \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} y_{kl} Y_{kl}(\omega). \quad (3.2)$$



Convergence of the first series is considered in  $\mathcal{D}'(\mathbb{B}(0, R_s)) = \mathcal{D}'((0, R_s), \mathcal{D}'(\mathbb{S}^{n-1}))$ , and the second one is considered in  $\mathcal{D}'(\mathbb{S}^{n-1})$ . A simple computation shows that

$$\begin{cases} \mathcal{A}_\gamma(\varphi, \eta) = \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} [\mathcal{L}_k(\varphi_{kl}) + b_k(r, \gamma)y_{kl}] Y_{kl}(\omega), \\ \mathcal{B}_\gamma(\varphi, \eta) = \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} [J_k(\varphi_{kl}) + \alpha_k(\gamma)y_{kl}] Y_{kl}(\omega), \end{cases} \quad (3.3)$$

where

$$\alpha_k(\gamma) = \left(1 - \frac{\lambda_k}{n-1}\right) \frac{k\gamma}{R_s^3} + g(1, 1) - \frac{\sigma'_s(R_s)}{R_s^{n+2k-1}} \int_0^{R_s} \rho^{n+2k-1} g_\sigma^*(\rho) u_k(\rho) d\rho, \quad (3.4)$$

$$\begin{aligned} b_k(r, \gamma) = & -\left(1 - \frac{\lambda_k}{n-1}\right) \gamma k R_s^{-k-2} r^{k-1} p'_s(r) - \sigma'_s(R_s) R_s^{-k} f_\sigma^*(r) r^k u_k(r) \\ & - \sigma'_s(R_s) R_s^{-k} r^{k-1} p'_s(r) \left[ \theta_k \int_r^{R_s} \rho g_\sigma^*(\rho) u_k(\rho) d\rho - \frac{1 - \theta_k}{r^{n+2(k-1)}} \int_0^r \rho^{n+2k-1} g_\sigma^*(\rho) u_k(\rho) d\rho \right. \\ & \left. - \frac{\theta_k}{R_s^{n+2(k-1)}} \int_0^{R_s} \rho^{n+2k-1} g_\sigma^*(\rho) u_k(\rho) d\rho \right], \end{aligned} \quad (3.5)$$

where  $\theta_k = \frac{k}{n+2(k-1)}$ , and for  $\phi = \phi(r)$ ,

$$\begin{aligned} \mathcal{L}_k(\phi) = & -v_s(r) \phi'(r) + f_p^*(r) \phi(r) + r^{k-1} p'_s(r) \left[ \theta_k \int_r^{R_s} \rho^{-k+1} g_p^*(\rho) \phi(\rho) d\rho \right. \\ & \left. - \frac{1 - \theta_k}{r^{n+2(k-1)}} \int_0^r \rho^{n+k-1} g_p^*(\rho) \phi(\rho) d\rho - \frac{\theta_k}{R_s^{n+2(k-1)}} \int_0^{R_s} \rho^{n+k-1} g_p^*(\rho) \phi(\rho) d\rho \right], \end{aligned} \quad (3.6)$$

$$J_k(\phi) = \frac{1}{R_s^{n+k-1}} \int_0^{R_s} \rho^{n+k-1} g_p^*(\rho) \phi(\rho) d\rho. \quad (3.7)$$

**Lemma 3.1.** *Given  $\gamma \in \mathbb{R}$ , the system (2.21) has a nontrivial solution if and only if there exists a nonnegative integer  $k$  such that the following system has a nontrivial solution:*

$$\begin{cases} \mathcal{L}_k(\phi_k) + b_k(r, \gamma)y_k = 0 & \text{for } 0 < r < R_s, \\ J_k(\phi_k) + \alpha_k(\gamma)y_k = 0. \end{cases} \quad (3.8)$$

**Proof.** Indeed, if  $(\phi_k, y_k)$  is a nontrivial solution of the above system, then from (3.3) we see that for any  $1 \leq l \leq d_k$ ,  $(\varphi(r, \omega), \eta(\omega)) = (\phi_k(r)Y_{kl}(\omega), y_k Y_{kl}(\omega))$  is a nontrivial solution of the system (2.12). Conversely, if  $(\varphi(r, \omega), \eta(\omega))$  is a nontrivial solution of the system (2.12), then by expanding  $\varphi(r, \omega)$  and  $\eta(\omega)$  into the expressions in (3.2), there must be a pair of  $k$  and  $l$  such that  $(\varphi_{kl}, y_{kl}) \neq (0, 0)$ . By (3.3), we see that  $(\phi_k, y_k) = (\varphi_{kl}, y_{kl})$  is a nontrivial solution of (3.8). This proves the lemma.  $\square$

For every  $k \in \mathbb{Z}_+$  we denote by  $\tilde{\mathcal{L}}_k$  the following linear differential-integral operator in  $(0, R_s)$ : for  $\phi = \phi(r)$ ,

$$\begin{aligned}\tilde{\mathcal{L}}_k(\phi) &= \mathcal{L}_k(\phi) + R_s^{-(k-1)} r^{k-1} p'_s(r) J_k(\phi) \\ &= -v_s(r) \phi'(r) + f_p^*(r) \phi(r) + r^{k-1} p'_s(r) \left[ \theta_k \int_r^{R_s} \rho^{-k+1} g_p^*(\rho) \phi(\rho) d\rho \right. \\ &\quad \left. + \frac{1 - \theta_k}{R_s^{n+2(k-1)}} \int_0^{R_s} \rho^{n+k-1} g_p^*(\rho) \phi(\rho) d\rho - \frac{1 - \theta_k}{r^{n+2(k-1)}} \int_0^r \rho^{n+k-1} g_p^*(\rho) \phi(\rho) d\rho \right],\end{aligned}\quad (3.9)$$

and let

$$\begin{aligned}\tilde{b}_k(r) &= b_k(r, \gamma) + R_s^{-(k-1)} r^{k-1} p'_s(r) \alpha_k(\gamma) \\ &= \frac{g(1, 1)}{R_s^{k-1}} r^{k-1} p'_s(r) - \frac{\sigma'_s(R_s)}{R_s^k} r^k f_\sigma^*(r) u_k(r) - \frac{\sigma'_s(R_s)}{R_s^k} r^{k-1} p'_s(r) \left[ \theta_k \int_r^{R_s} \rho g_\sigma^*(\rho) u_k(\rho) d\rho \right. \\ &\quad \left. + \frac{1 - \theta_k}{R_s^{n+2(k-1)}} \int_0^{R_s} \rho^{n+2k-1} g_\sigma^*(\rho) u_k(\rho) d\rho - \frac{1 - \theta_k}{r^{n+2(k-1)}} \int_0^r \rho^{n+2k-1} g_\sigma^*(\rho) u_k(\rho) d\rho \right].\end{aligned}\quad (3.10)$$

**Lemma 3.2.** For fixed  $\gamma \in \mathbb{R}$  and  $k \in \mathbb{Z}_+$ , the system (3.8) has a nontrivial solution  $(\phi_k, y_k)$  if and only if the following system has a solution  $\psi_k$ :

$$\tilde{\mathcal{L}}_k(\psi_k) + \tilde{b}_k(r) = 0, \quad (3.11)$$

$$J_k(\psi_k) + \alpha_k(\gamma) = 0. \quad (3.12)$$

More precisely, if  $\psi_k$  is a solution of the above system then for any nonzero constant  $c$ ,  $(\phi_k, y_k) = (c\psi_k, c)$  is a nontrivial solution of (3.8), and conversely, if  $(\phi_k, y_k)$  is a nontrivial solution of (3.8) then  $y_k \neq 0$  and  $\psi_k(r) = y_k^{-1} \phi_k(r)$  is a solution of the above system.

**Proof.** Later we shall see that the system of equations  $\mathcal{L}_k(\phi) = 0$  and  $J_k(\phi) = 0$  has only the trivial solution  $\phi = 0$  (see the remark following Lemma 4.4). It follows that if  $(\phi_k, y_k)$  is a nontrivial solution of the system (3.8), then  $y_k \neq 0$ . Let  $\psi_k(r) = y_k^{-1} \phi_k(r)$ . Then the system (3.8) reduces into the equation

$$\mathcal{L}_k(\psi_k) + b_k(r, \gamma) = 0 \quad \text{for } 0 < r < R_s \quad (3.13)$$

coupled by the equation (3.12). Multiplying (3.12) with  $R_s^{-(k-1)} r^{k-1} p'_s(r)$  and adding it into (3.13), we get (3.11). Conversely, it is easy to check that if  $\psi_k$  is a solution of the system (3.11)–(3.12) then for any nonzero constant  $c$ ,  $(\phi_k, y_k) = (c\psi_k, c)$  is a nontrivial solution of (3.8). This proves the lemma.  $\square$

We note that for fixed  $\gamma \in \mathbb{R}$  and  $k \in \mathbb{Z}_+$ , (3.11)–(3.12) is an over-determined system. Hence, later on for fixed  $k \in \mathbb{Z}_+$  we shall regard (3.8) as an eigenvalue problem by regarding  $\gamma$  as the eigenvalue variable. In the next section we shall prove that for every  $k \in \mathbb{Z}_+$ , the equation (3.11) has a unique solution  $\psi_k \in C[0, R_s]$ . It follows that the system (3.11)–(3.12) has a solution if and only if  $\gamma$  satisfies the equation (3.12). For each  $k \geq 2$  we let

$$\begin{aligned} \gamma_k = & \frac{(n-1)R_s^3}{(\lambda_k - n + 1)k} \left[ g(1, 1) - \frac{\sigma'_s(R_s)}{R_s^{n+2k-1}} \int_0^{R_s} \xi^{n+2k-1} g_\sigma^*(\xi) u_k(\xi) d\xi \right. \\ & \left. + \frac{1}{R_s^{n+k-1}} \int_0^{R_s} \xi^{n+k-1} g_p^*(\xi) \psi_k(\xi) d\xi \right]. \end{aligned} \quad (3.14)$$

Then

$$J_k(\psi_k) + \alpha_k(\gamma) = -\frac{(\lambda_k - n + 1)k}{(n-1)R_s^3}(\gamma - \gamma_k).$$

Hence, we have the following result:

**Lemma 3.3.** For  $k \geq 2$ , the system (3.8) has a nontrivial solution if and only if  $\gamma = \gamma_k$ .

**Proof.** See Corollary 4.6 in the next section.  $\square$

For  $k = 0, 1$  it is clear that  $\alpha_0, \alpha_1, b_0$  and  $b_1$  are independent of  $\gamma$ , so that the system (3.8) does not contain  $\gamma$  in these cases.

**Lemma 3.4.** For  $k = 1$  we have  $\psi_1(r) = -p'_s(r)$  and  $J_1(\psi_1) + \alpha_1 = 0$ .

**Proof.** Indeed, since  $u_1(r) = \frac{R_s c'_s(r)}{r c'_s(R_s)}$  (see Lemma 4.1 in the next section), by using the equations (2.10), (2.12), (2.14) and the equality  $v_s(r) = \frac{1}{r^{n-1}} \int_0^r \rho^{n-1} g^*(\rho) d\rho$  implied by (2.12), we see that

$$\begin{aligned} \tilde{\mathcal{L}}_1[-p'_s(r)] + \tilde{b}_1(r) &= v_s(r) p''_s(r) - f_p^*(r) p'_s(r) - f_\sigma^*(r) \sigma'_s(r) + g(1, 1) p'_s(r) \\ &\quad - p'_s(r) \left[ \theta_1 \int_r^{R_s} \frac{d}{d\rho} g^*(\rho) d\rho + \frac{1 - \theta_1}{R_s^n} \int_0^{R_s} \rho^n \frac{d}{d\rho} g^*(\rho) d\rho - \frac{1 - \theta_1}{r^n} \int_0^r \rho^n \frac{d}{d\rho} g^*(\rho) d\rho \right] \\ &= v_s(r) p''_s(r) - f_p^*(r) p'_s(r) - f_\sigma^*(r) \sigma'_s(r) + v'_s(r) p'_s(r) \\ &= [v_s(r) p'_s(r) - f^*(r)]' = 0. \end{aligned}$$

Hence  $\psi_1(r) = -p'_s(r)$ . Consequently, we have

$$J_1(\psi_1) + \alpha_1 = -\frac{1}{R_s^n} \int_0^{R_s} \rho^n g_p^*(\rho) p'_s(\rho) d\rho + g(1, 1) - \frac{1}{R_s^n} \int_0^{R_s} \rho^n g_\sigma^*(\rho) \sigma'_s(\rho) d\rho = 0.$$

This proves the lemma.  $\square$

The above lemma implies that in the case  $k = 1$ , the system (3.8) has nontrivial solutions for all  $\gamma \in \mathbb{R}$ . This is actually a restatement of the fact that (1.20) are nontrivial solutions of the system (1.12)–(1.18) for all  $\gamma \in \mathbb{R}$ .

**Lemma 3.5.** For  $k = 0$  the system (3.8) does not have a nontrivial solution.

**Proof.** Since as a stationary solution of the corresponding time-dependent system of (1.1)–(1.8),  $(\sigma_s, p_s, q_s, v_s, \varpi_s, \Omega_s)$  is asymptotically stable under radial perturbations, it follows that in the case  $k = 0$  the system (3.8) cannot have a nontrivial solution. This is an implicit proof. We can also give an explicit proof by repeating some arguments in [2]. To save spaces we omit it here.  $\square$

It remains to prove existence and uniqueness of a solution for (3.11). This is the task of the next section.

#### 4. Existence and uniqueness of the solution of (3.11)

In this section we prove existence and uniqueness of the solution of (3.11). We need the following preliminary lemma:

**Lemma 4.1.** *Let  $u_k(r)$  be the solution of the problem (3.1). We have the following assertions:*

- (1)  $u_k \in C^\infty[0, R_s]$ , and  $0 < u_k(r) \leq 1$  for  $0 \leq r \leq R_s$ .
- (2) There exists a constant  $C > 0$  independent of  $k$  such that

$$1 - \frac{C}{n+2k}(R_s - r) \leq u_k(r) \leq 1 \quad \text{for } 0 \leq r \leq R_s, \quad (4.1)$$

$$0 \leq u'_k(r) \leq \frac{Cr}{n+2k} \quad \text{for } 0 \leq r \leq R_s. \quad (4.2)$$

- (3)  $u_k(r)$  is monotone non-decreasing in  $k$ , i.e.,  $u_k(r) \geq u_l(r)$  for  $0 \leq r \leq R_s$  and  $k > l$ .

$$(4) \quad u_1(r) = \frac{R_s c'_s(r)}{r c'_s(R_s)}.$$

**Proof.** See Lemma 3.3 of [5].  $\square$

In the next lemma we shall use the following notations:

$$\alpha_0 = \frac{f_p^*(0)}{v'_s(0)}, \quad \alpha_1 = -\frac{f_p^*(R_s)}{v'_s(R_s)}.$$

Note that from (1.19) and (2.17) we have  $\alpha_0, \alpha_1 > 0$ .

**Lemma 4.2.** *For any  $h \in C[0, R_s]$ , the equation*

$$-v_s(r)\varphi'(r) + f_p^*(r)\varphi(r) = h(r) \quad \text{for } 0 < r < R_s \quad (4.3)$$

*has a unique solution  $\varphi \in C[0, R_s] \cap C^1(0, R_s)$ , with boundary values*

$$\varphi(0) = \frac{h(0)}{f_p^*(0)} \quad \text{and} \quad \varphi(R_s) = \frac{h(R_s)}{f_p^*(R_s)}. \quad (4.4)$$

*Moreover, there exists a constant  $C > 0$  independent of  $h$  such that*

$$\max_{0 \leq r \leq R_s} |\varphi(r)| \leq C \max_{0 \leq r \leq R_s} |h(r)|. \quad (4.5)$$

*If furthermore  $h(r) = O(r^\mu)$  as  $r \rightarrow 0^+$  for some constants  $\mu > 0$ , then*

$$|\varphi(r)| \leq C m_\mu(r) \quad \text{for } 0 < r < R_s, \quad (4.6)$$

*where*

$$m_\mu(r) = \begin{cases} r^{\alpha_0}, & \text{if } \mu > \alpha_0, \\ r^{\alpha_0} \ln\left(\frac{2R_s}{r}\right), & \text{if } \mu = \alpha_0, \\ r^\mu, & \text{if } \mu < \alpha_0. \end{cases} \quad (4.7)$$

Moreover, if  $h \in C^\infty(0, R_s]$  then also  $\varphi \in C^\infty(0, R_s]$ .

**Proof.** The first two assertions follow from Lemma 4.1 of [5]. Here we only give the proof of the last two assertions. Choose an  $r_0 \in (0, R_s)$  and set

$$W(r) = \exp\left(-\int_{r_0}^r \frac{f_p^*(\rho)}{v_s(\rho)} d\rho\right) \quad \text{for } 0 < r < R_s.$$

It is easy to see that  $W \in C^\infty(0, R_s)$ ,  $W(r) > 0$  for  $0 < r < R_s$ , and

$$W(r) = C_0 r^{-\alpha_0} (1 + o(1)) \quad \text{as } r \rightarrow 0^+, \quad (4.8)$$

$$W(r) = C_1 (R_s - r)^{\alpha_1} (1 + o(1)) \quad \text{as } r \rightarrow R_s^-, \quad (4.9)$$

where  $C_0, C_1$  are positive constants depending on the choice of  $r_0$ . From the proof of Lemma 4.1 of [5] we see that the unique solution of the equation (4.3) in the class  $C[0, R_s] \cap C^1(0, R_s)$  is given by (4.4) and

$$\varphi(r) = \frac{1}{W(r)} \int_r^{R_s} \frac{h(\eta)W(\eta)}{v_s(\eta)} d\eta \quad \text{for } 0 < r < R_s. \quad (4.10)$$

From (2.17), (4.8), (4.9) and the hypothesis that  $h(r) = O(r^\mu)$  as  $r \rightarrow 0^+$  we have

$$\left| \frac{h(r)W(r)}{v_s(r)} \right| \leq C r^{\mu-\alpha_0-1} (R_s - r)^{\alpha_1-1} \quad \text{for } 0 < r < R_s.$$

This implies that

$$\left| \int_r^{R_s} \frac{h(\eta)W(\eta)}{v_s(\eta)} d\eta \right| \leq \begin{cases} C, & \text{if } \mu > \alpha_0, \\ C \ln\left(\frac{2R_s}{r}\right), & \text{if } \mu = \alpha_0, \\ C r^{\mu-\alpha_0}, & \text{if } \mu < \alpha_0, \end{cases} \quad \text{for } 0 < r < R_s.$$

Hence, using (4.8) once again we obtain the estimate (4.6).

Next we assume that  $h \in C^1(0, R_s]$ . Then clearly the unique solution of (4.3) obtained above satisfies  $\varphi \in C^2(0, R_s)$ . To show that  $\varphi(r)$  is continuously differentiable at  $r = R_s$  we differentiate both sides of (4.3) to get

$$-v_s(r)[\varphi'(r)]' + [f_p^*(r) - v_s'(r)]\varphi'(r) = h_1(r) \quad \text{for } 0 < r < R_s,$$

where  $h_1(r) = h'(r) - f_p^*(r)\varphi(r)$ . It follows that

$$\varphi'(r) = \frac{1}{W_1(r)} \left[ c_1 - \int_{r_0}^r \frac{h_1(\eta)W_1(\eta)}{v_s(\eta)} d\eta \right] \quad \text{for } 0 < r < R_s,$$

where  $c_1 = \varphi'(r_0)$  and  $W_1(r) = \exp\left(-\int_{r_0}^r \frac{f_p^*(\rho) - v_s'(\rho)}{v_s(\rho)} d\rho\right)$ . It is easy to see that

$$W_1(r) = C(R_s - r)^{\alpha_1+1}(1 + o(1)) \quad \text{as } r \rightarrow R_s^-$$

for some constant  $C > 0$ . It follows that if  $c_1 \neq \int_{r_0}^{R_s} \frac{h_1(\eta)W_1(\eta)}{v_s(\eta)} d\eta$  then

$$\varphi'(r) = C'(R_s - r)^{-\alpha_1-1}(1 + o(1)) \quad \text{as } r \rightarrow R_s^-$$

for some nonzero constant  $C'$ , which will lead to the absurd conclusion that  $|\varphi(r)| \rightarrow \infty$  as  $r \rightarrow R_s^-$ . Hence

we must have  $c_1 = \int_{r_0}^{R_s} \frac{h_1(\eta)W_1(\eta)}{v_s(\eta)} d\eta$  and, consequently,

$$\lim_{r \rightarrow R_s^-} \varphi'(r) = - \lim_{r \rightarrow R_s^-} \frac{1}{W_1'(r)} \cdot \frac{h_1(r)W_1(r)}{v_s(r)} = \frac{h_1(R_s)}{f_p^*(R_s) - v_s'(R_s)},$$

i.e.,  $\varphi(r)$  is continuously differentiable at  $r = R_s$ . Using an induction method we can finally prove that if  $h \in C^\infty(0, R_s]$  then also  $\varphi \in C^\infty(0, R_s]$ . This completes the proof of Lemma 4.2.  $\square$

For every integer  $k \geq 2$ , we introduce a differential-integral operator  $\tilde{\mathcal{L}}_k^0$  in  $(0, R_s)$  as follows: For  $\varphi \in C(0, R_s] \cap C^1(0, R_s)$ ,

$$\begin{aligned} \tilde{\mathcal{L}}_k^0(\varphi) = & -v_s(r)\varphi'(r) + f_p^*(r)\varphi(r) + r^{k-1}p_s'(r) \left[ \theta_k \int_r^{R_s} \xi^{-k+1} g_p^*(\xi) \varphi(\xi) d\xi \right. \\ & \left. + \frac{1 - \theta_k}{r^{n+2(k-1)}} \int_r^{R_s} \xi^{n+k-1} g_p^*(\xi) \varphi(\xi) d\xi \right] \quad \text{for } 0 < r < R_s. \end{aligned}$$

**Lemma 4.3.** Let  $k \geq 2$ ,  $h \in C(0, R_s]$  and consider the equation

$$\tilde{\mathcal{L}}_k^0(\varphi) = h \quad \text{in } (0, R_s). \quad (4.11)$$

We have the following assertions:

- (1) The above equation has a solution  $\varphi \in C(0, R_s] \cap C^1(0, R_s)$  which is unique in the class  $L_{\text{loc}}^\infty(0, R_s]$ , and  $\varphi(R_s) = \frac{h(R_s)}{f_p^*(R_s)}$ .
- (2) If  $h \in C^\infty(0, R_s]$  then also  $\varphi \in C^\infty(0, R_s]$ .
- (3) If  $h(r) \geq 0$  for  $0 < r \leq R_s$  then  $\varphi(r) \leq 0$  for  $0 < r \leq R_s$ .
- (4) If  $|h(r)| \leq Cr^{-a}$  for  $0 < r \leq R_s$  for some  $a < n + k$ , then  $\int_0^{R_s} \xi^{n+k-1} |\varphi(\xi)| d\xi < \infty$  or more precisely,

$$\int_0^{R_s} \xi^{n+k-1} |\varphi(\xi)| d\xi \leq C \int_0^{R_s} \int_\xi^{R_s} \frac{\xi^{n+k-1} W(\eta) |h(\eta)|}{W(\xi) |v_s(\eta)|} d\eta d\xi < \infty. \quad (4.12)$$

Here  $C$  is a positive constant independent of  $k$ .

**Proof.** The proof uses some similar arguments as in the proof of Lemma 4.4 of [5]; but for completeness we write it below.

The equation (4.11) can be explicitly rewritten as follows:

$$\begin{aligned} -v_s(r)\varphi'(r) + f_p^*(r)\varphi(r) + \theta_k r^{k-1} p'_s(r) \int_r^{R_s} \xi^{-k+1} g_p^*(\xi) \varphi(\xi) d\xi \\ + \frac{(1-\theta_k)p'_s(r)}{r^{n+k-1}} \int_r^{R_s} \xi^{n+k-1} g_p^*(\xi) \varphi(\xi) d\xi = h(r). \end{aligned} \quad (4.13)$$

Let  $W(r)$  be as before. By rewriting the above equation in the form

$$\begin{aligned} \frac{d}{dr} \left( W(r) \varphi(r) \right) = \frac{W(r)}{v_s(r)} \left[ -h(r) + \theta_k r^{k-1} p'_s(r) \int_r^{R_s} \xi^{-k+1} g_p^*(\xi) \varphi(\xi) d\xi \right. \\ \left. + \frac{(1-\theta_k)p'_s(r)}{r^{n+k-1}} \int_r^{R_s} \xi^{n+k-1} g_p^*(\xi) \varphi(\xi) d\xi \right], \end{aligned}$$

we can apply a similar argument as in the proof of Theorem 5.3 (1) of [2] to show that, as far as solutions which are bounded near  $r = R_s$  are concerned, the differential-integral equation (4.13) is equivalent to the following integral equation:

$$\begin{aligned} \varphi(r) = -\frac{1}{W(r)} \int_r^{R_s} \frac{W(\eta)}{v_s(\eta)} \left[ -h(\eta) + \theta_k \eta^{k-1} p'_s(\eta) \int_\eta^{R_s} \xi^{-k+1} g_p^*(\xi) \varphi(\xi) d\xi \right. \\ \left. + \frac{(1-\theta_k)p'_s(\eta)}{\eta^{n+k-1}} \int_\eta^{R_s} \xi^{n+k-1} g_p^*(\xi) \varphi(\xi) d\xi \right] d\eta. \end{aligned} \quad (4.14)$$

It then follows from the standard contraction mapping argument that there exists a sufficiently small  $\delta > 0$  such that (4.13) has a unique bounded solution in the interval  $(R_s - \delta, R_s)$ , such that  $\varphi \in C(R_s - \delta, R_s] \cap C^1(R_s - \delta, R_s)$ , and

$$\varphi(R_s) = \lim_{r \rightarrow R_s^-} \frac{1}{W(r)} \int_r^{R_s} \frac{W(\eta)}{v_s(\eta)} h(\eta) d\eta = \frac{h(R_s)}{f_p^*(R_s)}.$$

Since  $v_s(r) \neq 0$  for  $0 < r < R_s$ , by standard ODE theory we can uniquely extend the solution to the whole interval  $(0, R_s)$ . This proves the assertion (1). The assertion (2) follows from a similar argument as in the proof of Lemma 4.2. The assertion (3) follows from (4.14) and a standard continuity argument; cf. the proof of Lemma 7.1 of [2]. To prove the assertion (4) we note that from (4.14) we have

$$\begin{aligned} |\varphi(r)| &\leq \frac{1}{W(r)} \int_r^{R_s} \frac{W(\eta)}{|v_s(\eta)|} \left[ |h(\eta)| + C \eta^{k-1} p'_s(\eta) \int_\eta^{R_s} \xi^{-k+1} |\varphi(\xi)| d\xi \right. \\ &\quad \left. + \frac{C p'_s(\eta)}{\eta^{n+k-1}} \int_\eta^{R_s} \xi^{n+k-1} |\varphi(\xi)| d\xi \right] d\eta \\ &\leq \frac{1}{W(r)} \int_r^{R_s} \frac{W(\eta)}{|v_s(\eta)|} \left[ |h(\eta)| + \frac{C p'_s(\eta)}{\eta^{n+k-1}} \int_\eta^{R_s} \xi^{n+k-1} |\varphi(\xi)| d\xi \right] d\eta. \end{aligned}$$

It follows that for any  $0 < r < r' \leq R_s$  we have

$$\begin{aligned} \int_r^{r'} \rho^{n+k-1} |\varphi(\rho)| d\rho &\leq \int_r^{r'} \int_\rho^{R_s} \frac{\rho^{n+k-1} W(\eta) |h(\eta)|}{W(\rho) |v_s(\eta)|} d\eta d\rho \\ &\quad + C \int_r^{r'} \int_\rho^{R_s} \int_\eta^{R_s} \frac{\rho^{n+k-1} W(\eta) p'_s(\eta)}{\eta^{n+k-1} W(\rho) |v_s(\eta)|} \xi^{n+k-1} |\varphi(\xi)| d\xi d\eta d\rho \\ &\leq \int_r^{r'} \int_\rho^{R_s} \frac{\rho^{n+k-1} W(\eta) |h(\eta)|}{W(\rho) |v_s(\eta)|} d\eta d\rho \\ &\quad + C \left( \int_r^{r'} \int_\rho^{R_s} \frac{W(\eta) p'_s(\eta)}{W(\rho) |v_s(\eta)|} d\eta d\rho \right) \left( \int_r^{R_s} \xi^{n+2k-1} |\varphi(\xi)| d\xi \right). \end{aligned}$$

By Lemma 5.2 of [2] we have

$$p'_s(r) = c_0 r^\sigma (1 + o(1)) \quad \text{as } r \rightarrow 0^+, \quad (4.15)$$

where  $c_0 > 0$  and  $-1 < \sigma \leq 1$ . Using (4.8), (4.9) and (4.15) we easily see that

$$\int_0^{R_s} \int_\rho^{R_s} \frac{W(\eta) p'_s(\eta)}{W(\rho) |v_s(\eta)|} d\eta d\rho < \infty.$$

Hence there exists a constant  $\delta > 0$  independent of  $k$  such that if  $0 < r' - r \leq \delta$  then

$$C \int_r^{r'} \int_\rho^{R_s} \frac{W(\eta) p'_s(\eta)}{W(\rho) |v_s(\eta)|} d\eta d\rho \leq \frac{1}{2},$$

which implies that

$$\int_r^{r'} \rho^{n+k-1} |\varphi(\rho)| d\rho \leq 2 \int_r^{r'} \int_\rho^{R_s} \frac{\rho^{n+k-1} W(\eta) |h(\eta)|}{W(\rho) |v_s(\eta)|} d\eta d\rho + \int_{r'}^{R_s} \rho^{n+k-1} |\varphi(\rho)| d\rho.$$

Hence, by dividing the interval  $[0, R_s]$  into finite number (independent of  $k$ ) of subintervals and using an iteration argument, we see that there exists a constant  $C > 0$  independent of  $k$  such that

$$\int_r^{R_s} \rho^{n+k-1} |\varphi(\rho)| d\rho \leq C \int_r^{R_s} \int_\rho^{R_s} \frac{\rho^{n+k-1} W(\eta) |h(\eta)|}{W(\rho) |v_s(\eta)|} d\eta d\rho \quad \text{for any } 0 < r < R_s.$$

From (4.8) and (4.9) we have

$$C_1 r^{-\alpha_0} (R_s - r)^{\alpha_1} \leq W(r) \leq C_2 r^{-\alpha_0} (R_s - r)^{\alpha_1} \quad \text{for } 0 < r < R_s, \quad (4.16)$$



where  $0 < C_1 < C_2$ . By this fact it is not hard to prove that if  $|h(r)| \leq Cr^{-a}$  for  $0 < r \leq R_s$  for some  $a < n+k$ , then  $\int_0^{R_s} \int_\rho^{R_s} \frac{\rho^{n+k-1} W(\eta) |h(\eta)|}{W(\rho) |v_s(\eta)|} d\eta d\rho < \infty$ . Hence we have the assertion (4). The proof of Lemma 4.3 is complete.  $\square$

**Lemma 4.4.** *Let  $k \geq 2$ . For any  $h \in C(0, R_s]$  such that  $|h(r)| \leq Cr^{-a}$  for  $0 < r \leq R_s$  for some  $a < n+k$ , the equation*

$$\tilde{\mathcal{L}}_k(\varphi) = h \quad \text{in } (0, R_s) \quad (4.17)$$

*has a solution  $\varphi \in C(0, R_s] \cap C^1(0, R_s)$  such that  $J_k(|\varphi|) < \infty$ , and the solution is unique in the class  $\{\varphi \in L_{\text{loc}}^\infty(0, R_s] : J_k(|\varphi|) < \infty\}$ .*

**Proof.** It is clear that

$$\tilde{\mathcal{L}}_k(\varphi) = \tilde{\mathcal{L}}_k^0(\varphi) - e_k(r)J_k(\varphi),$$

where

$$e_k(r) = \frac{n+k-2}{n+2(k-1)} \frac{(R_s^{n+2(k-1)} - r^{n+2(k-1)})p'_s(r)}{R_s^{k-1}r^{n+k-1}}.$$

Hence, the equation (4.17) is equivalent to the following system of equations for  $\varphi$  and  $\nu$ :

$$\tilde{\mathcal{L}}_k^0(\varphi) = h(r) + \nu e_k(r), \quad (4.18)$$

$$J_k(\varphi) = \nu. \quad (4.19)$$

Let  $\psi_k$  and  $\phi_k$  be respectively solutions of the following equations:

$$\tilde{\mathcal{L}}_k^0(\psi_k) = h(r), \quad (4.20)$$

$$\tilde{\mathcal{L}}_k^0(\phi_k) = e_k(r). \quad (4.21)$$

By Lemma 4.3, these solutions exist, belong to  $C(0, R_s] \cap C^1(0, R_s)$ , satisfy  $J_k(|\psi_k|) < \infty$  and  $J_k(|\phi_k|) < \infty$ , and are unique in the class  $\{\varphi \in L_{\text{loc}}^\infty(0, R_s] : J_k(|\varphi|) < \infty\}$ . Moreover, the assertion (3) of Lemma 4.3 ensures that  $\phi_k(r) < 0$  for  $0 < r < R_s$ . Let  $\varphi = \psi_k + \nu\phi_k$ , where

$$\nu = \frac{J_k(\psi_k)}{1 - J_k(\phi_k)} = \frac{J_k(\psi_k)}{1 + J_k(|\phi_k|)}. \quad (4.22)$$

Then a simple computation shows that  $(\varphi, \nu)$  satisfies the equations (4.18) and (4.19), so that  $\varphi$  is a solution of the equation (4.17). This proves existence. To prove uniqueness we assume that  $\varphi$  is a solution of (4.17) in the class  $\{\varphi \in L_{\text{loc}}^\infty(0, R_s] : J_k(|\varphi|) < \infty\}$  and set  $\nu = J_k(\varphi)$ . Then from (4.17) we see that  $\varphi$  is a solution of the equation (4.18). By uniqueness of the solution of this equation in the class  $\{\varphi \in L_{\text{loc}}^\infty(0, R_s] : J_k(|\varphi|) < \infty\}$ , we conclude that  $\varphi = \psi_k + \nu\phi_k$  and, consequently,  $\nu = J_k(\varphi) = J_k(\psi_k) + \nu J_k(\phi_k)$ , which implies that (4.22) holds. Hence  $\varphi$  coincides with the solution we constructed above. The proof is complete.  $\square$

**Remark.** As a corollary of the above lemma we see that the system of equations  $\mathcal{L}_k(\phi) = 0$  and  $J_k(\phi) = 0$  does not have a nontrivial solution. Indeed, from the first equality in (3.9) we see that any solution of this system is also a solution of the equation  $\tilde{\mathcal{L}}_k(\phi) = 0$ . Hence, by the uniqueness of the solution for this equation ensured by Lemma 4.4, we obtain the desired assertion.

By applying Lemma 4.4 to  $h(r) = -\tilde{b}_k(r)$ , we see that the equation (3.11) has a unique solution in the class  $C(0, R_s) \cap C^1(0, R_s) \cap \{\varphi \in L_{\text{loc}}^\infty(0, R_s) : J_k(|\varphi|) < \infty\}$ . However, apparently, the solution obtained in this approach might be unbounded at  $r = 0$ , or more precisely, we cannot exclude the possibility that the solution obtained above is unbounded at  $r = 0$ . In what follows we use a different approach to reconsider the equation (3.11). This new approach relies on the uniqueness assertion in Lemma 4.4.

We denote by  $B$  the following operator in  $C[0, R_s]$ : For any  $h \in C[0, R_s]$ ,

$$Bh = \text{the right-hand side of (4.10).}$$

By (4.5), this is a bounded linear operator in  $C[0, R_s]$ . Next let  $K$  be the following operator in  $C[0, R_s]$ : For any  $\phi \in C[0, R_s]$ ,

$$K\phi(r) = r^{k-1}p'_s(r) \left[ \theta_k \int_r^{R_s} \rho^{-k+1} g_p^*(\rho) \phi(\rho) d\rho + \frac{1 - \theta_k}{R_s^{n+2(k-1)}} \int_0^{R_s} \rho^{n+k-1} g_p^*(\rho) \phi(\rho) d\rho \right. \\ \left. - \frac{1 - \theta_k}{r^{n+2(k-1)}} \int_0^r \rho^{n+k-1} g_p^*(\rho) \phi(\rho) d\rho \right].$$

Using (4.15) we can easily prove that  $K$  is a bounded linear operator in  $C[0, R_s]$  and is compact. We rewrite the equation (3.11) as follows:

$$-v_s(r)\psi'_k(r) + f_p^*(r)\psi_k(r) + K\psi_k(r) + \tilde{b}_k(r) = 0 \quad \text{for } 0 < r < R_s. \quad (4.23)$$

Clearly, if  $w_k \in C[0, R_s]$  is a solution of the equation

$$w_k(r) + KBw_k(r) + \tilde{b}_k(r) = 0 \quad \text{for } 0 < r < R_s, \quad (4.24)$$

then  $\psi_k = Bw_k$  is a solution of (4.23). Note that  $KB$  is a compact operator in  $C[0, R_s]$  and  $\tilde{b}_k \in C[0, R_s]$ . Now, by uniqueness of the solution of (4.17) in the class  $\{v \in L_{\text{loc}}^\infty(0, R_s) : J_k(|v|) < \infty\}$  we easily see that the equation  $v + KBv = 0$  has only the trivial solution  $v = 0$  in  $C[0, R_s]$ . It follows by a well-known theorem for Fredholm operators that the equation (4.24) has a unique solution  $w_k \in C[0, R_s]$ . Letting  $\psi_k = Bw_k$ , we get a solution of (4.23) in the class  $C[0, R_s]$ . This proves the existence assertion of the following result:

**Theorem 4.5.** *For any  $k \geq 2$ , the equation (3.11) has a unique solution  $\psi_k \in C[0, R_s]$ . Moreover,  $\psi_k \in C^\infty(0, R_s]$ , and there exists  $0 < \mu_k \leq 1$  such that  $\psi_k \in C^{\mu_k}[0, R_s]$ .*

**Proof.** The equation (3.11) can be rewritten as follows:

$$\tilde{\mathcal{L}}_k^0(\psi_k) = -\tilde{b}_k(r) + J_k(\psi_k)e_k(r).$$

Since  $\tilde{b}_k, e_k \in C^\infty(0, R_s]$ , by the assertion (3) of Lemma 4.3 we see that  $\psi_k \in C^\infty(0, R_s]$ . Next, since

$$|K\psi_k(r)| \leq \tilde{b}_k r p'_s(r) \leq \tilde{b}_k r^{1+\sigma} \quad \text{and} \quad |\tilde{b}_k(r)| \leq \tilde{b}_k r^{k-1} p'_s(r) + \tilde{b}_k r^k \leq \tilde{b}_k r^{1+\sigma}$$

for  $0 < r \leq R_s$  (recall that  $-1 < \sigma \leq 1$  and  $k \geq 2$ ), using Lemma 4.2 to the equation (4.23) we see that  $|\psi_k(r)| \leq \tilde{b}_k r^{\mu_k}$  for  $0 < r \leq R_s$  for some constant  $0 < \mu_k \leq 1 + \sigma$ . Again by (4.23), it follows that  $|\psi'_k(r)| \leq \tilde{b}_k r^{\mu_k-1}$  for  $0 < r \leq R_s$ . Using this fact we easily deduce that  $|\psi_k(r) - \psi_k(s)| \leq \tilde{b}_k |r - s|^{\min\{\mu_k, 1\}}$  for  $r, s \in [0, R_s]$ . This completes the proof.  $\square$

**Remark.** A more delicate analysis shows that if we denote by  $m_k(r)$  the function  $m_\mu(r)$  given by (4.7) for  $\mu = k - 1 + \sigma$ , then the solution of (3.8) satisfies  $|\psi_k(r)| \leq C_k m_k(r)$  for  $0 < r \leq R_s$ . To prove this assertion we only need to consider the equation (4.24) in the class

$$\left\{ v \in C[0, R_s] : |v(r)| \leq C m_k(r) \text{ for some } C > 0, \text{ and } \frac{v(r)}{m_k(r)} \in C[0, R_s] \right\}.$$

Then a similar argument as before yields the desired assertion. Since we do not need this result later on, we omit the details of the proof.

**Corollary 4.6.** *Let  $k \geq 2$  and  $\gamma_k$  be defined by (3.14). For  $\gamma = \gamma_k$  the system (3.8) has a nontrivial solution  $(\phi_k, y_k) \in (C[0, R_s] \cap C^1(0, R_s)) \times \mathbb{R}$ , which is unique up to a nonzero factor. Moreover,  $\phi_k \in C^\infty(0, R_s]$ , and there exists  $0 < \mu_k \leq 1$  such that  $\phi_k \in C^{\mu_k}[0, R_s]$ . For  $\gamma \neq \gamma_k$  the system (3.8) does not have a nontrivial solution.*

## 5. Estimates of the nonlinear eigenvalues $\gamma_k$

In this section we study properties of the eigenvalues  $\gamma_k$ ,  $k = 2, 3, \dots$ .

Let  $\psi_k$  be the solution of the equation (3.8) and set

$$v_k(r) = \psi_k(r) - \frac{c'_s(R_s)}{R_s^k} \frac{g_c^*(r)}{g_p^*(r)} r^k u_k(r). \quad (5.1)$$

A simple computation shows that  $v_k$  satisfies the following equation:

$$\tilde{\mathcal{L}}_k(v_k) = d_k(r), \quad (5.2)$$

where

$$d_k(r) = -\frac{g(1, 1)}{R_s^{k-1}} r^{k-1} p'_s(r) + \frac{c'_s(R_s)}{R_s^k} v_s(r) \left( \frac{g_c^*(r)}{g_p^*(r)} r^k u_k(r) \right)' + \frac{c'_s(R_s)}{R_s^k} \frac{f_c^*(r) g_p^*(r) - f_p^*(r) g_c^*(r)}{g_p^*(r)} r^k u_k(r).$$

Since  $\tilde{\mathcal{L}}_k(v_k) = \tilde{\mathcal{L}}_k^0(v_k) - e_k(r) J_k(v_k)$ , by letting  $\tilde{v}_k = J_k(v_k)$ , from (5.2) we get

$$\tilde{\mathcal{L}}_k^0(v_k) = d_k(r) + \tilde{v}_k e_k(r). \quad (5.3)$$

Hence, by letting  $\tilde{\psi}_k$  be the solution of the equation

$$\tilde{\mathcal{L}}_k^0(\tilde{\psi}_k) = d_k(r), \quad (5.4)$$

we have

$$v_k = \tilde{\psi}_k + \tilde{v}_k \phi_k, \quad (5.5)$$

where  $\phi_k$  is as before, i.e.,  $\tilde{\phi}_k$  is the solution of the equation (4.21). Note that by Lemma 4.3, the equation (5.4) has a unique solution  $\tilde{\psi}_k \in C^\infty(0, R_s]$ .

**Lemma 5.1.** *Let  $k \geq 2$ . For  $\tilde{\psi}_k$  defined above we have the following assertions:*

- (1)  $\tilde{\psi}_k(R_s) = -\frac{c'_s(R_s) g_c^*(R_s)}{g_p^*(R_s)} - p'_s(R_s)$ .
- (2)  $J_k(|\tilde{\psi}_k|) \leq C k^{-1}$ , where  $C$  is a constant independent of  $k$ .

**Proof.** By the assertion (2) of Lemma 4.3 we have

$$\tilde{\psi}_k(R_s) = \frac{d_k(R_s)}{f_p^*(R_s)} = -\frac{c'_s(R_s)g_c^*(R_s)}{g_p^*(R_s)} - \frac{g(1,1)p'_s(R_s) - c'_s(R_s)f_c^*(R_s)}{f_p^*(R_s)}.$$

Note that

$$\begin{aligned} & g(1,1)p'_s(R_s) - c'_s(R_s)f_c^*(R_s) \\ &= g(1,1)p'_s(R_s) - \frac{d}{dr}[f(c_s(r), p_s(r))]\Big|_{r=R_s} + f_p^*(R_s)p'_s(R_s) \\ &= g(1,1)p'_s(R_s) - \frac{d}{dr}[v_s(r)p'_s(r)]\Big|_{r=R_s} + f_p^*(R_s)p'_s(R_s) \\ &= g(1,1)p'_s(R_s) - [v'_s(R_s)p'_s(R_s) + v_s(R_s)p''_s(R_s)] + f_p^*(R_s)p'_s(R_s) \\ &= f_p^*(R_s)p'_s(R_s). \end{aligned}$$

Here we have used the fact that  $v_s(R_s) = 0$  and  $v'_s(R_s) = g(1,1)$ . Hence the assertion (1) follows. Next, using (4.15) we easily see that

$$|d_k(r)| \leq Cp'_s(r) + Ck|v_s(r)| + Cr \leq Cr^\sigma + Ck|v_s(r)|.$$

Using (4.12), the above estimate and (4.16), we see that

$$\begin{aligned} J_k(|\tilde{\psi}_k|) &\leq \frac{C}{R_s^{n+k-1}} \int_0^{R_s} \int_\xi^{R_s} \frac{\xi^{n+k-1} W(\eta) |d_k(\eta)|}{W(\xi) |v_s(\eta)|} d\eta d\xi \\ &\leq \frac{C}{R_s^{n+k-1}} \int_0^{R_s} \int_\xi^{R_s} \frac{\xi^{n+k-1+\alpha_0} (R_s - \eta)^{\alpha_1-1}}{\eta^{\alpha_0-\sigma+1} (R_s - \xi)^{\alpha_1}} d\eta d\xi + \frac{Ck}{R_s^{n+k-1}} \int_0^{R_s} \int_\xi^{R_s} \frac{\xi^{n+k-1+\alpha_0} (R_s - \eta)^{\alpha_1}}{\eta^{\alpha_0} (R_s - \xi)^{\alpha_1}} d\eta d\xi \\ &\leq \frac{C}{R_s^{n+k-1}} \int_0^{R_s} \int_\xi^{R_s} \frac{\xi^{n+k+\sigma-2} (R_s - \eta)^{\alpha_1-1}}{(R_s - \xi)^{\alpha_1}} d\eta d\xi + \frac{Ck}{R_s^{n+k-1}} \int_0^{R_s} \int_0^\eta \frac{\xi^{n+k-1+\alpha_0} (R_s - \eta)^{\alpha_1}}{\eta^{\alpha_0} (R_s - \xi)^{\alpha_1}} d\xi d\eta \\ &\leq \frac{C}{R_s^{n+k-1}} \int_0^{R_s} \xi^{n+k+\sigma-3} d\xi + \frac{Ck}{R_s^{n+k-1}} \int_0^{R_s} \int_0^\eta \frac{\xi^{n+k-1+\alpha_0}}{\eta^{\alpha_0}} d\xi d\eta \\ &\leq \frac{C}{k} + \frac{Ck}{(n+k+\alpha_0)(n+k)} \\ &\leq \frac{C}{k} \quad \text{for } k \geq 2. \end{aligned}$$

This completes the proof.  $\square$

**Lemma 5.2.** Let  $k \geq 2$ . For  $\phi_k$ , the solution of (4.21), we have the following assertions:

- (1)  $\phi_k(R_s) = 0$ , and  $\phi_k(r) < 0$  for  $0 < r < R_s$ .
- (2)  $J_k(|\phi_k|) \leq Ck^{-\min\{\alpha_1, \frac{1}{2}\}+\varepsilon}$ , where  $C$  is a positive constant independent of  $k$ , and  $\varepsilon$  represents an arbitrarily small positive number.

**Proof.** The assertion (1) follows from the fact that  $e_k(R_s) = 0$  and  $e_k(r) > 0$  for  $0 < r < R_s$ . Next, by using (4.12), (4.16) and the fact that

$$0 \leq e_k(r) \leq \frac{R_s^{n+k-1} p'_s(r)}{r^{n+k-1}} \leq C R_s^{n+k-1} r^{-n-k+1+\sigma}$$

we have

$$\begin{aligned} J_k(|\phi_k|) &\leq \frac{C}{R_s^{n+k-1}} \int_0^{R_s} \int_{\xi}^{R_s} \frac{\xi^{n+k-1} W(\eta) e_k(\eta)}{W(\xi) |v_s(\eta)|} d\eta d\xi \\ &\leq C \int_0^{R_s} \int_0^{\eta} \frac{\xi^{n+k-1+\alpha_0} (R_s - \eta)^{\alpha_1-1}}{\eta^{n+k+\alpha_0-\sigma} (R_s - \xi)^{\alpha_1}} d\xi d\eta \\ &= C \left( \int_0^{\frac{R_s}{2}} \int_0^{\eta} + \int_{\frac{R_s}{2}}^{R_s} \int_0^{\frac{R_s}{2}} + \int_{\frac{R_s}{2}}^{R_s} \int_{\frac{R_s}{2}}^{\eta} \right) \frac{\xi^{n+k-1+\alpha_0} (R_s - \eta)^{\alpha_1-1}}{\eta^{n+k+\alpha_0-\sigma} (R_s - \xi)^{\alpha_1}} d\xi d\eta \\ &\leq C \int_0^{\frac{R_s}{2}} \int_0^{\eta} \frac{\xi^{n+k-1+\alpha_0}}{\eta^{n+k+\alpha_0-\sigma}} d\xi d\eta + C \left( \frac{2}{R_s} \right)^{n+k+\alpha_0-\sigma} \int_{\frac{R_s}{2}}^{R_s} \int_0^{\frac{R_s}{2}} \xi^{n+k-1+\alpha_0} (R_s - \eta)^{\alpha_1-1} d\xi d\eta \\ &\quad + C \int_{\frac{R_s}{2}}^{R_s} \int_{\xi}^{R_s} \frac{\xi^{n+k-1+\alpha_0} (R_s - \eta)^{\alpha_1-1}}{\eta^{n+k+\alpha_0-\sigma} (R_s - \xi)^{\alpha_1}} d\eta d\xi \\ &= I + II + III. \end{aligned}$$

It is immediate to see that

$$I \leq \frac{C}{k}, \quad II \leq \frac{C}{k} \quad \text{for } k \geq 2.$$

For *III* we let

$$p = \frac{1}{1 - \min\{\alpha, \frac{1}{2}\} + \varepsilon} \quad \text{and} \quad q = \frac{1}{\min\{\alpha, \frac{1}{2}\} - \varepsilon},$$

where  $\varepsilon$  is a sufficiently small positive number. Then by the Hölder inequality we have

$$III \leq \left( \int_{\frac{R_s}{2}}^{R_s} \int_{\xi}^{R_s} \frac{\xi^{nq+kq+\alpha_0q-q}}{\eta^{nq+kq+\alpha_0q-\sigma q}} d\eta d\xi \right)^{\frac{1}{q}} \left( \int_{\frac{R_s}{2}}^{R_s} \int_{\xi}^{R_s} \frac{(R_s - \eta)^{(\alpha-1)p}}{(R_s - \xi)^{\alpha p}} d\eta d\xi \right)^{\frac{1}{p}} \leq C k^{-\frac{1}{q}}.$$

Hence the assertion (2) follows. This completes the proof.  $\square$

**Theorem 5.3.** *Let  $k \geq 2$ . We have the following assertions:*

- (1)  $\gamma_k = \frac{C_n}{k^3} \left[ 1 + O\left(\frac{1}{k}\right) \right]$  as  $k \rightarrow \infty$ , where  $C_n$  is a positive constant independent of  $k$ .
- (2)  $\gamma_k > 0$  and  $\gamma_{k+1} < \gamma_k$  for  $k$  sufficiently large.

**Proof.** From (3.12) and (5.1) we see that

$$\gamma_k = \frac{(n-1)R_s^3}{(\lambda_k - n + 1)k} [g(1, 1) + J_k(v_k)] = \frac{(n-1)R_s^3}{(\lambda_k - n + 1)k} [g(1, 1) + \tilde{\nu}_k].$$

From (5.5) we have

$$\tilde{\nu}_k = J_k(v_k) = J_k(\psi_k) + \tilde{\nu}_k J_k(\phi_k).$$

Hence

$$\tilde{\nu}_k = \frac{J_k(\psi_k)}{1 - J_k(\phi_k)} = \frac{J_k(\psi_k)}{1 + J_k(|\phi_k|)}.$$

By Lemmas 5.1 and 5.2, it follows that

$$|\tilde{\nu}_k| \leq Ck^{-1}.$$

Hence

$$\gamma_k = \frac{(n-1)R_s^3 g(1,1)}{(\lambda_k - n + 1)k} \left[ 1 + O\left(\frac{1}{k}\right) \right] = \frac{C_n}{k^3} \left[ 1 + O\left(\frac{1}{k}\right) \right] \quad \text{as } k \rightarrow \infty,$$

where  $C_n = (n-1)R_s^3 g(1,1)$ . This proves the assertion (1). The assertion (2) is an immediate consequence of the assertion (1).  $\square$

By now, we have finished proving Theorem 1.1. Indeed, that theorem follows from Lemmas 3.1, 3.2, 3.3 and Theorems 4.5 and 5.3.

## 6. Invertibility of some operators

In this section we study invertibility of the linear operator  $(u, \eta) \mapsto (\mathcal{A}_\gamma(u, \eta), \mathcal{B}_\gamma(u, \eta))$  in suitable function spaces, or equivalently, solvability of the system of equations

$$\begin{cases} \mathcal{A}_\gamma(u, \eta) = h(x) & \text{for } x \in \mathbb{B}(0, R_s) \\ \mathcal{B}_\gamma(u, \eta) = \rho(\omega) & \text{for } \omega \in \mathbb{S}^{n-1} \end{cases} \quad (6.1)$$

for given functions  $h$  and  $\rho$  defined in  $\mathbb{B}(0, R_s)$  and  $\mathbb{S}^{n-1}$ , respectively.

In view of the Fourier expansion (3.3) of the operators  $\mathcal{A}_\gamma$  and  $\mathcal{B}_\gamma$ , we see that the above system is equivalent to the following series of systems of equations:

$$\begin{cases} \mathcal{L}_k(u_{kl}) + b_k(r, \gamma)y_{kl} = h_{kl}(r) & \text{for } 0 < r < R_s \\ J_k(u_{kl}) + \alpha_k(\gamma)y_{kl} = z_{kl} \end{cases} \quad (6.2)$$

( $k = 0, 1, 2, \dots$ ,  $l = 1, 2, \dots, d_k$ ), where  $u_{kl} = u_{kl}(r)$ ,  $y_{kl}$ ,  $h_{kl} = h_{kl}(r)$  and  $z_{kl}$  are the Fourier coefficients of the functions  $u = (x)$ ,  $\eta = \eta(\omega)$ ,  $h = h(x)$  and  $\rho = \rho(\omega)$ , respectively, with respect to the basis spherical harmonic functions  $\{Y_{kl}(\omega) : k = 0, 1, \dots, l = 1, 2, \dots, d_k\}$ .

We first consider the case  $\gamma \neq \gamma_k$  for all  $k \geq 2$ . Since for  $k = 1$  the homogeneous version of the system (6.2) has nontrivial solutions, so that for  $k = 1$  the system (6.2) is not generally solvable, in what follows we only consider the cases  $k = 0$  and  $k \geq 2$ . Hence, in what follows we study the following system of equations

$$\begin{cases} \mathcal{L}_k(\varphi) + b_k(r, \gamma)y = \zeta(r) & \text{for } 0 < r < R_s \\ J_k(\varphi) + \alpha_k(\gamma)y = z \end{cases} \quad (6.3)$$

for  $k = 0$  and  $k = 2, 3, \dots$ . Here  $\zeta$  is a given continuous function in  $[0, R_s]$ ,  $z$  is a given real constant, and  $\varphi, y$  are unknown variables. Note that from the expression of  $b_k(r, \gamma)$  (see (3.5)) we see that for  $k \neq 1$ , we have  $b_k(\cdot, \gamma) \in C[0, R_s]$ .

**Lemma 6.1.** Let  $k \in \mathbb{Z}_+$ ,  $k \neq 1$ , and assume that  $\gamma \neq \gamma_j$  for all  $j \geq 2$ . For any  $(\zeta, z) \in C[0, R_s] \times \mathbb{R}$ , the system (6.3) has a unique solution  $(\varphi, y) \in (C[0, R_s] \cap C^1(0, R_s)) \times \mathbb{R}$ . Moreover, there exists a constant  $C > 0$  independent of  $k$  and  $(\zeta, z)$  such that the following estimate holds:

$$\max_{0 \leq r \leq R_s} |\varphi(r)| + \max_{0 \leq r \leq R_s} |r(R_s - r)\varphi'(r)| + (1 + k)^3 |y| \leq C \left[ \max_{0 \leq r \leq R_s} |\zeta(r)| + |z| \right]. \quad (6.4)$$

**Proof.** Let  $L$  be the following unbounded linear operator in  $C[0, R_s]$  with domain  $C_\vee^1[0, R_s] = \{\phi \in C[0, R_s] \cap C^1(0, R_s) : r(R_s - r)\phi'(r) \in C[0, R_s]\}$ :

$$L\phi(r) = -v_s(r)\phi'(r) + f_p^*(r)\phi(r) \quad \text{for } \phi \in C_\vee^1[0, R_s].$$

For each  $k \in \mathbb{Z}_+$  let  $B_k$  be the following bounded linear operator in  $C[0, R_s]$ :

$$B_k\phi(r) = r^{k-1}p'_s(r) \left[ \theta_k \int_r^{R_s} \rho^{-k+1} g_p^*(\rho)\phi(\rho) d\rho - \frac{1 - \theta_k}{r^{n+2(k-1)}} \int_0^r \rho^{n+k-1} g_p^*(\rho)\phi(\rho) d\rho \right. \\ \left. - \frac{\theta_k}{R_s^{n+2(k-1)}} \int_0^{R_s} \rho^{n+k-1} g_p^*(\rho)\phi(\rho) d\rho \right] \quad \text{for } \phi \in C[0, R_s].$$

Then we have  $\mathcal{L}_k = L + B_k$ . By Lemma 4.2, the operator  $L : C_\vee^1[0, R_s] \rightarrow C[0, R_s]$  is invertible, and its inverse  $L^{-1}$  is a bounded linear operator in  $C[0, R_s]$ . Clearly, for  $k \geq 2$ ,  $B_k$  is a compact linear operator in  $C[0, R_s]$ . For  $k = 0$ ,  $B_0$  has the following form:

$$B_0\phi(r) = -rp'_s(r) \cdot \frac{1}{r^n} \int_0^r \rho^{n-1} g_p^*(\rho)\phi(\rho) d\rho \quad \text{for } \phi \in C[0, R_s].$$

From this expression it is clear that  $B_0$  is also a compact linear operator in  $C[0, R_s]$ . Now, by letting  $\tilde{\zeta}(r) = L^{-1}\zeta(r)$  and  $\tilde{b}_k(r, \gamma) = L^{-1}b_k(r, \gamma)$ , we see that the system (6.1) is equivalent to the following one:

$$\begin{cases} \varphi(r) + L^{-1}B_k\varphi(r) + \tilde{b}_k(r, \gamma)y = \tilde{\zeta}(r), & \text{for } 0 < r < R_s \\ J_k(\varphi) + \alpha_k(\gamma)y = z. \end{cases} \quad (6.5)$$

Since  $L^{-1}B_k$  is a compact operator in  $C[0, R_s]$ ,  $J_k$  is a continuous functional in  $C[0, R_s]$ , and  $\tilde{b}_k(\cdot, \gamma) \in C[0, R_s]$ , it follows that the operator

$$(\varphi, y) \mapsto (\varphi + L^{-1}B_k\varphi + \tilde{b}_k(\cdot, \gamma)y, J_k(\varphi) + \alpha_k(\gamma)y)$$

from  $C[0, R_s] \times \mathbb{R}$  to itself is a Fredholm operator of index zero. Hence, solvability of the system (6.5) in  $C[0, R_s] \times \mathbb{R}$  for any given  $(\tilde{\zeta}, z) \in C[0, R_s] \times \mathbb{R}$  is equivalent to uniqueness of the solution of this system. By equivalence of the two systems (6.3) and (6.5), we infer that solvability of the system (6.3) in  $C[0, R_s] \times \mathbb{R}$  for any given  $(\zeta, z) \in C[0, R_s] \times \mathbb{R}$  is equivalent to uniqueness of the solution of this system. Now, since  $\gamma \neq \gamma_j$  for all  $j \geq 2$  and by assumption we have  $k = 0$  or  $k \geq 2$ , by Lemmas 3.3 and 3.5 it follows that the system (6.3) with  $(\zeta, z) = (0, 0)$  does not have a nontrivial solution so that its solution is unique. Hence, the system (6.3) is uniquely solvable for any given  $(\zeta, z) \in C[0, R_s] \times \mathbb{R}$  and, furthermore, there exists a constant  $C_k > 0$  such that the following estimate holds:

$$\max_{0 \leq r \leq R_s} |\varphi(r)| + |y| \leq C_k \left[ \max_{0 \leq r \leq R_s} |\zeta(r)| + |z| \right]. \quad (6.6)$$

In what follows we prove that the constant  $C_k$  can be chosen to be independent of  $k$ .

For  $k \geq 2$ , we make a transformation of unknown variables  $(\varphi, y) \mapsto (\psi, y)$  as follows:

$$\psi(r) = \varphi(r) + R_s^{-(k-1)} r^{k-1} p'_s(r) y. \quad (6.7)$$

Note that since  $k \geq 2$ , we have that  $r^{k-1} p'_s(r) \in C[0, R_s]$ . Multiplying both sides of the second equation in (6.3) with  $R_s^{-(k-1)} r^{k-1} p'_s(r)$  and adding them into the respective sides of the first equation in (6.3), we see that the system (6.3) reduces into the following equivalent one:

$$\begin{cases} \tilde{\mathcal{L}}_k(\psi) + c_k(r)y = \hat{\zeta}(r) & \text{for } 0 < r < R_s \\ J_k(\psi) + \tilde{\alpha}_k(\gamma)y = z, \end{cases} \quad (6.8)$$

where  $\tilde{\mathcal{L}}_k$  is as before, i.e.,  $\tilde{\mathcal{L}}_k(\psi) = \mathcal{L}_k(\psi) + R_s^{-(k-1)} r^{k-1} p'_s(r) J_k(\psi)$  (see (3.14)),

$$\begin{aligned} c_k(r) &= b_k(r, \gamma) + \alpha_k(\gamma) R_s^{-(k-1)} r^{k-1} p'_s(r) - R_s^{-(k-1)} \tilde{\mathcal{L}}_k[r^{k-1} p'_s(r)] \\ &= \frac{r^{k-1}}{R_s^{k-1}} \left\{ [g(1, 1) - g^*(r)] p'_s(r) + \frac{n+k-2}{r} f^*(r) + f_c^*(r) \left[ c'_s(r) - c'_s(R_s) R_s^{-1} r u_k(r) \right] \right. \\ &\quad \left. - p'_s(r) \left[ \theta_k \int_r^{R_s} v_k(\rho) d\rho + \frac{1-\theta_k}{R_s^{n+2(k-1)}} \int_0^{R_s} \rho^{n+2(j-1)} v_k(\rho) d\rho \right. \right. \\ &\quad \left. \left. - \frac{1-\theta_k}{r^{n+2(k-1)}} \int_0^r \rho^{n+2(k-1)} v_k(\rho) d\rho \right] \right\}, \end{aligned} \quad (6.9)$$

where

$$v_k(r) = g_p^*(r) p'_s(r) + c'_s(R_s) R_s^{-1} g_c^*(r) r u_k(r), \quad (6.10)$$

$$\begin{aligned} \tilde{\alpha}_k(\gamma) &= \alpha_k(\gamma) - R_s^{-(k-1)} J_k(r^{k-1} p'_s(r)) \\ &= \left( 1 - \frac{\lambda_k}{n-1} \right) \frac{k\gamma}{R_s^3} + g(1, 1) - \frac{1}{R_s^{n+2(k-1)}} \int_0^{R_s} \rho^{n+2(j-1)} v_k(\rho) d\rho, \end{aligned} \quad (6.11)$$

and

$$\hat{\zeta}(r) = \zeta(r) + R_s^{-(k-1)} r^{k-1} p'_s(r) z. \quad (6.12)$$

Note that  $c_k, \hat{h} \in C[0, R_s]$ . By using Lemma 4.1, it is easy to see that

$$\max_{0 \leq r \leq R_s} |c_k(r)| \leq C(1+k), \quad k = 0, 1, 2, \dots, \quad (6.13)$$

where  $C$  is positive constant independent of  $k$ . Besides, from (6.9) we see that there exists integer  $k_0 = k_0(\gamma) \geq 2$  and constant  $C(\gamma) > 0$  such that for  $k \geq k_0$  we have

$$|\tilde{\alpha}_k(\gamma)| \geq C(\gamma) k^3. \quad (6.14)$$



In particular, this implies that  $\tilde{\alpha}_k(\gamma) \neq 0$  for sufficiently large  $k$ . Using this fact, we deduce from (6.8) the following equation for  $\psi$ :

$$\tilde{\mathcal{L}}_k(\psi) - \frac{c_k(r)}{\tilde{\alpha}_k(\gamma)} J_k(\psi) = \hat{\zeta}(r) - \frac{c_k(r)}{\tilde{\alpha}_k(\gamma)} z.$$

This equation can be rewritten as follows:

$$L\psi(r) + \tilde{B}_k\psi(r) = \hat{\zeta}(r) - \frac{c_k(r)}{\tilde{\alpha}_k(\gamma)} z, \quad (6.15)$$

where  $\tilde{B}_k$  is the following bounded linear operator in  $C[0, R_s]$ :

$$\tilde{B}_k\psi(r) = B_k\psi(r) + \left(\frac{r}{R_s}\right)^{k-1} p'_s(r) J_k(\psi) - \frac{c_k(r)}{\tilde{\alpha}_k(\gamma)} J_k(\psi).$$

It is easy to see that for  $k \geq 2$ ,

$$\max_{0 \leq r \leq R_s} |B_k\phi(r)| + \max_{0 \leq r \leq R_s} |J_k\phi(r)| \leq Ck^{-1} \max_{0 \leq r \leq R_s} |\phi(r)| \quad \text{for } \phi \in C[0, R_s],$$

where  $C$  is a positive constant independent of  $k$ . Moreover, from (6.13) and (6.14) we see that  $|c_k(r)/\tilde{\alpha}_k(\gamma)|$  is bounded by a constant independent of  $k$  and, since  $k \geq 2$ ,  $(r/R_s)^{k-1} p'_s(r) = (r/R_s)^{k-2} R_s^{-1} r p'_s(r)$  is also bounded by a constant independent of  $k$ . Hence, for sufficiently large  $k$  we have

$$\max_{0 \leq r \leq R_s} |\tilde{B}_k\phi(r)| \leq Ck^{-1} \max_{0 \leq r \leq R_s} |\phi(r)| \quad \text{for } \phi \in C[0, R_s].$$

Using this estimate and the boundedness of  $L^{-1}$  in  $C[0, R_s]$  we easily deduce from (6.15) that for sufficiently large  $k$ ,

$$\max_{0 \leq r \leq R_s} |\psi(r)| \leq C \left[ \max_{0 \leq r \leq R_s} |\zeta(r)| + |z| \right], \quad (6.16)$$

where  $C$  is a positive constant independent of  $k$ . Now, since  $y = [z - J_k(\psi)]/\tilde{\alpha}_k(\gamma)$  (by the second equation in (6.8)), from (6.7), (6.14) and (6.17) we see that there exists a constant  $C > 0$  such that

$$\max_{0 \leq r \leq R_s} |\varphi(r)| + |y| \leq C \left[ \max_{0 \leq r \leq R_s} |\zeta(r)| + |z| \right], \quad (6.17)$$

for sufficiently large  $k$ . Since (6.6) ensures that this estimate also holds for  $k$  in any finite interval and  $k \neq 1$ , we see that (6.17) holds for all  $k \geq 0$  and  $k \neq 1$ .

We now prove (6.4). Indeed, from (3.4) we see that a similar estimate as (6.14) also holds for  $\alpha_k(\gamma)$ . It follows from the second equation in (6.3) and (6.17) that

$$(1+k)^3 |y| \leq C \left[ \max_{0 \leq r \leq R_s} |\zeta(r)| + |z| \right] \quad (6.18)$$

for  $k \neq 1$ . By (3.5) we see that  $|b_k(r, \gamma)|$  is bounded by  $C(\gamma)(1+k)^3$ . Hence from the first equation in (6.3) and (6.17), (6.18) we get

$$\max_{0 \leq r \leq R_s} |r(R_s - r)\varphi'(r)| \leq C \left[ \max_{0 \leq r \leq R_s} |\zeta(r)| + |z| \right]. \quad (6.19)$$

Combining (6.17), (6.18) and (6.19) together, we see that (6.4) follows. This completes the proof of Lemma 6.1.  $\square$

For any  $1 \leq \alpha < \infty$ , we denote by  $X_\alpha$  the space of all measurable functions  $u(x)$  in the ball  $\mathbb{B}(0, R_s) \subseteq \mathbb{R}^n$  satisfying the following conditions:

$$u(x) = \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} u_{kl}(r) Y_{kl}(\omega) \text{ in } C([0, R_s], \mathcal{D}'(\mathbb{S}^{n-1})), \quad (6.20)$$

$$\|u\|_{X_\alpha} = \left[ \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} \left( \max_{0 \leq r \leq R_s} |u_{kl}(r)| \right)^\alpha \right]^{\frac{1}{\alpha}} < \infty.$$

The notations  $X_\infty$  denote the space defined by modifying the above definition in conventional sense. It is clear that for any  $1 \leq \alpha \leq \infty$ ,  $X_\alpha$  is a Banach space. We also introduce the Banach space

$$X_\alpha^1 = \{u \in X_\alpha : r(R_s - r) \partial_r u \in X_\alpha\},$$

with norm  $\|u\|_{X_\alpha^1} = \|u\|_{X_\alpha} + \|r(R_s - r) \partial_r u\|_{X_\alpha}$ . Note that for  $u$  given by (6.18) we have

$$\|u\|_{X_\alpha^1} \approx \left[ \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} \left( \max_{0 \leq r \leq R_s} |u_{kl}(r)| + \max_{0 \leq r \leq R_s} r(R_s - r) |u'_{kl}(r)| \right)^\alpha \right]^{\frac{1}{\alpha}}.$$

Next, for any  $1 \leq \alpha < \infty$ , we denote by  $Y_\alpha$  the space of all measurable functions  $\varphi(\omega)$  on the sphere  $\mathbb{S}^{n-1}$  satisfying the following conditions:

$$\varphi(\omega) = \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} a_{kl} Y_{kl}(\omega) \text{ in } \mathcal{D}'(\mathbb{S}^{n-1}), \quad \|\varphi\|_{Y_\alpha} = \left( \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} |a_{kl}|^\alpha \right)^{\frac{1}{\alpha}} < \infty. \quad (6.21)$$

The notation  $Y_\infty$  denotes the space by replacing the summation over  $k, l$  with supremum. It is clear that for any  $1 \leq \alpha \leq \infty$ ,  $Y_\alpha$  is a Banach space. We also denote by  $Y_\alpha^3$  the Banach space made by functions  $\varphi(\omega)$  on the sphere  $\mathbb{S}^{n-1}$  with the expansion (6.19) satisfying the following condition:

$$\|\varphi\|_{Y_\alpha^3} = \begin{cases} \left\{ \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} [(1+k)^3 |a_{kl}|]^\alpha \right\}^{\frac{1}{\alpha}} < \infty & \text{if } 1 \leq \alpha < \infty, \\ \sup_{k,l} (1+k)^3 |a_{kl}| < \infty & \text{if } \alpha = \infty. \end{cases}$$

It is clear that  $Y_\alpha^3$  ( $1 \leq \alpha \leq \infty$ ) are also Banach spaces.

Moreover, for every  $k \in \mathbb{Z}_+$  we denote by  $X_{\alpha,k}$  and  $Y_{\alpha,k}$  the following closed subspaces of  $X_\alpha$  and  $Y_\alpha$ , respectively:

$$X_{\alpha,k} = \{u \in X_\alpha : \text{the coefficients } u_{kl}(r) \text{ } (l = 1, 2, \dots, d_k) \text{ in (6.20) are identically zero}\},$$

$$Y_{\alpha,k} = \{\varphi \in Y_\alpha : \text{the coefficients } a_{kl} \text{ } (l = 1, 2, \dots, d_k) \text{ in (6.21) are identically zero}\},$$

and denote by  $X_{\alpha,k}^1$  and  $Y_{\alpha,k}^3$  similar closed subspaces of  $X_\alpha^1$  and  $Y_\alpha^3$ , respectively.

It is easy to see that the linear operator  $(u, \eta) \mapsto (\mathcal{A}_\gamma(u, \eta), \mathcal{B}_\gamma(u, \eta))$  maps  $X_\alpha^1 \times Y_\alpha^3$  into  $X_\alpha \times Y_\alpha$  boundedly, and when restricted to  $X_{\alpha,1}^1 \times Y_{\alpha,1}^3$ , it maps this space into  $X_{\alpha,1} \times Y_{\alpha,1}$  boundedly. From Lemma 6.1 we immediately get:

**Theorem 6.2.** Assume that  $\gamma \neq \gamma_k$  for all  $k \geq 2$  and let  $1 \leq \alpha \leq \infty$  be given. For any  $(h, \rho) \in X_{\alpha,1} \times Y_{\alpha,1}$ , the system (6.1) has a unique solution  $(u, \eta) \in X_\alpha^1 \times Y_\alpha^3$ . Moreover, there exists a constant  $C > 0$  depending on  $\gamma$  such that the following estimate holds:

$$\|u\|_{X_\alpha^1} + \|\eta\|_{Y_\alpha^3} \leq C[\|h\|_{X_\alpha} + \|\rho\|_{Y_\alpha}].$$

Using a similar argument, we can also prove the following result:

**Theorem 6.3.** *Assume that  $\gamma = \gamma_k$  for some  $k \geq 2$  and let  $1 \leq \alpha \leq \infty$  be given. Let*

$$\tilde{X}_{\alpha,k} \times \tilde{Y}_{\alpha,k} = \bigcap_{\gamma_j = \gamma_k} X_{\alpha,j} \times Y_{\alpha,j}, \quad \tilde{X}_{\alpha,k}^1 \times \tilde{Y}_{\alpha,k}^3 = \bigcap_{\gamma_j = \gamma_k} X_{\alpha,j}^1 \times Y_{\alpha,j}^3.$$

*For any  $(h, \rho) \in (\tilde{X}_{\alpha,k} \times \tilde{Y}_{\alpha,k}) \cap (X_{\alpha,1} \times Y_{\alpha,1})$ , the system (6.1) has a unique solution  $(u, \eta) \in (\tilde{X}_{\alpha,k}^1 \times \tilde{Y}_{\alpha,k}^3) \cap (X_\alpha^1 \times Y_\alpha^3)$ . Moreover, there exists a constant  $C_k > 0$  such that the following estimate holds:*

$$\|u\|_{X_\alpha^1} + \|\eta\|_{Y_\alpha^3} \leq C_k[\|h\|_{X_\alpha} + \|\rho\|_{Y_\alpha}].$$

We omit the proof of this result.

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