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A group theoretic analysis of the generalised Gardner equation with arbitrary order nonlinear terms

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ABSTRACT

In this paper, we analyse a Gardner equation with dual power law nonlinearity of any order which has been widely investigated due to its applications in quantum field theory, solid state, plasma and fluid physics. A group theoretic approach is used to perform a comprehensive and detailed analysis of the equation. We derive the symmetry generators of the equation in terms of its arbitrary parameters and used them to obtain symmetry reductions and exact solutions. Furthermore, the conservation laws of the equation are derived via the Noether approach after increasing the order and by the use of the multiplier method. The importance of these conservation laws in finding exact solutions is proved via double reduction theory. Most of these solutions are new and contain many known solutions as special cases. These solutions include important soliton solutions and nontrivial solutions in terms of special functions which are meromorphic in the entire complex plane. Since the Gardner equation can model a variety of wave phenomena in plasma, solid state and fluid physics, these solutions possess significant features in the nonlinear mechanics aspects of the work. They can also be used as a basis for solving other related model problems and assessing numerical and approximate analytical methods for nonlinear equations describing solitons in wave mechanics.

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1. Introduction

The study of nonlinear evolution equations (NLEEs) is one of the most important areas of research in science and engineering. NLEEs are widely used as models to describe complex physical behaviour in solid state physics, plasma physics, fluid mechanics and biology. One of the main foci of research in this area is the aspect of integrability of these NLEEs. There are various methods used to carry out the integration of these NLEEs. Among them are the homogeneous balance method [35], the ansatz method [17], the extended tanh method [15], the Jaccobi elliptic function method [25], the projective Riccati equation method [26], the

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direct method [13], the sine-cosine method [36] and the Lie group method [31,12] (here, hidden symmetries have played an interesting role [1,2]). Although considerable work has been done over the years, there is no unique method that can be used to tackle all types of NLEEs. The main focus of this paper is the integrability aspect of these equations.

Lie symmetries and conservation laws are among the most powerful tools for understanding the behaviour of physical systems and for finding exact solutions of many problems in mathematical physics. The existence of a large number of conservation laws for a partial differential equation indicates its integrability [12]. One of the oldest methods of finding conservation laws of differential equations (DEs) is Noether's theorem [9]. Noether's theorem provides an elegant and constructive way of finding conservation laws for any system of one or more DEs that admits a variational formulation in terms of a Lagrangian. However, the existence of many physically and mathematically interesting DEs that are not variational systems motivated much work in the past few decades to look for some generalisation of Noether's theorem which is applicable to non-variational DEs. Recently, Anco and Bluman [5–7] develop a complete method for both variational and non-variational DEs called the direct construction method (multiplier method), which also coincides with Noether's method for variational DEs. This direct method uses a characteristic form involving multipliers and is based on the standard result [28,34,31,4] that there is a direct correspondence between conservation laws and multipliers whenever a PDE system can be expressed in a solved form in terms of a set of leading derivatives.

Other methods for constructing conservation laws recently introduced include the partial Lagrangian method [21] and the nonlocal conservation theorem method [18] which are subsumed by the general version of the direct construction method.

Recently, the relationship between conservation laws with point symmetries [20] and non-local symmetries [33] has been used to reduce the numbers of variables and order of differential equations. A conservation law is invariant (or, more generally, homogeneous) under a symmetry or group of symmetries and this leads to the development of the double reduction method for PDEs with two independent variables [32]. The fundamental basis of this method is that when a conserved vector is invariant under a symmetry, a double reduction transformation exists. PDEs of order n with two independent variables are reduced to ODEs of order $n-1$, which are generally easier to solve. Hence, the invariance of a conserved vector under a symmetry firstly reduces the number of independent variables and secondly reduces the order of the differential equation. This is helpful in finding solutions of PDEs having a nontrivial conserved vector which is invariant under one or more symmetries. The invariance of conservation laws under the action of a given set of symmetries also serves as a useful feature for finding a generating subset or basis for the set of all conservation laws of the PDE system [3,8].

In this paper, we consider a NLEE that is widely used in various branches of physics, such as solid-state physics, plasma physics, fluid physics and quantum field theory and which describes a variety of wave phenomena in plasma and solid state physics. This is the generalised form of the Gardner equation with nonlinear terms of any order given by [16,24]

$$u_t + (a + bu^n + cu^{2n})u_x + ku_{xxx} = 0. \tag{1.1}$$

The profile $u(t,x)$ is the amplitude of the relevant wave mode and the variables x and t represent spatial and temporal variables respectively. The term u_t is the evolution term while b and c represent the coefficients of the dual-power law nonlinearities. Then a and k are the coefficients of dispersion terms. The parameter $n \geq 1$ represents the power law nonlinearity parameter. The dual power nonlinearity terms, u^n and u^{2n}, are introduced to mitigate the dispersive effects of u_{xxx} and obtain a rich variety of solutions with interesting properties. (We note that in many applications (see below), the ratio of the powers of these nonlinear terms is always two and so we maintain that relationship in our general equation.) This nonlinearity is added in a generalised setting that could represent the regular Korteweg-de Vries (KdV) equation or the modified

Korteweg-de Vries (mKdV) depending on the values of n and the constants of the equation. For example, when $n = 1, b \neq 0, c \neq 0$, we obtain the famous combined KdV-mKdV equation [16] given by

$$u_t + (a + bu + cu^2)u_x + ku_{xxx} = 0. \tag{1.2}$$

When $n = 1, b \neq 0, c = 0$, we obtain the KdV equation [22]

$$u_t + (a + bu)u_x + ku_{xxx} = 0, \tag{1.3}$$

and when $n = 1, b = 0, c \neq 0$ we have the mKdV equation

$$u_t + (a + cu^2)u_x + ku_{xxx} = 0. \tag{1.4}$$

The Gardner equation (1.1) has been investigated as a generalised KdV-mKdV equation with high-order nonlinear terms [24] and the Benjamin-Bona-Mahoney (BBM) equation, with dual-power law nonlinearity [10]. Different versions of equation (1.1) have been studied in order to obtain exact solutions [11,29,38,37].

Most of these studies are based on the "travelling wave" type solutions via some well known substitutions. Here the solutions will be obtained via the Lie symmetry approach and a double reduction which involves the invariance of conservation laws under symmetries. As such an invariance is not limited to the travelling wave symmetries, there exist some possibilities of additional solutions different from travelling wave solutions. This will be investigated with the scaling symmetries which might lead to new exact solutions. Finally it is worth noting that the improved generalised Riccati equation mapping method leads to important singular soliton solutions.

The organisation of the paper is as follows: In the next Section, we present a Lie symmetry analysis of the Gardner equation (1.1). In Section 3, the conservation laws of the equation are derived. In Section 4 the exact solutions of the equation are discussed. Constants of the motion are provided in Section 5 and concluding remarks are made in Section 6.

2. Lie symmetries of the Gardner equation

The Lie point symmetries admitted by (1.1) are generated by a vector field of the form

$$X = \xi^1(t, x, u)\frac{\partial}{\partial t} + \xi^2(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u}. \tag{2.1}$$

The operator X satisfies the Lie symmetry condition [31,12]

$$X^{[3]}\left[u_t + (a + bu^n + cu^{2n})u_x + ku_{xxx}\right]\Big|_{(1.1)} = 0, \tag{2.2}$$

where $X^{[3]}$ is the third prolongation of the operator X defined by

$$X^{[3]} = X + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{xxx}\frac{\partial}{\partial u_{xxx}} \tag{2.3}$$

and the coefficients ζ_t, ζ_x, and ζ_{xxx} are given by

$$\zeta_t = D_t(\eta) - u_tD_t(\xi^1) - u_xD_t(\xi^2), \tag{2.4}$$

$$\zeta_x = D_x(\eta) - u_tD_x(\xi^1) - u_xD_x(\xi^2), \tag{2.5}$$

$$\zeta_{xxx} = D_x(\zeta_{xx}) - u_{xxt}D_x(\xi^1) - u_{xxx}D_x(\xi^2). \tag{2.6}$$

Table 1
Commutator table of symmetry operators of (1.1) for Case (1) & Case (2).

[,]	X_1	X_2	X_3
X_1	0	0	$-3nX_1 - 2anX_2$
X_2	0	0	$-nX_2$
X_3	$3nX_1 + 2anX_2$	nX_2	0

Here D_x, D_t denote the total derivative operators defined by

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} + \cdots, D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_t} + \cdots. \tag{2.7}$$

Expansion and separation of (2.2) with respect to the powers of different derivatives of u yields an overdetermined system in the unknown coefficients ξ^1, ξ^2 and η. Solving the overdetermined system for arbitrary parameters gives the 2-dimensional trivial Lie algebra spanned by the vector fields of translation with respect to the independent variables

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}. \tag{2.8}$$

Next we find specifications of the arbitrary elements that extend the principal Lie algebra (2.8) and obtain only the following three cases:

Case (1) $n \neq 1$, $c = 0$, $b \neq 0$, $k \neq 0$. This condition reduces (1.1) to the generalised version of the KdV equation

$$u_t + (a + bu^n)u_x + ku_{xxx} = 0. \tag{2.9}$$

The additional operator obtained in this case is

$$X_3 = -3nt\frac{\partial}{\partial t} - (2ant + nx)\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}. \tag{2.10}$$

Case (2) $n \neq 1$, $c \neq 0$, $b = 0$, $k \neq 0$. In this case (1.1) becomes the generalised MKdV equation

$$u_t + (a + cu^{2n})u_x + ku_{xxx} = 0. \tag{2.11}$$

The additional operator is

$$X_3 = -3nt\frac{\partial}{\partial t} - (2ant + nx)\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}. \tag{2.12}$$

Case (3) $n = 1$, $c \neq 0$, $b \neq 0$, $k \neq 0$. This case corresponds to the combined KdV-mKdV equation given in (1.2). The additional operator is

$$X_3 = -6ct\frac{\partial}{\partial t} + (b^2t - 4act - 2cx)\frac{\partial}{\partial x} + (b + 2cu)\frac{\partial}{\partial u}. \tag{2.13}$$

The Lie symmetry algebra for Cases (1) and (2) exhibits the same commutative property. The commutator table for both cases are presented in Table 1 while the commutator table for Case (3) is given in Table 2.

Table 2
Commutator table of symmetry operators of (1.1) for Case (3).

[,]	X_1	X_2	X_3
X_1	0	0	$(b^2-4ac)\,X_2-6cX_1$
X_2	0	0	$-2cX_2$
X_3	$6cX_1-(b^2-4ac)\,X_2$	$2cX_2$	0

3. Conservation laws of the Gardner equation

A conserved vector corresponding to a conservation law of the Gardner equation (1.1) is a 2−tuple (T^t, T^x), such that

$$D_tT^t + D_xT^x = 0 \tag{3.1}$$

along the solutions of the equation. The Gardner equation (1.1) is a third order partial differential equation and its conservation laws cannot be computed directly by the Noether approach. Hence, the conservation laws for (1.1) will be derived by two approaches, namely (i) the multiplier approach and (ii) Noether approach (after increasing its order by one).

(i) The multiplier approach: Consider the multiplier Λ of order up to two, viz. $\Lambda = (t, x, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt})$ for (1.1). The conserved vector (T^t, T^x) of (1.1) satisfies the divergence relation

$$D_tT^t + D_xT^x = \Lambda(u_t + (a + bu^n + cu^{2n})u_x + ku_{xxx}) = 0. \tag{3.2}$$

Moreover, we have

$$\frac{\delta}{\delta u}(\Lambda)(u_t + (a + bu^n + cu^{2n})u_x + ku_{xxx}) = 0. \tag{3.3}$$

After a lengthy calculation, we observe that Λ is of order zero or two in derivatives with respect to u. The obtained forms of $\Lambda = \Lambda_i$ are

$$\Lambda_1 = 1, \tag{3.4}$$

$$\Lambda_2 = u, \tag{3.5}$$

$$\Lambda_3 = \frac{c}{k(2n+1)}u^{2n+1} + \frac{b}{k(n+1)}u^{n+1} + u_{xx}. \tag{3.6}$$

For $n = 1$, we obtain in addition to the three multipliers (3.4)−(3.6) above the following multipliers

$$\Lambda_4 = (2cu + b)(ta - x + tu(b + cu) + 6cktu_{xx}, \tag{3.7}$$

$$\begin{aligned}\Lambda_5 = -\frac{1}{36k}(&6c^2u^5 + 15bcu^4 + 12acu^3 + 10b^2u^3 + 24cku^2u_{xx}\\ &-12ckuu_x^2 + 18abu^2 + 24bkuu_{xx} - 6bku_x^2 - 36ku_{tx}),\end{aligned} \tag{3.8}$$

$$\begin{aligned}\Lambda_6 = -\frac{1}{144c^2k}\big(&40c^4u^7 + 140bc^3u^6 + 144a^5c^3u + 174b^2c^2u^5 + 264c^3ku^4uxx + 85b^3cu^4\\ &-48c^3ku^3u_x^2b + 360abc^2 + 528bc^2ku^3u_{xx} - 72bc^2ku^2u_x^2 + 144a^2c^2u^3 + 252ab^2cu^3\\ &+10b^4u^3 + 432ac^2ku^2u_{xx} + 288b^2cku^2u_{xx} - 36b^2ckuu_x^2 + 432c^2k^2uu_{xx}^2 - 144c^2k^2u_x^2\\ &-144c^2ku^2u_{tx} + 216a^2bcu^2 + 18a^2b^3u^2 + 288c^2kuu_tu_x + 432abckuu_{xx} + 24b^3kuu_{xx}\\ &-6b^3ku_x^2 + 216bck^2u_{xx}^2 - 144bckuu_{tx} + 144bcku_tu_x - 36b^2ku_{tx} + 432cku_{tt}\big).\end{aligned} \tag{3.9}$$

Now, we calculate the corresponding conserved density T^t and flux T^x by substituting (3.4)−(3.9) into (3.2). For Λ_1 we obtain the obvious conserved vector

$$\begin{aligned} T_1^t &= u, \\ T_1^x &= \frac{c}{2n+1}u^{2n+1} + \frac{b}{n+1}u^{n+1} + au + ku_{xx}. \end{aligned} \tag{3.10}$$

Λ_2 gives rise to

$$\begin{aligned} T_2^t &= \frac{1}{2}u^2, \\ T_2^x &= \frac{c}{2n+2}u^{2n+2} + \frac{b}{n+2}u^{n+2} + kuu_{xx} - \frac{1}{2}ku_x^2 + \frac{1}{2}au^2, \end{aligned} \tag{3.11}$$

whereas Λ_3 results in

$$\begin{aligned} T_3^t &= \frac{1}{k(2n+1)(2n+2)}cu^{2n+2} + \frac{1}{k(n+1)(n+2)}bu^{n+2} + \frac{1}{2}uu_{xx}, \\ T_3^x &= \frac{1}{2k}\left(ku_{xx} + \frac{b}{n+1}u^{n+1}\right)^2 + \frac{1}{2k(2n+1)^2}c^2u^{4n+2} + \frac{1}{k(2n+1)(n+1)}bcu^{3n+2} \\ &\quad + \frac{1}{k(2n+1)(2n+2)}acu^{2n+2} + \frac{1}{2n+1}cu^{2n+1}u_{xx} + \frac{1}{k(n+1)(n+2)}abu^{n+2} \\ &\quad + \frac{1}{2}au_x^2 + \frac{1}{2}u_tu_x - \frac{1}{2}uu_{tx}. \end{aligned} \tag{3.12}$$

Since

$$\frac{1}{2}uu_{xx} = D_x\left(\frac{1}{2}uu_x\right) - \frac{1}{2}u_x^2, \quad -\frac{1}{2}uu_{tx} = -D_t\left(\frac{1}{2}uu_x\right) + \frac{1}{2}u_tu_x$$

the conserved vector (T_3^t, T_3^x) becomes

$$\begin{aligned} T_3^t &= \frac{1}{k(2n+1)(2n+2)}cu^{2n+2} + \frac{1}{k(n+1)(n+2)}bu^{n+2} - \frac{1}{2}u_x^2, \\ T_3^x &= \frac{1}{2k}\left(ku_{xx} + \frac{b}{n+1}u^{n+1}\right)^2 + \frac{1}{2k(2n+1)^2}c^2u^{4n+2} + \frac{1}{k(2n+1)(n+1)}bcu^{3n+2} \\ &\quad + \frac{1}{k(2n+1)(2n+2)}acu^{2n+2} + \frac{1}{2n+1}cu^{2n+1}u_{xx} + \frac{1}{k(n+1)(n+2)}abu^{n+2} \\ &\quad + \frac{1}{2}au_x^2 + u_tu_x. \end{aligned} \tag{3.13}$$

These three conserved quantities (3.10), (3.11) and (3.13) respectively represent mass, momentum and energy for the Gardner equation (1.1).

We construct $(T_4^t, T_4^x) - (T_6^t, T_6^x)$ using the same approach. The multiplier method used here is the same as the "direct construction method".

(ii) The Noether approach: To apply Noether's method, we increase the order of equation (1.1) by one. Let $u = U_x$, then equation (1.1) becomes

$$U_{tx} + (a + bU_x^n + cU_x^{2n})U_{xx} + kU_{xxxx} = 0. \tag{3.14}$$

A standard Lagrangian for equation (3.14) is

$$L = \frac{1}{2}U_tU_x + \left(\frac{1}{2}a + \frac{1}{(n+1)(n+2)}bU_x^n + \frac{1}{(2n+1)(2n+2)}cU_x^{2n}\right)U_x^2 - \frac{1}{2}kU_{xx}^2 \tag{3.15}$$

and the associated Euler-Lagrange equation is

$$\frac{\delta L}{\delta U} = 0,$$

where $\frac{\delta}{\delta U}$ is defined by

$$\frac{\delta}{\delta U} = \frac{\partial}{\partial U} - D_t\frac{\partial}{\partial U_t} - D_x\frac{\partial}{\partial U_x} + D_t^2\frac{\partial}{\partial U_{tt}} + D_x^2\frac{\partial}{\partial U_{xx}} + D_xD_t\frac{\partial}{\partial U_{tx}}, \tag{3.16}$$

and

$$D_t = \frac{\partial}{\partial t} + U_t\frac{\partial}{\partial U},$$
$$D_x = \frac{\partial}{\partial x} + U_x\frac{\partial}{\partial U} + U_{xx}\frac{\partial}{\partial U_x}.$$

The point symmetry operator X defined in (2.1) is a Noether operator corresponding to the Lagrangian L of equation (3.15) if it satisfies [19]

$$X(L) + L(D_t\xi^1 + D_x\xi^2) = D_tB^1 + D_xB^2, \tag{3.17}$$

where $B^1(t,x,U), B^2(t,x,U)$ are the gauge terms.

Substituting (3.15) into (3.17) and solving yields the following Noether coefficient functions and gauge terms

$$\xi^1 = c_1,\ \xi^2 = c_2,\ \eta = m(t),\ B^1 = f(t,x),$$
$$B^2 = \frac{1}{2}m_t(t) + g(t,x),\ f_t(t,x) + g_x(t,x) = 0, \tag{3.18}$$

where c_1, c_2 are arbitrary constants and $m(t)$, $f(t,x)$, $g(t,x)$ are arbitrary gauge functions. We set $f(t,x)$ and $g(t,x)$ to zero as they contribute to the trivial part of the conserved vector. The conserved vectors for the second−order Lagrangian L is given by [30]

$$T^t = B^1 - L\xi^1 - w\frac{\delta L}{\delta U_t} - D_t(w)\frac{\delta L}{\delta Utt} - D_x(w)\frac{\delta L}{\delta U_{tx}}, \tag{3.19}$$

$$T^x = B^2 - L\xi^2 - w\frac{\delta L}{\delta U_x} - D_t(w)\frac{\delta L}{\delta Utx} - D_x(w)\frac{\delta L}{\delta U_{xx}}, \tag{3.20}$$

where $w = \eta - U_t\xi^1 - U_x\xi^2$ is the characteristic function.

Equations (3.19) and (3.20) together with (3.18) (reverting to the original variable $u = U_x$) yield the following conserved vectors for the Gardner equation (1.1):

$$T_{11}^t = \frac{1}{2}au^2 + \frac{1}{(n+1)(n+2)}bu^{n+2} + \frac{1}{(2n+1)(2n+2)}cu^{2n+2} - \frac{1}{2}ku_x^2,$$

$$T^x_{11} = \frac{1}{2}\left(au + \frac{1}{(n+1)}bu^{n+1} + \frac{1}{2n+1}cu^{2n+1} + ku_{xx}\right)^2 + ku_tu_x, \tag{3.21}$$

$$T^t_{21} = \frac{1}{2}u^2,$$

$$T^x_{21} = \frac{1}{2}au^2 + \frac{1}{n+2}bu^{n+2} + \frac{1}{2n+2}cu^{2n+2} + kuu_{xx} - \frac{1}{2}ku_x^2. \tag{3.22}$$

The arbitrary function $m(t)$, gives rise to a purely spatial conservation law given by

$$T^t_m = 0,$$

$$T^x_m = m(t)\left(D_t\int udx + au + \frac{1}{n+1}bu^{n+1} + \frac{1}{2n+1}cu^{2n+1} + ku_{xx}\right) \tag{3.23}$$

It is worth noting that the conservation laws (3.21)−(3.23) can also be obtained by applying the multiplier method to the potential equation (3.14). Further, since we only use point symmetries, the conservation laws which arise from the higher-order multipliers (3.7)−(3.9) (corresponding to higher-order symmetries) are missing. However, we have managed to obtain the genuine nonlocal conservation law (3.23) (which arises as a result of the potential equation (3.14)).

4. Travelling wave solutions of the Gardner equation

In this section we will discuss travelling wave solutions of the Gardner equation (1.1). The linear combination of symmetries (2.8) given by $X = X_1 + \lambda X_2$, where λ is a constant, can be used to find travelling wave solutions. The similarity variables for X obtained by solving the characteristic equation

$$\frac{dt}{1} = \frac{dx}{\lambda} = \frac{du}{u} \tag{4.1}$$

are given by

$$r = x - \lambda t, w(r) = u. \tag{4.2}$$

These similarity variables reduce the Gardner equation (1.1) to the third order ODE

$$(a-\lambda)w_r + cw_rw^{2n} + bw_rw^n + kw_{rrr} = 0. \tag{4.3}$$

We consider the solutions of the ODE (4.3) via:

(A) Direct integration. Integrating (4.3) twice with respect to r yields

$$\frac{(a-\lambda)}{2}w^2 + \frac{c}{(2n+1)(2n+2)}w^{2n+2} + \frac{b}{(n+1)(n+2)}w^{n+2} + \frac{k}{2}w_r^2 = wk_1 + k_2, \tag{4.4}$$

where k_1, k_2 are constants of integration. The solution for (4.4) in terms of the original variables is

$$\int \frac{du}{\sqrt{\frac{\lambda-a}{k}u^2 - \frac{2c}{k(2n+1)(2n+2)}u^{2n+2} - \frac{2b}{k(n+1)(n+2)}u^{n+2} - \frac{2k_1}{k}u + k_2}} = x - \lambda t + k_3, \tag{4.5}$$

where k_3 is a constant of integration. Setting the constants of integration to zero in (4.5) and simplifying we obtained the following solutions to the Gardner equation (1.1)

$$u_1(t,x)=\left\{\frac{(\lambda-a)(n+1)(n+2)sech\left(n\sqrt{\frac{\lambda-a}{k}}(x-\lambda t)\right)}{\pm\sqrt{\frac{b^2(2n+1)+c(\lambda-a)(n+1)(n+2)^2}{2n+1}}+bsech\left(n\sqrt{\frac{\lambda-a}{k}}(x-\lambda t)\right)}\right\}^{\frac{1}{n}},\lambda>a, \tag{4.6}$$

$$u_2(t,x)=\left\{\frac{(\lambda-a)csch^2\left(\frac{n}{2}\sqrt{\frac{\lambda-a}{k}}(x-\lambda t)\right)}{2\left(\frac{-b}{(n+1)(n+2)}\pm\sqrt{\frac{(a-\lambda)c}{(2n+1)(n+1)}}coth\left(\frac{n}{2}\sqrt{\frac{\lambda-a}{k}}(x-\lambda t)\right)\right)}\right\}^{\frac{1}{n}},\lambda>a, \tag{4.7}$$

$$u_3(t,x)=\left\{\frac{(a-\lambda)csc^2\left(\frac{n}{2}\sqrt{\frac{a-\lambda}{k}}(x-\lambda t)\right)}{2\left(\frac{-b}{(n+1)(n+2)}\pm\sqrt{\frac{(a-\lambda)c}{(2n+1)(n+1)}}\cot\left(\frac{n}{2}\sqrt{\frac{a-\lambda}{k}}(x-\lambda t)\right)\right)}\right\}^{\frac{1}{n}},\lambda<a. \tag{4.8}$$

(B) The improved generalised Riccati equation mapping method. Here we apply the improved generalised Riccati equation mapping method [40] to the reduced Gardner equation (4.3). Our main goal is to derive explicit or at least approximate solutions, if possible, for the ODE (4.3). This will allow us to plot the solutions as they will be in a more accessible form than (4.5).

We express the solution $w(r)$ of equation (4.3) in the finite series form

$$w(r)=\sum_{i=-m}^{m}a_i\psi^i, \tag{4.9}$$

where the new function $\psi=\psi(r)$ satisfies the Riccati equation

$$\psi'=\mu+\beta\psi+(\nu-1)\psi^2. \tag{4.10}$$

The parameters μ,β,ν are constants and both a_{-m}, a_m are not zero. The positive integer m can be determined by balancing the highest order derivatives with nonlinear terms appearing in ODE (4.3). Integrating (4.3) once and setting the constant of integration to zero results in

$$(a-\lambda)w+\frac{1}{2n+1}cw^{2n+1}+\frac{1}{n+1}bw^{n+1}+kw_{rr}=0. \tag{4.11}$$

Balancing w_{rr} with w^{2n+1} in (4.11), we obtain an irreducible fraction, $m=\frac{1}{n}$, for some n. Therefore we make the following transformation

$$w(r)=h(r)^{\frac{1}{n}} \tag{4.12}$$

and then substitute (4.12) into (4.11) to obtain

$$n^2(a-\lambda)h^2+\frac{1}{2n+1}n^2ch^4+\frac{1}{n+1}n^2bh^3+k(1-n)h_r^2+knhh_{rr}=0. \tag{4.13}$$

Now balancing $h(r)^4$ and hh_{rr} we find $m=1$. Thus, we assume that

$$h(r)=a_{-1}\psi^{-1}+a_0+a_1\psi. \tag{4.14}$$

Substituting equation (4.14) together with (4.10) into (4.13) yields an algebraic equation involving powers of ψ. Equating the coefficients of each power of ψ to zero gives a system of algebraic equations for a_{-1},a_0,a_1 with solution

$$a_0 = a_0, \quad a_1 = \frac{(\nu-1)a_0}{\beta}, \quad a_{-1} = \frac{\mu a_0}{\beta}, \quad a = \frac{n^2\lambda - k(\beta^2 - 4\mu(\nu-1))}{2\mu},$$
$$b = \frac{k(n+1)(n+2)\beta^2}{n^2 a_0}, \quad c = -\frac{k(n+1)(2n+1)\beta^2}{n^2 a_0}. \tag{4.15}$$

From (4.15), (4.14), (4.12) and (4.2), we obtain multiple explicit solutions of the Gardner equation (1.1). We follow [40], [27], [39] and consider four cases. These can be found in Appendix A. Note that we do not find a Type 4 solution.

Basic features. We discuss some basic features of the solutions in Section 4 B. It is well known that the presence of dispersion without nonlinearity kills the solitary wave as different Fourier harmonics start propagating at different group velocities. On the other hand, introducing nonlinearity without dispersion also destroys the propagation of solitary waves, since the pulse energy is frequently pumped into higher frequency modes. Thus, in order to maintain solitary waves both dispersion and nonlinearity must be introduced. We maintained a delicate balance between the nonlinearity effect of dual power law terms, u^{2n} and u^n, and the dispersive effect of u_{xxx} which resulted in solitons or solitary waves of the Gardner equation (1.1). The solitons are waves that are characterised by retaining their identity upon interacting with other solitons without change of shape and velocity properties. The type of travelling wave solutions depends on the values of the physical parameters. Solutions (A.1), (A.3), (A.6), (A.7), (A.8) and (A.10) describe the soliton solution [14]. Fig. 1 (a) shows the soliton obtained from solution (A.8). Solutions (A.9), (A.11) and (A.12) represent exact singular soliton solutions (as shown in Fig. 1 (b)). Solutions (A.25) and (A.26) have the exact solitary wave of kink type [23], graphically represented in Fig. 1(c). Solutions (A.2), (A.4) and (A.5) represent the solitary wave solutions of singular kink type (see Fig. 1 (d)). The solutions of (A.13), (A.15), (A.18) − (A.20) and (A.22) are periodic wave solutions. Fig. 1 (e) shows the periodic travelling wave solution obtained from (A.13). Solutions (A.14), (A.16), (A.17), (A.21), (A.23) and (A.24) describe singular periodic wave solutions as shown in Fig. 1 (f).

Note that if $\mu = 0$ in equations (A.1), (A.2), (A.4), (A.5), (A.9), (A.12) and when $\mu = 0$, with $F = 0$ in (A.6), our obtained solutions are identical to those solutions obtained by Li *et al.* [24] using the substitution method. Thus their solutions are a subset of our results.

5. Similarity solutions of the Gardner equation

The symmetry generator X_3 in (2.13) gives rise the invariants

$$u(x,t) = w(r)t^{-\frac{1}{3}} - \frac{b}{2c}, r = xt^{-\frac{1}{3}} - at^{\frac{2}{3}} + \frac{b^2 t^{\frac{2}{3}}}{4c}. \tag{5.1}$$

Substitution of this (5.1) into equation (1.2) results in the ODE

$$3kw_{rrr} + 3cw^2 w_r - rw_r - w = 0. \tag{5.2}$$

Integrating the above equation (5.2) gives

$$3kw_{rr} + cw^3 - rw = k_1, \tag{5.3}$$

where k_1 is a constant of integration. Equation (5.3) is known as the first Painlevé transcendent. Its solutions $w = \sigma(r)$ are meromorphic in the entire complex plane, but are essentially new functions that cannot be expressed in any standard form [31].

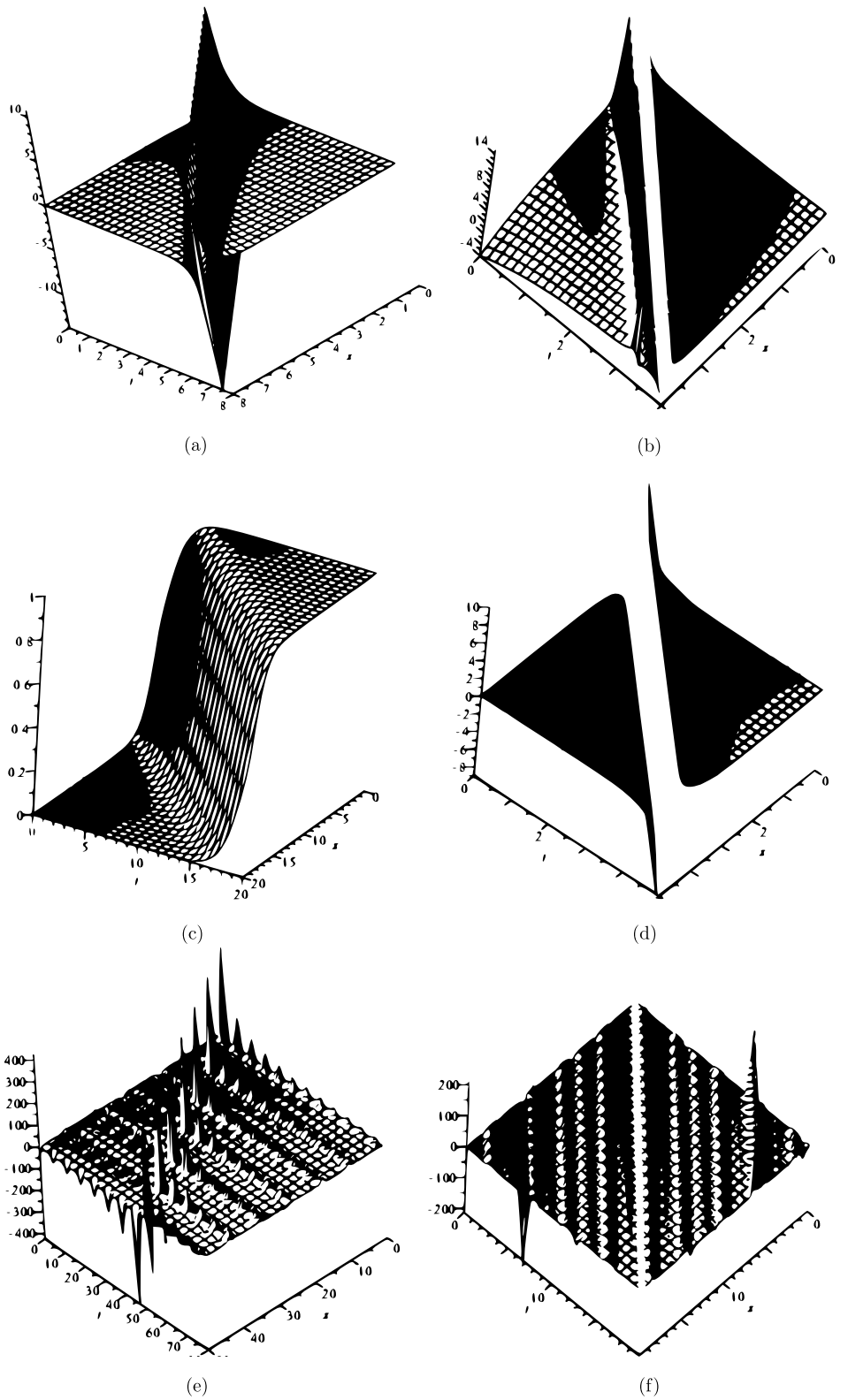


Fig. 1. (a) Soliton shaped solitary wave solution for (A.8), with $n = 1, a_0 = 1, \beta = 1, \lambda = 1, \mu = -1, \nu = 2$. (b) Singular soliton shaped solitary wave solution of (A.9) with same fixed values in (a). (c) Kink type solitary wave solution for (A.26) with $n = 1, a_0 = 1, \beta = 1, d = 1, \lambda = 1$. (d) Singular kink type solitary wave solution for (A.2) with same fixed values in (c). (e) Periodic wave solution of (A.13), with $n = 1, a_0 = 1, \beta = 1, \lambda = 1, \mu = 1, \nu = 2$. (f) Singular periodic wave solution of (A.16), with same fixed values in (e).

Remark. It should be noted that the reduced equation (5.3) is to some extent highly nonlinear and hence quite challenging to solve analytically. The next logical step would be to look at numerical solutions of the reduced equations.

Next we investigate the similarity solutions using double reduction theory. We examine the invariance of the conserved vector (T_1^t, T_1^x) (3.10) under the symmetry point X_3 in (2.13). This will lead to reduction of Gardner equation (1.1) to ODE form which can be easily solved to find exact solutions. A conserved vector (T^t, T^x) is invariant under a Lie point symmetry X if [20]

$$X^{[2]}\begin{pmatrix} T^t \\ T^x \end{pmatrix} + (D_t\xi_t^1 + D_x\xi_x^2)\begin{pmatrix} T^t \\ T^x \end{pmatrix} - \begin{pmatrix} D_t\xi_t^1 & D_x\xi_t^1 \\ D_t\xi_x^2 & D_x\xi_x^2 \end{pmatrix}\begin{pmatrix} T^t \\ T^x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.4}$$

In terms of the canonical variables r, s obtained by mapping X to $Y = \frac{\partial}{\partial s}$, the conservation laws can be expressed as [32]

$$D_r T^r + D_s T^s = 0, \tag{5.5}$$

with T^r and T^s given as

$$T^r = \frac{T^t D_t(r) + T^x D_x(r)}{D_t(r)D_x(s) - D_x(r)D_t(s)}, \tag{5.6}$$

$$T^s = \frac{T^t D_t(s) + T^x D_x(s)}{D_t(r)D_x(s) - D_x(r)D_t(s)}. \tag{5.7}$$

We obtain from (5.4)

$$X_3^{[2]}\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} - 8c\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} - \begin{pmatrix} -6c & 0 \\ b^2 - 4ac & -2c \end{pmatrix}\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} = \begin{pmatrix} b \\ ba \end{pmatrix},$$

where $X_3^{[2]}$ is given by

$$X_3^{[2]} = -6ct\frac{\partial}{\partial t} + (b^2 t - 4act - 2cx)\frac{\partial}{\partial x} + (b + 2cu)\frac{\partial}{\partial u} + 4cu_x\frac{\partial}{\partial u_x} + 6cu_{xx}\frac{\partial}{\partial u_{xx}}.$$

Hence, doubly reduced Gardner equation (1.1) using the scaling symmetry X_3 with the conserved vector (T_1^t, T_1^x) (3.10) is only obtainable when $b = 0$, $n = 1$. This means that the power law nonlinearity term must be quadratic.

We transform the generator X_3 to its canonical form $Y = \frac{\partial}{\partial s}$ via the invariants

$$r = \frac{x}{t^{\frac{1}{3}}} - at^{\frac{2}{3}}, \quad w(r) = ut^{\frac{1}{3}}, \quad s = -\frac{1}{3}\ln(t). \tag{5.8}$$

Using the above canonical variables (5.8) with the formulas (5.6) and (5.7), we obtained the reduced conserved vector for (3.10) written as $D_r T_1^r + D_s T_1^s = 0$ with

$$T_1^r = cw^3 + kw_{rr}, \tag{5.9}$$

$$T_1^s = w. \tag{5.10}$$

The reduced equation becomes

$$cw^3 + kw_{rr} = k_1, \tag{5.11}$$

where k_1 is a constant. The integration of the above equation (5.11) results in

$$\int \frac{dw}{\sqrt{\frac{2}{3k}wk_1 - \frac{c}{6k}w^4}} = r + k_2, \tag{5.12}$$

where k_2 is a constant of integration.

6. Constants of motion and asymptotic behaviour

The conserved integrals defining mass, momentum and energy for the Gardner equation (1.1) are given respectively by

$$I_1 = \int_{-\infty}^{\infty} T_1^t dx = \int_{-\infty}^{\infty} u dx, \tag{6.1}$$

$$I_2 = \int_{-\infty}^{\infty} T_2^t dx = \int_{-\infty}^{\infty} \frac{1}{2}u^2 dx, \tag{6.2}$$

$$I_3 = \int_{-\infty}^{\infty} T_3^t dx = \int_{-\infty}^{\infty} \left(\frac{1}{k(2n+1)(2n+2)} cu^{2n+2} + \frac{1}{k(n+1)(n+2)} bu^{n+2} - \frac{1}{2}u_x^2 \right) dx \tag{6.3}$$

which yield constants of motion for all smooth solutions $u(t,x)$ with sufficiently rapid decay $u \to 0$ as $x \to \pm\infty$.

For a given solution $u(t,x)$ of a Gardner equation (1.1), these integrated densities (6.1)–(6.3) will define a finite conserved (i.e. time–independent) quantity if and only if the densities T_i^t, $i = 1, ..., 3$ themselves have sufficiently rapid decay as $|x| \to \infty$ and also the corresponding fluxes T_i^x, $i = 1, ..., 3$ vanish for $x \to \pm\infty$. We will now determine which of the integrated densities yield a finite conserved quantity for the solutions (4.6)–(4.8) obtained in Section 4. To proceed we first write down the asymptotic behaviour of each solution such that $|r| \gg \frac{1}{\sqrt{\frac{\lambda - a}{k}}}$. Solution (4.6) has asymptotic behaviour

$$u_1 = O(1) \quad \text{as} \quad |x| \to \infty. \tag{6.4}$$

Similarly,

$$u_3 = O(x) \quad \text{as} \quad x \to 0. \tag{6.5}$$

Finally, solution (4.7) has asymptotic exponential decay

$$u_2 = O\left(e^{-\sqrt{\frac{\lambda - a}{k}}|x|} \right) \quad \text{as} \quad |x| \to \infty. \tag{6.6}$$

From this asymptotic behaviour, it is unambiguous to derive the conditions under which each integrated density (6.1)–(6.3) is conserved and finite for each of the solutions.

After some detailed calculations, it turns out that all the integrated density (6.1)–(6.3) are conserved and finite only for the solitary wave (4.6) in the special case $n = 2$, $c = 0$. The conserved mass, momentum and energy for this solution are given by

$$I_1 = \int_{-\infty}^{\infty} u_1 dx = \sqrt{\frac{6k}{b}}\pi, \tag{6.7}$$

$$I_2 = \frac{1}{2}\int_{-\infty}^{\infty} u_1^2 dx = \frac{6\sqrt{k(\lambda-a)}}{b}, \tag{6.8}$$

$$\begin{aligned} I_3 &= \int_{-\infty}^{\infty} \left(\frac{1}{k(n+1)(n+2)} b u_1^{n+2} - \frac{1}{2}u_{1x}^2\right) dx \\ &= \frac{2(\lambda-a)^{\frac{3}{2}}}{b\sqrt{k}}. \end{aligned} \tag{6.9}$$

Interestingly, the momentum I_2 and the energy I_3 satisfy the relation

$$I_3 = \frac{1}{3}\left(\frac{\lambda-a}{k}\right) \tag{6.10}$$

7. Conclusion

In this paper, using the group theoretic method, the third order Gardner equation with dual power law nonlinearity as a generalised setting was studied. In the sense of geometric symmetry, all the vector fields of the equation were presented. The Gardner equation possesses only translational symmetries. These symmetries reduce the nonlinear third order PDE to an integrable third order ODE and we obtained four independent solutions. As the Gardner equation is a third order PDE (and hence not derivable from a variational principle) its conservation laws were discussed via the multiplier method and Noether approach (after increasing the order by one). The importance of these conservation laws was demonstrated using double reduction theory which is based on the invariance of conserved vectors under symmetries. As a result, one independent solution was obtained. Furthermore, we derived additional nontrivial conservation laws and scaling symmetries for some special cases of the Gardner equation. Similar studies are carried out for these special cases. Moreover, the singular soliton and singular periodic solutions were extracted for the Gardner equation with the aid of the improved generalised Riccati method. Certain constraints or compatibility conditions of the parameters of the Gardner equation had to be satisfied to ensure the existence of these solutions. To understand the behaviour of the solutions, we plotted the graphs of the solution surfaces for some special parameter values. Most of the solutions presented have not been reported in literature and many known solutions are only special cases of our results. These solutions include important soliton solutions and nontrivial solutions in terms of special functions which are meromorphic in the entire complex plane. Remarkably, such solutions, having special functions together with arbitrary parameters can be used as a benchmark for solving other related model problems and assessing numerical and approximate analytical methods for nonlinear equations describing solitons in wave mechanics. Apart from complementing results in the literature, the solutions are also useful in the analysis of wave propagation on physical phenomena.

We note that a proper application of the multiplier method will provide additional conservation laws not obtainable from other methods. Work on this is ongoing.

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Appendix A. Explicit solutions of the Gardner equation using improved generalised Riccati equation mapping method

Type 1. $\Omega = \beta^2 - 4\mu(\nu-1) > 0$, $\beta(\nu-1) \neq 0$, (or $\mu(\nu-1) \neq 0$)

$$u_1(t,x) = a_0^{\frac{1}{n}} \left[\frac{-2\mu(\nu-1)}{\beta\left(\beta + \sqrt{\Omega}\tanh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})\right)} + 1 - \frac{1}{2\beta}\left(\beta + \sqrt{\Omega}\tanh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})\right)\right]^{\frac{1}{n}}, \tag{A.1}$$

$$u_2(t,x) = a_0^{\frac{1}{n}} \left[\frac{-2\mu(\nu-1)}{\beta\left(\beta + \sqrt{\Omega}\coth(\frac{\sqrt{\Omega}(x-\lambda t)}{2})\right)} + 1 - \frac{1}{2\beta}\left(\beta + \sqrt{\Omega}\coth(\frac{\sqrt{\Omega}(x-\lambda t)}{2})\right)\right]^{\frac{1}{n}}, \tag{A.2}$$

$$u_3(t,x) = a_0^{\frac{1}{n}} \left[\frac{-2\mu(\nu-1)}{\beta\left(\beta + \sqrt{\Omega}\left(\tanh(\sqrt{\Omega}(x-\lambda t)) \pm i\,\mathrm{sech}(\sqrt{\Omega}(x-\lambda t))\right)\right)} + 1 - \frac{1}{2\beta}\left(\beta + \sqrt{\Omega}\left(\tanh(\sqrt{\Omega}(x-\lambda t)) \pm i\,\mathrm{sech}(\sqrt{\Omega}(x-\lambda t))\right)\right)\right]^{\frac{1}{n}}, \tag{A.3}$$

$$u_4(t,x) = a_0^{\frac{1}{n}} \left[\frac{-2\mu(\nu-1)}{\beta\left(\beta + \sqrt{\Omega}\left(\coth(\sqrt{\Omega}(x-\lambda t)) \pm \mathrm{csch}(\sqrt{\Omega}(x-\lambda t))\right)\right)} + 1 - \frac{1}{2\beta}\left(\beta + \sqrt{\Omega}\left(\coth(\sqrt{\Omega}(x-\lambda t)) \pm \mathrm{csch}(\sqrt{\Omega}(x-\lambda t))\right)\right)\right]^{\frac{1}{n}}, \tag{A.4}$$

$$u_5(t,x) = a_0^{\frac{1}{n}} \left[\frac{-4\mu(\nu-1)}{\beta\left(2\beta + \sqrt{\Omega}\left(\tanh(\frac{\sqrt{\Omega}(x-\lambda t)}{4}) \pm \coth(\frac{\sqrt{\Omega}(x-\lambda t)}{4})\right)\right)} + 1 - \frac{1}{4\beta}\left(2\beta + \sqrt{\Omega}\left(\tanh(\frac{\sqrt{\Omega}(x-\lambda t)}{4}) \pm \coth(\frac{\sqrt{\Omega}(x-\lambda t)}{4})\right)\right)\right]^{\frac{1}{n}}, \tag{A.5}$$

$$u_6(t,x) = a_0^{\frac{1}{n}} \left[\frac{2\mu(\nu-1)}{\beta}\left(-\beta + \frac{\sqrt{\Omega(E^2+F^2)} - E\sqrt{\Omega}\cosh(\sqrt{\Omega}(x-\lambda t))}{E\sinh(\sqrt{\Omega}(x-\lambda t)) + F}\right)^{-1} + 1 + \frac{1}{2\beta}\left(-\beta + \frac{\sqrt{\Omega(E^2+F^2)} - E\sqrt{\Omega}\cosh(\sqrt{\Omega}(x-\lambda t))}{E\sinh(\sqrt{\Omega}(x-\lambda t)) + F}\right)\right]^{\frac{1}{n}}, \tag{A.6}$$

$$u_7(t,x) = a_0^{\frac{1}{n}} \left[\frac{2\mu(\nu-1)}{\beta}\left(-\beta - \frac{\sqrt{\Omega(F^2-E^2)} + E\sqrt{\Omega}\cosh(\sqrt{\Omega}(x-\lambda t))}{E\sinh(\sqrt{\Omega}(x-\lambda t)) + F}\right)^{-1} + 1 + \frac{1}{2\beta}\left(-\beta - \frac{\sqrt{\Omega(F^2-E^2)} + E\sqrt{\Omega}\cosh(\sqrt{\Omega}(x-\lambda t))}{E\sinh(\sqrt{\Omega}(x-\lambda t)) + F}\right)\right]^{\frac{1}{n}}, \tag{A.7}$$

where E and F are two non-zero real constants satisfying $F^2 - E^2 > 0$,

$$u_8(t,x) = a_0^{\frac{1}{n}} \left[\frac{\sqrt{\Omega}\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{2}) - \beta\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})}{2\beta\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})} + 1 \right.$$
$$\left. + \frac{2\mu(\nu-1)}{\beta}\left(\frac{\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})}{\sqrt{\Omega}\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{2}) - \beta\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})} \right) \right]^{\frac{1}{n}}, \tag{A.8}$$

$$u_9(t,x) = a_0^{\frac{1}{n}} \left[-\frac{\beta\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{2}) - \sqrt{\Omega}\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})}{2\beta\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})} + 1 \right.$$
$$\left. - \frac{2\mu(\nu-1)}{\beta}\left(\frac{\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})}{\beta\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{2}) - \sqrt{\Omega}\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})} \right) \right]^{\frac{1}{n}}, \tag{A.9}$$

$$u_{10}(t,x) = a_0^{\frac{1}{n}} \left[\frac{\sqrt{\Omega}\sinh(\sqrt{\Omega}(x-\lambda t)) - \beta\cosh(\sqrt{\Omega}(x-\lambda t)) \pm i\sqrt{\Omega}}{2\beta\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})} + 1 \right.$$
$$\left. + \frac{2\mu(\nu-1)}{\beta}\left(\frac{\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})}{\sqrt{\Omega}\sinh(\sqrt{\Omega}(x-\lambda t)) - \beta\cosh(\sqrt{\Omega}(x-\lambda t)) + \pm i\sqrt{\Omega}} \right) \right]^{\frac{1}{n}}, \tag{A.10}$$

$$u_{11}(t,x) = a_0^{\frac{1}{n}} \left[\frac{-\beta\sinh(\sqrt{\Omega}(x-\lambda t)) + \sqrt{\Omega}\cosh(\sqrt{\Omega}(x-\lambda t)) \pm \sqrt{\Omega}}{2\beta\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})} + 1 \right.$$
$$\left. + \frac{2\mu(\nu-1)}{\beta}\left(\frac{\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{2})}{-\beta\sinh(\sqrt{\Omega}(x-\lambda t)) + \sqrt{\Omega}\cosh(\sqrt{\Omega}(x-\lambda t)) \pm \sqrt{\Omega}} \right) \right]^{\frac{1}{n}}, \tag{A.11}$$

$$u_{12}(t,x) = a_0^{\frac{1}{n}} \left[\frac{-2\beta\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{4})\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{4}) + 2\sqrt{\Omega}\cosh^2(\frac{\sqrt{\Omega}(x-\lambda t)}{4}) - \sqrt{\Omega}}{4\beta\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{4})\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{4})} + 1 \right.$$
$$\left. + \frac{4\mu(\nu-1)}{\beta}\left(\frac{\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{4})\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{4})}{-2\beta\sinh(\frac{\sqrt{\Omega}(x-\lambda t)}{4})\cosh(\frac{\sqrt{\Omega}(x-\lambda t)}{4}) + 2\sqrt{\Omega}\cosh^2(\frac{\sqrt{\Omega}(x-\lambda t)}{4}) - \sqrt{\Omega}} \right) \right]^{\frac{1}{n}}. \tag{A.12}$$

Type 2. $\Omega < 0$, $\beta(\nu-1) \neq 0$, (or $\mu(\nu-1) \neq 0$)

$$u_{13}(t,x) = a_0^{\frac{1}{n}} \left[\frac{2\mu(\nu-1)}{\beta\left(-\beta + \sqrt{-\Omega}\tan\left(\frac{\sqrt{-\Omega}(x-\lambda t)}{2}\right)\right)} + 1 \right.$$
$$\left. + \frac{1}{2\beta}\left(-\beta + \sqrt{-\Omega}\tan\left(\frac{\sqrt{-\Omega}(x-\lambda t)}{2}\right)\right) \right]^{\frac{1}{n}}, \tag{A.13}$$

$$u_{14}(t,x) = a_0^{\frac{1}{n}} \left[-\frac{2\mu(\nu-1)}{\beta\left(\beta + \sqrt{-\Omega}\cot\left(\frac{\sqrt{-\Omega}(x-\lambda t)}{2}\right)\right)} + 1 \right.$$
$$\left. - \frac{1}{2\beta}\left(\beta + \sqrt{-\Omega}\cot\left(\frac{\sqrt{-\Omega}(x-\lambda t)}{2}\right)\right) \right]^{\frac{1}{n}}, \tag{A.14}$$

$$u_{15}(t,x) = a_0^{\frac{1}{n}} \left[\frac{2\mu(\nu-1)}{\beta\left(-\beta + \sqrt{-\Omega}\left(\tan(\sqrt{-\Omega}(x-\lambda t)) \pm \sec(-\sqrt{\Omega}(x-\lambda t))\right)\right)} \right.$$

$$+1+\frac{1}{2\beta}\left(-\beta+\sqrt{-\Omega}\left(\tan(\sqrt{-\Omega}(x-\lambda t))\pm\sec(-\sqrt{\Omega}(x-\lambda t))\right)\right)\Bigg]^{\frac{1}{n}}, \tag{A.15}$$

$$u_{16}(t,x)=a_0^{\frac{1}{n}}\left[-\frac{2\mu(\nu-1)}{\beta\left(\beta+\sqrt{-\Omega}\left(\cot(\sqrt{-\Omega}(x-\lambda t))\pm\csc(-\sqrt{\Omega}(x-\lambda t))\right)\right)}\right.$$

$$\left.+1-\frac{1}{2\beta}\left(\beta+\sqrt{-\Omega}\left(\cot(\sqrt{-\Omega}(x-\lambda t))\pm\csc(-\sqrt{\Omega}(x-\lambda t))\right)\right)\right]^{\frac{1}{n}}, \tag{A.16}$$

$$u_{17}(t,x)=a_0^{\frac{1}{n}}\left[\frac{4\mu(\nu-1)}{\beta\left(-2\beta+\sqrt{-\Omega}\left(\tan\left(\frac{\sqrt{-\Omega}}{4}(x-\lambda t)\right)-\cot\left(\frac{\sqrt{-\Omega}}{4}(x-\lambda t)\right)\right)\right)}\right.$$

$$\left.+1+\frac{1}{4\beta}\left(-2\beta+\sqrt{-\Omega}\left(\tan\left(\frac{\sqrt{-\Omega}}{4}(x-\lambda t)\right)-\cot\left(\frac{\sqrt{-\Omega}}{4}(x-\lambda t)\right)\right)\right)\right]^{\frac{1}{n}}, \tag{A.17}$$

$$u_{18}(t,x)=a_0^{\frac{1}{n}}\left[\frac{2\mu(\nu-1)}{\beta}\left(-\beta+\frac{\pm\sqrt{-\Omega(E^2-F^2)}-E\sqrt{-\Omega}\cos(\sqrt{-\Omega}(x-\lambda t))}{E\sin(\sqrt{-\Omega}(x-\lambda t))+F}\right)^{-1}\right.$$

$$\left.+1+\frac{1}{2\beta}\left(-\beta+\frac{\pm\sqrt{-\Omega(E^2-F^2)}-E\sqrt{-\Omega}\cos(\sqrt{-\Omega}(x-\lambda t))}{E\sin(\sqrt{-\Omega}(x-\lambda t))+F}\right)\right]^{\frac{1}{n}}, \tag{A.18}$$

$$u_{19}(t,x)=a_0^{\frac{1}{n}}\left[\frac{2\mu(\nu-1)}{\beta}\left(-\beta-\frac{\pm\sqrt{-\Omega(E^2-F^2)}+E\sqrt{-\Omega}\cos(\sqrt{-\Omega}(x-\lambda t))}{E\sin(\sqrt{-\Omega}(x-\lambda t))+F}\right)^{-1}\right.$$

$$\left.+1+\frac{1}{2\beta}\left(-\beta-\frac{\pm\sqrt{-\Omega(E^2-F^2)}+E\sqrt{-\Omega}\cos(\sqrt{-\Omega}(x-\lambda t))}{E\sin(\sqrt{-\Omega}(x-\lambda t))+F}\right)\right]^{\frac{1}{n}}, \tag{A.19}$$

where E and F are two non-zero real constants satisfying $E^2-F^2>0$,

$$u_{20}(t,x)=a_0^{\frac{1}{n}}\left[-\frac{\sqrt{-\Omega}\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})+\beta\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})}{2\beta\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})}\right.$$

$$\left.+1-\frac{2\mu(\nu-1)}{\beta}\left(\frac{\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})}{\sqrt{-\Omega}\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})+\beta\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})}\right)\right]^{\frac{1}{n}}, \tag{A.20}$$

$$u_{21}(t,x)=a_0^{\frac{1}{n}}\left[\frac{-\beta\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})+\sqrt{-\Omega}\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})}{2\beta\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})}\right.$$

$$\left.+1+\frac{2\mu(\nu-1)}{\beta}\left(\frac{\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})}{-\beta\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})+\sqrt{-\Omega}\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})}\right)\right]^{\frac{1}{n}}, \tag{A.21}$$

$$u_{22}(t,x)=a_0^{\frac{1}{n}}\left[-\frac{\sqrt{-\Omega}\sin(\sqrt{-\Omega}(x-\lambda t))+\beta\cos(\sqrt{-\Omega}(x-\lambda t))\pm\sqrt{-\Omega}}{2\beta\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})}\right.$$

$$\left.+1-\frac{2\mu(\nu-1)}{\beta}\left(\frac{\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})}{\sqrt{-\Omega}\sin(\sqrt{-\Omega}(x-\lambda t))+\beta\cos(\sqrt{-\Omega}(x-\lambda t))\pm\sqrt{-\Omega}}\right)\right]^{\frac{1}{n}}, \tag{A.22}$$

$$u_{23}(t,x)=a_0^{\frac{1}{n}}\left[\frac{-\beta\sin(\sqrt{-\Omega}(x-\lambda t))+\sqrt{-\Omega}\cos(\sqrt{-\Omega}(x-\lambda t))\pm\sqrt{-\Omega}}{2\beta\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})}\right.$$

$$+1+\frac{2\mu(\nu-1)}{\beta}\left(\frac{\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{2})}{-\beta\sin(\sqrt{-\Omega}(x-\lambda t))+\sqrt{-\Omega}\cos(\sqrt{-\Omega}(x-\lambda t))\pm\sqrt{-\Omega}}\right)\Bigg]^{\frac{1}{n}}, \tag{A.23}$$

$$u_{24}(t,x)=a_0^{\frac{1}{n}}\left[\frac{-2\beta\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{4})\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{4})+2\sqrt{-\Omega}\cos^2(\frac{\sqrt{-\Omega}(x-\lambda t)}{4})-\sqrt{-\Omega}}{4\beta\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{4})\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{4})}\right.$$

$$\left.+1+\frac{4\mu(\nu-1)}{\beta}\left(\frac{\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{4})\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{4})}{-2\beta\sin(\frac{\sqrt{-\Omega}(x-\lambda t)}{4})\cos(\frac{\sqrt{-\Omega}(x-\lambda t)}{4})+2\sqrt{-\Omega}\cos^2(\frac{\sqrt{-\Omega}(x-\lambda t)}{4})-\sqrt{-\Omega}}\right)\right]^{\frac{1}{n}}. \tag{A.24}$$

Type 3. $\mu=0$ $\beta(\nu-1)\neq 0$,

$$u_{25}(t,x)=a_0^{\frac{1}{n}}\left[1-\left(\frac{d}{d+\cosh(\beta(x-\lambda t)-\sinh(\beta(x-\lambda t)}\right)\right]^{\frac{1}{n}}, \tag{A.25}$$

$$u_{26}(t,x)=a_0^{\frac{1}{n}}\left[1-\left(\frac{\cosh(\beta(x-\lambda t)+\sinh(\beta(x-\lambda t)}{d+\cosh(\beta(x-\lambda t)+\sinh(\beta(x-\lambda t)}\right)\right]^{\frac{1}{n}}, \tag{A.26}$$

where d is an arbitrary constant.

Type 4. $\mu=\beta=0$, $\nu\neq 1$. There is no solution to the Gardner equation (1.1) for Type 4.

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