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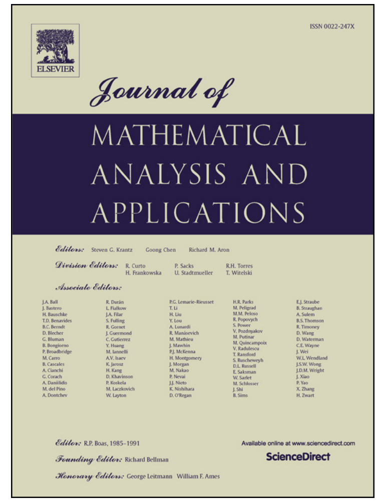

Universality of automorphisms on the ball of bounded holomorphic functions on the polydisk

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UNIVERSALITY OF AUTOMORPHISMS ON THE BALL OF BOUNDED HOLOMORPHIC FUNCTIONS ON THE POLYDISK

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ABSTRACT. Given a sequence of automorphisms of the polydisk, we show that the associated composition semigroup homomorphisms on the ball of bounded holomorphic functions on the polydisk admit a universal inner function if a certain condition on the automorphisms are satisfied.

1. INTRODUCTION

Let X be a separable, metrizable topological space. A sequence of continuous maps $\{T_n : X \to X | n \geq 1\}$ is said to be universal if there exists $x \in X$ so that $\{T_n x : n \geq 1\}$ is dense in X. Such an x is called a universal element of $\{T_n\}$. Let $\Omega \subset \mathbb{C}^n$ be a domain. We let $H(\Omega)$ represent the set of all holomorphic functions on Ω. When equipped with the compact open topology, $H(\Omega)$ is a Frechet space. Recall that a sequence $\{f_j\} \subset H(\Omega)$ converges to f in the compact open topology if and only if $f_j \to f$ uniformly on compact subsets of Ω as $j \to \infty$. We define

$$\mathbb{D}^n := \{(z_1, ..., z_n) \in \mathbb{C}^n : |z_j| < 1\,, j = 1, 2, ..., n\}$$

to be the unit polydisk and let

$$\mathbb{T}^n := \{(z_1, ..., z_n) \in \mathbb{C}^n : |z_j| = 1\,, j = 1, ..., n\}$$

be the distinguished boundary. Then $H^\infty(\mathbb{D}^n)$ denotes the space of bounded holomorphic functions on $\mathbb{D}^n$ equipped with the compact open topology. Then we define the ball of $H^\infty(\mathbb{D}^n)$ to be

$$\overline{Ball}(H^\infty(\mathbb{D}^n)) := \{h \in H^\infty(\mathbb{D}^n) : \|h\|_{L^\infty(\mathbb{D}^n)} \leq 1\}.$$

We will show the non-Euclidean analog of Seidel-Walsh theorem (see [GS87] and [SW41]) for the polydisk in $\mathbb{C}^n$ and then give necessary and sufficient conditions for a sequence of automorphisms of the disk and the polydisk to have a universal element on the ball of $H^\infty(\mathbb{D}^n)$. Since $\overline{Ball}(H^\infty(\mathbb{D}^n))$ is a semigroup and not a vector space, one cannot use vector space techniques. Instead, we show that the semigroup universality criterion in [CW19] is satisfied for certain sequences of composition operators (actually semigroup homomorphisms) on $\overline{Ball}(H^\infty(\mathbb{D}^n))$. On the unit ball in $\mathbb{C}^n$, [AG07] studies whether a sequence of automorphisms

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on the unit ball in $\mathbb{C}^n$ admits a universal element. On the polydisk, the main result in [Che79] shows there exists a sequence of automorphisms of the polydisk with a universal element, but does not consider arbitrary sequences of automorphisms. We give a natural condition on a sequence of automorphisms of the polydisk that ensures the associated composition semigroup homomorphisms admit a universal element. The following theorem classifies automorphisms of the unit polydisk. Let S_n be the symmetry group of n elements. Recall the following theorem appearing in [Rud69].

Theorem 1. [Rud69] *Let ϕ be an automorphism of the polydisk $\mathbb{D}^n \subset \mathbb{C}^n$. Then there exists $(\alpha_1, ..., \alpha_n) \in \mathbb{D}^n$, $(\theta_1, ..., \theta_n) \in \mathbb{R}^n$, and $p \in S_n$ so that*

$$\phi(z_1, z_2, ..., z_n) = \left(e^{i\theta_1} \frac{\alpha_1 - z_{p(1)}}{1 - \overline{\alpha_1} z_{p(1)}}, ..., e^{i\theta_n} \frac{\alpha_n - z_{p(n)}}{1 - \overline{\alpha_n} z_{p(n)}} \right).$$

2. Inner functions on the Polydisk

Recall that on the disk in $\mathbb{C}$, an inner function is any bounded holomorphic function on the disk $g : \mathbb{D} \to \mathbb{C}$ so that

$$\lim_{r \to 1^-} |f(re^{i\theta})| = 1$$

almost everywhere. We can extend this definition to the polydisk in a natural way. That is, we say a bounded holomorphic function on the polydisk $f : \mathbb{D}^n \to \mathbb{C}$ is an inner function if

$$\lim_{r \to 1^-} |f(re^{i\theta_1}, re^{i\theta_2}, ..., re^{i\theta_n})| = 1$$

almost everywhere. We say an inner function $g : \mathbb{D}^n \to \mathbb{C}$ is a good inner function (see [Che79]) if

$$\int_{b\mathbb{D}^n} \log |g(r\zeta)| d\sigma(\zeta) \to 0$$

as $r \to 1^-$.

The following proposition shows that if f is a good inner function then for any automorphism of the polydisk, ϕ, $f \circ \phi$ is also a good inner function. That is, the action of the composition operator C_ϕ preserves good inner functions.

Proposition 1. *Let $g : \mathbb{D}^n \to \mathbb{C}$ be a good inner function and $\phi : \mathbb{D}^n \to \mathbb{D}^n$ be an automorphism of the polydisk. Then, $g \circ \phi$ is a good inner function.*

Proof. First we will show if $g \in H^\infty(\mathbb{D}^n)$ is a good inner function and ϕ an automorphism of $\mathbb{D}^n$, then $g \circ \phi$ is an inner function. Then $|g^*(w)| = 1$ for almost every $w \in \mathbb{T}^n$ where g^* denotes the boundary values (radial limits) of g on $\mathbb{T}^n$. By Theorem 1, we may assume $\phi(z_1, ..., z_n) = (\sigma_1(z_1), \sigma_2(z_2), ..., \sigma_n(z_n))$ where

$$\sigma_j(z_j) = \frac{\alpha_j - \lambda_j z_j}{1 - \overline{\alpha_j} \lambda_j z_j}$$

is an inner function on $\mathbb{D}$ for $\alpha_j \in \mathbb{D}$ and $|\lambda_j| = 1$ and $j = 1, 2, ..., n$. Then an application of [Saw77, Theorem 1.2.4] allows us to write $|(g \circ \phi)^*| = |g^* \circ \phi^*| = 1$ almost everywhere on $\mathbb{T}^n$. Thus $g \circ \phi$ is inner.

Now let $r < 1$ be sufficiently large so that

$$r > \max\{|\alpha_j| : j = 1, 2, ..., n\}.$$

Then the image $\phi(r\mathbb{T}^n)$ can be represented as

$$\prod_{j=1}^{n} \mathbb{T}_{a_j(r), r_j(r)}.$$

Here,

$$\mathbb{T}_{a_j(r), r_j(r)}$$

is a circle centered at $a_j(r)$ with radius $r_j(r)$ where $a_j(r) \to 0$ and $r_j(r) \to 1$ as $r \to 1^-$. For some fixed $M > \sup\{|J(\phi)(z)|^{-1} : z \in \overline{\mathbb{D}^n}\}$ we have,

$$\begin{aligned}
&(1) \qquad |\int_{\mathbb{T}^n} \log|g \circ \phi(r\zeta)| d\sigma(\zeta)| \\
&(2) \qquad = |\int_{(r\mathbb{T}^n)} \log|g \circ \phi(\zeta)| r^{-n} d\sigma(\zeta)| \\
&(3) \qquad = |\int_{\phi((r\mathbb{T}^n))} \log|g(w)||J(\phi)(w)|^{-1} r^{-n} d\sigma(w)| \\
&(4) \qquad = |\int_{\prod_{j=1}^{n} \mathbb{T}_{a_j(r), r_j(r)}} \log|g(w)||J(\phi)(w)|^{-1} r^{-n} d\sigma(w)| \\
&(5) \qquad \le \int_0^{2\pi} \int_0^{2\pi} ... \int_0^{2\pi} |\log|g(a_1(r) + r_1(r)e^{i\theta_1}, ..., a_n(r) + r_n(r)e^{i\theta_n})| \\
&(6) \qquad - \log|g(r_1(r)e^{i\theta_1}, ..., r_n(r)e^{i\theta_n})||Mr^{-n} d\theta_1 ... d\theta_n \\
&(7) \qquad + \int_0^{2\pi} ... \int_0^{2\pi} |\log|g(r_1(r)e^{i\theta_1}, ..., r_n(r)e^{i\theta_n})| - \log|g(Re^{i\theta_1}, ..., Re^{i\theta_n})||Mr^{-n} d\theta_1 ... d\theta_n \\
&(8) \qquad + \int_{\zeta \in \mathbb{T}^n} - \log|g(R\zeta)| r^{-n} M d\sigma(\zeta)
\end{aligned}$$

Then using a continuity argument and using the Lebesgue dominated convergence theorem, one can make lines 5 and 6 sufficiently small when $r < 1$ is sufficiently large and $R > r$ is sufficiently close to r. Furthermore, using the assumption that g is a good inner function, one can make the line 7 integral arbitrarily small if $r < 1$ is sufficiently large. Then the line 8 integral is arbitrarily small for all $r < 1$ sufficiently large since g is a good inner function.

Thus we have shown that

$$|\int_{\mathbb{T}^n} \log|g \circ \phi(r\zeta)| d\sigma(\zeta)| \to 0$$

as $r \to 1^-$ for any automorphism of the polydisk ϕ and good inner function g.

□

The following proposition uses a result in [Rud69] about the density of inner functions continuous up to $\overline{\mathbb{D}^n}$ in the ball of $H^\infty(\mathbb{D}^n)$.

Proposition 2. *Let $\overline{Ball}(H^\infty(\mathbb{D}^n))$ be the ball of $H^\infty(\mathbb{D}^n)$ as defined previously. Suppose $\lambda := (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{T}^n$. Then there exists a collection of inner functions $\{G_{j,\lambda}\}_{j\in\mathbb{N}} \subset \overline{Ball}(H^\infty(\mathbb{D}^n))$ so that*

1. *$G_{j,\lambda} \in C(\overline{\mathbb{D}^n})$ for all $j \in \mathbb{N}$.*
2. *$\{G_{j,\lambda}\}_{j\in\mathbb{N}}$ is dense in $\overline{Ball}(H^\infty(\mathbb{D}^n))$ equipped with the compact-open topology.*
3. *$G_{j,\lambda}(\lambda_1, \lambda_2, ..., \lambda_n) = 1$ for all $j \in \mathbb{N}$.*

Proof. By [Rud69, Theorem 5.5.1], there exists a collection of inner functions $\{A_j\}_{j\in\mathbb{N}} \subset \overline{Ball}(H^\infty(\mathbb{D}^n))$ that is dense in $\overline{Ball}(H^\infty(\mathbb{D}^n))$ equipped with the compact open topology and $A_j \in C(\overline{\mathbb{D}^n})$ for all $j \in \mathbb{N}$. To construct the sequence $G_{j,\lambda}$ we modify the sequence A_j as follows. By [Che79], (see also [Hei54]) there exists $\Psi_j \in C^\infty(\overline{\mathbb{D}^n}) \cap H^\infty(\mathbb{D}^n)$ so that for all $j \in \mathbb{N}$, Ψ_j is inner,

$$\Psi_j(1,1,...,1) = \frac{1}{A_j(1,1,...,1)},$$

and $\Psi_j(0,0,...,0) = 1 - 2^{-j}$. We set

$$\Psi_{j,\lambda}(z_1,...,z_n) := \Psi_j(\lambda_1^{-1}z_1,...,\lambda_n^{-1}z_n),$$
$$A_{j,\lambda}(z_1,...,z_n) := A_j(\lambda_1^{-1}z_1,...,\lambda_n^{-1}z_n),$$

and $G_{j,\lambda} := A_{j,\lambda}\Psi_{j,\lambda}$. Then for all $j \in \mathbb{N}$, $G_{j,\lambda}(\lambda_1,...,\lambda_n) = 1$ and $G_{j,\lambda} \in C(\overline{\mathbb{D}^n})$. Let $f \in \overline{Ball}(H^\infty(\mathbb{D}^n))$. Then there exists a subsequence A_{j_k} so that $A_{j_k} \to f(\lambda_1 z_1,...,\lambda_n z_n)$ uniformly on compact subsets of $\mathbb{D}^n$ as $k \to \infty$. So $A_{j_k,\lambda} \to f$ uniformly on compact subsets of $\mathbb{D}^n$ as $k \to \infty$. Then $\{\Psi_{j_k,\lambda}\}$ is a normal family, so by passing to a subsequence if necessary, we may assume by the maximum modulus principle that $\Psi_{j_k,\lambda} \to 1$ uniformly on compact subsets of $\mathbb{D}^n$ as $k \to \infty$. So $G_{j_k,\lambda} \to f$ uniformly on compact subsets of $\mathbb{D}^n$ as $k \to \infty$. Thus $\{G_{j,\lambda} : j \in \mathbb{N}\}$ is dense in $\overline{Ball}(H^\infty(\mathbb{D}^n))$ equipped with the compact open topology.

□

Proposition 3. *Let ϕ be an automorphism of $\mathbb{D}^n$, and $C_\phi : H(\mathbb{D}^n) \to H(\mathbb{D}^n)$ be its associated composition operator. Then C_ϕ restricted to $\overline{Ball}(H^\infty(\mathbb{D}^n))$ is surjective onto $\overline{Ball}(H^\infty(\mathbb{D}^n))$.*

Proof. Let $f \in \overline{Ball}(H^\infty(\mathbb{D}^n))$. Then it is clear that $C_{\phi^{-1}}f \in \overline{Ball}(H^\infty(\mathbb{D}^n))$ and $f = C_\phi\left(C_{\phi^{-1}}f\right)$. □

3. Universality Criterion for Semigroups

We use the semigroup analog of the following theorem in a significant way, as vector space techniques are not applicable to the ball of $H^\infty(\mathbb{D}^n)$.

Theorem 2. [GS87] *Suppose T is a continuous linear operator on a separable F-space X. Suppose there exists a dense subset $\mathcal{D}$ of X and a right inverse S for T so that $\|T^n x\| \to 0$ and $\|S^n x\| \to 0$ for all $x \in \mathcal{D}$, then X has T-universal vectors.*

The following is a consequence of Theorem 2 as mentioned in [GS87, pp. 283].

Corollary 1. [GS87] *Suppose $\mathcal{D}$ is a dense subset of X and $\{T_j\}$ is a sequence of continuous linear operators on X for which $T_j \to 0$ pointwise on $\mathcal{D}$ suppose for each j the operator T_j has a right inverse S_j and $S_j \to 0$ pointwise on $\mathcal{D}$. Then the set $\{T_j x : j \geq 0\}$ is dense in X for a dense G_δ set of vectors $x \in X$.*

This semigroup analog appears in [CW19] and is stated here for the convenience of the reader.

Theorem 3. [CW19] *Let X be a separable, metrizable, complete, topological semigroup with identity element e. Suppose $\{T_n : X \to X\}_{n\in\mathbb{N}}$ is a collection of continuous semigroup homomorphisms. If there exist dense sets $\mathcal{D}_0 \subset X$, $\mathcal{D}_1 \subset X$, and a sequence of mappings $\{R_n : \mathcal{D}_1 \to X\}_{n\in\mathbb{N}}$ so that*

(1) *$T_n\mathcal{D}_0 \to e$ as $n \to \infty$.*
(2) *$R_n\mathcal{D}_1 \to e$ as $n \to \infty$.*
(3) *$T_n R_n f \to f$ for all $f \in \mathcal{D}_1$.*

Then there exists $\{x_j\} \subset \mathcal{D}_0$ so that $x := \prod_j x_j$ is universal for $\{T_n\}$. That is, the orbit $\{T_n x : n \in \mathbb{N}\}$ is dense in X.

4. Main Result and Proof

Now we will study when an arbitrary sequence of automorphisms of the polydisk has a universal element.

This next theorem is a new result and improves [Che79], and is valid for the ball of $H^\infty(\mathbb{D}^n)$ for $n = 1, 2, \ldots$.

Theorem 4. *Let*

$$\phi_k(z_1, \ldots, z_n) := \left(e^{i\theta_1^k} \frac{\alpha_1^k - z_{p_k(1)}}{1 - \overline{\alpha_1^k} z_{p_k(1)}}, \ldots, e^{i\theta_n^k} \frac{\alpha_n^k - z_{p_k(n)}}{1 - \overline{\alpha_n^k} z_{p_k(n)}} \right)$$

be a sequence of automorphisms of the polydisk $\mathbb{D}^n \subset \mathbb{C}^n$. If $(\alpha_1^k, \ldots, \alpha_n^k) \to (\alpha_1, \ldots, \alpha_n) \in \mathbb{T}^n$ as $k \to \infty$ then there exists an inner function x so that $\{x \circ \phi_k : k \in \mathbb{N}\}$ is dense in $\overline{Ball}(H^\infty(\mathbb{D}^n))$.

Furthermore, the universal element x has the form

$$x = \prod_{j=1}^{\infty} x_j$$

for some sequence of continuous (up to $\overline{\mathbb{D}^n}$) inner functions $\{x_j\}_{j\in\mathbb{N}}$.

Proof. We first note that the theorem is actually stronger than is stated in that we only require a subsequence of $(\alpha_1^k, ..., \alpha_n^k)$ to converge to $\mathbb{T}^n$ as $k \to \infty$. So, without loss of generality, we may assume $(\alpha_1^k, ..., \alpha_n^k) \to (\alpha_1, ..., \alpha_n) \in \mathbb{T}^n$ as $k \to \infty$. Then by Proposition 3, $C_{\phi_k} : \overline{Ball}(H^\infty(\mathbb{D}^n)) \to \overline{Ball}(H^\infty(\mathbb{D}^n))$ is surjective, so has a right inverse, denoted by $C_{\phi_k}^{-1}$. We let ϕ_{k_l} be a subsequence of ϕ_k with the following properties.

(1) $(\theta_1^{k_l}, ..., \theta_n^{k_l}) \to (\theta_1, ..., \theta_n)$ as $l \to \infty$.
(2) $p_{k_l} = p_{k_{l+1}} := p$ for all $l \in \mathbb{N}$.

We note that

$$\phi_{k_l}^{-1} = \left(e^{i\theta^{k_l}_{p^{-1}(1)}} \frac{a^{k_l}_{p^{-1}(1)} - z_{p^{-1}(1)}}{1 - \overline{a^{k_l}_{p^{-1}(1)}} z_{p^{-1}(1)}}, ..., e^{i\theta^{k_l}_{p^{-1}(n)}} \frac{a^{k_l}_{p^{-1}(n)} - z_{p^{-1}(n)}}{1 - \overline{a^{k_l}_{p^{-1}(n)}} z_{p^{-1}(n)}} \right).$$

Since ϕ_{k_l} and $\phi_{k_l}^{-1}$ are not necessarily equal, we need to define dense (in the ball of $H^\infty(\mathbb{D}^n)$) sets $\mathcal{D}_0$ and $\mathcal{D}_1$ so that $C_{\phi_{k_l}}\mathcal{D}_0 \to 1$ and $C_{\phi_{k_l}^{-1}}\mathcal{D}_1 \to 1$ as $l \to \infty$. Then we can apply the semigroup universality criterion seen in Theorem 3.

One can show that for any $g \in C(\overline{\mathbb{D}^n})$, $g \circ \phi_{k_l} \to g(\lambda)$ uniformly on compact subsets of $\mathbb{D}^n$ as $l \to \infty$ where

$$\lambda := (e^{i\theta_1}\alpha_1, ..., e^{i\theta_n}\alpha_n) = \lim_{l\to\infty} \left(e^{i\theta_1^{k_l}} \alpha_1^{k_l}, ..., e^{i\theta_n^{k_l}} \alpha_n^{k_l} \right).$$

Furthermore, $g \circ \phi_{k_l}^{-1} \to g(\gamma)$ uniformly on compact subsets of $\mathbb{D}^n$ as $l \to \infty$, where

$$\gamma := (e^{i\theta_{p^{-1}(1)}}\alpha_{p^{-1}(1)}, ..., e^{i\theta_{p^{-1}(n)}}\alpha_{p^{-1}(n)}) = \lim_{l\to\infty} \left(e^{i\theta^{k_l}_{p^{-1}(1)}}\alpha^{k_l}_{p^{-1}(1)}, ..., e^{i\theta^{k_l}_{p^{-1}(n)}}\alpha^{k_l}_{p^{-1}(n)} \right).$$

For $G_{j,\lambda}$ and $G_{j,\gamma}$ as defined in Proposition 2, we define

$$\mathcal{D}_0 := \{G_{j,\lambda} : j \in \mathbb{N}\}$$

and

$$\mathcal{D}_1 := \{G_{j,\gamma} : j \in \mathbb{N}\}.$$

Then if $C_{\phi_{k_l}}$ is defined as the composition operator on $H(\mathbb{D}^n)$, one can show that

$$C_{\phi_{k_l}}\mathcal{D}_0 \to 1$$

and

$$C_{\phi_{k_l}}^{-1}\mathcal{D}_1 \to 1$$

uniformly on compact subsets of $\mathbb{D}^n$ as $l \to \infty$. By Proposition 2, $\mathcal{D}_0$ and $\mathcal{D}_1$ are dense in the ball of $H^\infty(\mathbb{D}^n)$ equipped with the compact-open topology. Thus we can conclude by Theorem 3 that there exists a universal inner function x for C_{ϕ_k}. Furthermore, $x = \prod_{j\in\mathbb{N}} x_j$ for some $\{x_j\} \subset \mathcal{D}_0$, implying $x_j \in C(\overline{\mathbb{D}^n})$ for all $j \in \mathbb{N}$.

□

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