



Stochastic generalized magnetohydrodynamics equations with not regular multiplicative noise: Well-posedness and invariant measure

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ABSTRACT

In this paper we study a stochastic magnetohydrodynamics (MHD) system with fractional diffusion and resistivity $(-\Delta)^\alpha$, $\alpha > 0$, in \mathbb{R}^d , $d = 2, 3$. Our main goal is to identify the conditions on α under which we can prove the existence of a martingale solution, the pathwise uniqueness of solution and the existence of invariant measure when the noises are multiplicative and take values in functional space bigger than the space of square integrable functions. Roughly speaking, we prove that if $\alpha \geq 1$, $\theta \in (0, \alpha)$ and the driving noises take values in $\mathcal{H}^{-\theta}$, then the stochastic system has at least a weak martingale solution. We also establish the pathwise uniqueness of solution whenever $\alpha \geq \frac{d}{2}$. Finally, under the latter condition and under the addition of linear damping to the equations we are able to establish the existence of an invariant measure.

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1. Introduction

Since the pioneering work of Bensoussan and Temam [3] the stochastic Navier–Stokes equations have been the subject of intensive mathematical studies. The analysis of the existence of solution as well as the qualitative properties and long time behavior of their solutions has generated several important results, see, for instance, [9], [7], [8], [18], [6,15,17,19,20,25], [34] to cite a few papers. In recent years prominent mathematicians have made considerable effort in extending the known results related to stochastic Navier–Stokes equations and the stochastic magnetohydrodynamics (MHD) equations which are basically the coupling of the Navier–Stokes equations and the Maxwell’s equations. However, in contrast to the stochastic Navier–Stokes equations there are still only few papers related to the mathematical theory of the stochastic MHD equations. Among these few papers one can cite the articles [2], [12], [30], [32], [33], [35], [36] which give various results related to the existence and uniqueness of solution and ergodicity of the classical MHD driven

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by Gaussian and Lévy noise. Closely related models, such as stochastic MHD-Hall and magneto-micropolar equations, in periodic domain were treated in [45], [44] and [43].

In this paper we study the stochastic modified magnetohydrodynamics (MHD) equations in \mathbb{R}^d , $d = 2, 3$, driven by multiplicative Gaussian noise. More precisely, we consider the following stochastic system

$$du + [(-\Delta)^\alpha u + u \cdot \nabla u - m \cdot \nabla m + \gamma_0 u + \nabla p]dt = G(u)dW_1, \quad (1.1a)$$

$$dm + [(-\Delta)^\alpha m + u \cdot \nabla m - m \cdot \nabla u + \gamma_1 m]dt = F(m)dW_2, \quad (1.1b)$$

$$\operatorname{div} u = \operatorname{div} m = 0, \quad (1.1c)$$

$$u(0) = u_0 \text{ and } m(0) = m_0, \quad (1.1d)$$

where $\alpha > 0$, $\gamma_0, \gamma_1 \geq 0$ are given constants, u_0 and m_0 are given initial conditions, F and G are given nonlinear maps satisfying several technical assumptions that will be fixed later on. The unknowns are the random vector fields u , m and the random scalar field p which represent the velocity of the fluid, the magnetic field and the fluid pressure, respectively. The stochastic processes W_1 and W_2 are basically independent cylindrical Wiener processes evolving on separable Hilbert spaces K_1 and K_2 , respectively. In many parts of the paper, W_1 and W_2 along with a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ will be part of the solution. Loosely speaking, if we denote by \mathcal{H} the subspace of $\mathbb{L}^2 \times \mathbb{L}^2$ consisting of divergence free functions, by $P_L : \mathbb{L}^2 \times \mathbb{L}^2 \rightarrow \mathcal{H}$ the Leray–Helmoltz projection and $A = -P_L(\Delta)$ the Stokes operator (see Section 2 for the rigorous definition of these objects), then the above system can be written as an abstract stochastic evolution equation in \mathcal{H} and taking the form

$$d\mathbf{u} + [\mathbf{A}_\alpha \mathbf{u} + \mathbf{B}(\mathbf{u}) + \mathbf{R}_\gamma \mathbf{u}]dt = \mathbf{G}(\mathbf{u})dW, \quad \mathbf{u}(0) = \mathbf{u}_0 \quad (1.2)$$

where $\mathbf{u} = (u, m)$, $\mathbf{u}_0 = (u_0, m_0)$ and $\mathbf{A}_\alpha = \operatorname{diag}(A^\alpha, A^\alpha)$, $\mathbf{G}(\mathbf{u}) = \operatorname{diag}(G(u), F(m))$, $\mathbf{R}_\gamma = \operatorname{diag}(\gamma_0, \gamma_1)$, $W = (W_1, W_2)$ and $\mathbf{B}(\mathbf{u})$ represents the nonlinear terms in (1.1). In order to avoid too much repetition we refer to Section 2 for the precise definitions of these notations. Our modest goal in this paper is to identify the conditions on α and the regularity of the noises $F(m)dW_2$ and $G(u)dW_1$ under which we can prove the existence of a martingale solution, pathwise uniqueness of the solution and the existence of an invariant measure. In order to describe our results let us denote by $C([0, T], \mathcal{H}_w)$ the space of functions $\mathbf{u} : [0, T] \rightarrow \mathcal{H}_w$ which are weakly continuous, by $\gamma(X, Y)$ the space of γ -radonifying operators from a Hilbert spaces X into a Banach space Y . We also denote by $\mathcal{H}^{-\theta, 4}$ the subspace of the dual of the Sobolev space $\mathcal{H}^{\theta, \frac{3}{4}} \subset \mathbb{H}^{\theta, \frac{3}{4}} \times \mathbb{H}^{\theta, \frac{3}{4}}$ consisting of divergence free functions $\mathbf{u} = (u, m)$. Roughly speaking, the main results in this paper can be summarized in the following theorem.

Theorem 1.1.

- (1) If $\alpha \geq 1$, $\gamma_0, \gamma_1 \geq 0$, $\mathbf{u}_0 \in \mathcal{H}$ and the nonlinear map $\mathbf{G} : \mathcal{H} \rightarrow \mathcal{L}_2(K, \mathcal{H}^{-\theta, 4})$, with $\theta \in (0, \alpha)$, is bounded and satisfies several continuity conditions, then (1.1) has at least a weak martingale solution with $\mathbf{u} \in C([0, T], \mathcal{H}_w)$. Furthermore, if $\alpha \geq \frac{d}{2}$ then $\mathbf{u} \in C([0, T]; \mathcal{H})$, i.e., $\mathbf{u} : [0, T] \rightarrow \mathcal{H}$ is strongly continuous.
- (2) Let the conditions of the second item hold and $\alpha \geq \frac{d}{2}$. If in addition \mathbf{G} is globally Lipschitz, then any two solutions \mathbf{u}_1 and \mathbf{u}_2 of (1.1) defined on the same filtered probability space and with the same noise $W = (W_1, W_2)$ and initial condition $\mathbf{u}_0 \in \mathcal{H}$ satisfy

$$\mathbb{P}(\mathbf{u}_1(t) = \mathbf{u}_2(t), t \in [0, T]) = 1.$$

- (3) Finally, under the assumption of the third item the stochastic MHD equations (1.1) has at least an invariant measure provided that $\gamma_0, \gamma_1 > 0$.

The precise assumptions and statement of items (1), (2) and (3) can be found in Theorems 3.1, 3.2 and 4.1, respectively. The proof of the existence of solutions in items (1) and (3) follow the following steps: we first construct an approximated auxiliary problem by using a Yosida approximation to smoothen the noises. This auxiliary problem has at least a weak martingale thanks to the result in [1]. We then derive several uniform estimates for the approximated solution. These uniform estimates will be used to prove the tightness of the laws family of the approximated solutions. After using Prokhorov's and Jakubowski–Skorokhod's theorem [23] we finally pass to the limit in the approximated problem. For these steps we closely follow Brzeźniak and Ferrario's paper [7]. The uniqueness stated in item (2) follows from a careful estimate of the nonlinear term, the application of Itô formula and a trick due to Schmalfuß [34]. The standard tool to prove the existence of an invariant measure of a Markov semigroup is the Bogolyubov–Krylov theorem which relies on some compactness assumption of the phase space or in the case of Markov solution of SPDEs the compactness of various spaces where the solution lives. Since we are working in \mathbb{R}^d we have a lack of compactness between the spaces where our solution lives, and hence we cannot rely on the method based on Bogolyubov–Krylov's theorem. To overcome this issue and to prove item (3), we will use the new approach initially elaborated by Maslowski and Seidler in [27] and further developed in [8], [11] and [10] amongst others. This new technique is a modified version of Bogolyubov–Krylov's theorem and relies on the sequentially weakly Feller property of the semigroup and a boundedness type in probability of the solution of the stochastic problem. To implement this method to our framework we closely follow [8].

Before we proceed to the presentation of the layout of the current paper, we should note that the deterministic counterpart, *i.e.*, when $F \equiv 0$ and $G \equiv 0$, of (1.1) has been intensively studied and there is a vast amount of obtained results, see, for instance, [16], [22], [39], [40], [41], [46] and the references therein. For the stochastic case we refer for instance to [22] for results related to the well-posedness and the random attractor of (1.1) in a bounded domain of \mathbb{R}^2 . Closely related work is the paper by Ali [1] where the existence of martingale solutions to stochastic generalized MHD (SGMHD) system has been investigated. It should be noted that the author of [1] studied SGMHD systems driven by cylindrical Wiener noise which is more regular than the noise considered in the present paper. We also note that our results do not follow from [2], [32], [33], [35], [36] and the general frameworks of [12] and [30]. In fact, these papers study stochastic version of classical MHD or hydrodynamical systems which correspond to $\alpha = 1$ in our paper and almost all of them treat the case of noise taking values in \mathcal{H} , see for instance [2], [12], [30], [32], [35] and [36]. Last but not the least, the only paper which gives result about invariant measure of stochastic MHD on a bounded domain $\mathcal{O} \subset \mathbb{R}^2$ is [2], but our result does not follow from the result in [2]. The papers [9], [7], [8] and [18] treat the stochastic Navier–Stokes equations and our results are not included in theirs. Moreover, the techniques for the derivation of various estimates in our paper is quite different to theirs as we have to take into consideration the role of the parameter α and the regularity of the noise on the behavior of sample paths of the solution to (1.1).

Let us now close this introduction with the layout of the paper. In Section 2 we introduce several notations which enable us to reformulate the problem (1.1) into the abstract stochastic evolution equations (1.2). In Section 2.2, we give the precise assumptions, the statement and the proof of the result outlined in Theorem 1.1. Section 3 is devoted to the statements and the proofs of the existence and uniqueness results sketched in items (1) and (2) of Theorem 1.1. Section 4 is devoted to the proof of the existence of invariant measure for (1.1). The paper also contains an appendix recalling several tightness criteria frequently used in the paper.

2. Preliminaries, notations and MHD with smooth noise

2.1. Preliminaries and notations

This section is mainly borrowed from Ali's paper [1] which in turn closely follows [7]. For $p \in [1, \infty]$ we denote by $\mathbb{L}^p := [L^p(\mathbb{R}^d)]^d$ the Lebesgue spaces of \mathbb{R}^d -valued functions which are p -integrable. For $p \in [1, \infty)$ the norm of \mathbb{L}^p is defined by

$$\|u\|_{\mathbb{L}^p} = \left(\sum_{i=1}^d \|u^i\|_{L^p(\mathbb{R}^d)}^p \right)^{\frac{1}{p}},$$

and for $p = \infty$

$$\|u\|_{\mathbb{L}^\infty} = \sum_{i=1}^d \|u^i\|_{L^\infty(\mathbb{R}^d)},$$

where $u = (u^1, \dots, u^d)$. We set $J^s = (I - \Delta)^{s/2}$ and for $s \in \mathbb{R}$ and $1 \leq p \leq \infty$ we define the following generalized Sobolev spaces

$$\mathbb{H}^{s,p} = \{u \in [\mathcal{S}'(\mathbb{R}^d)]^d : \|J^s u\|_{\mathbb{L}^p} < \infty\},$$

where $\mathcal{S}'(\mathbb{R}^d)$ is the dual of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. It is known, see, for instance, [4], that J^σ is an isomorphism between $\mathbb{H}^{s,p}$ and $\mathbb{H}^{s-\sigma,p}$, the embedding $\mathbb{H}^{s_2,p} \subset \mathbb{H}^{s_1,p}$, $s_1 < s_2$, is continuous, and the dual of $\mathbb{H}^{s,p}$ is $\mathbb{H}^{-s,q}$ with $1 < q \leq \infty$ satisfying $1/p + 1/q = 1$. The duality pairing between $\mathbb{H}^{s,p}$ and its dual $\mathbb{H}^{-s,q}$ is defined by

$$\langle u, v \rangle = \sum_{i=1}^d \int_{\mathbb{R}^d} (J^s u^i)(x) (J^{-s} v^i)(x) dx.$$

We also use $\langle \cdot, \cdot \rangle$ to denote the duality pairing between any Sobolev space, and in general any Banach space, X and its dual X' .

For $s \in \mathbb{R}$ and $p \in [1, \infty]$ we also set

$$\mathbb{H}_{\text{sol}}^{s,p} = \{u \in \mathbb{H}^{s,p} : \nabla \cdot u = 0\}.$$

In the above formula the divergence is understood in the weak sense. For $p = 2$ we simply write $\mathbb{H} = \mathbb{H}_{\text{sol}}^{0,2}$ and $\mathbb{H}_{\text{sol}}^s = \mathbb{H}_{\text{sol}}^{s,2}$ for $s \neq 0$.

For $s \in \mathbb{R}$ we set $\Lambda^s = (-\Delta)^{s/2}$. We note that for $s \geq 0$ the norm $\|J^s \cdot\|_{\mathbb{L}^p}$ is equivalent to the norm $\|\cdot\|_{\mathbb{L}^p} + \|\Lambda^s \cdot\|_{\mathbb{L}^p}$, i.e., there exist two constants $c_0, c_1 > 0$ such that

$$c_0 \|J^s \cdot\|_{\mathbb{L}^p} \leq \|\cdot\|_{\mathbb{L}^p} + \|\Lambda^s \cdot\|_{\mathbb{L}^p} \leq c_1 \|J^s \cdot\|_{\mathbb{L}^p}. \quad (2.3)$$

For $s < 0$ it is not difficult to see that there exists a positive constant $C > 0$ such that

$$\|J^s \cdot\|_{\mathbb{L}^p} \leq c_2 \|\Lambda^s \cdot\|_{\mathbb{L}^p}. \quad (2.4)$$

For two Banach spaces X and Y the product space $X \times Y$ is again a Banach space when equipped with the product space norm

$$\|u\|_{X \times Y} := (\|u_1\|_X^2 + \|u_2\|_Y^2)^{\frac{1}{2}}, \quad u = (u_1, u_2) \in X \times Y.$$

For the sake of simplicity we set

$$\mathcal{H} = \mathbb{H} \times \mathbb{H},$$

$$\mathcal{H}^{\theta,p} = \mathbb{H}_{\text{sol}}^{\theta,p} \times \mathbb{H}_{\text{sol}}^{\theta,p}, \quad \theta \in \mathbb{R}, p \in (1, \infty].$$

We equip these spaces with their respective product space norms. We denote by $\|\cdot\|_{\mathcal{H}^\theta}$ and $\|\cdot\|_{\mathcal{H}^{\theta,p}}$ the norms of \mathcal{H}^θ and $\mathcal{H}^{\theta,p}$, $\theta \in \mathbb{R}$, $p \in [1, \infty]$, respectively. To simplify the notation we denote by $\|\cdot\|$ the norm of \mathcal{H} . The norm $\|\cdot\|_{\mathcal{H}^{\theta,p}}$ is equivalent to the norm $\|\mathbf{J}^\theta \cdot\|_{\mathbb{L}^p \times \mathbb{L}^p}$ where

$$\mathbf{J}^\theta \mathbf{u} = \begin{pmatrix} J^\theta & 0 \\ 0 & J^\theta \end{pmatrix} \mathbf{u}, \quad \theta \in \mathbb{R}. \quad (2.5)$$

Observe also that the spaces $\mathcal{H}^{s,p}$ satisfy the same properties satisfied by the Sobolev spaces $\mathbb{H}^{s,p}$, $s \in \mathbb{R}$, $p \in [1, \infty]$.

Now, from [21, Lemma 2.5], see also [9, Lemma C.1], we infer that there exists a separable Hilbert space \mathbb{U} such that for $s > \frac{d}{2} + \alpha + 1$ we have

$$\mathbb{U} \hookrightarrow \mathcal{H}^s \hookrightarrow \mathcal{H}^\theta \hookrightarrow \mathcal{H}^\alpha \hookrightarrow \mathcal{H} \simeq \mathcal{H}' \hookrightarrow \mathcal{H}^{-\alpha} \hookrightarrow \mathbb{U}'$$

where the notation \hookrightarrow (resp. $\hookrightarrow\hookrightarrow$) stands for the continuous (resp. compact) embedding. In particular, the embedding $\mathbb{U} \subset \mathcal{H}$ is compact and dense. These observations are very important for our analysis.

Given two Banach spaces K and H , we denote by $\mathcal{L}(K, H)$ the space of bounded linear operators. For two Hilbert spaces K and H we denote by $\mathcal{L}_2(K, H)$ the Hilbert space of all Hilbert–Schmidt operators from K to H , [14]. For $K = H$ we just write $\mathcal{L}(K)$ instead of $\mathcal{L}(K, K)$.

In (1.1a), it is convenient to eliminate the pressure p by applying the Helmholtz–Leray orthogonal projector P_L which projects into divergence free vectors and annihilates gradients. The projector P_L is formally defined by $P_L = (I - \partial_j \partial_k \Delta^{-1})$ which is a pseudo-differential operator with symbol $(\delta_{jk} - \xi_j \xi_k / |\xi|^2)$. One of the remarkable properties of P_L is that as a consequence of Mihlin’s theorem, see for e.g., [28] $P_L \in \mathcal{L}(\mathbb{H}^s, \mathbb{H}^s)$, $s > 0$. We will frequently use this property without further notice. Hereafter, we denote by $A := -P_L \Delta$ the celebrated Stokes operator. Note that $Ay = -\Delta y$, hence $A^\alpha y = (-\Delta)^\alpha y$, for any divergence free function y . Hence, if there is no risk of confusion we will just use the abusive notation $A = -\Delta$.

Now, we define a trilinear form $b(\cdot, \cdot, \cdot)$ by

$$b(u, v, w) = \sum_{i,j=1}^d \int_{\mathbb{R}^d} u^{(i)} \frac{\partial v^{(j)}}{\partial x_i} w^{(j)} dx, \quad u \in \mathbb{L}^p, v \in \mathbb{H}^{1,q}, \text{ and } w \in \mathbb{L}^r,$$

with numbers $p, q, r \in [1, \infty]$ satisfying

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1.$$

The map b is the trilinear form used in the mathematical analysis of the Navier–Stokes equations, see for instance [37, Chapter II, Section 1.2]. It is well known, see for e.g. [37, Chapter II, Section 1.2], that one can define a continuous bilinear map B on $\mathbb{H} \times \mathbb{H}$ with values in \mathbb{H}^{-1} such that

$$\langle B(u, v), w \rangle = b(u, v, w) \text{ for } w \in \mathbb{H}^1, \text{ and } u \in \mathbb{H}, v \in \mathbb{H}^1.$$

Note that there exists a constant $C > 0$ such that for any $u, v \in \mathbb{H}_{\text{sol}}^{0,4}$ and $w \in \mathbb{H}^1$

$$|b(u, v, w)| \leq C \|u\|_{\mathbb{L}^4} \|v\|_{\mathbb{L}^4} \|w\|_{\mathbb{H}^1}. \quad (2.6)$$

The following property also holds for any $u \in \mathbb{H}_{\text{sol}}^{0,4}$ and $v \in \mathbb{H}^1$

$$b(u, v, v) = 0. \quad (2.7)$$

The two properties (2.6) and (2.7) of the trilinear form $b(\cdot, \cdot, \cdot)$ are proved first for smooth functions with compact support and are extended to less regular functions by density argument. The property (2.6) enables us to extend the bilinear map $B(\cdot, \cdot)$ to $\mathbb{H}_{\text{sol}}^{0,4} \times \mathbb{H}_{\text{sol}}^{0,4}$; the extension of B to $\mathbb{H}_{\text{sol}}^{0,4} \times \mathbb{H}_{\text{sol}}^{0,4}$ is still denoted by B .

For $\mathbf{u}_1 = (u_1, m_1) \in \mathcal{H}^{0,4}$, $\mathbf{u}_2 = (u_2, m_2) \in \mathcal{H}^{0,4}$ we define

$$\mathbf{B}(\mathbf{u}_1, \mathbf{u}_2) = \begin{pmatrix} B(u_1, u_2) - B(m_1, m_2) \\ B(u_1, m_2) - B(m_1, u_2) \end{pmatrix}.$$

From (2.6) we easily derive that there exists a constant $C > 0$ such that for any $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{H}^{0,4}$

$$\|\mathbf{B}(\mathbf{u}_1, \mathbf{u}_2)\|_{\mathcal{H}^{-1}} \leq C \|\mathbf{u}_1\|_{\mathcal{H}^{0,4}} \|\mathbf{u}_2\|_{\mathcal{H}^{0,4}}. \quad (2.8)$$

Using the property (2.7) we obtain

$$(\mathbf{B}(\mathbf{u}_1, \mathbf{u}_2), \mathbf{u}_2) = 0, \text{ for } \mathbf{u}_1 \in \mathcal{H}^{0,4} \text{ and } \mathbf{u}_2 \in \mathcal{H}^1. \quad (2.9)$$

From the estimate (2.8) and the interpolation inequality

$$\|u\|_{\mathbb{L}^4} \leq \|u\|_{\mathbb{H}}^{1-\frac{d}{4\alpha}} \|u\|_{\mathbb{H}^\alpha}^{\frac{d}{4\alpha}} \quad (2.10)$$

we infer that there exists a constant $C > 0$ such that for any $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{H}^\alpha$, $\alpha > \frac{d}{4}$,

$$\|\mathbf{B}(\mathbf{u}_1, \mathbf{u}_2)\|_{\mathcal{H}^{-1}} \leq C \|\mathbf{u}_1\|^{1-\frac{d}{4\alpha}} \|\mathbf{u}_2\|^{1-\frac{d}{4\alpha}} \|\mathbf{u}_1\|_{\mathcal{H}^\alpha}^{\frac{d}{4\alpha}} \|\mathbf{u}_2\|_{\mathcal{H}^\alpha}^{\frac{d}{4\alpha}}. \quad (2.11)$$

Throughout this paper we set

$$\mathbf{B}(\mathbf{u}) = \mathbf{B}(\mathbf{u}, \mathbf{u}) \text{ for any } \mathbf{u} \in \mathcal{H}^{0,4}.$$

For any $\kappa \in \mathbb{R}$ we also set

$$\mathbf{A}_\kappa = \begin{pmatrix} \mathbf{A}^\kappa & 0 \\ 0 & \mathbf{A}^\kappa \end{pmatrix}.$$

The following notations will also be frequently used

$$\mathbf{R}_\gamma \mathbf{u} = \begin{pmatrix} \gamma_0 & 0 \\ 0 & \gamma_1 \end{pmatrix} \mathbf{u},$$

for $\mathbf{u} = (u, m)$.

With these notations we can rewrite the stochastic MHD as a stochastic evolution equation in \mathcal{H}

$$d\mathbf{u} + [\mathbf{A}_\alpha \mathbf{u} + \mathbf{B}(\mathbf{u}) + \mathbf{R}_\gamma \mathbf{u}]dt = \mathbf{G}(\mathbf{u})dW, \quad (2.12)$$

with $\mathbf{u} = (u, m)$,

$$\mathbf{G}(\mathbf{u}) = \begin{pmatrix} G(u) & 0 \\ 0 & F(m) \end{pmatrix} \text{ and } W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}.$$

Here W_1 and W_2 are cylindrical Wiener processes evolving on separable Hilbert spaces K_1 and K_2 , respectively. Note that W is a cylindrical Wiener process evolving on $K = K_1 \times K_2$ and it (hence W_1 and W_2) along with a filtered probability space will be part of the solution of (2.12), see, for instance, Definition 2.1.

2.2. A short review of the results for modified MHD driven by regular multiplicative noise

In this section we recall the results related to (2.12) driven by regular multiplicative noise. The proof of these results can be found in [1]. To start with this short review we introduce the following assumptions.

Assumption 2.1. The maps $G : \mathbb{H} \rightarrow \mathcal{L}_2(K_1, \mathbb{H})$ and $F : \mathbb{H} \rightarrow \mathcal{L}_2(K_2, \mathbb{H})$ are continuous. Furthermore, there exist two constants $C_0, C_1 > 0$ such that for any $u \in \mathbb{H}$

$$\begin{aligned}\|G(u)\|_{\mathcal{L}_2(K_1, \mathbb{H})} &\leq C_0(1 + \|u\|_{\mathbb{H}}), \\ \|F(u)\|_{\mathcal{L}_2(K_2, \mathbb{H})} &\leq C_1(1 + \|u\|_{\mathbb{H}}).\end{aligned}$$

Remark 2.1. Observe that Assumption 2.1 implies that $\mathbf{G} : \mathcal{H} \rightarrow \mathcal{L}_2(K, \mathcal{H})$ is continuous and satisfies: there exists a constant $C_2 > 0$ such that for any $\mathbf{u} = (u, m) \in \mathcal{H}$

$$\|\mathbf{G}(\mathbf{u})\|_{\mathcal{L}_2(K, \mathcal{H})} \leq C_2(1 + \|\mathbf{u}\|).$$

Assumption 2.2. For every $\psi \in \mathcal{V}$, where \mathcal{V} is the space of divergence-free functions (u, m) which are infinitely many differentiable and with compact support,

$$\text{the mapping } \mathbb{H} \ni u \mapsto \langle G(u), \psi \rangle := G(u)^* \psi \in K_1 \text{ is continuous,} \quad (2.13)$$

if in the space \mathbb{H} we consider the Fréchet topology inherited from the space $\mathbb{L}_{loc}^2(\mathbb{R}^d, \mathbb{R}^d)$. Similarly, for any $\psi \in \mathcal{V}$

$$\text{the mapping } \mathbb{H} \ni u \mapsto \langle F(u), \psi \rangle := F(u)^* \psi \in K_2 \text{ is continuous,} \quad (2.14)$$

if in the space \mathbb{H} we consider the Fréchet topology inherited from the space $\mathbb{L}_{loc}^2(\mathbb{R}^d, \mathbb{R}^d)$.

Remark 2.2. Assumption 2.2 implies that for every $\varphi \in \mathcal{D} := \mathcal{V} \times \mathcal{V}$

$$\text{the mapping } \mathcal{H} \ni \mathbf{u} \mapsto \langle \mathbf{G}(\mathbf{u}), \varphi \rangle := \mathbf{G}(\mathbf{u})^* \varphi \in K \text{ is continuous,} \quad (2.15)$$

if in the space \mathcal{H} we consider the Fréchet topology inherited from the space $[L_{loc}^2(\mathbb{R}^d, \mathbb{R}^d)]^2$.

Next, we define the concept of weak solution to the problem (2.12).

Definition 2.1. A martingale solution to the problem (2.12) is a triple $(\check{\mathcal{U}}, (\check{W}_1, \check{W}_2), (\check{u}, \check{m}))$ where

- (1) $\check{\mathcal{U}} = (\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{P}})$ is a complete filtered probability spaces equipped with a filtration $\check{\mathbb{F}} = \{\check{\mathcal{F}}_t, t \geq 0\}$ satisfying the usual conditions.
- (2) The process $\check{W} = (\check{W}_1, \check{W}_2)$ is a cylindrical Wiener process evolving on $K = (K_1, K_2)$,
- (3) The process $\check{\mathbf{u}} = (\check{u}, \check{m}) : [0, T] \times \check{\Omega} \rightarrow \mathcal{H}$ is a progressively measurable process such that $\check{\mathbb{P}}$ -a.s.

$$\check{\mathbf{u}} \in \mathbb{L}^2(0, T; \mathcal{H}^\alpha) \cap C([0, T]; \mathcal{H}_w),$$

and $\check{\mathbb{P}}$ -a.s. for all $t \in [0, T]$ and for $\Phi \in \mathbb{U}$

$$\begin{aligned}
& (\check{\mathbf{u}}(t), \Phi) + \int_0^t \langle \mathbf{A}_\alpha \check{\mathbf{u}}(s) + \mathbf{R}_\gamma \check{\mathbf{u}}(s) + \mathbf{B}(\check{\mathbf{u}})(s), \Phi \rangle ds \\
& = (\check{\mathbf{u}}_0, \Phi) + \int_0^t (\mathbf{G}(\check{\mathbf{u}}(s)), \Phi) d\check{W}(s).
\end{aligned} \tag{2.16}$$

Remark 2.3. In the above definition $C([0, T]; \mathcal{H}_w)$ is the space of all \mathcal{H} -valued functions which are weakly continuous.

Note that with $\Phi = (\varphi, \phi) \in \mathbb{U}$ the identity (2.16) is equivalent to

$$\begin{aligned}
& (\check{u}(t), \varphi) + \int_0^t \langle \mathbf{A}^{\frac{\alpha}{2}} \check{u}(s), \mathbf{A}^{\frac{\alpha}{2}} \varphi \rangle ds + \int_0^t \langle \gamma_0 \check{u}(s) + B(\check{u})(s) - B(\check{m})(s), \varphi \rangle ds \\
& = (\check{u}_0, \varphi) + \int_0^t (G(\check{u}(s)), \varphi) d\check{W}_1, \\
& (\check{m}(t), \phi) + \int_0^t \langle \mathbf{A}^{\frac{\alpha}{2}} \check{m}(s), \mathbf{A}^{\frac{\alpha}{2}} \phi \rangle ds + \int_0^t \langle \gamma_1 \check{u}(s) + B(\check{u}, \check{m})(s) - B(\check{m}, \check{u})(s), \phi \rangle ds \\
& = (\check{m}_0, \phi) + \int_0^t (F(\check{m}(s)), \phi) d\check{W}_2.
\end{aligned}$$

With the above definition in mind we now recall the following result which was proved in [1].

Theorem 2.1. Let $\mathbf{u}_0 = (u_0, m_0) \in \mathcal{H}$, $\alpha > \frac{d}{4}$ and $\gamma_0, \gamma_1 \geq 0$. If the nonlinear maps F and G satisfy Assumptions 2.1 and 2.2, then there exists at least a weak martingale solution to the problem (2.12).

3. Stochastic generalized MHD with not regular multiplicative noise

In this section, we will consider noises which are less regular than in the previous sections. We should however note that the results of the current section do not generalize the ones in Section 2.2. To start with let us fix some notations. For a Banach space X and a Hilbert space K we denote by $\gamma(K, X)$ the spaces of all γ -radonifying X -valued linear maps defined on K ; see, e.g., [7] and [38] for more information on γ -radonifying operators. In this section we impose the following set of conditions on F and G .

Assumption 3.1. There exists $\theta \in (0, \alpha)$ such that

- (1) The maps $\Lambda^{-\theta} G : \mathbb{H} \rightarrow \gamma(K_1, \mathbb{H})$ and $\Lambda^{-\theta} F : \mathbb{H} \rightarrow \gamma(K_2, \mathbb{H})$ are well-defined, continuous and uniformly bounded, i.e., there exist two constants (depending on θ) $M_1, M_2 > 0$ such that

$$\begin{aligned}
\sup_{u \in \mathbb{H}} \|\Lambda^{-\theta} G(u)\|_{\gamma(K_1, \mathbb{H})} & \leq M_1, \\
\sup_{u \in \mathbb{H}} \|\Lambda^{-\theta} F(u)\|_{\gamma(K_2, \mathbb{H})} & \leq M_2.
\end{aligned}$$

- (2) The maps $\Lambda^{-\theta} G : \mathbb{H} \rightarrow \gamma(K_1, \mathbb{H}^{0,4})$ and $\Lambda^{-\theta} F : \mathbb{H} \rightarrow \gamma(K_2, \mathbb{H}^{0,4})$ are well-defined, continuous and uniformly bounded, i.e., there exist two constants (depending on θ) $M_3, M_4 > 0$ such that

$$\sup_{u \in \mathbb{H}} \|\Lambda^{-\theta} G(u)\|_{\gamma(K_1, \mathbb{H}^{0,4})} \leq M_3,$$

$$\sup_{u \in \mathbb{H}} \|\Lambda^{-\theta} F(u)\|_{\gamma(K_2, \mathbb{H}^{0,4})} \leq M_4.$$

(3) If F and G satisfy item (1), then for every $\psi \in \mathbb{H}^{-\theta}$

$$\text{the mapping } \mathbb{H} \ni u \mapsto \langle G(u), \psi \rangle := G(u)^* \psi \in K_1 \text{ is continuous,} \quad (3.17)$$

if in the space \mathbb{H} we consider the Fréchet topology inherited from the space $\mathbb{L}_{loc}^2(\mathbb{R}^d, \mathbb{R}^d)$. Similarly, for any $\psi \in \mathbb{H}^{-\theta}$

$$\text{the mapping } \mathbb{H} \ni u \mapsto \langle F(u), \psi \rangle := F(u)^* \psi \in K_2 \text{ is continuous} \quad (3.18)$$

if in the space \mathbb{H} we consider the Fréchet topology inherited from the space $\mathbb{L}_{loc}^2(\mathbb{R}^d, \mathbb{R}^d)$.

Before we proceed to the statement of an important remark we recall that $A = -\Delta$ is a sectorial operator on $L^p(\mathbb{R}^d)$, $p > 1$, of angle 0. Moreover, the family of $z(z - A)^{-1}$, $z \in \mathbb{C} \setminus \{0\}$ with $0 < |\arg(z)| < \pi$ is bounded in $\mathcal{L}(L^p(\mathbb{R}^d))$. For these results, see [26, Section 2.3]. Hence, the operator A satisfies the multiplicativity property, see for instance [26, Theorem 5.4.3],

$$(A^\alpha)^\beta = A^{\alpha\beta}, \text{ for } \alpha > 0, \beta > 0. \quad (3.19)$$

Remark 3.1. Using the operator \mathbf{A}_α and (3.19) we can reformulate the first two items of Assumption 3.1 as follows:

(1) the mapping $\mathbf{A}_\alpha^{-\frac{\theta}{2\alpha}} \mathbf{G} : \mathcal{H} \rightarrow \gamma(K, \mathcal{H})$ is well-defined and uniformly bounded, *i.e.*, there exists a constant (depending on θ) $\tilde{M}_1 > 0$ such that

$$\sup_{\mathbf{u} \in \mathcal{H}} \|\mathbf{A}_\alpha^{-\frac{\theta}{2\alpha}} \mathbf{G}(\mathbf{u})\|_{\gamma(K, \mathcal{H})} \leq \tilde{M}_1.$$

(2) The map $\mathbf{A}_\alpha^{-\frac{\theta}{2\alpha}} \mathbf{G} : \mathcal{H} \rightarrow \gamma(K, \mathcal{H}^{0,4})$ is well-defined and uniformly bounded, *i.e.*, there exists a constant (depending on θ) $\tilde{M}_2 > 0$ such that

$$\sup_{\mathbf{u} \in \mathcal{H}} \|\mathbf{A}_\alpha^{-\frac{\theta}{2\alpha}} \mathbf{G}(\mathbf{u})\|_{\gamma(K, \mathcal{H}^{0,4})} \leq \tilde{M}_2.$$

Since the noises are now less regular than in the paper by Ali [1], the solution will surely be less regular than those in [1]. Therefore, we need to redefine the concept of our solution, and for doing so we adopt [7, Definition 2.4].

Definition 3.1. A martingale solution to the problem (2.12) is a triple $(\mathcal{U}, (W_1, W_2), (u, m))$ where

- (1) $\mathcal{U} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a complete filtered probability spaces equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ satisfying the usual conditions.
- (2) The process $W = (W_1, W_2)$ is a cylindrical Wiener process evolving on $K = K_1 \times K_2$,
- (3) The process $\mathbf{u} = (u, m) : [0, T] \times \Omega \rightarrow \mathcal{H}$ is a progressively measurable process such that \mathbb{P} -a.s.

$$\mathbf{u} \in \mathbb{L}^2(0, T; \mathcal{H}^{0,4}) \cap C([0, T]; \mathcal{H}_w),$$

and \mathbb{P} -a.s. for all $t \in [0, T]$ the identity (2.16) holds for any $\Phi \in \mathbb{U}$. That is \mathbb{P} -a.s. for all $t \in [0, T]$ and for any $\Phi \in \mathbb{U}$,

$$\begin{aligned} (\mathbf{u}(t), \Phi) + \int_0^t \langle \mathbf{A}_\alpha \mathbf{u}(s) + \mathbf{R}_\gamma \mathbf{u}(s) + \mathbf{B}(\mathbf{u}(s)), \Phi \rangle ds \\ = (\mathbf{u}_0, \Phi) + \left(\int_0^t \mathbf{G}(\mathbf{u}(s)) dW(s), \Phi \right). \end{aligned} \quad (3.20)$$

Theorem 3.1. Let $\mathbf{u}_0 = (u_0, m_0) \in \mathcal{H}$, $\alpha \geq 1$ and $\gamma_0, \gamma_1 \geq 0$. If the nonlinear maps F and G satisfy Assumption 3.1, then there exists at least a weak martingale solution of the problem (2.12). Moreover, \mathbf{u} satisfies

$$\mathbf{u} \in \mathbb{L}^{\frac{4\alpha}{\alpha-\theta}}(0, T; \mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}) \cap \mathbb{L}^{\frac{8\alpha}{d}}(0, T; \mathcal{H}^{0,4}) \quad \mathbb{P}\text{-a.s.}$$

If additionally $\alpha \geq \frac{d}{2}$, then

$$\mathbf{u} \in C([0, T]; \mathcal{H}) \quad \mathbb{P}\text{-a.s.}$$

The proof of this theorem will be given in subsections 3.1 and 3.2.

In the next theorem we will prove the pathwise uniqueness of the solution under additional assumptions on α and the maps F and G .

Assumption 3.2. There exists a constant $M_6 > 0$ such that for any $u_1, u_2 \in \mathbb{H}$

$$\begin{aligned} \|F(u_1) - F(u_2)\|_{\gamma(K_2, \mathbb{H}^{-\theta})} &< M_6 \|u_1 - u_2\|, \\ \|G(u_1) - G(u_2)\|_{\gamma(K_1, \mathbb{H}^{-\theta})} &< M_6 \|u_1 - u_2\|. \end{aligned}$$

Theorem 3.2. Let $\alpha \geq \frac{d}{2}$, $\gamma_0, \gamma_1 \geq 0$, and assume that F, G satisfy Assumptions 3.1 and 3.2 and \mathbf{u}_1 and \mathbf{u}_2 are solutions to problem (2.12) defined on the same probability space and with the same initial condition, then

$$\mathbb{P}(\mathbf{u}_1(t) = \mathbf{u}_2(t) \text{ for all } t \in [0, T]) = 1.$$

The proof of this theorem will be given in Subsection 3.3.

To close this subsection we reformulate the following remark.

Remark 3.2. Using (2.4) and the operator \mathbf{J} defined in (2.5), the first two items of the above assumption imply:

- (1) the map $\mathbf{J}^{-\theta} \mathbf{G} : \mathcal{H} \rightarrow \gamma(K, \mathcal{H})$ is well-defined and uniformly bounded, i.e., there exists a constant (depending on θ) $M_5 > 0$ such that

$$\sup_{\mathbf{u} \in \mathcal{H}} \|\mathbf{J}^{-\theta} \mathbf{G}(\mathbf{u})\|_{\gamma(K, \mathcal{H})} \leq M_5.$$

- (2) the mapping $\mathbf{J}^{-\theta} \mathbf{G} : \mathcal{H} \rightarrow \gamma(\mathbf{K}, \mathcal{H}^{0,4})$ is well-defined and uniformly bounded, i.e., there exists a constant (depending on θ) $M_6 > 0$ such that

$$\sup_{\mathbf{u} \in \mathcal{H}} \|\mathbf{J}^{-\theta} \mathbf{G}(\mathbf{u})\|_{\gamma(\mathbf{K}, \mathcal{H}^{0,4})} \leq M_6.$$

- (3) Assumption 2.2 implies that for every $\varphi \in \mathcal{H}^\theta$

$$\text{the mapping } \mathcal{H} \ni \mathbf{u} \mapsto \langle \mathbf{G}(\mathbf{u}), \varphi \rangle := \mathbf{G}(\mathbf{u})^* \varphi \in \mathbf{K} \text{ is continuous,} \quad (3.21)$$

if in the space \mathcal{H} we consider the Fréchet topology inherited from the space $[L_{loc}^2(\mathbb{R}^d, \mathbb{R}^d)]^2$.

These observation show that our assumptions are weaker than the assumption in [1], but they are stronger than the assumptions of [7]. We were really inspired by [7], hence our assumptions and methods are very similar to the ones in the aforementioned paper. Our modest contribution in the present paper is to identify the ranges of α for which we get existence, uniqueness and strong continuity in time of solutions to the fractional magnetohydrodynamic model. A difference between our paper and [7] is also the proof of the key Lemma 3.3 which was proved using variational approach in [7] and using semigroup approach in the present paper.

3.1. Proof of Theorem 3.1: the existence of a martingale solution

Before we proceed to the proof of Theorem 3.1, we should mention that since the Stokes operator generates an analytic semigroup on $\mathbb{H}^{0,p}$, $1 \leq p < \infty$, so does $\mathbf{A}_\alpha = \mathbf{A}^\alpha$, see [24]. This remark implies that \mathbf{A}_α also generates on $\mathcal{H}^{0,p}$ an analytic semigroup $e^{-t\mathbf{A}_\alpha}$ which satisfies: for any $\gamma \geq \kappa \geq 0$ there exists a constant $C_0 > 0$ such that

$$\|\mathbf{A}_\alpha^\gamma e^{-t\mathbf{A}_\alpha} \mathbf{u}\|_{0,p} \leq \frac{C_0}{t^{-(\gamma-\kappa)}} \|\mathbf{A}_\alpha^\kappa \mathbf{u}\|_{0,p}, \text{ for any } \mathbf{u} \in \mathcal{H}^{\kappa,p}. \quad (3.22)$$

The above inequality clearly implies that for any $\gamma \geq \kappa$ there exists a constant $C_0 > 0$ such that

$$\|e^{-t\mathbf{A}_\alpha} \mathbf{u}\|_{\gamma,p} \leq C_0 \left(1 + \frac{1}{t^{-\frac{\gamma-\kappa}{2\alpha}}}\right) \|\mathbf{u}\|_{\kappa,p}, \text{ for any } \mathbf{u} \in \mathcal{H}^{\kappa,p}. \quad (3.23)$$

Now, we can proceed to the proof of Theorem 3.1 which will be divided in 3 steps. Here, we will follow [7, Section 3].

Step 1: Constructing and solving an approximated auxiliary problem

The scheme of the proof of Theorem 3.1 very similar to the idea used in the proof of Theorem 2.1 carried out in [1], but instead of using Galerkin approximation we use Yosida like approximation. As we mentioned in Remark 3.2 we were inspired by [7] and hence we will closely follow the approach used in [7, Subsections 3.1 and 3.2]. More precisely, for any $n \in \mathbb{N}$ we set

$$R_n = n(nI + \mathbf{A})^{-\alpha}, \quad F_n = R_n F, \quad G_n = R_n G, \text{ and } \mathbf{G}_n = \begin{pmatrix} G_n & 0 \\ 0 & F_n \end{pmatrix},$$

and consider the following problem

$$d\mathbf{u} + [\mathbf{A}_\alpha \mathbf{u} + \mathbf{R}_\gamma \mathbf{u} + \mathbf{B}(\mathbf{u})]dt = \mathbf{G}_n(\mathbf{u})dW, \quad \mathbf{u}(0) = \mathbf{u}_0. \quad (3.24)$$

Thanks to Remark 3.2 and the regularization effect of the operator

$$\mathcal{R}_n = \begin{pmatrix} R_n & 0 \\ 0 & R_n \end{pmatrix}$$

we have

$$\|\mathbf{G}_n(\mathbf{u})\|_{\mathcal{L}_2(\mathbf{K}, \mathcal{H})} \leq \|\mathcal{R}_n \mathbf{J}^\theta\|_{\mathcal{L}(\mathcal{H})} \|\mathbf{J}^{-\theta} \mathbf{G}(\mathbf{u})\|_{\gamma(\mathbf{K}, \mathcal{H})}. \quad (3.25)$$

From this observation, Assumption 3.1 and Theorem 2.1 we infer that for any $n \in \mathbb{N}$, the problem (3.24) has a martingale solution $(\mathcal{U}_n, W_n, \mathbf{u}_n)$, where $\mathcal{U}_n = (\Omega_n, \mathcal{F}_n, \mathbb{F}_n, \mathbb{P}_n)$ is a family of complete filtered probability spaces with the family of filtrations $\mathbb{F}_n = \{\mathcal{F}_t^n, t \geq 0\}$ satisfying the usual conditions, $W^n = (W_1^n, W_2^n)$ is a family of cylindrical Wiener processes evolving on \mathbf{K} . The family of processes $\mathbf{u}_n = (u_n, m_n)$ satisfies the following inequality: for each $n \in \mathbb{N}$ there exists $C > 0$ such that

$$\mathbb{E}_n \sup_{t \in [0, T]} \|\mathbf{u}_n(t)\|^2 + \mathbb{E}_n \int_0^T \|\mathbf{u}_n(t)\|_{\mathcal{H}^\alpha}^2 dt \leq C \quad (3.26)$$

Before we proceed further, let us make an important remark about (3.26). Using the Fourier characterization of Sobolev spaces we infer that

$$\|\mathcal{R}_n \mathbf{u}\|_{\mathcal{H}^{\lambda+\gamma}} = \|\mathbf{J}^\gamma \mathcal{R}_n \mathbf{J}^\lambda \mathbf{u}\| = \left\| n \frac{(1+|\xi|^2)^{\frac{\gamma}{2}}}{(n+|\xi|^2)^\alpha} (1+|\xi|^2)^{\frac{\lambda}{2}} \hat{\mathbf{u}}(\xi) \right\| \leq B_n \|\mathbf{u}\|_{\mathcal{H}^\lambda}$$

where $\hat{\mathbf{u}} = \hat{\mathbf{u}}(\xi)$ is the Fourier transform of \mathbf{u} and $B_n := \left\| n \frac{(1+|\xi|^2)^{\frac{\gamma}{2}}}{(n+|\xi|^2)^\alpha} \right\|_{\mathbb{L}^\infty} = C_\gamma \frac{n}{(n-1)^{\alpha-\frac{\gamma}{2}}}$, given $\gamma \in (0, 2\alpha)$. For $n > 1$ we have

$$B_n \leq C_\gamma n^{\frac{\gamma}{2}}.$$

Therefore,

$$\|\mathcal{R}_n \mathbf{u}\|_{\mathcal{H}^{\lambda+\gamma}} \leq C_\gamma n^{\frac{\gamma}{2}} \|\mathbf{u}\|_{\mathcal{H}^\lambda},$$

which along with (3.25) and Assumption 3.2 implies that there exist two constants $C_0, C_\theta > 0$ such the for any $n \in \mathbb{N}$

$$\begin{aligned} \|\mathbf{G}_n(\mathbf{u})\|_{\gamma(\mathbf{K}, \mathcal{H}^{-\theta, p})} &\leq C_0 \|\mathbf{G}(\mathbf{u})\|_{\gamma(\mathbf{K}, \mathcal{H}^{-\theta, p})}, p \in \{0, 4\}, \\ \|\mathbf{G}_n(\mathbf{u})\|_{\mathcal{L}_2(\mathbf{K}, \mathcal{H})} &\leq C_\theta n^{\frac{\theta}{2}} \tilde{M}_1. \end{aligned} \quad (3.27)$$

The latter estimate implies that $\mathbf{G}_n(\mathbf{u})$ is more regular than $\mathbf{G}(\mathbf{u})$, but it also shows that the inequalities in (3.26) are not uniform with respect to n . In the next step we will show how to overcome this difficulty.

Step 2: Derivation of uniform estimates for the approximated solutions

To overcome the difficulty outlined at the end of the previous step we will derive uniform estimates for the families of stochastic convolutions \mathbf{y}_n and a sequence of processes \mathbf{v}_n that will be defined later. Let $\mathbf{y}_n = (y_n, z_n)$ be a solution to

$$d\mathbf{y}_n + \mathbf{A}_\alpha \mathbf{y}_n dt = \mathbf{G}_n(\mathbf{u}_n) dW, \quad \mathbf{y}_n(0) = 0.$$

Hereafter, we also set $\mathbf{v}_n = \mathbf{u}_n - \mathbf{y}_n$, $n \in \mathbb{N}$, where \mathbf{u}_n , $n \in \mathbb{N}$, is a martingale solution to (3.24) (see previous step). Clearly, for each $n \in \mathbb{N}$, \mathbf{v}_n is a solution of the modified generalized MHD:

$$\partial_t \mathbf{v}_n + \mathbf{A}_\alpha \mathbf{v}_n + \mathbf{B}(\mathbf{u}_n) + \mathbf{R}_\gamma \mathbf{u}_n = 0, \quad \mathbf{v}_n(0) = \mathbf{u}_0. \quad (3.28)$$

Few properties of the stochastic convolution are stated and proved in the following lemma.

Lemma 3.1. *Let $\theta_0 \in [\theta, \alpha)$ and $\varepsilon = \theta_0 - \theta$. For any $k \geq 2$ there exists a constant $C > 0$ such that*

$$\sup_{n \in \mathbb{N}} \mathbb{E}_n \|\mathbf{A}_\alpha^{\frac{\varepsilon}{2\alpha}} \mathbf{y}_n(t)\|_{L^k(0,T;\mathcal{H}^{0,4})}^k < C.$$

Proof. Let $\varepsilon = \theta_0 - \theta$. From an application of [5, Corollary 3.5], see also [5, Theorem 3.2], we infer that

$$\begin{aligned} \mathbf{A}_\alpha^{\frac{\varepsilon}{2\alpha}} \mathbf{y}_n(t) &= \int_0^t \mathbf{A}_\alpha^{\frac{\theta_0 - \theta}{2\alpha}} e^{-(t-s)\mathbf{A}_\alpha} \mathbf{G}_n(\mathbf{u}_n(s)) dW \\ &= \int_0^t \mathbf{A}_\alpha^{\frac{\theta_0}{2\alpha}} e^{-(t-s)\mathbf{A}_\alpha} \mathbf{A}_\alpha^{-\frac{\theta}{2\alpha}} \mathbf{G}_n(\mathbf{u}_n(s)) dW. \end{aligned}$$

Invoking Remark 3.1 we obtain

$$\begin{aligned} \mathbb{E}_n \|\mathbf{A}_\alpha^{\frac{\varepsilon}{2\alpha}} \mathbf{y}_n(t)\|_{\mathcal{H}^{0,4}}^k &\leq \mathbb{E}_n \left[\int_0^t (t-s)^{-\frac{\theta_0}{\alpha}} \|\mathbf{A}_\alpha^{-\frac{\theta}{2\alpha}} \mathbf{G}_n(\mathbf{u}_n)\|_{\gamma(\mathbf{K}, \mathcal{H}^{0,4})}^2 ds \right]^{\frac{k}{2}} \\ &\leq \tilde{M}_2^k (t^{1-\frac{\theta_0}{\alpha}})^{\frac{k}{2}} \end{aligned}$$

from which we easily derive the desired result. \square

We will also need the following results.

Lemma 3.2. *Let $\beta \in [0, \frac{1}{2}(1 - \frac{\theta}{\alpha})]$. Then, there exists a modification $\tilde{\mathbf{y}}_n$ of \mathbf{y}_n such that for any $p \geq 2$ and $\delta \geq 0$ satisfying*

$$\beta + \frac{\delta}{2\alpha} + \frac{1}{p} < \frac{1}{2} \left(1 - \frac{\theta}{\alpha} \right) \quad (3.29)$$

we have

$$\sup_{n \in \mathbb{N}} \mathbb{E}_n \|\tilde{\mathbf{y}}_n\|_{C^\beta([0,T];\mathcal{H}^\delta)}^p \leq C_0$$

for some constant C_0 independent of n .

Proof. Noticing that $\mathbb{H}_{\text{sol}}^\lambda \simeq D(I + \mathbf{A}_\alpha^{\frac{\lambda}{2\alpha}})$, $\lambda \geq 0$, the proof easily follows from [5, Corollary 3.5]. \square

Hereafter we will identify \mathbf{y}_n with its continuous modification $\tilde{\mathbf{y}}_n$. We have the following corollary.

Corollary 3.1. a) For $\beta \in [0, \frac{1}{2}(1 - \frac{\theta}{\alpha})]$ there exists a constant $\tilde{C}_1 > 0$ such that

$$\sup_{n \in \mathbb{N}} \mathbb{E}_n \|y_n\|_{C^\beta([0,T];\mathcal{H})} \leq \tilde{C}_1.$$

b) There exists a constant $\tilde{C}_2 > 0$ such that

$$\sup_{n \in \mathbb{N}} \mathbb{E}_n \|y_n\|_{C([0,T];\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}})} \leq \tilde{C}_2.$$

Proof. a) In Lemma 3.2 take $\delta = 0$ and p big enough so that $\beta + \frac{1}{p} < \frac{1}{2}(1 - \frac{\theta}{\alpha})$ and

$$\mathbb{E}_n \|y_n\|_{C^\beta([0,T];\mathcal{H})} < \left(\mathbb{E}_n \|y_n\|_{C^\beta([0,T];\mathcal{H})}^p \right)^{\frac{1}{p}} < \tilde{C}_1.$$

b) Similarly, in Lemma 3.2 put $\beta = 0$, $\delta = \frac{1}{2}(1 - \frac{\theta}{\alpha})$ and p big enough so that

$$\frac{\delta}{2\alpha} + \frac{1}{p} < \frac{1}{2} \left(1 - \frac{\theta}{\alpha} \right)$$

and

$$\mathbb{E}_n \|y_n\|_{C^\beta([0,T];\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}})} < \left(\mathbb{E}_n \|y_n\|_{C^\beta([0,T];\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}})}^p \right)^{\frac{1}{p}} < \tilde{C}_2. \quad \square$$

We also have the following two corollaries which can be proved with the same argument as used in [7, Lemma 3.3 and Proposition 3.4], hence we omit the detail of the proofs. The first one easily follows from Lemma 3.1, Lemma 3.2 and Corollary 3.1 and we omit its proof.

Corollary 3.2. Under the assumptions on β in Corollary 3.1, there exist positive constants \tilde{C}_3 , \tilde{C}_4 and \tilde{C}_5 such that

$$\begin{aligned} \sup_{n \in \mathbb{N}} \mathbb{E}_n \|y_n\|_{\mathbb{L}^k(0,T;\mathcal{H}^{0,4})}^k &\leq \tilde{C}_3, \quad k \geq 2, \\ \sup_{n \in \mathbb{N}} \mathbb{E}_n \|y_n\|_{C([0,T];\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}})} &\leq \tilde{C}_4, \\ \sup_{n \in \mathbb{N}} \mathbb{E}_n \|y_n\|_{C^\beta([0,T];\mathcal{H}^\delta)}^p &\leq \tilde{C}_5, \end{aligned}$$

where p , β and δ are the constants in Lemma 3.2.

Corollary 3.3. For any $\varepsilon > 0$ there exist positive constants \tilde{C}_3 , \tilde{C}_4 and \tilde{C}_5 such that

$$\begin{aligned} \sup_{n \in \mathbb{N}} \mathbb{P}_n \left(\|y_n\|_{\mathbb{L}^k(0,T;\mathcal{H}^{0,4})}^k \geq \varepsilon \right) &\leq \frac{\tilde{C}_3}{\varepsilon}, \\ \sup_{n \in \mathbb{N}} \mathbb{P}_n \left(\|y_n\|_{C([0,T];\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}})} \geq \varepsilon \right) &\leq \frac{\tilde{C}_4}{\varepsilon}, \\ \sup_{n \in \mathbb{N}} \mathbb{P}_n \left(\|y_n\|_{C^\beta([0,T];\mathcal{H}^\delta)}^p \geq \varepsilon \right) &\leq \frac{\tilde{C}_5}{\varepsilon}. \end{aligned}$$

Proof. The proof of this corollary uses Corollary 3.2 and Chebychev's inequality. This is easy and we omit the proof. \square

In order to get uniform estimates for \mathbf{u}_n we now need to establish uniform estimates for the solution \mathbf{v}_n of (3.28). More precisely, we state and prove the following lemma which is similar to [7, Proposition 3.4]. However, we should note that, in contrast to [7, Proposition 3.4] which uses variational approach, in our proof we use the semigroup approach. Moreover, our problem is α -dependent and hence we have to be careful with respect to the choice of functional spaces.

Lemma 3.3. *For any $\varepsilon > 0$ there exist constants $K_i > 0$, $i = 1, \dots, 5$ such that*

$$\begin{aligned} \sup_{n \in \mathbb{N}} \mathbb{P}_n \left(\|\mathbf{v}_n\|_{\mathbb{L}^\infty(0,T;\mathcal{H})} > K_1 \right) &\leq \varepsilon, \\ \sup_{n \in \mathbb{N}} \mathbb{P}_n \left(\|\mathbf{v}_n\|_{\mathbb{L}^2(0,T;\mathcal{H}^\alpha)} > K_2 \right) &\leq \varepsilon, \\ \sup_{n \in \mathbb{N}} \mathbb{P}_n \left(\|\mathbf{v}_n\|_{\mathbb{L}^{\frac{4\alpha}{\alpha-\theta}}(0,T;\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}})} > K_3 \right) &\leq \varepsilon, \\ \sup_{n \in \mathbb{N}} \mathbb{P}_n \left(\|\mathbf{v}_n\|_{\mathbb{L}^{\frac{8\alpha}{d}}(0,T;\mathcal{H}^{0,4})} > K_4 \right) &\leq \varepsilon, \\ \sup_{n \in \mathbb{N}} \mathbb{P}_n \left(\|\mathbf{v}_n\|_{C^{1-\mu(\alpha)}(0,T;\mathcal{H}^{-\alpha})} > K_5 \right) &\leq \varepsilon, \end{aligned}$$

where

$$\mu(\alpha) = \begin{cases} \frac{d}{4\alpha} & \text{if } \alpha \in [1, \frac{d}{2}) \\ 2 & \text{if } \alpha \geq \frac{d}{2}. \end{cases}$$

Proof. To prove the lemma we start with the study of the nonlinear term

$$\tilde{\mathbf{B}}(\mathbf{u}_n) := \mathbf{B}(\mathbf{u}_n) + \mathbf{R}_\gamma \mathbf{u}_n.$$

Let us assume that $\alpha \in [1, \frac{d}{2})$. From (2.11) we obtain

$$\int_0^T \|\mathbf{B}(\mathbf{u}_n(s))\|_{\mathcal{H}^{-1}}^{\frac{4\alpha}{d}} ds \leq C \sup_{s \in [0,T]} \|\mathbf{u}_n(s)\|^{\frac{2}{d}(4\alpha-d)} \int_0^T \|\mathbf{u}_n(s)\|_{\mathcal{H}^\alpha}^2 ds,$$

from which along with the definition of $\mathbf{R}_\gamma \mathbf{u}_n$ and the fact that $\mathbf{u}_n \in \mathbb{L}^\infty(0,T;\mathcal{H}) \cap \mathbb{L}^2(0,T;\mathcal{H}^\alpha)$ a.s. we infer that there exists a constant $C > 0$ such that

$$\int_0^T \|\tilde{\mathbf{B}}(\mathbf{u}_n(s))\|_{\mathcal{H}^{-1}}^{\frac{4\alpha}{d}} < C. \quad (3.30)$$

Notice that the inequality (3.30) holds with $\frac{4\alpha}{d}$ replaced by 2 when $\alpha \geq \frac{d}{2}$.

Now, since \mathbf{A}_α is the generator of an analytic semigroup of bounded linear operators, by [29, Theorem 2.6] it satisfies the maximal regularity property on \mathcal{H} . Thus, since, by (3.30), $\mathbf{A}^{-\frac{1}{2\alpha}} \tilde{\mathbf{B}}(\mathbf{u}_n) \in \mathbb{L}^{\frac{4\alpha}{d}}(0,T;\mathcal{H})$ we infer from an application of maximal regularity, see [29], that the function $\tilde{\mathbf{v}}_n(\cdot) := \int_0^\cdot e^{-(\cdot-s)\mathbf{A}_\alpha} \mathbf{A}_\alpha^{-\frac{1}{2\alpha}} \tilde{\mathbf{B}}(\mathbf{u}_n(s)) ds$ satisfies

$$\mathbf{A}_\alpha \tilde{\mathbf{v}}_n \in \mathbb{L}^{\frac{4\alpha}{d}}(0,T;\mathcal{H}). \quad (3.31)$$

Observe also that $\|\mathbf{A}_\alpha^{1-\frac{1}{2\alpha}} e^{-t\mathbf{A}_\alpha} \mathbf{u}_0\| \leq t^{\frac{1-2\alpha}{2\alpha}} \|\mathbf{u}_0\|$ for any $t > 0$ which along with $d \in \{2, 3\}$ and $\alpha \geq 1$ implies that $e^{-\cdot \mathbf{A}_\alpha} \mathbf{u}_0 \in \mathbb{L}^1(0, T; \mathcal{H}^{2\alpha-1})$. This observation together with (3.31) yields that

$$\mathbf{A}_\alpha^{1-\frac{1}{2\alpha}} \mathbf{v}_n \in \mathbb{L}^1(0, T; \mathcal{H}), \quad (3.32)$$

because $\mathbf{v}_n(\cdot) = e^{-\cdot \mathbf{A}_\alpha} \mathbf{u}_0 + \mathbf{A}_\alpha^{\frac{1}{2\alpha}} \tilde{\mathbf{v}}_n(\cdot)$. Hence, from (3.32), the last observation and the assumption $\alpha \geq 1$ we infer that $\mathbf{v}_n \in \mathbb{L}^1(0, T; \mathcal{H}^\alpha)$. Hence $\mathbf{v}_n(t) \in \mathcal{H}^\alpha$ a.e. $t \in [0, T]$. This last fact enables us to multiply (with respect to the \mathcal{H} -scalar product) the first identity in (3.28) by \mathbf{v}_n . This multiplication procedure and the definition of \mathbf{R}_γ imply that there exists a constant $K_0 > 0$ such that

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{v}_n\|^2 + \|\mathbf{A}_{\frac{\alpha}{2}} \mathbf{v}_n\|^2 \leq -(\mathbf{B}(\mathbf{u}_n, \mathbf{v}_n + \mathbf{y}_n), \mathbf{v}_n) + K_0(\|\mathbf{v}_n\|^2 + \|\mathbf{y}_n\|^2).$$

Now using (2.8), (2.9) and (2.10) we infer that

$$\begin{aligned} |\langle \mathbf{B}(\mathbf{u}_n, \mathbf{v}_n + \mathbf{y}_n), \mathbf{v}_n \rangle| &= |\langle \mathbf{B}(\mathbf{u}_n, \mathbf{v}_n), \mathbf{y}_n \rangle| \\ &\leq C \|\mathbf{u}_n\|_{\mathcal{H}^{0,4}} \|\mathbf{v}_n\|_{\mathcal{H}^1} \|\mathbf{y}_n\|_{\mathcal{H}^{0,4}} \\ &\leq C (\|\mathbf{v}_n\|_{\mathcal{H}^{0,4}} \|\mathbf{v}_n\|_{\mathcal{H}^1} \|\mathbf{y}_n\|_{\mathcal{H}^{0,4}} + \|\mathbf{y}_n\|_{\mathcal{H}^{0,4}}^2 \|\mathbf{v}_n\|_{\mathcal{H}^1}) \\ &\leq C \left(\|\mathbf{v}_n\|^{1-\frac{d}{4\alpha}} \|\mathbf{v}_n\|_{\mathcal{H}^\alpha}^{\frac{d}{4\alpha}} \|\mathbf{v}_n\|_{\mathcal{H}^1} \|\mathbf{y}_n\|_{\mathcal{H}^{0,4}} \right) + C \|\mathbf{v}_n\|_{\mathcal{H}^1} \|\mathbf{y}_n\|_{\mathcal{H}^{0,4}}^2. \end{aligned} \quad (3.33)$$

Since $\mathcal{H}^\alpha \subset \mathcal{H}^1$ for $\alpha \geq 1$ and the norm of \mathcal{H}^α is equivalent to $\|\cdot\| + \|\mathbf{A}_{\frac{\alpha}{2}} \cdot\|$, we infer that there exists a constant $C > 0$ such that

$$\begin{aligned} &|(\mathbf{B}(\mathbf{u}_n, \mathbf{v}_n + \mathbf{y}_n), \mathbf{v}_n)| \\ &\leq C \|\mathbf{v}_n\|^{1-\frac{d}{4\alpha}} (\|\mathbf{v}_n\| + \|\mathbf{A}_{\frac{\alpha}{2}} \mathbf{v}_n\|)^{1+\frac{d}{4\alpha}} \|\mathbf{y}_n\|_{\mathcal{H}^{0,4}} + (\|\mathbf{v}_n\| + \|\mathbf{A}_{\frac{\alpha}{2}} \mathbf{v}_n\|) \|\mathbf{y}_n\|_{\mathcal{H}^{0,4}}^2 \\ &\leq \frac{1}{2} \|\mathbf{A}_{\frac{\alpha}{2}} \mathbf{v}_n\|^2 + C \|\mathbf{y}_n\|_{\mathcal{H}^{0,4}}^4 + C \|\mathbf{v}_n\|^2 \left(1 + \|\mathbf{y}_n\|_{\mathcal{H}^{0,4}}^{\frac{8\alpha}{4\alpha-d}} \right) + C \|\mathbf{y}_n\|^2, \end{aligned} \quad (3.34)$$

where we have used Young's inequality to obtain the last line. Hence,

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{v}_n\|^2 + \frac{1}{2} \|\mathbf{A}_{\frac{\alpha}{2}} \mathbf{v}_n\|^2 \leq C \|\mathbf{y}_n\|_{\mathcal{H}^{0,4}}^4 + C \|\mathbf{y}_n\|^2 + C \|\mathbf{v}_n\|^2 \left(1 + \|\mathbf{y}_n\|_{\mathcal{H}^{0,4}}^{\frac{8\alpha}{4\alpha-d}} \right) + C \|\mathbf{y}_n\|^2. \quad (3.35)$$

Observe that by Lemma 3.1 $\|\mathbf{y}_n\|^2$, $\|\mathbf{y}_n\|_{\mathcal{H}^{0,4}}^4$ and $1 + \|\mathbf{y}_n\|_{\mathcal{H}^{0,4}}^{\frac{8\alpha}{4\alpha-d}}$ are uniformly bounded in $L^1(0, T)$. Thus an application of Gronwall's lemma yields

$$\begin{aligned} \sup_{t \in [0, T]} \|\mathbf{v}_n\|^2 &\leq \|\mathbf{v}_n(0)\|^2 \exp \left(CT + C \int_0^T \|\mathbf{y}_n(t)\|_{\mathcal{H}^{0,4}}^{\frac{8\alpha}{4\alpha-d}} dt \right) \\ &\quad \times \left(1 + \int_0^T \|\mathbf{y}_n(t)\|_{\mathcal{H}^{0,4}}^4 dt + \sup_{t \in [0, T]} \|\mathbf{y}_n(t)\|^2 \right) \\ &\leq C_1 \|\mathbf{u}_0\|^2 + C_2, \end{aligned} \quad (3.36)$$

for some constants $C_1, C_2 > 0$ which do not depend on n .

From (3.35) and (3.36) we infer that there exist constants $C_3 > 0$, $C_4 > 0$ such that

$$\int_0^T \|\mathbf{v}_n(t)\|_{\mathcal{H}^\alpha}^2 dt \leq C_3 \|\mathbf{u}_0\|^2 + C_4 \quad (3.37)$$

From Corollary 3.3 we infer that for any $\varepsilon > 0$ there exist constants $\eta_1, \eta_2 > 0$ and $\eta_3 > 0$ such that

$$\begin{aligned} \sup_{n \in \mathbb{N}} \mathbb{P}_n \left(\int_0^T \|\mathbf{y}_n(t)\|_{\mathcal{H}^{0,4}}^4 dt > \eta_1 \right) &\leq \varepsilon, \\ \sup_{n \in \mathbb{N}} \mathbb{P}_n \left(\int_0^T \left(1 + \|\mathbf{y}_n(t)\|_{\mathcal{H}^{0,4}}^{\frac{8\alpha}{4\alpha-d}} \right) dt > \eta_2 \right) &\leq \varepsilon, \\ \sup_{n \in \mathbb{N}} \mathbb{P}_n \left(\sup_{t \in [0,T]} \|\mathbf{y}_n(t)\|^2 > \eta_3 \right) &\leq \varepsilon, \end{aligned}$$

from which along with (3.36) and (3.37) we infer that for any $\varepsilon > 0$ there exist two constants $R_1 > 0$ and $R_2 > 0$ such that

$$\sup_{n \in \mathbb{N}} \mathbb{P}_n \left(\|\mathbf{v}_n\|_{\mathbb{L}^\infty(0,T;\mathcal{H})} > R_1 \right) \leq \varepsilon, \quad (3.38)$$

$$\sup_{n \in \mathbb{N}} \mathbb{P}_n \left(\|\mathbf{v}_n\|_{\mathbb{L}^2(0,T;\mathcal{H}^\alpha)} > R_2 \right) \leq \varepsilon, \quad (3.39)$$

which is the first two inequalities in Lemma 3.3.

Using the interpolation $[\mathcal{H}, \mathcal{H}^\alpha]_{\frac{\alpha-\theta}{2\alpha}} = \mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}$ we have

$$\|h\|_{\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}} \leq \|h\|^{1-\frac{(\alpha-\theta)}{2\alpha^2}} \|h\|_{\mathcal{H}^\alpha}^{\frac{\alpha-\theta}{2\alpha^2}}.$$

Hence

$$\int_0^T \|\mathbf{v}_n\|_{\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}}^{\frac{4\alpha^2}{\alpha-\theta}} dt \leq C \|\mathbf{v}_n\|_{\mathbb{L}^\infty(0,T;\mathcal{H})}^{4\frac{\alpha^2}{\alpha-\theta}-2} \|\mathbf{v}_n\|_{\mathbb{L}^2(0,T;\mathcal{H}^\alpha)}^2,$$

which along with (3.38) and (3.39) implies that for any $\varepsilon > 0$ there exists a constant $R_3 > 0$ such that

$$\sup_{n \in \mathbb{N}} \mathbb{P}_n \left(\|\mathbf{v}_n\|_{\mathbb{L}^{\frac{4\alpha^2}{\alpha-\theta}}(0,T;\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}})} > R_3 \right) \leq \varepsilon.$$

This proves the third inequality in Lemma 3.3.

From (2.10), (3.38) and (3.39) we infer that for any $\varepsilon > 0$ there exists a constant $R_4 > 0$ such that

$$\sup_{n \in \mathbb{N}} \mathbb{P}_n \left(\|\mathbf{v}_n\|_{\mathbb{L}^{\frac{8\alpha}{d}}(0,T;\mathcal{H}^{0,4})} > R_4 \right) \leq \varepsilon. \quad (3.40)$$

This is exactly the fourth inequality in Lemma 3.3.

We now prove that for any $\varepsilon > 0$ there exists a constant $R_6 > 0$ such that

$$\sup_{n \in \mathbb{N}} \mathbb{P}_n (|\dot{\mathbf{v}}_n|_{Y_\alpha} > R_6) \leq \varepsilon, \quad (3.41)$$

where $\dot{\mathbf{v}}_n = \frac{d\mathbf{v}_n}{dt}$ and $Y_\alpha = \mathbb{L}^{\mu(\alpha)}(0, T; \mathcal{H}^{-\alpha})$.

We will only prove (3.41) for the case $\alpha \in [1, d/2]$ as the other case $\alpha \geq \frac{d}{2}$ is easy. For this purpose and the sake of simplicity we set $Y = \mathbb{L}^{\frac{4\alpha}{d}}(0, T; \mathcal{H}^{-\alpha})$.

Before we proceed further let us note that there exists a constant $C > 0$ such that for any $n \in \mathbb{N}$

$$\int_0^T \|\mathbf{A}_\alpha \mathbf{v}_n(s)\|_{\mathcal{H}^{-\alpha}} ds \leq C \left(\int_0^T \|\mathbf{v}_n(s)\|_{\mathcal{H}^\alpha}^2 ds \right)^{\frac{1}{2}}. \quad (3.42)$$

In fact, from the definition of $\|\cdot\|_{\mathcal{H}^{-\alpha}}$ and the Hölder's inequality we infer that

$$\begin{aligned} \int_0^T \|\mathbf{A}_\alpha \mathbf{v}_n(s)\|_{\mathcal{H}^{-\alpha}} ds &= \int_0^T \sup_{\|\varphi\|_{\mathcal{H}^\alpha} < 1} |\langle \mathbf{A}_\alpha \mathbf{v}_n(s), \varphi \rangle| ds \\ &\leq C \int_0^T \sup_{\|\varphi\|_{\mathcal{H}^\alpha} < 1} \|\mathbf{v}_n(s)\|_{\mathcal{H}^\alpha} \|\varphi\|_{\mathcal{H}^\alpha} ds, \end{aligned}$$

which completes the proof of (3.42).

Now, from (3.42) and (2.11) we infer that there exists a constant $C > 0$ such that for any $n \in \mathbb{N}$

$$\begin{aligned} \|\dot{\mathbf{v}}_n\|_Y &\leq \|\mathbf{A}_\alpha \mathbf{v}_n\|_Y + \|\mathbf{B}(\mathbf{u}_n, \mathbf{u}_n)\|_Y + \|\mathbf{R}_\gamma \mathbf{u}_n\|_Y \\ &\leq C \|\mathbf{A}_\alpha \mathbf{v}_n\|_{\mathbb{L}^{\frac{4\alpha}{d}}(0, T; \mathcal{H}^{-\alpha})} + C(\|\mathbf{u}_n\|_Y^2 + \|\mathbf{u}_n\|_{\mathbb{L}^{\frac{8\alpha}{d}}(0, T; \mathcal{H}^{0,4})}) \\ &\leq \|\mathbf{v}_n\|_{\mathbb{L}^2(0, T; \mathcal{H}^\alpha)} + C \|\mathbf{v}_n\|_{\mathbb{L}^\infty(0, T; \mathcal{H})}^{2-\frac{d}{2\alpha}} \|\mathbf{v}_n\|_{\mathbb{L}^2(0, T; \mathcal{H}^\alpha)}^{\frac{d}{2\alpha}} + C \|\mathbf{y}_n\|_{\mathbb{L}^{\frac{8\alpha}{d}}(0, T; \mathcal{H}^{0,4})}^2 \\ &\quad + \|\mathbf{v}_n\|_{\mathbb{L}^{\frac{8\alpha}{d}}(0, T; \mathcal{H}^{0,4})} + \|\mathbf{y}_n\|_{\mathbb{L}^{\frac{8\alpha}{d}}(0, T; \mathcal{H}^{0,4})}. \end{aligned}$$

Hence, from Corollary 3.3, inequalities (3.39) and (3.40) we easily derive (3.41).

Now, from (3.41) and the Sobolev embeddings

$$W^{1, \mu(\alpha)}(0, T) \subset C^{1-\mu(\alpha)}(0, T) \text{ and } W^{1,2}(0, T) \subset C^{\frac{1}{2}}(0, T)$$

we derive that for any $\varepsilon > 0$ there exists a constant $R_7 > 0$ such that

$$\sup_{n \in \mathbb{N}} \mathbb{P}_n (|\mathbf{v}_n|_{C^{1-\mu(\alpha)}(0, T; \mathcal{H}^{-\alpha})} > R_7) \leq \varepsilon. \quad (3.43)$$

This proves the fifth estimate in Lemma 3.3. \square

Step 3: Tightness property of the approximated solutions, passage to the limits and end of the proof of the existence

In this final step we will exploit the uniform estimates in the previous step to establish the tightness of the approximated solution \mathbf{u}_n in appropriate topological space. We will also pass to the limit in the nonlinear

terms of the approximated problems to complete the proof of the existence of martingale solution. Again, we will adopt the approach and notation similar to the ones used in [7, Subsection 3.3].

For this purpose we note that since $\mathbf{v}_n = \mathbf{u}_n - \mathbf{y}_n$ we infer that $\mathbf{u}_n = \mathbf{v}_n + \mathbf{y}_n$ from which along with, Corollary 3.3 and Lemma 3.3 we easily derive the following proposition.

Proposition 3.1. *For any $\varepsilon > 0$ there exist constants K_i , $i = 1, \dots, 5$ such that*

$$\begin{aligned} \sup_{n \in \mathbb{N}} \mathbb{P}_n \left(\|\mathbf{u}_n\|_{\mathbb{L}^\infty(0,T;\mathcal{H})} > K_1 \right) &\leq \varepsilon, \\ \sup_{n \in \mathbb{N}} \mathbb{P}_n \left(\|\mathbf{u}_n\|_{\mathbb{L}^2(0,T;\mathcal{H}^\delta)} > K_2 \right) &\leq \varepsilon, \\ \sup_{n \in \mathbb{N}} \mathbb{P}_n \left(\|\mathbf{u}_n\|_{\mathbb{L}^{\frac{4\alpha^2}{\alpha-\theta}}(0,T;\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}})} > K_3 \right) &\leq \varepsilon, \\ \sup_{n \in \mathbb{N}} \mathbb{P}_n \left(\|\mathbf{u}_n\|_{\mathbb{L}^{\frac{8\alpha}{d}}(0,T;\mathcal{H}^{0,4})} > K_4 \right) &\leq \varepsilon, \\ \sup_{n \in \mathbb{N}} \mathbb{P}_n \left(\|\mathbf{u}_n\|_{C^\lambda(0,T;\mathcal{H}^{-\alpha})} > K_5 \right) &\leq \varepsilon \end{aligned}$$

where $\lambda = \min\{\beta, 1 - \frac{d}{4\alpha}\}$ with β and δ satisfying the inequality (3.29) which implies that $\beta \in (0, \frac{1}{2})$ and $\delta \in (0, \alpha)$.

With this proposition at hand we state the following crucial result.

Lemma 3.4. *The family of measures $(\mathcal{L}(\mathbf{u}_n))_{n \in \mathbb{N}}$ is tight in $\tilde{\mathcal{Z}}$, where*

$$\tilde{\mathcal{Z}} = \mathbb{L}_w^{\frac{8\alpha}{d}}(0, T; \mathcal{H}^{0,4}) \cap C(0, T; \mathbb{U}') \cap \mathbb{L}^2(0, T; \mathcal{H}_{loc}) \cap C([0, T]; \mathcal{H}_w)$$

and $\tilde{\mathcal{T}}$ is the supremum of the corresponding topologies.

Proof. Thanks to Proposition 3.1, the tightness of $(\mathcal{L}(\mathbf{u}_n))_{n \in \mathbb{N}}$ in $\tilde{\mathcal{Z}}$ follows from the application of [7, Lemma 5.5] (see also Lemma Appendix A.2). \square

In order to complete the proof of Theorem 3.1 we need to carry out a passage to the limit procedure very similar to the proof of Theorem 3.6 in [7].

From Lemma 3.4 and Jakubowski–Skorokhod’s theorem we can find a subsequence $(\tilde{\mathbf{u}}_{n_k})_{k \geq 1}$, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which a $\tilde{\mathcal{Z}}$ -valued random variable $\tilde{\mathbf{u}}$ and a sequence of $\tilde{\mathcal{Z}}$ -valued random variables $(\tilde{\mathbf{u}}_k)_{k \geq 1}$ are defined. The random variables satisfy:

$$\mathcal{L}(\mathbf{u}_{n_k}) = \mathcal{L}(\tilde{\mathbf{u}}_k) \text{ on } \tilde{\mathcal{Z}} \quad (3.44)$$

$$\tilde{\mathbf{u}}_k \longrightarrow \tilde{\mathbf{u}} \text{ in } \tilde{\mathcal{Z}}, \mathbb{P}\text{-a.s.}, \quad (3.45)$$

as $k \rightarrow \infty$.

We also have the following lemma.

Lemma 3.5. *For any $s, t \in [0, T]$ with $s \leq t$ and $\phi \in \mathbb{U}$, we have*

$$\begin{aligned} (\tilde{\mathbf{u}}_k(t), \phi) &\longrightarrow (\tilde{\mathbf{u}}(t), \phi), \quad \tilde{\mathbb{P}}\text{-a.s.}, \\ \int_s^t \langle \mathbf{A}_\alpha \tilde{\mathbf{u}}_k(r), \phi \rangle dr &\longrightarrow \int_s^t \langle \mathbf{A}_\alpha \tilde{\mathbf{u}}(r), \phi \rangle dr, \quad \tilde{\mathbb{P}}\text{-a.s.}, \end{aligned}$$

$$\int_s^t \langle \mathbf{R}_\gamma \tilde{\mathbf{u}}_k(r), \phi \rangle dr \longrightarrow \int_s^t \langle \mathbf{R}_\gamma \tilde{\mathbf{u}}(r), \phi \rangle dr, \quad \tilde{\mathbb{P}}\text{-a.s.},$$

$$\int_s^t \langle \mathbf{B}(\tilde{\mathbf{u}}_k(r)), \phi \rangle dr \longrightarrow \int_s^t \langle \mathbf{B}(\tilde{\mathbf{u}}(r)), \phi \rangle dr, \quad \tilde{\mathbb{P}}\text{-a.s.},$$

as $n \rightarrow \infty$.

Proof. Thanks to the convergence (3.45) one can prove the above convergences by the same idea as used in [1]. \square

As in [1], we set

$$\tilde{\mathcal{M}}_k(t) = \tilde{\mathbf{u}}_k(t) - \mathbf{u}_k(0) + \int_0^t \mathbf{A}_\alpha \tilde{\mathbf{u}}_k(s) ds + \int_0^t \mathbf{B}(\tilde{\mathbf{u}}_k(s)) ds + \int_0^t \mathbf{R}_\gamma \tilde{\mathbf{u}}_k(s) ds.$$

Lemma 3.6. *The process $\mathcal{M}_k(t)$, $t \geq 0$ is a square integrable martingale w.r.t. the filtration $\tilde{\mathbb{F}}_k = (\tilde{\mathcal{F}}_{k,t})$, where $\tilde{\mathcal{F}}_{k,t} = \sigma\{\tilde{\mathbf{u}}_k(s); s \leq t\}$, with quadratic variation*

$$[\mathcal{M}_k]_t = \int_0^t \mathbf{G}_k(\mathbf{u}_k(s)) \mathbf{G}_k(\mathbf{u}_k(s))^* ds, \quad t \in [0, T].$$

Proof. We will closely follow the proof given by [1] and [9]. Since \mathbf{u}_{n_k} and u_k have the same law, for all $s, t \in [0, T]$, $s \leq t$, all (real-valued) function h bounded continuous on $C([0, s]; \mathbb{U}')$, and all $\psi, \zeta \in \mathbb{U}$, we have

$$[\langle \mathcal{M}_k(t) - \mathcal{M}_k(s), \psi \rangle h(\tilde{\mathbf{u}}_k|_{[0,s]})] = 0 \quad \tilde{\mathbb{P}}\text{-a.s. for any } \psi \in \mathbb{U}, \quad (3.46)$$

and

$$\mathbb{E} \left[\left(\langle \mathcal{M}_k(t), \psi \rangle \langle \mathcal{M}_k(t), \zeta \rangle - \langle \mathcal{M}_k(s), \psi \rangle \langle \mathcal{M}_k(s), \zeta \rangle \right. \right. \\ \left. \left. - \int_s^t \left(\mathbf{G}(\tilde{\mathbf{u}}_k(\sigma))^* \psi, \mathbf{G}(\tilde{\mathbf{u}}_k(\sigma))^* \zeta \right)_K d\sigma \right) \cdot h(\tilde{\mathbf{u}}_k|_{[0,s]}) \right] = 0. \quad (3.47)$$

Here, $\langle \cdot, \cdot \rangle$ stands for the dual pairing between \mathbb{U}' and \mathbb{U} . This completes the proof of the lemma. \square

Lemma 3.7. *For all $s, t \in [0, T]$ such that $0 \leq s \leq t$, all $\psi \in \mathbb{U}$ and all (real-valued) function h bounded continuous on $C([0, s]; \mathbb{U}')$ we have*

$$\lim_{k \rightarrow \infty} \mathbb{E} \left[\langle \mathcal{M}_k(t) - \mathcal{M}_k(s), \psi \rangle h(\tilde{\mathbf{u}}_k|_{[0,s]}) \right] = \mathbb{E} \left[\langle \mathcal{M}(t) - \mathcal{M}(s), \psi \rangle h(\tilde{\mathbf{u}}|_{[0,s]}) \right].$$

Proof. For any $t, s \in [0, T]$ with $s \leq t$ and $\psi \in \mathbb{U}$, we have

$$\langle \mathcal{M}_k(t) - \mathcal{M}_k(s), \psi \rangle = (\tilde{\mathbf{u}}_k(t), \psi) - (\tilde{\mathbf{u}}_k(s), \psi) + \int_s^t \langle \mathbf{A}_\alpha \tilde{\mathbf{u}}_k(\tau), \psi \rangle d\tau \\ + \int_s^t \langle \mathbf{B}(\tilde{\mathbf{u}}_k(\tau)), \psi \rangle d\tau + \int_s^t \langle \mathbf{R}_\gamma \tilde{\mathbf{u}}_k(\tau), \psi \rangle d\tau,$$

which along with Lemma 3.6 implies

$$\lim_{n \rightarrow \infty} \langle \mathcal{M}_k(t) - \mathcal{M}_k(s), \psi \rangle = \langle \mathcal{M}(t) - \mathcal{M}(s), \psi \rangle, \quad \tilde{\mathbb{P}}\text{-a.s..}$$

In particular,

$$\lim_{n \rightarrow \infty} \langle \mathcal{M}_k(t), \psi \rangle = \langle \mathcal{M}(t), \psi \rangle, \quad \tilde{\mathbb{P}}\text{-a.s..} \quad (3.48)$$

One should also note that

$$h(\tilde{\mathbf{u}}_{k|[0,s]}) \longrightarrow h(\tilde{\mathbf{u}}_{|[0,s]}) \quad \tilde{\mathbb{P}}\text{-a.s. as } n \rightarrow \infty,$$

from which and the previous convergence we infer that

$$\langle \mathcal{M}_k(t) - \mathcal{M}_k(s), \psi \rangle h(\tilde{\mathbf{u}}_{k|[0,s]}) \longrightarrow \langle \mathcal{M}(t) - \mathcal{M}(s), \psi \rangle h(\tilde{\mathbf{u}}_{|[0,s]}) \quad \tilde{\mathbb{P}}\text{-a.s..} \quad (3.49)$$

Similarly, passing to the limit in (3.47) we have

$$\begin{aligned} \tilde{\mathbb{E}} \left[\left(\langle \mathcal{M}(t), \psi \rangle \langle \mathcal{M}(t), \zeta \rangle - \langle \mathcal{M}(s), \psi \rangle \langle \mathcal{M}(s), \zeta \rangle \right. \right. \\ \left. \left. - \int_s^t \left(\mathbf{G}(\tilde{\mathbf{u}}(\sigma))^* \psi, \mathbf{G}(\tilde{\mathbf{u}}(\sigma))^* \zeta \right)_{\mathbf{K}} d\sigma \right) h(\tilde{\mathbf{u}}_{|[0,s]}) \right] = 0. \end{aligned} \quad (3.50)$$

We will now show that

$$\tilde{\mathbb{E}} |\langle \mathcal{M}_k(t) - \mathcal{M}_k(s), \psi \rangle h(\tilde{\mathbf{u}}_{k|[0,s]})|^2 < \infty, \quad (3.51)$$

which along with (3.49) and the Lebesgue Dominated Convergence Theorem (LDCT) completes the proof of the lemma.

Hence, it remains to prove (3.51). To this end, observe that $\mathcal{M}_k(t)$ is a continuous martingale with quadratic variation

$$[\mathcal{M}_k]_t = \int_0^t \mathbf{G}_k(\tilde{\mathbf{u}}_k(s)) \mathbf{G}_k(\tilde{\mathbf{u}}_k(s))^* ds, \quad t \in [0, T].$$

By Burkholder–Gundy–Davis’ inequality, (3.27) and Remark 3.2 we have

$$\tilde{\mathbb{E}} \sup_{t \in [0, T]} \|\mathcal{M}_k(t)\|_{\mathcal{H}^{-\theta}}^{2r} \leq C \tilde{\mathbb{E}} \left[\int_0^T \|\mathbf{J}^{-\theta} \mathbf{G}(\tilde{\mathbf{u}}_k(t))\|_{\gamma(\mathbf{K}, \mathcal{H})}^2 dt \right]^r \leq C, \quad (3.52)$$

for $r \geq 1$ and some constant $C > 0$ independent of k .

Next, from Cauchy–Schwarz’s inequality and the continuous embedding $\mathbb{U} \subset \mathcal{H}$, we easily infer that for any bounded function $h : C([0, s]; \mathbb{U}) \rightarrow \mathbb{R}$, $s \leq t$,

$$\begin{aligned} \tilde{\mathbb{E}} |\langle \mathcal{M}_k(t) - \mathcal{M}_k(s), \psi \rangle h(\tilde{\mathbf{u}}_k)|^2 &\leq \tilde{\mathbb{E}} \left(|\langle \mathcal{M}_k(t) - \mathcal{M}_k(s), \psi \rangle|^2 |h(\tilde{\mathbf{u}}_{n|[0,s]})|^2 \right) \\ &\leq \tilde{\mathbb{E}} \left(\|\mathcal{M}_k(t) - \mathcal{M}_k(s)\|_{\mathcal{H}^{-\theta}}^2 \|\psi\|^2 \|h\|_{L^\infty}^2 \right) \\ &\leq \tilde{\mathbb{E}} \left(\|\mathcal{M}_k(t)\|_{\mathcal{H}^{-\theta}}^2 + \|\mathcal{M}_k(s)\|_{\mathcal{H}^{-\theta}}^2 \right) \|\psi\|_{\mathbb{U}}^2 \|h\|_{L^\infty}^2, \end{aligned}$$

from which along with (3.52) we easily complete the proof of (3.51) and the Lemma. \square

The next step will be stated in the following lemma.

Lemma 3.8. *The quadratic variation $[\tilde{\mathcal{M}}_k]_t$ satisfies*

$$\int_0^t (\mathbf{G}_k(\tilde{\mathbf{u}}_k(s))^* \phi, \mathbf{G}_k(\tilde{\mathbf{u}}_k(s))^* \zeta)_K ds \longrightarrow \int_0^t (\mathbf{G}(\tilde{\mathbf{u}}(s))^* \phi, \mathbf{G}(\tilde{\mathbf{u}}(s))^* \zeta)_K ds, \forall \phi, \zeta \in \mathbb{U}$$

Proof. The proof is quite similar to the proof of [7, Eq. (3.35)]. To prove the lemma we observe that from (3.17)–(3.21) and (3.27) we infer that

$$\begin{aligned} & \int_0^t |(\mathbf{G}_k(\tilde{\mathbf{u}}_k(s))^* \phi, \mathbf{G}_k(\tilde{\mathbf{u}}_k(s))^* \zeta)_K - (\mathbf{G}(\tilde{\mathbf{u}}(s))^* \phi, \mathbf{G}(\tilde{\mathbf{u}}(s))^* \zeta)_K| ds \\ & \leq \int_0^t |(\mathbf{G}_k(\tilde{\mathbf{u}}_k(s))^* \phi - \mathbf{G}(\tilde{\mathbf{u}}(s))^* \phi, \mathbf{G}_k(\tilde{\mathbf{u}}_k(s))^* \zeta)_K| ds + \\ & \quad + \int_0^t |(\mathbf{G}(\tilde{\mathbf{u}}(s))^* \phi, \mathbf{G}_k(\tilde{\mathbf{u}}_k(s))^* \zeta - \mathbf{G}(\tilde{\mathbf{u}}(s))^* \zeta)_K| ds \\ & \leq \int_0^t \|\mathbf{G}_k(\tilde{\mathbf{u}}_k(s))^* \phi - \mathbf{G}(\tilde{\mathbf{u}}(s))^* \phi\|_K \times \|\mathbf{G}_k(\tilde{\mathbf{u}}_k(s))^* \zeta\|_K ds + \\ & \quad + \int_0^t \|\mathbf{G}(\tilde{\mathbf{u}}(s))^* \phi\|_K \|\mathbf{G}_k(\tilde{\mathbf{u}}_k(s))^* \zeta - \mathbf{G}(\tilde{\mathbf{u}}(s))^* \zeta\|_K ds \\ & \leq \int_0^t \|\mathbf{G}_k(\tilde{\mathbf{u}}_k(s))^* \phi - \mathbf{G}(\tilde{\mathbf{u}}(s))^* \phi\|_K \|\zeta\|_{\mathcal{H}^{-\theta}} \sup_{\mathbf{u} \in \mathcal{H}} \|\mathbf{G}_k(\tilde{\mathbf{u}}_k(s))\|_{\gamma(K, \mathcal{H}^{-\theta})} ds + \\ & \quad + \int_0^t \|\mathbf{G}_k(\tilde{\mathbf{u}}_k(s))^* \zeta - \mathbf{G}(\tilde{\mathbf{u}}(s))^* \zeta\|_K \|\phi\|_{\mathcal{H}^{-\theta}} \sup_{\mathbf{u} \in \mathcal{H}} \|\mathbf{G}(\tilde{\mathbf{u}}(s))\|_{\gamma(K, \mathcal{H}^{-\theta})} ds \end{aligned}$$

which converges to zero because

$$\begin{aligned} & \int_0^t \|\mathbf{G}_k(\tilde{\mathbf{u}}_k(s))^* \phi - \mathbf{G}(\tilde{\mathbf{u}}(s))^* \phi\|_K \|\zeta\|_{\mathcal{H}^{-\theta}} ds \\ & \leq C \int_0^t \|\mathbf{G}(\tilde{\mathbf{u}}_k(s))^* \phi - \mathbf{G}(\tilde{\mathbf{u}}(s))^* \phi\|_K ds + C \int_0^t \|\mathbf{G}(\tilde{\mathbf{u}}_k(s))^* \mathcal{R}_k \phi - \mathbf{G}(\tilde{\mathbf{u}}(s))^* \phi\|_K ds \\ & \leq C \int_0^t \|\mathbf{G}(\tilde{\mathbf{u}}_k(s))^* \phi - \mathbf{G}(\tilde{\mathbf{u}}(s))^* \phi\|_K ds + C \|\mathcal{R}_k \phi - \phi\|_{\mathcal{H}^{-\theta}} \longrightarrow 0 \text{ as } k \rightarrow \infty \end{aligned}$$

by means of (3.21), (3.45) and the Lebesgue Dominated Convergence Theorem. \square

From Lemma 3.7, 3.6 and 3.8 we infer that the process $\tilde{\mathcal{M}}(\cdot)$ satisfies that for any $\phi \in \mathbb{U}$

$$\langle \tilde{\mathcal{M}}(t), \phi \rangle = \int_0^t \langle \mathbf{G}(\tilde{\mathbf{u}}(s))^* \phi, d\tilde{\mathcal{W}} \rangle = \left\langle \phi, \int_0^t \mathbf{G}(\tilde{\mathbf{u}}(s)) d\tilde{\mathcal{W}} \right\rangle$$

which implies that $\tilde{\mathcal{M}}(\cdot)$ is a \mathbb{U}' -valued martingale.

Now, let $L_{\mathcal{H}} =: \tilde{L} : D(\tilde{L}) \subset \mathcal{H} \rightarrow \mathbb{U}$ be the restriction to \mathcal{H} of the isomorphism $L : \mathbb{U}' \rightarrow \mathbb{U}$ and $\tilde{L}' : \mathbb{U}' \rightarrow \mathcal{H}$ its dual. Note that we have used the identification $\mathcal{H} \simeq \mathcal{H}'$ in the definition of \tilde{L}' . Taking $\psi = \tilde{L}(\varphi)$ and $\zeta = \tilde{L}(\phi)$ with $\varphi, \phi \in \mathcal{H}$ in (3.49) and (3.50) we infer that $\tilde{L}'\mathcal{M}$ is a \mathcal{H} -valued martingale with quadratic variation

$$[\tilde{L}'\mathcal{M}]_t = \int_0^t \tilde{L}'G(\tilde{\mathbf{u}}(s))(G(\tilde{\mathbf{u}}(s))\tilde{L}')^* ds.$$

Since $\mathbf{u} \in C([0, T]; \mathbb{U}')$, it follows that $\tilde{L}'\mathcal{M} \in C([0, T]; \mathbb{U}')$. Now, applying the Martingale Representation Theorem, see [14], we infer that there exist

- (i) a stochastic basis $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{F}}, \check{\mathbb{P}})$,
- (ii) a cylindrical Wiener process \check{W} defined on this stochastic basis, and a progressively measurable process $\check{\mathbf{u}}$ such that

$$\tilde{L}'\check{\mathbf{u}}(t) - \tilde{L}'\check{\mathbf{u}}(0) + \int_0^t \tilde{L}'[\mathbf{A}_{\alpha}\check{\mathbf{u}} + \mathbf{B}(\check{\mathbf{u}}) + \mathbf{R}_{\gamma}(\check{\mathbf{u}})](s) ds = \int_0^t \tilde{L}'\mathbf{G}(\check{\mathbf{u}}(s)) d\check{W}(s).$$

Since

$$\int_0^t \tilde{L}'\mathbf{G}(\check{\mathbf{u}}(s)) d\check{W}(s) = \tilde{L}' \int_0^t \mathbf{G}(\check{\mathbf{u}}(s)) d\check{W}(s),$$

we infer that for all $t \in [0, T]$ and $\psi \in \mathbb{U}$

$$(\check{\mathbf{u}}(t) - \check{\mathbf{u}}(0), \psi) + \int_0^t \langle [\mathbf{A}_{\alpha}\check{\mathbf{u}} + \mathbf{B}(\check{\mathbf{u}}) + \mathbf{R}_{\gamma}(\check{\mathbf{u}})](s), \psi \rangle ds = \left\langle \int_0^t \mathbf{G}(\check{\mathbf{u}}(s)) d\check{W}(s), \psi \right\rangle.$$

Hence, $\{(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{F}}, \check{\mathbb{P}}), \check{W}, \check{\mathbf{u}}\}$ is a weak martingale solution to our problem. This completes the proof of Theorem 2.1.

3.2. Proof of Theorem 3.1: the time regularity when $\alpha \geq \frac{d}{2}$

To complete the proof of Theorem 3.1, we will prove that if $\alpha \geq \frac{d}{2}$, then $\mathbf{u} \in C([0, T]; \mathcal{H})$ with probability 1. To this end, for a martingale solution as \mathbf{u} we set $\mathbf{v} = \mathbf{u} - \mathbf{y}$ where \mathbf{y} is a solution to

$$d\mathbf{y} + \mathbf{A}_{\alpha}\mathbf{y} = \mathbf{G}(\mathbf{u})dW, \quad \mathbf{y}(0) = 0.$$

It is clear that \mathbf{v} solves

$$\frac{d\mathbf{v}}{dt} + \mathbf{A}_{\alpha}\mathbf{v} + \mathbf{B}(\mathbf{u}) + \mathbf{R}_{\gamma}\mathbf{u} = 0, \quad \mathbf{v}(0) = \mathbf{u}_0.$$

Notice that since $\mathbf{u} \in \mathbb{L}^{\frac{8\alpha}{d}}(0, T; \mathcal{H}^{0,4})$ and $\alpha \geq \frac{d}{2}$ we easily infer from (2.8) that $\mathbf{B}(\mathbf{u}) \in \mathbb{L}^2(0, T; \mathcal{H}^{-1})$. With this in mind one can argue as in Lemma 3.3 to show that $\mathbf{v} \in \mathbb{L}^\infty(0, T; \mathcal{H}) \cap \mathbb{L}^2(0, T; \mathcal{H}^\alpha)$. Now, with these observations and the following identity

$$\frac{d\mathbf{v}}{dt} = -(\mathbf{A}_\alpha \mathbf{v} + \mathbf{B}(\mathbf{u}) + \mathbf{R}_\gamma \mathbf{u}),$$

it is not difficult to prove that $\frac{d\mathbf{v}}{dt} \in \mathbb{L}^2(0, T; \mathcal{H}^{-\alpha})$. Thus, applying [37, Lemma III.1.2] we infer that $\mathbf{v} \in C([0, T]; \mathcal{H})$. Now, arguing as in Corollary 3.1 we infer that $\mathbf{y} \in C([0, T]; \mathcal{H})$ with probability 1. Hence, with the identity $\mathbf{u} = \mathbf{v} + \mathbf{y}$ we infer that $\mathbf{u} \in C([0, T]; \mathcal{H})$ with probability 1.

3.3. Proof of Theorem 3.2: the pathwise uniqueness

In this subsection we will prove the pathwise uniqueness of our solution. We use similar argument as in the proof of [7, Theorem 4.1]. However, we should note that in contrast to [7, Theorem 4.1] which treats the pathwise uniqueness of two dimensional stochastic Navier–Stokes equations, in the present paper we give uniqueness result of stochastic fractional MHD in both $d = 2, 3$ under the assumption that $\alpha \geq \frac{d}{2}$.

Let \mathbf{u}_1 and \mathbf{u}_2 be two solutions of (2.12) with the same initial data, i.e., $\mathbf{u}_1(0) = \mathbf{u}_2(0) = \mathbf{u}_0$. The processes $\mathbf{u}_1, \mathbf{u}_2$ satisfy

$$\mathbf{u}_1, \mathbf{u}_2 \in C([0, T]; \mathcal{H}_w) \cap \mathbb{L}^{\frac{4\alpha}{\alpha-\theta}}(0, T; \mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}) \cap \mathbb{L}^{\frac{8\alpha}{d}}(0, T; \mathcal{H}^{0,4}).$$

Now let us set $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$. Since \mathbf{B} is bilinear we have

$$\mathbf{B}(\mathbf{u}_1, \mathbf{u}_1) - \mathbf{B}(\mathbf{u}_2, \mathbf{u}_2) = \mathbf{B}(\mathbf{w}, \mathbf{u}_1) + \mathbf{B}(\mathbf{u}_2, \mathbf{w}),$$

and it is easy to see that \mathbf{w} solves

$$d\mathbf{w} + [\mathbf{A}_\alpha \mathbf{w} + \mathbf{B}(\mathbf{w}, \mathbf{u}_1) + \mathbf{B}(\mathbf{u}_2, \mathbf{w}) + \mathbf{R}_\gamma \mathbf{w}] dt = [\mathbf{G}(\mathbf{u}_1) - \mathbf{G}(\mathbf{u}_2)] dW. \quad (3.53)$$

Before we proceed to the proof of the uniqueness of solution we should make few observations. First, knowing that

$$(\mathbf{J}^{-\theta} \mathbf{A}_\alpha \mathbf{w}, \mathbf{J}^{-\theta} \mathbf{w}) = (\mathbf{J}^{-\theta} \mathbf{A}_{\alpha/2} \mathbf{w}, \mathbf{J}^{-\theta} \mathbf{A}_{\alpha/2} \mathbf{w}) = \|\mathbf{J}^{-\theta} \mathbf{A}_{\alpha/2} \mathbf{w}\|^2,$$

we infer that there exists a constant \tilde{K}_0 such that

$$(\mathbf{J}^{-\theta} (\mathbf{R}_\gamma \mathbf{w} + \mathbf{A}_\alpha \mathbf{w}), \mathbf{J}^{-\theta} \mathbf{w}) \geq \tilde{K}_0 (\|\mathbf{J}^{-\theta} \mathbf{A}_{\frac{\alpha}{2}} \mathbf{w}\|^2 + \|\mathbf{J}^{-\theta} \mathbf{w}\|^2).$$

Second, we state and prove the following lemma whose proof is inspired by [7, Proof of (4.2), p. 74].

Lemma 3.9. *There exists a constant \tilde{K}_1 such that for any $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}$ and $\mathbf{w} \in \mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}$*

$$\begin{aligned} & |(\mathbf{J}^{-\theta} \mathbf{B}(\mathbf{w}, \mathbf{u}_1), \mathbf{J}^{-\theta} \mathbf{w})| + |(\mathbf{J}^{-\theta} \mathbf{B}(\mathbf{u}_2, \mathbf{w}), \mathbf{J}^{-\theta} \mathbf{w})| \\ & \leq \tilde{K}_1 \left(\frac{4}{\tilde{K}_0} \right)^{\frac{1+\lambda}{1-\lambda}} \left[\|\mathbf{u}_1\|_{\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}} + \|\mathbf{u}_2\|_{\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}} \right]^{\frac{2}{1-\lambda}} \|\mathbf{w}\|_{\mathcal{H}^{-\theta}}^2 \\ & \quad + \frac{\tilde{K}_0}{2} (\|\mathbf{J}^{-\theta} \mathbf{A}_{\frac{\alpha}{2}} \mathbf{w}\|^2 + \|\mathbf{J}^{-\theta} \mathbf{w}\|^2), \end{aligned} \quad (3.54)$$

where

$$\lambda = \frac{\alpha - \theta}{\alpha^2} + \frac{\theta}{\alpha}.$$

Since

$$\frac{\alpha - \theta}{\alpha^2} + \frac{\theta}{\alpha} = 1 - \left(1 - \frac{1}{2\alpha}\right) \frac{\alpha - \theta}{\alpha},$$

it is clear that

$$\lambda \in (0, 1).$$

Proof. We first observe that

$$(\mathbf{J}^{-\theta} \mathbf{B}(\mathbf{w}, \mathbf{u}_1), \mathbf{J}^{-\theta} \mathbf{w}) = (\mathbf{J}^{-\alpha-\theta} \mathbf{B}(\mathbf{w}, \mathbf{u}_1), \mathbf{J}^{\alpha-\theta} \mathbf{w}). \quad (3.55)$$

Second, let

$$p_1 = p_3 = \frac{2\alpha d}{\alpha(d-1) + \theta}, \quad p_2 = \frac{\alpha d}{\alpha - \theta} \text{ and } \delta = \frac{d}{2} + \frac{\theta}{\alpha} - 1.$$

It is not difficult to check that $\sum_{i=1}^3 p_i^{-1} = 1$ and, by assumption, $\alpha + \theta \geq 1 + \delta$. Thus, using Hölder's inequality, the Sobolev embeddings $\mathbb{L}^{p_1}, \mathbb{L}^{p_3} \supset \mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}$, $\mathbb{L}^{p_2} \supset \mathcal{H}^\delta$ and $\mathcal{H}^{\alpha+\theta} \subset \mathcal{H}^{1+\delta}$ we deduce the following chain of inequalities

$$\begin{aligned} \|\mathbf{B}(\mathbf{w}, \mathbf{u}_1)\|_{-\alpha-\theta} &= \sup_{\varphi \in \mathcal{H}^{\alpha+\theta}, \|\varphi\|_{\mathcal{H}^{\alpha+\theta}} \leq 1} \left| \int_{\mathbb{R}^d} \mathbf{w} \cdot \nabla \mathbf{u}_1 \varphi dx \right| \\ &\leq C \sup_{\varphi \in \mathcal{H}^{\alpha+\theta}, \|\varphi\|_{\mathcal{H}^{\alpha+\theta}} \leq 1} \left| \int_{\mathbb{R}^d} \mathbf{w} \cdot \nabla \varphi \mathbf{u}_1 dx \right| \\ &\leq C \sup_{\varphi \in \mathcal{H}^{\alpha+\theta}, \|\varphi\|_{\mathcal{H}^{\alpha+\theta}} \leq 1} \|\mathbf{w}\|_{\mathbb{L}^{p_1}} \|\nabla \varphi\|_{\mathbb{L}^{p_2}} \|\mathbf{u}_1\|_{\mathbb{L}^{p_3}} \\ &\leq C \sup_{\varphi \in \mathcal{H}^{\alpha+\theta}, \|\varphi\|_{\mathcal{H}^{\alpha+\theta}} \leq 1} \|\mathbf{w}\|_{\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}} \|\nabla \varphi\|_{\mathcal{H}^\delta} \|\mathbf{u}_1\|_{\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}} \\ &\leq C \sup_{\varphi \in \mathcal{H}^{\alpha+\theta}, \|\varphi\|_{\mathcal{H}^{\alpha+\theta}} \leq 1} \|\mathbf{w}\|_{\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}} \|\varphi\|_{\mathcal{H}^{1+\delta}} \|\mathbf{u}_1\|_{\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}}, \\ &\leq C \sup_{\varphi \in \mathcal{H}^{\alpha+\theta}, \|\varphi\|_{\mathcal{H}^{\alpha+\theta}} \leq 1} \|\mathbf{B}(\mathbf{w}, \mathbf{u}_1)\|_{\mathcal{H}^{-\alpha-\theta}} \\ &\leq C \|\mathbf{w}\|_{\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}} \|\varphi\|_{\mathcal{H}^{\alpha+\theta}} \|\mathbf{u}_1\|_{\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}}. \end{aligned}$$

Hence,

$$\|\mathbf{B}(\mathbf{w}, \mathbf{u}_1)\|_{\mathcal{H}^{-\alpha-\theta}} \leq C \|\mathbf{B}(\mathbf{w}, \mathbf{u}_1)\|_{\mathcal{H}^{-\alpha-\theta}} \leq \|\mathbf{w}\|_{\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}} \|\mathbf{u}_1\|_{\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}},$$

which along with the interpolation identity

$$\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}} = [\mathcal{H}^{-\theta}, \mathcal{H}^{\alpha-\theta}]_\lambda,$$

yields

$$\|\mathbf{B}(\mathbf{w}, \mathbf{u}_1)\|_{\mathcal{H}^{-(\alpha+\theta)}} \leq C \|\mathbf{w}\|_{\mathcal{H}^{-\theta}}^{1-\lambda} \|\mathbf{w}\|_{\mathcal{H}^{\alpha-\theta}}^{\lambda} \|\mathbf{u}_1\|_{\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}}$$

In a similar way we prove that

$$\|\mathbf{B}(\mathbf{u}_2, \mathbf{w})\|_{\mathcal{H}^{-(\alpha+\theta)}} \leq C \|\mathbf{w}\|_{\mathcal{H}^{-\theta}}^{1-\lambda} \|\mathbf{w}\|_{\mathcal{H}^{\alpha-\theta}}^{\lambda} \|\mathbf{u}_2\|_{\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}}.$$

It now follows from (3.55) and the two last inequalities that

$$\begin{aligned} & |(\mathbf{J}^{-\theta} \mathbf{B}(\mathbf{w}, \mathbf{u}_1), \mathbf{J}^{-\theta} \mathbf{w})| + |(\mathbf{J}^{-\theta} \mathbf{B}(\mathbf{u}_2, \mathbf{w}), \mathbf{J}^{-\theta} \mathbf{w})| \\ & \leq C \left[\|\mathbf{u}_1\|_{\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}} + \|\mathbf{u}_2\|_{\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}} \right] \|\mathbf{w}\|_{\mathcal{H}^{-\theta}}^{1-\lambda} \|\mathbf{w}\|_{\mathcal{H}^{\alpha-\theta}}^{1+\lambda} \\ & \leq C \left[\|\mathbf{u}_1\|_{\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}} + \|\mathbf{u}_2\|_{\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}} \right] \|\mathbf{w}\|_{\mathcal{H}^{-\theta}}^{1-\lambda} \left(\|\mathbf{J}^{-\theta} \mathbf{A}_{\frac{\alpha}{2}} \mathbf{w}\| + \|\mathbf{J}^{-\theta} \mathbf{w}\| \right)^{1+\lambda}, \end{aligned}$$

from which along with Young's inequality with the conjugate exponents $p = \frac{2}{1+\lambda}$ and $q = \frac{2}{1-\lambda}$ we easily complete the proof of the lemma. \square

We now proceed to the proof of the uniqueness of the solution. For this aim, we apply $\mathbf{J}^{-\theta}$ to both sides of (3.53) and apply Itô's formula to the function $\|\mathbf{J}^{-\theta} \mathbf{w}\|^2$ to derive

$$\begin{aligned} & d\|\mathbf{J}^{-\theta} \mathbf{w}\|^2 + 2(\mathbf{J}^{-\theta} \mathbf{A}_{\alpha} \mathbf{w}, \mathbf{J}^{-\theta} \mathbf{w}) + 2(\mathbf{J}^{-\theta} \mathbf{R}_{\gamma} \mathbf{w}, \mathbf{J}^{-\theta} \mathbf{w}) + 2(\mathbf{J}^{-\theta} \mathbf{B}(\mathbf{w}, \mathbf{u}_1) \\ & \quad = -\mathbf{J}^{-\theta} \mathbf{B}(\mathbf{u}_2, \mathbf{w}), \mathbf{J}^{-\theta} \mathbf{w}) + \frac{1}{2} \|\mathbf{G}(\mathbf{u}_1) - \mathbf{G}(\mathbf{u}_2)\|_{\gamma(\mathbf{K}, H^{-\theta})}^2 \\ & \quad + \langle \mathbf{J}^{-\theta} [\mathbf{G}(\mathbf{u}_1) - \mathbf{G}(\mathbf{u}_2)] dW, \mathbf{J}^{-\theta} \mathbf{w} \rangle. \end{aligned}$$

Let us set

$$\Sigma(t) = e^{-\int_0^t \varrho(s) ds},$$

where

$$\varrho(s) = \tilde{K}_1 \left(\frac{4}{\tilde{K}_0} \right)^{\frac{1+\lambda}{1-\lambda}} \left[\|\mathbf{u}_1(s)\|_{\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}} + \|\mathbf{u}_2(s)\|_{\mathcal{H}^{\frac{\alpha-\theta}{2\alpha}}} \right]^{\frac{2}{1-\lambda}},$$

and \tilde{K}_0 is the constant from (3.54). From Itô's formula we infer that

$$d \left[e^{-\int_0^t \varrho(s) ds} \|\mathbf{w}\|_{\mathcal{H}^{-\theta}}^2 \right] = e^{-\int_0^t \varrho(s) ds} d\|\mathbf{w}\|_{\mathcal{H}^{-\theta}}^2 - \varrho(s) e^{-\int_0^t \varrho(s) ds} \|\mathbf{w}\|_{\mathcal{H}^{-\theta}}^2,$$

which is equivalent to

$$\begin{aligned} d(\Sigma(t) \|\mathbf{w}\|_{\mathcal{H}^{-\theta}}^2) & = -\Sigma(t) (\mathbf{J}^{-\theta} [\mathbf{R}_{\gamma} \mathbf{w} + \mathbf{A}_{\alpha} \mathbf{w}], \mathbf{J}^{-\theta} \mathbf{w}) dt - \varrho(t) \Sigma(t) \|\mathbf{w}\|_{\mathcal{H}^{-\theta}}^2 dt \\ & \quad - \Sigma(t) (\mathbf{J}^{-\theta} [\mathbf{B}(\mathbf{w}, \mathbf{u}_1) + \mathbf{B}(\mathbf{u}_2, \mathbf{w})], \mathbf{J}^{-\theta} \mathbf{w}) dt \\ & \quad + \Sigma(t) (\mathbf{J}^{-\theta} [\mathbf{G}(\mathbf{u}_1) - \mathbf{G}(\mathbf{u}_2)] dW, \mathbf{J}^{-\theta} \mathbf{w}) \\ & \quad + \Sigma(t) \|\mathbf{G}(\mathbf{u}_1) - \mathbf{G}(\mathbf{u}_2)\|_{\gamma(\mathbf{K}, \mathcal{H}^{-\theta})}^2 dt. \end{aligned}$$

This identity along with (3.54) imply

$$\begin{aligned}
& d\left(\Sigma(t)\|\mathbf{w}\|_{\mathcal{H}^{-\theta}}^2\right) + \tilde{K}_0 \Sigma(t)\left(\|\mathbf{J}^{-\theta} \mathbf{A}_{\frac{\alpha}{2}} \mathbf{w}\|^2 + \|\mathbf{J}^{-\theta} \mathbf{w}\|^2\right) dt \\
& \leq \frac{\tilde{K}_0}{2} \Sigma(t)\left(\|\mathbf{J}^{-\theta} \mathbf{A}_{\frac{\alpha}{2}} \mathbf{w}\|^2 + \|\mathbf{J}^{-\theta} \mathbf{w}\|^2\right) + \Sigma(t)\|\mathbf{G}(\mathbf{u}_1) - \mathbf{G}(\mathbf{u}_2)\|_{\gamma(\mathbf{K}, \mathcal{H}^{-\theta})}^2 dt \\
& \quad + \Sigma(t)\left(\mathbf{J}^{-\theta}[\mathbf{G}(\mathbf{u}_1) - \mathbf{G}(\mathbf{u}_2)]dW, \mathbf{J}^{-\theta} \mathbf{w}\right).
\end{aligned}$$

Absorbing the first term of the right-hand side to the last left-hand side implies that

$$\begin{aligned}
& d\left(\Sigma(t)\|\mathbf{w}\|_{\mathcal{H}^{-\theta}}^2\right) + \frac{\tilde{K}_0}{2} \Sigma(t)\left(\|\mathbf{J}^{-\theta} \mathbf{A}_{\frac{\alpha}{2}} \mathbf{w}\|^2 + \|\mathbf{J}^{-\theta} \mathbf{w}\|^2\right) dt \\
& \leq \Sigma(t)\|\mathbf{G}(\mathbf{u}_1) - \mathbf{G}(\mathbf{u}_2)\|_{\gamma(\mathbf{K}, \mathcal{H}^{-\theta})}^2 dt + \Sigma(t)\left(\mathbf{J}^{-\theta}[\mathbf{G}(\mathbf{u}_1) - \mathbf{G}(\mathbf{u}_2)]dW, \mathbf{J}^{-\theta} \mathbf{w}\right). \quad (3.56)
\end{aligned}$$

We now prove that the term in the right hand side of (3.56),

$$\mathcal{M}(t) := \int_0^t \Sigma(s)\left(\mathbf{J}^{-\theta}[\mathbf{G}(\mathbf{u}_1) - \mathbf{G}(\mathbf{u}_2)], \mathbf{J}^{-\theta} \mathbf{w}\right)(s) dW$$

is a local and square integrable martingale. For this purpose, we define the stopping time

$$\tau_N := T \wedge \inf\{t \in [0, T] : \|\mathbf{w}\|_{\mathcal{H}^{-\theta}} > N\}.$$

Integrating both sides of (3.56) over the interval $[0, t \wedge \tau_N]$ and taking mathematical expectation we obtain

$$\begin{aligned}
& \mathbb{E}\left(\Sigma(t \wedge \tau_N)\|\mathbf{w}(t \wedge \tau_N)\|_{\mathcal{H}^{-\theta}}^2\right) \\
& \leq \mathbb{E}(\mathcal{M}(t \wedge \tau_N)) + C \mathbb{E} \int_0^{t \wedge \tau_N} \left(\Sigma(t \wedge \tau_N)\|\mathbf{G}(\mathbf{u}_1(s)) - \mathbf{G}(\mathbf{u}_2(s))\|_{\gamma(\mathbf{K}, \mathcal{H}^{-\theta})}^2\right) ds.
\end{aligned}$$

By Burkholder–Gundy’s inequality and the Lipschitz assumption on \mathbf{G} , we have

$$\begin{aligned}
\mathbb{E} \mathcal{M}(t \wedge \tau_N) & \leq \mathbb{E} \left| \int_0^{t \wedge \tau_N} \Sigma(s)\left(\mathbf{J}^{-\theta}[\mathbf{G}(\mathbf{u}_1(s)) - \mathbf{G}(\mathbf{u}_2(s))], \mathbf{J}^{-\theta} \mathbf{w}(s)\right) dW \right| \\
& \leq \left[\mathbb{E} \left| \int_0^{t \wedge \tau_N} \Sigma(s)\left(\mathbf{J}^{-\theta}[\mathbf{G}(\mathbf{u}_1(s)) - \mathbf{G}(\mathbf{u}_2(s))], \mathbf{J}^{-\theta} \mathbf{w}(s)\right) dW \right|^2 \right]^{\frac{1}{2}} \\
& \leq C \left[\mathbb{E} \int_0^{t \wedge \tau_N} \Sigma^2(s)\|\mathbf{w}(s)\|_{\mathcal{H}^{-\theta}}^2 \|\mathbf{G}(\mathbf{u}_1(s)) - \mathbf{G}(\mathbf{u}_2(s))\|_{\gamma(\mathbf{K}, \mathcal{H}^{-\theta})}^2 ds \right]^{\frac{1}{2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}|\mathcal{M}(t \wedge \tau_N)|^2 & \leq \mathbb{E} \int_0^{t \wedge \tau_N} \Sigma^2(s)\|\mathbf{w}(s)\|_{\mathcal{H}^{-\theta}}^2 \|\mathbf{G}(\mathbf{u}_1(s)) - \mathbf{G}(\mathbf{u}_2(s))\|_{\gamma(\mathbf{K}, \mathcal{H}^{-\theta})}^2 ds \\
& \leq C \mathbb{E} \int_0^{t \wedge \tau_N} \|\mathbf{w}(s)\|^4 ds \\
& \leq CN^4 \mathbb{E}(t \wedge \tau_N) \leq CN^4 \quad (3.57)
\end{aligned}$$

from which we deduce that the sequence $\mathcal{M}(\cdot \wedge \tau_N)$ is a square integrable martingale satisfying $\mathbb{E}(\mathcal{M}(t \wedge \tau_N)) = 0$ for any $t \in [0, T]$.

Now, owing to Assumption 3.2 and (3.57), we infer that

$$\begin{aligned} & \mathbb{E} \left(\Sigma(t \wedge \tau_N) \|\mathbf{w}(t \wedge \tau_N)\|_{\mathcal{H}^{-\theta}}^2 \right) \\ & \leq C \mathbb{E} \int_0^{t \wedge \tau_N} \left(\Sigma(s) \|\mathbf{G}(\mathbf{u}_1(s)) - \mathbf{G}(\mathbf{u}_2(s))\|_{\gamma(\mathbf{K}, \mathcal{H}^{-\theta})}^2 \right) ds \\ & \leq C \mathbb{E} \int_0^{t \wedge \tau_N} \Sigma(s) \|\mathbf{w}(s)\|_{\mathcal{H}^{-\theta}}^2 ds, \end{aligned}$$

from which we deduce using Gronwall's inequality that

$$\mathbb{E} \left(\Sigma(t \wedge \tau_N) \|\mathbf{w}(t \wedge \tau_N)\|_{\mathcal{H}^{-\theta}}^2 \right) \leq 0.$$

Hence

$$\mathbb{E} \left(\Sigma(t \wedge \tau_N) \|\mathbf{w}(t \wedge \tau_N)\|_{\mathcal{H}^{-\theta}}^2 \right) = 0 \quad \mathbb{P}\text{-a.s.}$$

Since $\lim_{N \rightarrow \infty} \tau_N = T$ a.s., Fatou's lemma and a passage to the limit implies that

$$\mathbb{E} \left(\Sigma(t) \|\mathbf{w}(t)\|_{\mathcal{H}^{-\theta}}^2 \right) = 0.$$

Now, arguing as in [7], we can prove that

$$\mathbb{P}(\mathbf{u}_1(t) = \mathbf{u}_2(t), \text{ for all } t \in [0, T]) = 1.$$

This completes the proof of Theorem 3.2.

4. Invariant measure

In this section we study the long time behavior of the problem (2.12), in particular the existence of an invariant measure. Due to the lack of compactness between \mathcal{H} and \mathcal{H}^α we cannot rely on the method based on Bogolyubov–Krylov's theorem. We will instead use the new approach initially elaborated by Maslowski and Seidler [27] and developed further by [8], [11] and [10] amongst others. This new technique is a modified version of Bogolyubov–Krylov's theorem and relies on the sequentially weakly Feller property of the semigroup and boundedness type in probability of the solution of the stochastic problem. In this part of the paper we will mainly adapt the techniques elaborated in [8].

In order to introduce the main result of this theorem we recall that a function $\phi : \mathcal{H} \rightarrow \mathbb{R}$ is in the space $SC(\mathcal{H}_w)$ if and only if it is sequentially continuous with respect to weak topology on \mathcal{H} . The space $SC_b(\mathcal{H}_w)$ consists of functions $\phi \in SC(\mathcal{H}_w)$ which are bounded. A Markov \mathcal{P}_t , $t \geq 0$, satisfies the sequentially weakly Feller property if and only if for any $t \geq 0$ $\mathcal{P}_t \phi \in SC_b(\mathcal{H}_w)$, $\phi \in SC_b(\mathcal{H}_w)$, see [8].

Now, we say that a probability measure μ on \mathcal{H} is an invariant measure for a given sequentially weakly Feller semigroup on \mathcal{H} if for any $t \geq 0$

$$\int_{\mathcal{H}} \mathcal{P}_t \phi d\mu = \int_{\mathcal{H}} \phi d\mu \text{ for any } \phi \in C_b(\mathcal{H}_w).$$

Note that both sides of the above identity are meaningful, see [8].

We now recall that under the assumption of Theorem 3.1 and Theorem 3.2 for each initial $\xi \in \mathcal{H}$ the stochastic problem (2.12) has a weak martingale solution which satisfies a pathwise uniqueness. Thus, in view of Watanabe–Yamada’s theorem, see [42], for each initial condition $\xi \in \mathcal{H}$ the problem (2.12) has a unique strong solution $\mathbf{u}(\cdot, \xi)$ which should be understood in the sense of Stochastic Calculus. Moreover, the solution paths satisfy $\mathbf{u} \in C([0, T]; \mathcal{H})$ which along with [31, Theorem 27] implies that the solution defines a Markov process with associated Markov semigroup \mathcal{P}_t , $t \geq 0$, defined by

$$\mathcal{P}_t \phi(\xi) = \mathbb{E} \phi(\mathbf{u}(t, \xi)), \text{ for any } \phi \in B_b(\mathcal{H}), t \geq 0 \text{ and } \xi \in \mathcal{H},$$

where $B_b(\mathcal{H})$ denotes the space of functions $\phi : \mathcal{H} \rightarrow \mathbb{R}$ which are bounded and Borel measurable.

We can now state the main result of this section.

Theorem 4.1. *If $\gamma_0, \gamma_1 > 0$ and all the assumptions of Theorem 3.2 hold, then there exists at least an invariant measure for the Markov semigroup \mathcal{P}_t , $t \geq 0$, associated to the unique solution of problem (2.12).*

Remark 4.1. As mentioned in the introduction we will closely follow [8] to prove Theorem 4.1. Unlike [8] we prove the existence of invariant measure in both $d = 2$ and $d = 3$, but we need that $\alpha \geq \frac{d}{2}$.

In order to prove Theorem 4.1 we need to establish several auxiliary results whose proof are postponed to the end of this section. For this purpose we state and prove the following three lemma and corollary. They correspond to some results in [8] which will be precised in the proofs.

Lemma 4.1. *Under the assumption of the above theorem the Markov semigroup \mathcal{P}_t , $t \geq 0$, associated to the unique solution of problem (2.12) satisfies the sequentially weakly Feller property.*

Now, for each $\lambda > 0$ let

$$\mathbf{y} = \int_0^t e^{-(t-s)(\mathbf{R}_\gamma + \lambda I)} e^{-(t-s)\mathbf{A}_\alpha} \mathbf{G}(\mathbf{u}(s)) dW(s)$$

be the unique solution to

$$d\mathbf{y} + [\mathbf{A}_\alpha \mathbf{y} + (\mathbf{R}_\gamma + \lambda I)\mathbf{y}]dt = \mathbf{G}(\mathbf{u})dW.$$

Lemma 4.2. *Let all the assumptions of Theorem 4.1 hold. Then, for any $\lambda > 0$, $q \geq 2$ there exist constants C_λ , $C_{\lambda,q} > 0$ such that*

$$\mathbb{E} \|\mathbf{y}^\lambda(t)\|_{\mathcal{H}^{0,4}}^q \leq C_{\lambda,q},$$

for any $t > 0$. Moreover,

$$\lim_{\lambda \rightarrow \infty} C_\lambda = 0 = \lim_{\lambda \rightarrow \infty} C_{\lambda,q}.$$

We also have

$$\frac{1}{T} \int_0^T \mathbb{P}(\|\mathbf{y}^\lambda(t)\| > R) dt \leq \frac{C_\lambda}{R^2}$$

For any $\lambda > 0$ we set $\mathbf{v}^\lambda = \mathbf{u} - \mathbf{y}^\lambda$, where \mathbf{u} is the unique solution to (2.12). It is clear that \mathbf{v}^λ solves

$$\frac{d\mathbf{v}^\lambda}{dt} + \mathbf{A}_\alpha \mathbf{v}^\lambda + \mathbf{R}_\gamma \mathbf{v}^\lambda + \mathbf{B}(\mathbf{u}, \mathbf{u}) = \lambda \mathbf{y}^\lambda, \quad \mathbf{v}^\lambda(0) = 0. \quad (4.58)$$

Lemma 4.3. *Let all the assumptions of Lemma 4.2 hold and \mathbf{v}^λ be the solution to (4.58). Then, for any $\varepsilon > 0$ there exist two constants $\lambda, R > 0$ such that*

$$\frac{1}{T} \int_0^T \mathbb{P}(\|\mathbf{v}^\lambda(t)\| > R) dt < \varepsilon$$

for any $T > 0$.

From Lemma 4.2 and Lemma 4.3 we derive the following result.

Corollary 4.1. *For any $\varepsilon > 0$ there exists $R > 0$ such that*

$$\sup_{T \geq 1} \frac{1}{T} \int_0^T \mathbb{P}(\|\mathbf{u}(t; 0)\| > R) dt < \varepsilon.$$

This corollary corresponds to [8, (3.16)].

With these results in mind we now give the proof of the main theorem of the section.

Proof of Theorem 4.1. Thanks to Lemma 4.1 and Corollary 4.1 we easily infer the existence of an invariant measure for \mathcal{P}_t , $t \geq 0$, from Maslowski–Seidler theorem, see [27] or [8, Theorem 3.3]. \square

To close this section let us prove the auxiliary results stated above.

Proof of Lemma 4.1. In order to prove Lemma 4.1 we follow the idea in [8, Subsection 3.1]. In order to prove the sequentially weakly Feller property we fix a sequence of initial condition $(\xi_k)_{k \in \mathbb{N}} \subset \mathcal{H}$ such that

$$\xi_k \rightharpoonup \xi \text{ in } \mathcal{H},$$

and prove that for any $t \geq 0$ and $\phi \in SC_b(\mathcal{H}_w)$

$$\mathcal{P}_t \phi(\xi_k) \rightarrow \mathcal{P}_t \phi(\xi). \quad (4.59)$$

The proof of this claim is very similar to the proof of Theorem 3.1: derive several uniform estimates for \mathbf{u}_k , prove the tightness of the sequence of laws of \mathbf{u}_k , and finally show (4.59). To carry out this program we proceed as in Subsection 3.1. Let $\mathbf{u}_k := \mathbf{u}(\cdot, \xi_k)$ be the unique solution to (2.12) with initial condition ξ_k . Let \mathbf{y}_k be the solution of

$$d\mathbf{y}_k + (\mathbf{R}_\gamma + \mathbf{A}_\alpha) \mathbf{y}_k dt = \mathbf{G}(\mathbf{u}_k) dW, \quad \mathbf{y}_k(0) = 0,$$

and $\mathbf{v}_k = \mathbf{u}_k - \mathbf{y}_k$. It is clear that \mathbf{v}_k satisfies

$$\partial_t \mathbf{v}_k + \mathbf{A}_\alpha \mathbf{v}_k + \mathbf{B}(\mathbf{u}_k) + \mathbf{R}_\gamma \mathbf{v}_k = 0, \quad \mathbf{v}_k(0) = \xi_k. \quad (4.60)$$

Note that

$$\mathbf{y}_k(t) = \int_0^t e^{-(t-s)\mathbf{R}_\gamma} e^{-(t-s)\mathbf{A}_\alpha} \mathbf{G}(\mathbf{u}_k(s)) dW(s),$$

where

$$e^{-t\mathbf{R}_\gamma} = \begin{pmatrix} e^{-\gamma_0 t} & 0 \\ 0 & e^{-\gamma_1 t} \end{pmatrix}.$$

Bearing this remark in mind and the fact that the semigroup generated by \mathbf{A}_α and that by $(\mathbf{R}_\gamma + \mathbf{A}_\alpha)$ share the same properties, we infer that \mathbf{y}_k uniformly satisfies on $(\Omega, \mathcal{F}, \mathbb{P})$ all the estimates stated in Corollary 3.3. Now, we observe that (4.60) has a unique solution $\mathbf{v}_k \in C([0, T]; \mathcal{H}) \cap \mathbb{L}^2(0, T; \mathcal{H}^\alpha)$. Also, since $\xi_k \rightarrow \xi$ in \mathcal{H} one can find a constant $C > 0$ such that

$$\sup_{k \in \mathbb{N}} \|\xi_k\| \leq C.$$

Keeping these observations in mind and noticing that $(\mathbf{R}_\gamma \mathbf{v}_k, \mathbf{v}_k) \geq 0$, one can multiply (4.60) by \mathbf{v}_k and follow the same idea as used in the proof of Lemma 3.3 to deduce that \mathbf{v}_k uniformly satisfies on $(\Omega, \mathcal{F}, \mathbb{P})$ the estimates in Lemma 3.3. It is then easy to conclude that \mathbf{u}_k satisfies on $(\Omega, \mathcal{F}, \mathbb{P})$ Proposition 3.1. Hence, by applying [8, Lemma 5.5] we deduce that the family of laws of \mathbf{u}_k form a tight family on $\tilde{\mathcal{Z}}$, where $\tilde{\mathcal{Z}}$ is defined in Lemma 3.4. Now, as in [8], we invoke the Jakubowski–Skorokhod’s theorem to infer that there exists a subsequence $\check{\mathbf{u}}_{k_\ell}$ of $\tilde{\mathcal{Z}}$ -valued processes, a $\tilde{\mathcal{Z}}$ -valued stochastic process $\check{\mathbf{u}}$ defined on a new probability space $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{P}})$, equipped with a filtration $\check{\mathbb{F}}$, such that

$$\text{the processes } \mathbf{u}_k \text{ and } \check{\mathbf{u}}_{k_\ell} \text{ have the same law on } \tilde{\mathcal{Z}}, \quad (4.61)$$

$$\check{\mathbf{u}}_{k_\ell} \rightarrow \check{\mathbf{u}} \text{ in } \tilde{\mathcal{Z}} \text{ a.s.} \quad (4.62)$$

Following the lines of the proof of Theorem 3.1 we can prove that $\check{\mathbf{u}}$ is a weak martingale solution to problem (2.12) with initial condition ξ . Using the result of Theorem 3.2 and Watanabe–Yamada’s theorem, see [42], we infer that $\check{\mathbf{u}}$ is the unique strong solution with initial data ξ of (2.12). In view of (4.62) we have $\check{\mathbf{u}}_{k_\ell}(t, \xi_{k_\ell}) \rightarrow \check{\mathbf{u}}(t, \xi)$ $\check{\mathbb{P}}$ -a.s. which, in its turn, yields that $\phi(\check{\mathbf{u}}_{k_\ell}(t, \xi_{k_\ell})) \rightarrow \phi(\check{\mathbf{u}}(t, \xi))$ $\check{\mathbb{P}}$ -a.s. for any $\phi \in SC_b(\mathcal{H}_w)$. Due to the boundedness of ϕ we can apply Lebesgue Dominated Convergence theorem and infer that $\check{\mathbb{E}}\phi(\check{\mathbf{u}}_{k_\ell}(t, \xi_{k_\ell})) \rightarrow \check{\mathbb{E}}(\phi(\check{\mathbf{u}}(t, \xi)))$. Owing to (4.61) and the equality of laws of $\check{\mathbf{u}}(\cdot, \xi)$ and $\mathbf{u}(\cdot, \xi)$ we easily infer that $\mathbb{E}\phi(\mathbf{u}_{k_\ell}(t, \xi_{k_\ell})) \rightarrow \mathbb{E}(\phi(\mathbf{u}(t, \xi)))$. Now the uniqueness of solutions implies that this convergence is in fact true for the whole sequence. This completes the proof of Lemma 4.1. \square

Let us now proceed to the proof of Lemma 4.2.

Proof of Lemma 4.2. We will use the same argument as in [8, Lemma 3.5] to establish the proof of Lemma 4.2.

First, we have

$$\|e^{-(t-s)\mathbf{A}_\alpha}\|_{\mathcal{L}(\mathcal{H}^{s,p}, \mathcal{H}^{s',p})} \leq M \left(1 + t^{-(s'-s)/2\alpha}\right)$$

and

$$\begin{aligned} & \mathbb{E} \|\mathbf{y}^\lambda(t)\|^2 \\ & \leq \mathbb{E} \left\| \int_0^t e^{-(t-s)(\mathbf{R}_\gamma + \lambda I)} e^{-(t-s)\mathbf{A}_\alpha} \mathbf{G}(\mathbf{u}(s)) dW(s) \right\|^2 \end{aligned}$$

$$\begin{aligned}
&\leq \mathbb{E} \left| \int_0^t e^{-(t-s)(\mathbf{R}_\gamma + \lambda I)} \mathbf{A}_\alpha^{\frac{\theta}{2\alpha}} e^{-(t-s)\mathbf{A}_\alpha} \mathbf{A}_\alpha^{-\frac{\theta}{2\alpha}} \mathbf{G}(\mathbf{u}(s)) dW(s) \right|^2 \\
&\leq \mathbb{E} \int_0^t \left\| e^{-(t-s)(\mathbf{R}_\gamma + \lambda I)} \right\|_{\mathcal{L}(\mathcal{H}, \mathcal{H})}^2 \left\| \mathbf{A}_\alpha^{\frac{\theta}{2\alpha}} e^{-(t-s)\mathbf{A}_\alpha} \mathbf{A}_\alpha^{-\frac{\theta}{2\alpha}} \mathbf{G}(\mathbf{u}(s)) \right\|^2 ds \\
&\leq M \int_0^t \left\| e^{-(t-s)(\mathbf{R}_\gamma + \lambda I)} \right\|_{\mathcal{L}(\mathcal{H}, \mathcal{H})}^2 \left(1 + (t-s)^{-\theta/\alpha} \right) ds \\
&\leq M \int_0^t \left(e^{-2(\gamma_0 + \lambda)(t-s)} + e^{-2(\gamma_1 + \lambda)(t-s)} \right) \left(1 + (t-s)^{-\theta/\alpha} \right) ds \\
&\leq M \int_0^t \left(e^{-2(\gamma_0 + \lambda)r} + e^{-2(\gamma_1 + \lambda)r} \right) \left(1 + r^{-\theta/\alpha} \right) dr.
\end{aligned}$$

Now, set

$$C_\lambda = M \int_0^\infty \left(e^{-2(\gamma_0 + \lambda)r} + e^{-2(\gamma_1 + \lambda)r} \right) \left(1 + r^{-\theta/\alpha} \right) dr.$$

Since $\frac{\theta}{\alpha} < 1$ we have $C_\lambda \rightarrow 0$ as $\lambda \rightarrow \infty$. Hence

$$\sup_{t \in [0, \infty)} \mathbb{E} \|\mathbf{y}^\lambda(t)\|^2 \leq C_\lambda$$

with $C_\lambda \rightarrow 0$ as $\lambda \rightarrow \infty$.

Next, we show that

$$\sup_{t \in [0, \infty)} \mathbb{E} \|\mathbf{y}^\lambda(t)\|_{\mathcal{H}^{0,4}}^q \leq C_{\lambda,q}$$

with $C_{\lambda,q} \rightarrow 0$ as $\lambda \rightarrow \infty$. Using the same argument as above and (3.22) we have

$$\mathbb{E} \|\mathbf{y}^\lambda(t)\|_{\mathcal{H}^{0,4}}^q \leq M^q C \left[\int_0^t \left(e^{-2(\gamma_0 + \lambda)(t-s)} + e^{-2(\gamma_1 + \lambda)(t-s)} \right) \left(1 + (t-s)^{-\theta/\alpha} \right) ds \right]^{\frac{q}{2}}.$$

Hence as above we show that

$$\sup_{t \in [0, \infty)} \mathbb{E} \|\mathbf{y}^\lambda(t)\|_{\mathcal{H}^{0,4}}^q \leq M^q C_{\lambda,q}$$

where

$$C_{\lambda,q} = \left[\int_0^\infty \left(e^{-2(\gamma_0 + \lambda)r} + e^{-2(\gamma_1 + \lambda)r} \right) \left(1 + r^{-\theta/\alpha} \right) dr \right]^{\frac{q}{2}}.$$

Note that $C_{\lambda,q} \rightarrow 0$ as $\lambda \rightarrow \infty$.

It now remains to prove the last part of the lemma. Using Chebychev's inequality we infer that

$$\sup_{t \geq 0} \mathbb{P} (||\mathbf{y}^\lambda(t)|| > R) \leq \frac{C_\lambda}{R^2},$$

from which we complete the proof of the lemma. \square

We now give the proof of Lemma 4.3.

Proof of Lemma 4.3. Lemma 4.3 will be established in the same way as Brzeźniak and Ferrario proved Proposition 3.6 in [8]. For doing so, we first observe that $||\mathbf{v}^\lambda(\cdot)||$ solves

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} ||\mathbf{v}^\lambda||^2 + \langle \mathbf{A}_\alpha \mathbf{v}^\lambda, \mathbf{v}^\lambda \rangle + \langle \mathbf{R}_\gamma \mathbf{v}^\lambda, \mathbf{v}^\lambda \rangle \\ &= -\langle \mathbf{B}(\mathbf{v}^\lambda + \mathbf{y}^\lambda, \mathbf{v}^\lambda + \mathbf{y}^\lambda), \mathbf{v}^\lambda \rangle + \lambda \langle \mathbf{y}^\lambda, \mathbf{v}^\lambda \rangle \\ &= -\langle \mathbf{B}(\mathbf{v}^\lambda, \mathbf{v}^\lambda), \mathbf{y}^\lambda \rangle + \langle \mathbf{B}(\mathbf{y}^\lambda, \mathbf{v}^\lambda), \mathbf{y}^\lambda \rangle + \lambda \langle \mathbf{y}^\lambda, \mathbf{v}^\lambda \rangle, \end{aligned}$$

from which along with the estimates (2.6), (2.8), (3.33) and (3.34) we infer that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} ||\mathbf{v}^\lambda||^2 + \langle \mathbf{A}_\alpha \mathbf{v}^\lambda, \mathbf{v}^\lambda \rangle + \langle \mathbf{R}_\gamma \mathbf{v}^\lambda, \mathbf{v}^\lambda \rangle \\ & \leq ||\mathbf{v}^\lambda||_{\mathcal{H}^{0,4}} ||\mathbf{v}^\lambda||_{\mathcal{H}^1} ||\mathbf{y}^\lambda||_{\mathcal{H}^{0,4}} + ||\mathbf{y}^\lambda||_{\mathcal{H}^{0,4}}^2 ||\mathbf{v}^\lambda||_{\mathcal{H}^1} + \lambda ||\mathbf{y}^\lambda|| ||\mathbf{v}^\lambda|| \\ & \leq ||\mathbf{v}^\lambda||_{\mathcal{H}^{0,4}}^{1-\kappa} ||\mathbf{v}^\lambda||_{\mathcal{H}^\alpha}^{1+\kappa} ||\mathbf{y}^\lambda||_{\mathcal{H}^{0,4}} + C ||\mathbf{y}^\lambda||_{\mathcal{H}^{0,4}}^2 ||\mathbf{v}^\lambda||_{\mathcal{H}^\alpha} + C \lambda^2 ||\mathbf{y}^\lambda||^2 + \frac{\gamma_0 + \gamma_1}{2} ||\mathbf{v}^\lambda||^2, \end{aligned}$$

where $\kappa = \frac{d}{4\alpha}$.

By Young's inequality we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} ||\mathbf{v}^\lambda||^2 + ||\mathbf{A}_{\frac{\alpha}{2}} \mathbf{v}^\lambda||^2 + (\gamma_0 + \gamma_1) ||\mathbf{v}^\lambda||^2 \\ & \leq \frac{1}{2} (||\mathbf{v}^\lambda||^2 + ||\mathbf{A}_{\frac{\alpha}{2}} \mathbf{v}^\lambda||^2) + C ||\mathbf{v}^\lambda||^2 ||\mathbf{y}^\lambda||_{\mathcal{H}^{0,4}}^{2/(1-\kappa)} + C ||\mathbf{y}^\lambda||_{\mathcal{H}^{0,4}}^4 \\ & \quad + \frac{\gamma_0 + \gamma_1}{2} ||\mathbf{v}^\lambda||^2 + C \lambda^2 ||\mathbf{y}^\lambda||^2. \end{aligned}$$

Hence

$$\frac{d}{dt} ||\mathbf{v}^\lambda||^2 \leq -\frac{1}{2} (\gamma_0 + \gamma_1) ||\mathbf{v}^\lambda||^2 + C_0 \left(||\mathbf{v}^\lambda||^2 (1 + ||\mathbf{y}^\lambda||_{\mathcal{H}^{0,4}}^{2/(1-\kappa)}) + ||\mathbf{y}^\lambda||_{\mathcal{H}^{0,4}}^4 + \lambda^2 ||\mathbf{y}^\lambda||^2 \right).$$

Now, let $R > 0$ be arbitrary. Using Da Prato–Gatarek's argument [13] we obtain

$$\begin{aligned} & \frac{d}{dt} \ln (||\mathbf{v}^\lambda||^2 \vee R) \\ &= I_{\{||\mathbf{v}^\lambda||^2 > R\}} \frac{1}{||\mathbf{v}^\lambda||^2} \frac{d}{dt} ||\mathbf{v}^\lambda||^2 \\ & \leq I_{\{||\mathbf{v}^\lambda||^2 > R\}} \left(-(\gamma_0 + \gamma_1) + C_0 \left(1 + ||\mathbf{y}^\lambda||_{\mathcal{H}^{0,4}}^{\frac{2}{1-\kappa}} \right) \right) \\ & \quad + I_{\{||\mathbf{v}^\lambda||^2 > R\}} \frac{C_0}{||\mathbf{v}^\lambda||^2} (||\mathbf{y}^\lambda||_{\mathcal{H}^{0,4}}^4 + \lambda^2 ||\mathbf{y}^\lambda||^2). \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{dt} \ln (||\mathbf{v}^\lambda||^2 \vee R) &\leq I_{\{||\mathbf{v}^\lambda||^2 > R\}} \left(-(\gamma_0 + \gamma_1) + C_0 \left(1 + ||\mathbf{y}^\lambda||^{\frac{2}{1-\kappa}} \right) \right) \\ &\quad + \frac{C_0}{R} (||\mathbf{y}^\lambda||_{\mathcal{H}^{0,4}}^4 + \lambda^2 ||\mathbf{y}^\lambda||^2), \end{aligned}$$

from which we deduce that

$$\begin{aligned} \frac{\gamma_0 + \gamma_1}{T} \int_0^T \mathbb{E} I_{\{||\mathbf{v}^\lambda|| > R\}} dt &\leq \frac{C_0}{R} \left(1 + \sup_{t \geq 0} \mathbb{E} ||\mathbf{y}^\lambda(t)||_{\mathcal{H}^{0,4}}^4 + \lambda^2 \sup_{t \geq 0} \mathbb{E} ||\mathbf{y}^\lambda(t)||^2 \right) \\ &\quad + C_0 \sup_{t \geq 0} \mathbb{E} ||\mathbf{y}^\lambda(t)||_{\mathcal{H}^{0,4}}^4 \\ &\leq C_0 \left(C_{\lambda, \frac{2}{1-\kappa}} + \frac{C_{\lambda,4} + \lambda^2 C_\lambda}{R} \right). \end{aligned}$$

Thus, since $C_\lambda \rightarrow 0$, $C_{\lambda, \frac{2}{1-\kappa}} \rightarrow 0$, $C_{\lambda,4} \rightarrow 0$ as $\lambda \rightarrow \infty$, for any $\varepsilon > 0$ one can find $\lambda > 0$ and R such that

$$\frac{C_0}{\gamma_0 + \gamma_1} \left(C_{\lambda, \frac{2}{1-\kappa}} + \frac{C_{\lambda,4} + \lambda^2 C_\lambda}{R} \right) < \varepsilon.$$

Hence

$$\frac{1}{T} \int_0^T \mathbb{P} (||\mathbf{v}^\lambda(t)||^2 > R) dt < \varepsilon$$

which proves the lemma. \square

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Appendix A. Compactness result and tightness criteria

Throughout this section \mathcal{H} , \mathcal{H}^α and \mathbb{U} are the space introduced in Section 2. Our aim is to recall two tightness criteria which were very crucial for our analysis. To this end we first denote by \mathcal{H}_{loc} the space \mathcal{H} endowed with the topology generated by the family of seminorms $||\cdot||_{\mathcal{H}_N}$, $N \in \mathbb{N}$, defined by

$$||\mathbf{u}||_{\mathcal{H}_N} := \left(\int_{|x| < N} [|u(x)|^2 + |b(x)|^2] dx \right)^{\frac{1}{2}}, \quad \mathbf{u} = (u, b) \in \mathcal{H}, N \in \mathbb{N}.$$

In a similar way, the Fréchet space $\mathbb{L}^2(0, T; \mathcal{H}_{loc})$ is the space $\mathbb{L}^2(0, T; \mathcal{H})$ equipped with the topology generated by the seminorms

$$||\mathbf{u}||_{\mathbb{L}^2(0, T; \mathcal{H}_N)} := \left(\int_0^T ||\mathbf{u}(t)||^2 dt \right)^{\frac{1}{2}}, \quad N \in \mathbb{N}.$$

Here for each N we have set $\mathcal{H}_N := \{\mathbf{u}|_{B_N} : \mathbf{u} \in \mathcal{H}\}$ with B_N denoting the open ball of \mathbb{R}^d centered at 0 and with radius N . By \mathcal{H}_w we denote the space \mathcal{H} endowed with the weak topology and $C([0, T]; \mathcal{H}_w)$ is the space of weakly continuous functions $u : [0, T] \rightarrow \mathcal{H}$ endowed with the topology of uniform weak convergence on $[0, T]$. Notice that $\mathbf{u}(t) \in \mathcal{H}$ for any $t \in [0, T]$ if $\mathbf{u} \in C([0, T]; \mathcal{H}_w)$. The space $\mathbb{L}_w^2(0, T; \mathcal{H}^\alpha)$ is the space $\mathbb{L}^2(0, T; \mathcal{H}^\alpha)$ endowed with the weak topology. The space $C([0, T]; \mathbb{U}')$ is the space of continuous functions $u : [0, T] \rightarrow \mathbb{U}'$ with the topology induced by the norm

$$\|u\|_{C([0, T]; \mathbb{U}')} := \sup_{t \in [0, T]} \|u(t)\|_{\mathbb{U}'}.$$

We now consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ with the filtration $\mathbb{F} := \{\mathcal{F}_t : t \in [0, T]\}$ satisfying the usual conditions. On this probability space we consider $(X_n)_{n \geq 1}$ a sequence of continuous \mathbb{F} -adapted \mathbb{S} -valued processes and formulate the following definition.

Definition Appendix A.1. A sequence $(X_n)_{n \geq 1}$ satisfies the Aldous condition **[A]** in \mathbb{U}' iff for any $\varepsilon, \eta > 0$, there exists $\delta > 0$ such that for every sequence $(\tau_n)_{n \geq 1}$ of \mathcal{F}_t -stopping times with $\tau_n \leq T$, we have

$$\sup_{n \in \mathbb{N}} \sup_{\theta \in [0, \delta]} \mathbb{P}(\|X_n(\tau_n + \theta) - X_n(\tau_n)\|_{\mathbb{U}'} \geq \eta) \leq \varepsilon.$$

In Definition [Appendix A.1](#) and throughout we understand that X_n is extended to zero outside the interval $[0, T]$. Now, let

$$\mathcal{Z} := C([0, T]; \mathbb{U}') \cap \mathbb{L}^2(0, T; \mathcal{H}_{loc}) \cap C([0, T]; \mathcal{H}_w) \cap \mathbb{L}_w^2([0, T]; \mathcal{H}^\alpha), \quad (\text{A.1})$$

be endowed with the supremum \mathcal{T} of the corresponding topologies. In the next lemma we will state a tightness criteria for stochastic processes with paths in \mathcal{Z} .

Lemma Appendix A.1. Let $(X_n)_{n \geq 1}$ be a sequence of continuous \mathcal{F}_t -adapted \mathbb{U}' -valued processes satisfying

(a) there exists a constant $C_1 > 0$ such that

$$\sup_{n \geq 1} \mathbb{E} \sup_{t \in [0, T]} \|X_n(t)\|^2 \leq C_1,$$

(b) there exists a constant $C_2 > 0$ such that

$$\sup_{n \geq 1} \mathbb{E} \left[\int_0^T \|X_n(t)\|_{\mathcal{H}^\alpha}^2 dt \right] \leq C_2$$

(c) $(X_n)_{n \geq 1}$ satisfies the condition **[A]** in \mathbb{U}' .

Let $\tilde{\mathbb{P}}_n$ be the law of X_n on \mathcal{Z} . Then for any $\varepsilon > 0$, there exists a compact set $\mathbb{K}_\varepsilon \subset \mathcal{Z}$ such that

$$\sup_{n \geq 1} \tilde{\mathbb{P}}_n(\mathbb{K}_\varepsilon) \geq 1 - \varepsilon.$$

Proof. This lemma is just [\[9, Corollary 3.9\]](#) with $V = \mathcal{H}^\alpha$, $H = \mathcal{H}$ and $U = \mathbb{U}$. \square

In the next few lines we will recall also a tightness criteria for stochastic processes with paths in $\tilde{\mathcal{Z}}$ where

$$\tilde{\mathcal{Z}} = \mathbb{L}_w^{\frac{8\alpha}{d}}(0, T; \mathcal{H}^{0,4}) \cap C(0, T; \mathbb{U}') \cap \mathbb{L}^2(0, T; \mathcal{H}_{loc}) \cap C([0, T]; \mathcal{H}_w)$$

is endowed with the supremum of the corresponding topologies. Note that $\alpha \geq 1$ is as in previous sections.

Lemma Appendix A.2. *Let $\lambda > 0$, $\delta > 0$ and $\alpha \geq 1$ be given parameters and $(X_n)_{n \geq 1}$ be a sequence of continuous \mathcal{F}_t -adapted \mathbb{U}' -valued processes. Let \mathcal{L}_n be the law of X_n on $\tilde{\mathcal{Z}}$. If for any $\varepsilon > 0$ there exist constants K_i , $i = 1, \dots, 4$ such that*

$$\begin{aligned} \sup_n \mathbb{P} \left(\|X_n\|_{\mathbb{L}^\infty(0,T;\mathcal{H})} > K_1 \right) &\leq \varepsilon, \\ \sup_n \mathbb{P} \left(\|X_n\|_{\mathbb{L}^2(0,T;\mathcal{H}^\delta)} > K_2 \right) &\leq \varepsilon, \\ \sup_n \mathbb{P} \left(\|X_n\|_{\mathbb{L}^{\frac{8\alpha}{d}}(0,T;\mathcal{H}^{0,4})} > K_3 \right) &\leq \varepsilon, \\ \sup_n \mathbb{P} \left(\|X_n\|_{C^\lambda(0,T;\mathcal{H}^{-\alpha})} > K_4 \right) &\leq \varepsilon, \end{aligned}$$

then the sequence $(\mathcal{L}_n)_{n \in \mathbb{N}}$ is tight on $\tilde{\mathcal{Z}}$.

Proof. We omit the proof because it is the same as the proof of [7, Lemma 5.5]. \square

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