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Regularity and stability analysis for a class of semilinear nonlocal differential equations in Hilbert spaces

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We deal with a class of semilinear nonlocal differential equations in Hilbert spaces which is a general model for some anomalous diffusion equations. By using the theory of integral equations with completely positive kernel together with local estimates, some existence, regularity and stability results are established. An application to nonlocal partial differential equations is shown to demonstrate our abstract results.

1. Introduction

Let H be a separable Hilbert space. Consider the following problem

$$\frac{d}{dt}[k*(u-u_0)](t) + Au(t) = f(u(t)),\ t>0, \tag{1.1}$$

$$u(0) = u_0, \tag{1.2}$$

where the unknown function u takes values in H, the kernel $k \in L^1_{loc}(\mathbb{R}^+)$, A is an unbounded linear operator, and $f : H \to H$ is a given function. Here $*$ denotes the Laplace convolution, i.e., $(k*v)(t) = \int_0^t k(t-s)v(s)ds$.

It should be mentioned that, nonlocal equations have been employed to model different problems related to processes in materials with memory (see, e.g., [3,4,6,16]). In particular, when the kernel $k(t) = g_{1-\alpha}(t) := t^{-\alpha}/\Gamma(1-\alpha), \alpha \in (0,1)$, equation (1.1) is in the form of fractional differential equations as the term $\dfrac{d}{dt}[k*(u-u_0)]$ represents the Caputo fractional derivative of order α, and this equation has been a subject of an extensive study. In a specific setting, for example, when $H = L^2(\Omega), \Omega \subset \mathbb{R}^N$, and $A = -\Delta$ is the Laplace

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operator associated with a boundary condition of Dirichlet/Neumann type, equation (1.1) with a class of kernel functions is utilized to describe anomalous diffusion phenomena including slow/ultra-slow diffusion, which were remarked in [18].

Our motivation for the present work is that, up to our knowledge, no attempt has been made to establish regularity results for (1.1)-(1.2). Moreover, the stability analysis in the sense of Lyapunov for (1.1) has been less known. In the special case when $k = g_{1-\alpha}$, we refer to some results on stability analysis given in [1,7,8]. In the case of multi-term fractional derivatives, i.e. $k = \sum_{i=1}^{m} \mu_i g_{1-\alpha_i}, \mu_i > 0, \alpha_i \in (0,1)$, some results on regularity and long-time estimate were established for linear equations in [2,12,13]. An asymptotic estimate result was also made for linear ultra-slow diffusion equations in [11]. In the recent paper [19], Vergara and Zacher investigated a concrete model of type (1.1), which is a nonlocal semilinear partial differential equation (PDE). Using a maximum principle for the linearized equation, they proved the asymptotic stability for zero solution of this equation. It is worth noting that, the technique used in [19] does not work for the abstract equation (1.1). In this paper, the regularity and asymptotic stability of solutions to (1.1) will be analyzed by using a new representation of solutions together with a new Gronwall type inequality. In order to deal with (1.1), we make the following standing hypotheses.

(A) *The operator $A : D(A) \to H$ is densely defined, self-adjoint, and positively definite with compact resolvent.*

(K) *The kernel function $k \in L^1_{loc}(\mathbb{R}^+)$ is nonnegative and nonincreasing, and there exists a function $l \in L^1_{loc}(\mathbb{R}^+)$ such that $k * l = 1$ on $(0, \infty)$.*

(F) *The nonlinear function $f : H \to H$ is locally Lipschitzian, i.e., for each $\rho > 0$ there is a nonnegative number $\kappa(\rho)$ such that*

$$\|f(v_1) - f(v_2)\| \le \kappa(\rho)\|v_1 - v_2\|,\ \forall v_1, v_2 \in B_\rho,$$

where B_ρ is the closed ball in H with center at origin and radius ρ.

Noting that, the hypothesis (K) was used in a lot of works, e.g. [9,10,17–19,21]. This enables us to transform equations of type (1.1) to a Volterra integral equation with completely positive kernel, which is a main subject discussed in [16]. In this case, one writes $(k, l) \in \mathcal{PC}$. Some typical examples of (k, l) were given in [18], e.g.,

- $k(t) = g_{1-\alpha}(t)$ and $l(t) = g_\alpha(t), t > 0$: slow diffusion (fractional order) case.
- $k(t) = \int\limits_0^1 g_\beta(t)d\beta$ and $l(t) = \int\limits_0^\infty \frac{e^{-pt}}{1+p}dp, t > 0$: ultra-slow diffusion (distributed order) case.
- $k(t) = g_{1-\alpha}(t)e^{-\gamma t}, \gamma \ge 0$, and $l(t) = g_\alpha(t)e^{-\gamma t} + \gamma \int\limits_0^t g_\alpha(s)e^{-\gamma s}ds, t > 0$: tempered fractional order case.

For more examples on (K), we refer the reader to [17].

Owing these hypotheses, we are able to derive, in the next section, a variation-of-parameter formula as well as the concept of mild solution for inhomogeneous equations. We show that a mild solution is also a weak solution, and it is a strong solution if the external force function is Hölder continuous and the kernel function l is sectorial and smooth enough. Section 3 is devoted to the semilinear equations, in which we prove the local/global solvability and asymptotic stability for (1.1). In addition, we show that, the mild solution of semilinear problem is also Hölder continuous. Finally, we present in the last section an application of the abstract results, where we show a concrete condition ensuring the Hölder regularity and asymptotic stability of solutions to multi-term fractional in time PDEs.

2. Preliminaries

For $\mu \in \mathbb{R}^+$, consider the following scalar integral equations

$$s(t) + \mu(l * s)(t) = 1,\ t \geq 0, \tag{2.1}$$

$$r(t) + \mu(l * r)(t) = l(t),\ t > 0. \tag{2.2}$$

The existence and uniqueness of s and r were examined in [5, Section 20.4] (see also [14]). In the case $l(t) = g_\alpha(t)$, following from the Laplace transform of $s(\cdot)$ and $r(\cdot)$, we know that $s(t) = E_{\alpha,1}(-\mu t^\alpha)$ and $r(t) = t^{\alpha-1}E_{\alpha,\alpha}(-\mu t^\alpha)$, here $E_{\alpha,\beta}$ is the Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)},\ z \in \mathbb{C}.$$

Recall that the kernel function l is said to be completely positive iff $s(\cdot)$ and $r(\cdot)$ take nonnegative values for every $\mu > 0$. The complete positivity of l is equivalent to that (see [3]), there exist $\alpha \geq 0$ and $k \in L^1_{loc}(\mathbb{R}^+)$ nonnegative and nonincreasing which satisfy $\alpha l + l * k = 1$. In particular, the hypothesis (K) ensures that l is completely positive.

Denote by $s(\cdot,\mu)$ and $r(\cdot,\mu)$ the solutions of (2.1) and (2.2), respectively. We collect some properties of these functions.

Proposition 2.1. *Let the hypothesis (K) hold. Then for every $\mu > 0$, $s(\cdot,\mu), r(\cdot,\mu) \in L^1_{loc}(\mathbb{R}^+)$. In addition, we have:*

(1) *The function $s(\cdot,\mu)$ is nonnegative and nonincreasing. Moreover,*

$$s(t,\mu)\left[1 + \mu\int_0^t l(\tau)d\tau\right] \leq 1,\ \forall t \geq 0. \tag{2.3}$$

(2) *The function $r(\cdot,\mu)$ is nonnegative and the following relations hold*

$$s(t,\mu) = 1 - \mu\int_0^t r(\tau,\mu)d\tau = k * r(\cdot,\mu)(t),\ t \geq 0. \tag{2.4}$$

(3) *For each $t > 0$, the functions $\mu \mapsto s(t,\mu)$ and $\mu \mapsto r(t,\mu)$ are nonincreasing.*

Proof. The justification for (2.3) and (2.4) can be found in [3]. The last statement was proved in [15, Lemma 5.1 and Lemma 5.3]. $\square$

Remark 2.1.

(1) As mentioned in [19], the functions $s(\cdot,\mu)$ and $r(\cdot,\mu)$ take nonnegative values even in the case $\mu \leq 0$.

(2) Equation (2.1) is equivalent to the problem

$$\frac{d}{dt}[k*(s-1)]+\mu s=0,\ s(0)=1.$$

This can be seen by convoluting both sides of equation $(s-1)+\mu l*s=0$ with k and using $k*l=1$.

(3) Let $v(t)=s(t,\mu)v_0+(r(\cdot,\mu)*g)(t)$, here $g\in L^1_{loc}(\mathbb{R}^+)$. Then v solves the problem

$$\frac{d}{dt}[k*(v-v_0)](t)+\mu v(t)=g(t),\ v(0)=v_0.$$

Indeed, by formulation and the relation $k*r=s$, we have

$$\begin{aligned}k*(v-v_0)&=k*(s-1)v_0+k*r*g\\&=k*(s-1)v_0+s*g.\end{aligned}$$

So

$$\begin{aligned}\frac{d}{dt}[k*(v-v_0)]&=\frac{d}{dt}[k*(s-1)]v_0+s(0,\mu)g+s'(\cdot,\mu)*g\\&=-\mu s(\cdot,\mu)v_0+g-\mu r(\cdot,\mu)*g\\&=-\mu[s(\cdot,\mu)v_0+r(\cdot,\mu)*g]+g\\&=-\mu v+g,\end{aligned}$$

thanks to the fact that $s(0,\mu)=1$ and $s'(t,\mu)=-\mu r(t,\mu),\ t>0$.

(4) We deduce from (2.3) that, if $l\notin L^1(\mathbb{R}^+)$ then $\lim\limits_{t\to\infty}s(t,\mu)=0$ for every $\mu>0$.

(5) It follows from (2.4) that $\int_0^t r(\tau,\mu)d\tau\le\mu^{-1},\ \forall t>0$. If $l\notin L^1(\mathbb{R}^+)$ then $\int_0^\infty r(\tau,\mu)d\tau=\mu^{-1}$ for every $\mu>0$.

We are now in a position to prove a Gronwall type inequality, which plays an important role in our analysis.

Proposition 2.2. *Let v be a nonnegative function satisfying*

$$v(t)\le s(t,\mu)v_0+\int_0^t r(t-\tau,\mu)[\alpha v(\tau)+\beta(\tau)]d\tau,\ t\ge 0, \tag{2.5}$$

for $\mu>0,\alpha>0,v_0\ge 0$, and $\beta\in L^1_{loc}(\mathbb{R}^+)$. Then

$$v(t)\le s(t,\mu-\alpha)v_0+\int_0^t r(t-\tau,\mu-\alpha)\beta(\tau)d\tau.$$

Particularly, if β is constant and $\alpha<\mu$ then

$$v(t)\le s(t,\mu-\alpha)v_0+\frac{\beta}{\mu-\alpha}(1-s(t,\mu-\alpha)).$$

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Proof. Let $w(t)$ be the expression in the right hand side of (2.5). Then $v(t) \le w(t)$ for $t \ge 0$, and w solves the problem

$$\frac{d}{dt}[k * (w - v_0)](t) + \mu w(t) = \alpha v(t) + \beta(t),$$
$$w(0) = v_0,$$

thanks to Remark 2.1 (2). This is equivalent to

$$\frac{d}{dt}[k * (w - v_0)](t) + (\mu - \alpha) w(t) = \alpha(v(t) - w(t)) + \beta(t),$$
$$w(0) = v_0,$$

which implies

$$\begin{aligned} w(t) &= s(t, \mu - \alpha) v_0 + \int_0^t r(t - \tau, \mu - \alpha)[\alpha(v(\tau) - w(\tau)) + \beta(\tau)] d\tau \\ &\le s(t, \mu - \alpha) v_0 + \int_0^t r(t - \tau, \mu - \alpha) \beta(\tau) d\tau, \end{aligned}$$

in accordance with $v(\tau) - w(\tau) \le 0$ for $\tau \ge 0$ and the positivity of r.

Finally, if β is constant, we employ (2.4) to get

$$\int_0^t r(t - \tau, \mu - \alpha) \beta d\tau = \beta \int_0^t r(\tau, \mu - \alpha) d\tau = \frac{\beta}{\mu - \alpha}(1 - s(t, \mu - \alpha)),$$

which completes the proof. $\square$

Let us mention that, the hypothesis (A) ensures the existence of an orthonormal basis of H consisting of eigenfunctions $\{e_n\}_{n=1}^{\infty}$ of the operator A and we have

$$Av = \sum_{n=1}^{\infty} \lambda_n v_n e_n,$$

where $\lambda_n > 0$ is the eigenvalue corresponding to the eigenfunction e_n of A,

$$D(A) = \{v = \sum_{n=1}^{\infty} v_n e_n : \sum_{n=1}^{\infty} \lambda_n^2 v_n^2 < \infty\}.$$

We can assume that $0 < \lambda_1 \le \lambda_2 \le ... \le \lambda_n \to \infty$ as $n \to \infty$.

For $\gamma \in \mathbb{R}$, one can define the fractional power of A as follows

$$D(A^\gamma) = \left\{ v = \sum_{n=1}^{\infty} v_n e_n : \sum_{n=1}^{\infty} \lambda_n^{2\gamma} v_n^2 < \infty \right\},$$
$$A^\gamma v = \sum_{n=1}^{\infty} \lambda_n^\gamma v_n e_n.$$

Let $V_\gamma = D(A^\gamma)$. Then V_γ is a Banach space endowed with the norm

$$\|v\|_\gamma = \|A^\gamma v\| = \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma} v_n^2\right)^{\frac{1}{2}}.$$

Furthermore, for $\gamma > 0$, we can identify the dual space V_γ^* of V_γ with $V_{-\gamma}$.

We now define the following operators

$$S(t)v = \sum_{n=1}^{\infty} s(t, \lambda_n) v_n e_n,\ t \geq 0, v \in H, \tag{2.6}$$

$$R(t)v = \sum_{n=1}^{\infty} r(t, \lambda_n) v_n e_n,\ t > 0, v \in H. \tag{2.7}$$

It is easily seen that $S(t)$ and $R(t)$ are linear. We show some basic properties of these operators in the following lemma.

Lemma 2.3. *Let $\{S(t)\}_{t\geq 0}$ and $\{R(t)\}_{t>0}$, be the families of linear operators defined by (2.6) and (2.7), respectively. Then*

(1) *For each $v \in H$ and $T > 0$, $S(\cdot)v \in C([0,T]; H)$, $S(t)v \in D(A)$ for $t > 0$, and $AS(\cdot)v \in C((0,T]; H)$. Moreover,*

$$\|S(t)v\| \leq s(t, \lambda_1)\|v\|,\ t \in [0,T], \tag{2.8}$$

$$\|AS(t)v\| \leq \frac{\|v\|}{(1 * l)(t)},\ t \in (0,T]. \tag{2.9}$$

(2) *Let $v \in H, T > 0$ and $g \in C([0,T]; H)$. Then $R(\cdot)v \in C((0,T]; H)$, $R * g \in C([0,T]; H)$ and $A(R * g) \in C([0,T]; V_{-\frac{1}{2}})$. Furthermore,*

$$\|R(t)v\| \leq r(t, \lambda_1)\|v\|,\ t \in (0,T], \tag{2.10}$$

$$\|(R * g)(t)\| \leq \int_0^t r(t-\tau, \lambda_1)\|g(\tau)\| d\tau,\ t \in [0,T], \tag{2.11}$$

$$\|A(R * g)(t)\|_{-\frac{1}{2}} \leq \left(\int_0^t r(t-\tau, \lambda_1)\|g(\tau)\|^2 d\tau\right)^{\frac{1}{2}},\ t \in [0,T]. \tag{2.12}$$

Proof. (1) For the first statement, we observe that

$$\|S(t)v\|^2 = \sum_{n=1}^{\infty} s^2(t, \lambda_n) v_n^2. \tag{2.13}$$

Since $s(t, \lambda_n) \leq 1$ for every $t \geq 0, n \in \mathbb{N}$, this series is uniformly convergent on $[0,T]$. So is series (2.6). Due to the fact that $s(\cdot, \lambda_n)$ is continuous, we get $S(\cdot)v \in C([0,T]; H)$. Estimate (2.8) is deduced from (2.13) by using $s(t, \lambda_n) \leq s(t, \lambda_1)$ for all $n > 1$.

Considering

$$AS(t)v = \sum_{n=1}^{\infty} \lambda_n s(t, \lambda_n) v_n e_n, \tag{2.14}$$

we have

$$\|AS(t)v\|^2 = \sum_{n=1}^{\infty} \lambda_n^2 s^2(t, \lambda_n) v_n^2. \tag{2.15}$$

In view of (2.3), we get

$$\lambda_n s(t, \lambda_n) \leq \frac{\lambda_n}{1 + \lambda_n (1 * l)(t)} \leq \frac{1}{(1 * l)(t)}, \forall t > 0.$$

Substituting into (2.15), we have $\|AS(t)v\| \leq \dfrac{\|v\|}{(1 * l)(t)}$, for every $t > 0$. In addition, for any δ such that $0 < \delta < T$, one has $\lambda_n s(t, \lambda_n) \leq \dfrac{1}{(1 * l)(\delta)}$ for $t \geq \delta$, which implies that the convergence of (2.15) as well as (2.14) is uniform on $[\delta, T]$. That is, $AS(\cdot)v \in C([\delta, T]; H)$.

(2) Recall that $r(\cdot, \mu)$ is continuous on $(0, \infty)$ (see, e.g. [14]). So for any $\delta \in (0, T)$ and $\mu > 0$, $r(\cdot, \mu) \in C([\delta, T])$. This ensures that the series

$$\|R(t)v\|^2 = \sum_{n=1}^{\infty} r^2(t, \lambda_n) v_n^2 \tag{2.16}$$

is uniformly convergent on $[\delta, T]$. So is series (2.7). Inequality (2.10) follows from (2.16) since $r(t, \cdot)$ is nonincreasing.

We now prove that $R * g \in C([0, T]; H)$. Denoting $g_n(t) = (g(t), e_n)$, we first check that

$$(R * g)(t) = \sum_{n=1}^{\infty} [r(\cdot, \lambda_n) * g_n](t) e_n. \tag{2.17}$$

Indeed, since $g \in C([0, T]; H)$, the series $\|g(t)\| = \sum\limits_{n=1}^{\infty} |g_n(t)|^2$ is uniformly convergent on $[\delta, T]$. Given $\epsilon > 0$, for $\delta \leq \tau \leq t \leq T$ and $p \in \mathbb{N}$, we have

$$\|\sum_{k=n}^{n+p} r(\tau, \lambda_k) g_k(t - \tau) e_k\| \leq r(\tau, \lambda_1) \left(\sum_{k=n}^{n+p} |g_k(t - \tau)|^2 \right)^{\frac{1}{2}} < \epsilon,$$

provided that n is large enough. So the series $\sum\limits_{n=1}^{\infty} r(\tau, \lambda_n) g_n(t - \tau) e_n$ converges uniformly on $[\delta, T]$ and one can take integration term by term on $[\delta, t]$, i.e.

$$\int_{\delta}^{t} R(t - \tau) g(\tau) d\tau = \sum_{n=1}^{\infty} \left(\int_{\delta}^{t} r(\tau, \lambda_n) g_n(t - \tau) d\tau \right) e_n.$$

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Fix $t > 0$ and put $h_n(\delta) = \int_\delta^t r(\tau, \lambda_n) g_n(t-\tau) d\tau$. Arguing as above for the uniform convergence of the series $\sum_{n=1}^\infty h_n(\delta) e_n$ on $[0, t]$, we can pass to the limit as $\delta \to 0$ to get (2.17). Taking (2.17) into account, by the Hölder inequality, one has

$$
\begin{aligned}
|[r(\cdot, \lambda_n) * g_n](t)| &\le \int_0^t \sqrt{r(t-\tau, \lambda_n)} \sqrt{r(t-\tau, \lambda_n)} |g_n(\tau)| d\tau \\
&\le \left(\int_0^t r(t-\tau, \lambda_n) d\tau \right)^{\frac{1}{2}} \left(\int_0^t r(t-\tau, \lambda_n) |g_n(\tau)|^2 d\tau \right)^{\frac{1}{2}} \\
&\le \left(\frac{1}{\lambda_n} (1 - s(t, \lambda_n)) \right)^{\frac{1}{2}} \left(\int_0^t r(t-\tau, \lambda_1) |g_n(\tau)|^2 d\tau \right)^{\frac{1}{2}} \\
&\le \frac{1}{\lambda_n^{\frac{1}{2}}} \left(\int_0^t r(t-\tau, \lambda_1) |g_n(\tau)|^2 d\tau \right)^{\frac{1}{2}}, \qquad (2.18)
\end{aligned}
$$

thanks to (2.4) and the monotonicity of $r(t, \cdot)$. Then it follows

$$
\begin{aligned}
\sum_{k=n}^{n+p} |[r(\cdot, \lambda_k) * g_k](t)|^2 &\le \frac{1}{\lambda_1} \int_0^t r(t-\tau, \lambda_1) \left(\sum_{k=n}^{n+p} |g_k(\tau)|^2 \right) d\tau \\
&\le \frac{\epsilon}{\lambda_1} \int_0^t r(t-\tau, \lambda_1) d\tau \le \frac{\epsilon}{\lambda_1^2},
\end{aligned}
$$

for n large, thanks to the uniform convergence of $\sum_{n=1}^\infty |g_n(t)|^2$ on $[0, T]$ and relation (2.4). Hence (2.17) is uniformly convergent on $[0, T]$ and then $R * g \in C([0, T]; H)$. Estimate (2.11) takes place by employing (2.10).

Finally, we show that $A(R * g) \in C([0, T]; V_{-\frac{1}{2}})$. Noticing that

$$
A(R * g)(t) = \sum_{n=1}^\infty \lambda_n [r(\cdot, \lambda_n) * g_n](t) e_n,
$$

we obtain

$$
\|A(R * g)(t)\|_{-\frac{1}{2}}^2 = \|A^{\frac{1}{2}} (R * g)(t)\|^2 = \sum_{n=1}^\infty \left(\lambda_n^{\frac{1}{2}} [r(\cdot, \lambda_n) * g_n](t) \right)^2. \qquad (2.19)
$$

Using estimate (2.18), one can claim the uniform convergence of (2.19) on $[0, T]$ and estimate (2.12) follows. Thus $A(R * g) \in C([0, T]; V_{-\frac{1}{2}})$ as desired.

The proof is complete. □

Remark 2.2.

(1) Obviously, $S(0)v = v$ for every $v \in H$.

(2) We have $(R * g)(0) = 0$. Indeed, it follows from (2.11) that

$$\begin{aligned}\|(R * g)(t)\| &\le \sup_{\tau\in[0,T]} \|g(\tau)\| \int_0^t r(t-\tau,\lambda_1)d\tau \\ &= \sup_{\tau\in[0,T]} \|g(\tau)\|\lambda_1^{-1}(1 - s(t,\lambda_1)) \to 0 \text{ as } t \to 0.\end{aligned}$$

(3) Lemma 2.3 implies that $A^{\frac{1}{2}}S(\cdot)v, A^{\frac{1}{2}}(R * g) \in C((0,T];H)$ for every $v \in H$ and $g \in C([0,T];H)$. Equivalently, $S(\cdot)v, R * g \in C((0,T];V_{\frac{1}{2}})$.

Given $g \in C([0,T];H)$ and $u_0 \in H$, consider the linear problem

$$\frac{d}{dt}[k * (u - u_0)](t) + Au(t) = g(t), t \in (0,T], \tag{2.20}$$

$$u(0) = u_0. \tag{2.21}$$

Based on the operators $S(t)$ and $R(t)$, we introduce the following definition of mild solutions to (2.20)-(2.21).

Definition 2.1. A function $u \in C([0,T];H)$ is called a mild solution to the problem (2.20)-(2.21) on $[0,T]$ iff

$$u(t) = S(t)u_0 + \int_0^t R(t-s)g(s)ds,\ t \in [0,T]. \tag{2.22}$$

3. Weak solution and regularity

3.1. Existence and uniqueness

In the sequel, we will define weak solution for (2.20)-(2.21) and show that a mild solution is also a weak solution.

Definition 3.1. Let (A) and (K) hold, $g \in C([0,T];H)$ and $u_0 \in H$ be given. A function $u \in C([0,T];H) \cap C((0,T];V_{\frac{1}{2}})$ is said to be a weak solution to (2.20)-(2.21) on $[0,T]$ iff $u(0) = u_0$ and equation (2.20) holds in $V_{-\frac{1}{2}}$.

Theorem 3.1. *If u is a mild solution to the problem (2.20)-(2.21), then it is a weak solution.*

Proof. Let u be defined by (2.22). Then Lemma 2.3 ensures that $S(\cdot)u_0$ and $R * g$ belong to $C([0,T];H)$, so $u = S(\cdot)u_0 + R * g \in C([0,T];H)$. By Remark 2.2, we get $u(0) = u_0$ and $u \in C((0,T];V_{\frac{1}{2}})$.

By formulation, we have

$$\begin{aligned}k(\tau)(u(t-\tau) - u_0) &= \sum_{n=1}^{\infty} k(\tau)[s(t-\tau,\lambda_n) - 1]u_{0n}e_n \\ &\quad + \sum_{n=1}^{\infty} k(\tau)[r(\cdot,\lambda_n) * g_n](t-\tau)e_n\end{aligned}$$

for $\delta \le \tau \le t \le T$, where $\delta \in (0,T)$, and these series are uniformly convergent on $[\delta,t]$. So one has

$$\int_\delta^t k(\tau)(u(t-\tau)-u_0)d\tau = \sum_{n=1}^\infty \int_\delta^t k(\tau)[s(t-\tau,\lambda_n)-1]d\tau u_{0n}e_n + \sum_{n=1}^\infty \int_\delta^t k(\tau)[r(\cdot,\lambda_n)*g_n](t-\tau)d\tau e_n. \tag{3.1}$$

For fixed $t\in(0,T]$, put

$$h_n(\delta) = \int_\delta^t k(\tau)[s(t-\tau,\lambda_n)-1]d\tau\, u_{0n} + \int_\delta^t k(\tau)[r(\cdot,\lambda_n)*g_n](t-\tau)d\tau.$$

Obviously, h_n is continuous on $[0,t]$ for all n, and the function $\delta\mapsto h(\delta)=\int_\delta^t k(\tau)(u(t-\tau)-u_0)d\tau$ is also continuous on $[0,t]$. Then the series $\sum_{n=1}^\infty h_n(\delta)e_n$ converges uniformly on $[0,t]$, which enables us to pass to the limit in (3.1) to obtain

$$\begin{aligned} k*(u-u_0)(t) &= \sum_{n=1}^\infty k*(s(\cdot,\lambda_n)-1)(t)u_{0n}e_n + \sum_{n=1}^\infty k*[r(\cdot,\lambda_n)*g_n](t)e_n \\ &= \sum_{n=1}^\infty k*(s(\cdot,\lambda_n)-1)(t)u_{0n}e_n + \sum_{n=1}^\infty [s(\cdot,\lambda_n)*g_n](t)e_n, \end{aligned} \tag{3.2}$$

thanks to (2.4). We testify that, it is possible to take differentiation term by term in (3.2). It suffices to prove that the series

$$\sum_{n=1}^\infty \frac{d}{dt}[k*(s(\cdot,\lambda_n)-1)](t)u_{0n}e_n + \sum_{n=1}^\infty \frac{d}{dt}[s(\cdot,\lambda_n)*g_n](t)e_n \tag{3.3}$$

is uniformly convergent on $[\delta,T]$ for any $\delta\in(0,T)$. Indeed, by Remark 2.1 we have

$$\frac{d}{dt}[k*(s(\cdot,\lambda_n)-1)](t) = -\lambda_n s(t,\lambda_n),$$
$$\frac{d}{dt}[s(\cdot,\lambda_n)*g_n](t) = g_n(t)-\lambda_n[r(\cdot,\lambda_n)*g](t).$$

Therefore, (3.3) becomes

$$\begin{aligned} &-\sum_{n=1}^\infty \lambda_n s(t,\lambda_n)u_{0n}e_n - \sum_{n=1}^\infty \lambda_n[r(\cdot,\lambda_n)*g](t)e_n + \sum_{n=1}^\infty g_n(t)e_n \\ &= -AS(t)u_0 - A(R*g)(t)+g(t), \end{aligned}$$

which are uniformly convergent on $[\delta,T]$ as shown in Lemma 2.3. Hence, we can take differentiation in (3.2) and get the equation

$$\frac{d}{dt}[k*(u-u_0)](t) = -AS(t)u_0 - A(R*g)(t)+g(t) = -Au(t)+g(t), t\in(0,T],$$

which holds in $V_{-\frac{1}{2}}$. The proof is complete. □

We are in a position to prove the uniqueness of weak solution.

Theorem 3.2. *Problem* (2.20)-(2.21) *has a unique weak solution.*

Proof. It remains to show the uniqueness. Let $h_\mu = -s'_\mu = \mu r$, then h_μ is nonnegative and solves the equation

$$h_\mu(t) + \mu(h_\mu * l)(t) = \mu l(t),\ t > 0, \mu > 0.$$

In addition, for $1 \le p < \infty$, $f \in L^p(0,T)$, one has $h_n * f \to f$ in $L^p(0,T)$ as $n \to \infty$ ([20]). Put $k_\mu = k * h_\mu$, then $k_\mu = \mu k * r = \mu s_\mu$, thanks to (2.4). Hence $k_\mu \in W^{1,1}(0,T)$. This enables us to employ the fundamental identity ([18, Lemma 2.3])

$$\begin{aligned}(v(t), (k_\mu * v)'(t)) &= \frac{1}{2}(k_\mu * \|v(\cdot)\|^2)'(t) + \frac{1}{2}k_\mu(t)\|v(t)\|^2 \\ &\quad + \frac{1}{2}\int_0^t \|v(t) - v(t-s)\|^2[-k'_\mu(s)]ds,\ t \in [0,T], v \in C([0,T];H).\end{aligned}$$

Therefore

$$(v(t), (k_\mu * v)'(t)) \ge \frac{1}{2}(k_\mu * \|v(\cdot)\|^2)'(t),\ t \in [0,T], v \in C([0,T];H), \tag{3.4}$$

thanks to the fact that k_μ is nonincreasing.

Let u_1 and u_2 be weak solutions of (1.1)-(1.2). Put $v = u_2 - u_2$, then we have

$$((k * v)'(t), w) + (Av(t), w) = 0,\ \forall t \in (0,T],\ w \in V_{\frac{1}{2}},$$
$$v(0) = 0.$$

Then

$$((h_n * k * v)(t), w) + (h_n * 1 * Av(t), w) = 0,\ \forall t \in (0,T],\ w \in V_{\frac{1}{2}},$$

which is equivalent to

$$((k_n * v)'(t), w) + (h_n * Av(t), w) = 0,\ \forall t \in (0,T],\ w \in V_{\frac{1}{2}}.$$

Taking $w = v(t)$ and using (3.4) yields

$$\frac{1}{2}(k_n * \|v(\cdot)\|^2)'(t) + (h_n * Av(t), v(t)) \le 0,\ \forall t \in (0,T].$$

Let $q(t) = \frac{1}{2}(k_n * \|v(\cdot)\|^2)'(t) + (h_n * Av(t), v(t))$, then $q(t) \le 0,\ \forall t \in (0,T]$. Noting that, the relation

$$\frac{1}{2}(k_n * \|v(\cdot)\|^2)'(t) = q_n(t) := q(t) - (h_n * Av(t), v(t))$$

is equivalent to (see [18, Lemma 2.4])

$$\frac{1}{2}\|v(t)\|^2 = \frac{1}{n}q_n(t) + l * q_n(t), t \in (0,T]. \tag{3.5}$$

Since $q_n(t) \to q(t) - (Av(t), v(t))$ as $n \to \infty$, for $t \in (0,T]$, we obtain

$$\frac{1}{2}\|v(t)\|^2 = l * [q(\cdot) - \|A^{\frac{1}{2}}v(\cdot)\|^2](t) \le 0,\ t \in (0,T].$$

Thus $v = 0$ and the proof is complete. □

3.2. Regularity

By using (K), the problem (2.20)-(2.21) can be transformed to the integral equation

$$u(t) + l * Au(t) = u_0 + l * g(t),\ t \in [0,T].$$

This allows us to employ the resolvent theory in [16] for regularity analysis. Noting that the solution operator for the equation

$$u(t) + l * Au(t) = u_0,\ t \in [0,T], \tag{3.6}$$

is given by $S(t)u_0 = u(t)$, where $S(t)$ is defined by (2.6). It should be mentioned that (2.6) is a representation for the resolvent of (3.6) stated in [16, Theorem 1.1], in the case that A has a discrete spectrum. We refer to $S(\cdot)$ as the resolvent family.

We recall some notions and facts stated in [16].

Definition 3.2. Let $l \in L^1_{loc}(\mathbb{R}^+)$ be a function of subexponential growth, i.e. $\int\limits_0^\infty |l(t)|e^{-\epsilon t}dt < \infty$ for every $\epsilon > 0$.

- Suppose that $\hat{l}(\lambda) \ne 0$ for all $\mathrm{Re}\lambda > 0$. For $\theta > 0$, l is said to be θ-sectorial if $|\arg \hat{l}(\lambda)| \le \theta$ for all $\mathrm{Re}\lambda > 0$.
- For given $m \in \mathbb{N}$, l is called m-regular if there exists a constant $c > 0$ such that

$$|\lambda^n \hat{l}^{(n)}(\lambda)| \le c|\hat{l}(\lambda)| \quad \text{for all } \mathrm{Re}\lambda > 0, 0 \le n \le m.$$

Definition 3.3. Equation (3.6) is called parabolic if the following conditions hold:

(1) $\hat{l}(\lambda) \ne 0$, $1/\hat{l}(\lambda) \in \rho(-A)$ for all $\mathrm{Re}\lambda \ge 0$.
(2) There is a constant $M \ge 1$ such that $U(\lambda) = \lambda^{-1}(I + \hat{l}(\lambda)A)^{-1}$ satisfies

$$\|U(\lambda)\| \le \frac{M}{|\lambda|} \quad \text{for all } \mathrm{Re}\lambda > 0.$$

Denote by $\Sigma(\omega,\theta)$ the open sector with vertex $\omega \in \mathbb{R}$ and angle 2θ in the complex plane, i.e.

$$\Sigma(\omega,\theta) = \{\lambda \in \mathbb{C} : |\arg(\lambda - \omega)| < \theta\}.$$

We have the following sufficient condition for (3.6) to be parabolic.

Proposition 3.3. *[16, Proposition 3.1] Assume that $l \in L^1_{loc}(\mathbb{R}^+)$ is of subexponential growth and is θ-sectorial for some $\theta < \pi$. If A is closed linear densely defined, such that $\rho(-A) \supset \Sigma(0,\theta)$, and*

$$\|(\lambda I + A)^{-1}\| \le \frac{M}{|\lambda|} \text{ for all } \lambda \in \Sigma(0,\theta), \tag{3.7}$$

then (3.6) is parabolic.

Let us mention that, by the assumption (A), $-A$ generates a contraction C_0-semigroup in H, which is given by

$$e^{-tA}v = \sum_{n=1}^{\infty} e^{-t\lambda_n}(v, e_n)e_n, \ v \in H,$$

and (3.7) holds for $M = 1$ and for any $\theta < \pi$.

The following result on the resolvent family for (3.6) plays an important role in our analysis.

Proposition 3.4. *[16, Theorem 3.1] Assume that (3.6) is parabolic and the kernel function l is m-regular for some $m \geq 1$. Then there is a resolvent family $S(\cdot) \in C^{(m-1)}((0,\infty);\mathcal{L}(H))$ for (3.6), and a constant $M \geq 1$ such that*

$$\|t^n S^{(n)}(t)\| \leq M, \quad \textit{for all } t > 0, n \leq m-1.$$

In order to obtain the differentiability of the resolvent family, we replace (K) by a stronger assumption.

(K*) *The assumption (K) is satisfied with l being 2-regular and θ-sectorial with some $\theta < \pi$.*

Employing Proposition 3.4, we have the following statement.

Lemma 3.5. *Let (A) and (K*) hold. Then the resolvent family $S(\cdot)$ defined by (2.6) is differentiable on $(0,\infty)$, the relation*

$$S'(t) = -AR(t), \ t \in (0,\infty), \tag{3.8}$$

and the estimate

$$\|S'(t)\| \leq \frac{M}{t}, \ t \in (0,\infty), \tag{3.9}$$

hold for some $M \geq 1$.

Proof. The assumption (A) ensures that $-A$ generates a bounded analytic semigroup. So (3.6) is parabolic, according to Proposition 3.3. Therefore, it follows from Proposition 3.4 that $S(\cdot)$ is differentiable on $(0,\infty)$ and estimate (3.9) takes place. Finally, it is deduced from the formulation of S and R given by (2.6)-(2.7) that

$$\begin{aligned} S'(t)v &= \sum_{n=1}^{\infty} \partial_t s(t,\lambda_n)(v,e_n)e_n \\ &= \sum_{n=1}^{\infty} -\lambda_n r(t,\lambda_n)(v,e_n)e_n = -AR(t)v, \ t > 0, v \in H, \end{aligned}$$

thanks to (2.4), which proves (3.8). $\square$

Denote by $C^\gamma([a,b];H)$, $\gamma \in (0,1)$, the space of Hölder continuous functions on $[a,b]$, that is, $f \in C^\gamma([a,b];H)$ iff

$$\|f\|_{C^\gamma} = \sup_{t_1,t_2\in[a,b]} \frac{\|f(t_1)-f(t_2)\|}{|t_1-t_2|^\gamma} < \infty.$$

Definition 3.4. Let (A) and (K) hold, $g \in C([0,T];H)$ and $u_0 \in H$ be given. A function $u \in C([0,T];H)$ is said to be a strong solution to (2.20)-(2.21) on $[0,T]$ iff $u(0) = u_0$, $u(t) \in D(A)$ for $t > 0$, and equation (2.20) holds in H.

Theorem 3.6. *Let the hypotheses of Lemma 3.5 hold. Assume that the function g in (2.20) belongs to $C^\gamma([0,T];H)$, and u is the weak solution of (2.20)-(2.21). Then $u \in C([0,T];H) \cap C^\gamma([\delta,T];H)$ for any $0 < \delta < T$, and u is a strong solution.*

Proof. Recall that the unique weak solution of (2.20)-(2.21) is given by

$$u(t) = S(t)u_0 + (R * g)(t) = u_1(t) + u_2(t),\ t \in [0,T]. \tag{3.10}$$

We first show that u_2 is Hölder continuous on $[\delta,T]$. Indeed, for $t \in [\delta,T), h > 0, t+h \le T$, we have

$$\begin{aligned}\|u_2(t+h) - u_2(t)\| &\le \int_0^t \|R(\tau)\|\|g(t+h-\tau) - g(t-\tau)\|d\tau \\ &\quad + \int_t^{t+h} \|R(\tau)\|\|g(t+h-\tau)\|d\tau \\ &= I_1 + I_2.\end{aligned}$$

Considering I_1, one gets

$$\begin{aligned}I_1 &\le \int_0^t r(\tau,\lambda_1)\|g\|_{C^\gamma} h^\gamma d\tau = \|g\|_{C^\gamma} h^\gamma \lambda_1^{-1}(1 - s(t,\lambda_1)) \\ &\le \|g\|_{C^\gamma}\lambda_1^{-1}h^\gamma.\end{aligned}$$

Concerning I_2, the relation $S'(t) = -AR(t)$ for $t > 0$ implies

$$\begin{aligned}I_2 &\le \|(-A)^{-1}\| \int_t^{t+h} \|S'(\tau)\|\|g(t+h-\tau)\|d\tau \\ &\le \|A^{-1}\|\|g\|_\infty M \int_t^{t+h} \frac{d\tau}{\tau} = \|A^{-1}\|\|g\|_\infty M \ln\left(1 + \frac{h}{t}\right) \\ &\le \|A^{-1}\|\|g\|_\infty M\gamma^{-1}\left(\frac{h}{t}\right)^\gamma \\ &\le \|A^{-1}\|\|g\|_\infty M\gamma^{-1}\delta^{-\gamma}h^\gamma,\end{aligned}$$

here we utilize the inequality

$$\ln(1+r) \le \frac{r^\gamma}{\gamma} \text{ for } r > 0, \gamma \in (0,1).$$

So we have proved that $\|u_2(t+h) - u_2(t)\| \le Ch^\gamma$ with

$$C = \|g\|_{C^\gamma}\lambda_1^{-1} + \|A^{-1}\|\|g\|_\infty M\gamma^{-1}\delta^{-\gamma}.$$

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It remains to show that $u_1 \in C^\gamma([\delta, T]; H)$. Let $0 < \delta \le t < T$ and $h > 0$. Using the mean value formula

$$S(t+h)v - S(t)v = h \int_0^1 S'(t+\theta h)v d\theta,\ v \in H,$$

we have

$$\begin{aligned}
\|u_1(t+h) - u_1(t)\| &= \|S(t+h)u_0 - S(t)u_0\| \\
&\le h \int_0^1 \|S'(t+\theta h\|\|v\| d\theta \\
&\le M\|v\| h \int_0^1 \frac{d\theta}{t+\theta h} = M\|v\| \ln\left(1+\frac{h}{t}\right) \\
&\le M\|v\|\gamma^{-1}\delta^{-\gamma}h^\gamma.
\end{aligned}$$

Finally, we have to show that $u = S(\cdot)u_0 + R * g$ is a strong solution to (2.20)-(2.21). In the proof of Theorem 3.1, we have testified that u fulfills (2.20) in $V_{-\frac{1}{2}}$ by reasoning that $A(R * g)(t) \in V_{-\frac{1}{2}}$ for $t > 0$. In fact, by Lemma 2.3, $AS(t)u_0 \in H$ for $t > 0$. So it suffices to prove $A(R * g)(t) \in H$ for $t > 0$ under the assumption that g is Hölder continuous. Indeed, using the relation $S'(t) = -AR(t)$ for $t > 0$ again, we obtain

$$\begin{aligned}
A(R * g)(t) &= \int_0^t AR(t-\tau)g(\tau)d\tau = -\int_0^t S'(t-\tau)g(\tau)d\tau \\
&= -\int_0^t S'(t-\tau)[g(\tau) - g(t)]d\tau + [I - S(t)]g(t).
\end{aligned}$$

Then

$$\begin{aligned}
\|A(R * g)(t)\| &\le \int_0^t \|S'(t-\tau)\|\|g(\tau) - g(t)\| d\tau + \|[I - S(t)]g(t)\| \\
&\le M\|g\|_{C^\gamma} \int_0^t (t-\tau)^{\gamma-1} d\tau + \|[I - S(t)]g(t)\| \\
&\le M\|g\|_{C^\gamma}\gamma^{-1}T^\gamma + 2\|g\|_\infty, \quad \text{for } 0 < t \le T,
\end{aligned}$$

which completes the proof. □

4. Stability and regularity for semilinear equations

Definition 4.1. A function $u \in C([0, T]; H)$ is called a mild solution of the problem (1.1)-(1.2) on $[0, T]$ iff

$$u(t) = S(t)u_0 + \int_0^t R(t-s)f(u(s))ds,$$

for every $t \in [0,T]$, where $S(\cdot)$ and $R(\cdot)$ are given by (2.6)-(2.7).

In the next theorem, we prove a local solvability result.

Theorem 4.1. *Let (A), (K) and (F) be satisfied. Then there exists $t^* > 0$ such that the problem (1.1)-(1.2) has a unique mild solution defined on $[0,t^*]$. Moreover, $u(t) \in V_{\frac{1}{2}}$ for all $t \in (0,t^*]$.*

Proof. We make use of the contraction mapping principle. For given $\zeta \in (0,T]$ and $u_0 \in H$, let $\Phi : C([0,\zeta];H) \to C([0,\zeta];H)$ be the mapping defined by

$$\Phi(u)(t) = S(t)u_0 + \int_0^t R(t-\tau)f(u(\tau))d\tau, t \in [0,\zeta]. \tag{4.1}$$

Taking $\rho > \|u_0\|$ and assuming that $u \in \mathbb{B}_\rho$, the closed ball in $C([0,\zeta];H)$ with center at origin and radius ρ, we have

$$\begin{aligned}
\|\Phi(u)(t)\| &\le \|S(t)\|\|u_0\| + \int_0^t \|R(t-\tau)\|\|f(u(\tau))\|d\tau \\
&\le s(t,\lambda_1)\|u_0\| + \int_0^t r(t-\tau,\lambda_1)[\kappa(\rho)\|u(\tau)\| + \|f(0)\|]d\tau \\
&\le \|u_0\| + [\kappa(\rho)\rho + \|f(0)\|]\lambda_1^{-1}(1-s(t,\lambda_1)),\ t \in [0,\zeta],
\end{aligned}$$

here we employ the hypothesis (F), Proposition 2.1 and Lemma 2.3. Since $s(\cdot,\lambda_1) \in AC([0,\zeta])$ and $s(0,\lambda_1) = 1$, one can choose ζ such that the last expression is smaller than ρ as long as $t \in [0,\zeta]$. That is, $\Phi(\mathbb{B}_\rho) \subset \mathbb{B}_\rho$.

Using (F) again, one gets

$$\begin{aligned}
\|\Phi(u_1)(t) - \Phi(u_2)(t)\| &\le \int_0^t r(t-\tau,\lambda_1)\|f(u_1(\tau) - f(u_2(\tau))\|d\tau \\
&\le \int_0^t r(t-\tau,\lambda_1)\kappa(\rho)\|u_1(\tau) - u_2(\tau)\|d\tau \\
&\le \kappa(\rho)\|u_1 - u_2\|_\infty \lambda_1^{-1}(1-s(t,\lambda_1)),\ t \in [0,\zeta],
\end{aligned}$$

where $\|\cdot\|_\infty$ is the sup norm in $C([0,\zeta];H)$. Taking $t^* \le \zeta$ such that $\kappa(\rho)(1-s(t^*,\lambda_1)) < \lambda_1$, we see that Φ is a contraction as a map from $\mathbb{B}_\rho$ into itself, with $\mathbb{B}_\rho$ now in $C([0,t^*];H)$. So the problem (1.1)-(1.2) has a unique solution defined on $[0,t^*]$. In addition, since $t \mapsto g(t) = f(u(t))$ is a continuous function, $\Phi(u)(t) \in D(A^{\frac{1}{2}})$ for $t > 0$ due to Remark 2.2. So $u(t) \in V_{\frac{1}{2}}$ for $t > 0$. The proof is complete. □

We now discuss some circumstances, in which solutions exist globally.

Theorem 4.2. *Let (A) and (K) hold. For any $T > 0$, if the nonlinear function f is globally Lipschitzian, that is, $\kappa(\rho) = \kappa_0$ is a constant, then the problem (1.1)-(1.2) has a unique global mild solution $u \in C([0,T];H) \cap C((0,T];V_{\frac{1}{2}})$. If, in addition, that $\kappa_0 < \lambda_1$ and $l \notin L^1(\mathbb{R}^+)$, then every mild solution to (1.1) is globally bounded and asymptotically stable.*

Proof. Let $\beta > 0$ and $\|u\|_\beta = \sup_{t\in[0,T]} e^{-\beta t}\|u(t)\|$. Then $\|\cdot\|_\beta$ is equivalent to the sup norm in $C([0,T];H)$. From the estimate

$$\|\Phi(u_1)(t) - \Phi(u_2)(t)\| \le \int_0^t r(t-\tau,\lambda_1)\kappa_0\|u_1(\tau) - u_2(\tau)\|d\tau,$$

we get

$$\begin{aligned} e^{-\beta t}\|\Phi(u_1)(t) - \Phi(u_2)(t)\| &\le \left(\kappa_0 \int_0^t r(t-\tau,\lambda_1)e^{-\beta(t-\tau)}d\tau\right)\|u_1 - u_2\|_\beta \\ &\le \left(\kappa_0 \int_0^T r(t,\lambda_1)e^{-\beta t}dt\right)\|u_1 - u_2\|_\beta. \end{aligned}$$

Choosing $\beta > 0$ such that

$$\kappa_0 \int_0^T r(t,\lambda_1)e^{-\beta t}dt < 1,$$

we obtain that Φ is a contraction map from $C([0,T];H)$ endowed with the norm $\|\cdot\|_\beta$ into itself, which ensures the existence and uniqueness of solution to (1.1)-(1.2). In addition, we have $u(t) \in V_{\frac{1}{2}}$ for $t \in (0,T]$, by the same reasoning as in the proof of Theorem 4.1.

Now assume that $\kappa_0 < \lambda_1$. Let u be a solution of (1.1)-(1.2), then we have

$$\|u(t)\| \le s(t,\lambda_1)\|u_0\| + \int_0^t r(t-\tau,\lambda_1)[\kappa_0\|u(\tau)\| + \|f(0)\|]d\tau,\ \forall t \ge 0.$$

Using the Gronwall type inequality given in Proposition 2.2, we get

$$\begin{aligned} \|u(t)\| &\le s(t,\lambda_1-\kappa_0)\|u_0\| + \frac{1}{\lambda_1-\kappa_0}\|f(0)\|(1 - s(t,\lambda_1-\kappa_0)) \\ &\le \|u_0\| + \frac{1}{\lambda_1-\kappa_0}\|f(0)\|,\ \forall t \ge 0, \end{aligned}$$

which yields the global boundedness of u.

Let u and v be solutions of (1.1), then we have

$$\|u(t) - v(t)\| \le s(t,\lambda_1)\|u(0) - v(0)\| + \int_0^t r(t-\tau,\lambda_1)\kappa_0\|u(\tau) - v(\tau)\|d\tau,$$

thanks to (F) and Lemma 2.8. Employing Proposition 2.2 again, we obtain

$$\|u(t) - v(t)\| \le s(t,\lambda_1-\kappa_0)\|u(0) - v(0)\|,\ \forall t \ge 0.$$

Since $l \notin L^1(\mathbb{R}^+)$, it follows from Proposition 2.1(1) that $s(t,\lambda_1-\kappa_0) \to 0$ as $t \to \infty$, which completes the proof. $\square$

The following theorems show the main results of this section.

Theorem 4.3. *Let (A), (K) and (F) hold. If $f(0) = 0$ and $\limsup_{\rho\to 0} \kappa(\rho) = \alpha$ with $\alpha \in [0, \lambda_1)$, then there exists $\delta > 0$ such that the problem (1.1)-(1.2) admits a unique global mild solution $u \in C([0, T]; H) \cap C((0, T]; V_{\frac{1}{2}})$, provided that $\|u_0\| \le \delta$.*

Proof. By assumption, for $\theta \in (0, \lambda_1 - \alpha)$, there exists $\eta > 0$ such that $\|f(v)\| = \|f(v) - f(0)\| \le \kappa(\eta)\|v\| \le (\alpha + \theta)\|v\|$ as long as $\|v\| \le \eta$. Now we consider the solution map $\Phi : \mathbb{B}_\eta \to C([0, T]; H)$ defined by (4.1). We see that

$$\begin{aligned}
\|\Phi(u)(t)\| &\le s(t, \lambda_1)\|u_0\| + \int_0^t r(t - \tau, \lambda_1)(\alpha + \theta)\|u(\tau)\| d\tau \\
&\le s(t, \lambda_1)\|u_0\| + (\alpha + \theta)\eta\lambda_1^{-1}(1 - s(t, \lambda_1)) \\
&\le s(t, \lambda_1)[\|u_0\| - (\alpha + \theta)\lambda_1^{-1}\eta] + (\alpha + \theta)\lambda_1^{-1}\eta \\
&\le \eta, \ \forall t \in [0, T],
\end{aligned}$$

provided that $\|u_0\| \le \alpha\lambda_1^{-1}\eta$, thanks to the fact that $(\alpha + \theta)\lambda_1^{-1} < 1$. Fixing an θ and η mentioned above, for $\delta = \alpha\lambda_1^{-1}\eta$, we have shown that $\Phi(\mathbb{B}_\eta) \subset \mathbb{B}_\eta$ as $\|u_0\| \le \delta$. It remains to show that $\Phi : \mathbb{B}_\eta \to \mathbb{B}_\eta$ is a contraction mapping. Indeed, let $u_1, u_2 \in \mathbb{B}_\eta$, then

$$\begin{aligned}
\|\Phi(u_1)(t) - \Phi(u_2)(t)\| &\le \int_0^t r(t - s, \lambda_1)\kappa(\eta)\|u_1(s) - u_2(s)\| ds \\
&\le (\alpha + \theta)\lambda_1^{-1}(1 - s(t, \lambda_1))\|u_1 - u_2\|_\infty, \ \forall t \in [0, T],
\end{aligned}$$

which implies the assertion. The uniqueness follows from the Gronwall type inequality stated in Proposition 2.2. The proof is complete. □

Theorem 4.4. *Let the hypotheses of Theorem 4.3 hold. If $l \notin L^1(\mathbb{R}^+)$, then the zero solution of (1.1)-(1.2) is asymptotically stable.*

Proof. Taken θ and δ from the proof of Theorem 4.3, for $\|u_0\| \le \delta$ and a corresponding solution u of (1.1)-(1.2), we have

$$\|u(t)\| \le s(t, \lambda_1)\|u_0\| + \int_0^t r(t - \tau, \lambda_1)(\alpha + \theta)\|u(\tau)\| d\tau.$$

Using Proposition 2.2, we get

$$\|u(t)\| \le s(t, \lambda_1 - \alpha - \theta)\|u_0\|, \ \forall t \ge 0.$$

Since $l \notin L^1(\mathbb{R}^+)$ and $\lambda_1 - \alpha - \theta > 0$, we have $s(t, \lambda_1 - \alpha - \theta) \to 0$ as $t \to \infty$, and the last inequality ensures the stability and attractivity of the zero solution. The proof is complete. □

We now present a linearized stability result as a consequence of Theorem 4.4.

Corollary 4.5. *Let (A) and (K) hold. Assume that the nonlinearity f is continuously differentiable such that $f(0)=0$ and $A-f'(0)$ remains positively definite. Then the zero solution of (1.1) is asymptotically stable.*

Proof. Denote $\tilde{A}=A-f'(0)$, $\tilde{f}(v)=f(v)-f'(0)v$. Then equation (1.1) is equivalent to

$$\frac{d}{dt}[k*(u-u_0)](t)+\tilde{A}u(t)=\tilde{f}(u(t)),\ t>0. \tag{4.2}$$

By assumption, $\tilde{A}$ fulfills (A). Furthermore, $\tilde{f}$ is also continuously differentiable, so it is locally Lipschitzian and, therefore, $\tilde{f}$ satisfies (F). Specifically, let $v_1,v_2\in B_\rho$, then by using the mean value formula, we have

$$\begin{aligned}\|\tilde{f}(v_2)-\tilde{f}(v_1)\| &= \|\int_0^1 \tilde{f}'(v_1+(1-t)(v_2-v_1))(v_2-v_1)dt\| \\ &\le \left(\int_0^1 \|\tilde{f}'(v_1+(1-t)(v_2-v_1))\|dt\right)\|v_2-v_1\| \\ &\le \sup_{\|v\|\le 2\rho}\|\tilde{f}'(v)\|\|v_2-v_1\|.\end{aligned}$$

Taking $\kappa(\rho)=\sup_{\|v\|\le 2\rho}\|\tilde{f}'(v)\|$, we see that $\lim_{\rho\to 0}\kappa(\rho)=0$, thanks to the fact that $\tilde{f}'(v)=f'(v)-f'(0)\to 0$ as $v\to 0$. So one can apply Theorem 4.4 for (4.2) (with $\alpha=0$) to get the conclusion. □

To end this section, we prove the Hölder continuity of the mild solution to (1.1)-(1.2).

Theorem 4.6. *Let (A), (K*) and (F) hold. Then the mild solution to (1.1)-(1.2) is Hölder continuous on $[\delta,T]$ for every $0<\delta<T$.*

Proof. Let u be the mild solution to (1.1)-(1.2). Then

$$\begin{aligned}u(t) &= S(t)u_0+\int_0^t R(t-\tau)f(u(\tau))d\tau \\ &= u_1(t)+u_2(t).\end{aligned}$$

By the same reasoning as in the proof of Theorem 3.6, we have $u_1\in C^\gamma([\delta,T];H)$ for every $0<\delta<T$ and $\gamma\in(0,1)$.

Regarding u_2, let $\rho=\|u_2\|_\infty$ and $0<\delta\le t\le T$, then we see that

$$\begin{aligned}\|u_2(t+h)-u_2(t)\| &\le \int_0^t \|R(\tau)\|\|f(u_2(t+h-\tau))-f(u_2(t-\tau))\|d\tau \\ &\quad +\int_t^{t+h}\|R(\tau)\|\|f(u_2(t+h-\tau))\|d\tau \\ &\le \int_0^t r(\tau,\lambda_1)\kappa(\rho)\|u_2(t+h-\tau)-u_2(t-\tau)\|d\tau\end{aligned}$$

$$+\|A^{-1}\|\int_t^{t+h}\|S'(\tau)\|(\|f(0)\|+\kappa(\rho)\rho)d\tau$$

$$\le \int_0^t r(t-\tau,\lambda_1)\kappa(\rho)\|u_2(\tau+h)-u_2(\tau)\|d\tau$$

$$+\|A^{-1}\|M(\|f(0)\|+\kappa(\rho)\rho)\gamma^{-1}\delta^{-\gamma}h^\gamma,$$

here we use (F) and the arguments as in the proof of Theorem 3.6 for estimating the second integral.

Applying Proposition 2.2 for $v(t)=\|u_2(t+h)-u_2(t)\|$, one gets

$$\|u_2(t+h)-u_2(t)\|\le \|A^{-1}\|M(\|f(0)\|+\kappa(\rho)\rho)\gamma^{-1}\delta^{-\gamma}s(t,\lambda_1-\kappa(\rho))h^\gamma,$$

which implies $u_2\in C^\gamma([\delta,T];H)$. □

5. Application

Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. We apply the obtained results to the following two-term time-fractional PDE:

$$\partial_t^\alpha u(t,x)+\mu\,\partial_t^\beta u(t,x)+(-\Delta)^\gamma u(t,x)=F\left(\int_\Omega u^2(t,x)dx\right)G(x,u(t,x)), \tag{5.1}$$

$$\text{for } t>0, x\in\Omega,$$

$$u(t,x)=0,\quad \text{for } t\ge 0,\ x\in\partial\Omega, \tag{5.2}$$

$$u(0,x)=u_0(x),\quad \text{for } x\in\Omega, \tag{5.3}$$

where $0<\alpha<\beta<1$, $\mu\ge 0,\gamma>0$, ∂_t^α and ∂_t^β stand for the Caputo fractional derivatives of order α and β in t, respectively; Δ is the Laplacian with the domain $D(\Delta)=H^2(\Omega)\cap H_0^1(\Omega)$. Let $H=L^2(\Omega)$ with the inner product $(u,v)=\int_\Omega u(x)v(x)dx$. Put

$$k(t)=g_{1-\alpha}(t)+\mu\,g_{1-\beta}(t), \tag{5.4}$$

$$A=(-\Delta)^\gamma,$$

$$f(v)(x)=F\left(\int_\Omega v^2(x)dx\right)G(x,v(x)),\ v\in L^2(\Omega).$$

Then the problem (5.1)-(5.3) is in the form of (1.1)-(1.2). Observe that, the kernel function k is completely monotonic, i.e. $(-1)^n k^{(n)}(t)\ge 0$ for $t\in(0,\infty)$. As mentioned in [18], k admits a resolvent function l such that $k*l=1$ on $(0,\infty)$ and in this case, $(1*l)(t)\sim g_{1+\alpha}(t)$ as $t\to\infty$. Thus

$$s(t,\mu)\le \frac{1}{1+\mu(1*l)(t)}\to 0 \text{ as } t\to\infty,\ \text{for any } \mu>0.$$

Let $\lambda_\triangle$ be the first eigenvalue of $-\Delta$, that is $\lambda_\triangle=\inf\limits_{u\in H_0^1(\Omega)}\{\|\nabla u\|^2:\|u\|=1\}$. This implies that the first eigenvalue λ_1 of $A=(-\Delta)^\gamma$ is given by $\lambda_1=\lambda_\triangle^\gamma$.

Noting that, the nonlinearity in (5.1) can be seen as a perturbation depending not only on the state but also on the energy of the system. We assume that

(H1) $F \in C^1(\mathbb{R})$ obeys the estimate $|F(r)| \le a + b|r|^\nu$, for some nonnegative numbers a, b and ν.
(H2) $G : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and satisfies the Lipschitz condition in the second variable, i.e.

$$|G(x, y_1) - G(x, y_2)| \le h(x)|y_1 - y_2|, \forall x \in \Omega, y_1, y_2 \in \mathbb{R},$$

here $h \in L^\infty(\Omega)$ is a nonnegative function. In addition, assume that $G(x, 0) = 0$ for a.e. $x \in \Omega$.

Theorem 5.1. *Let (H1)-(H2) holds. Assume that*

$$a\|h\|_\infty < \lambda_\Delta^\gamma \text{ if } \nu > 0, \tag{5.5}$$

$$(a + b)\|h\|_\infty < \lambda_\Delta^\gamma \text{ if } \nu = 0, \tag{5.6}$$

where $\|h\|_\infty = \text{ess sup}_{x\in\Omega}|h(x)|$. *Then the problem* (5.1)-(5.3) *has a unique global mild solution. Furthermore, the zero solution of* (5.1) *is asymptotically stable.*

Proof. We first verify that f maps $L^2(\Omega)$ into itself. Indeed, using (H1)-(H2) we get

$$\begin{aligned}\|f(v)\| &= F\left(\int_\Omega v^2(x)dx\right)\left(\int_\Omega |G(x, v(x))|^2 dx\right)^{\frac{1}{2}} \\ &\le F\left(\int_\Omega v^2(x)dx\right)\left(\int_\Omega h^2(x)v^2(x)dx\right)^{\frac{1}{2}} \\ &\le (a + b\|v\|^{2\nu})\|h\|_\infty\|v\|.\end{aligned}$$

In addition, we can check that f is locally Lipschitzian due to the assumption that F' is continuous and G is Lipschitzian. Specifically, for $v_1, v_2 \in L^2(\Omega)$, $\|v_1\|, \|v_2\| \le \rho$, we see that

$$\begin{aligned}\|f(v_1) - f(v_2)\| &\le |F(\|v_1\|^2) - F(\|v_2\|^2)|\left(\int_\Omega |G(x, v_1(x))|^2 dx\right)^{\frac{1}{2}} \\ &\quad + |F(\|v_2\|^2)|\left(\int_\Omega |G(x, v_1(x)) - G(x, v_2(x))|^2 dx\right)^{\frac{1}{2}} \\ &\le |F'\left(\theta\|v_1\|^2 + (1-\theta)\|v_2\|^2\right)| \cdot |\|v_1\|^2 - \|v_2\|^2| \cdot \|h\|_\infty\|v_1\| \\ &\quad + (a + b\|v_2\|^{2\nu})\|h\|_\infty\|v_1 - v_2\| \\ &\le \kappa(\rho)\|v_1 - v_2\|,\end{aligned}$$

where

$$\kappa(\rho) = 2\rho^2\|h\|_\infty \sup_{r\in[0,\rho^2]} |F'(r)| + (a + b\rho^{2\nu})\|h\|_\infty.$$

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Hence $\lim_{\rho\to 0}\kappa(\rho) = a\|h\|_\infty$ if $\nu > 0$ and $\lim_{\rho\to 0}\kappa(\rho) = (a+b)\|h\|_\infty$ as $\nu = 0$. Using (5.5)-(5.6) and Theorem 4.3, we have the conclusion that, the problem (5.1)-(5.3) has a unique global mild solution. In addition, the zero solution of (5.1) is asymptotically stable, due to Theorem 4.4. $\square$

Let us mention that, the mild solution for (5.1)-(5.3) is Hölder continuous on $[\delta, T]$ for every $0 < \delta < T$. Indeed, the Laplace transform $\hat{l}$ is given by $\hat{l}(\lambda) = (\lambda^\alpha + \mu\lambda^\beta)^{-1}$. Obviously,

$$\begin{aligned} |\arg \hat{l}(\lambda)| &= |\arg (\lambda^\alpha + \mu\lambda^\beta)^{-1}| \\ &= |\arg (\lambda^\alpha + \mu\lambda^\beta)| \\ &< \frac{\pi}{2} \ \text{ for all } \operatorname{Re}\lambda > 0. \end{aligned}$$

So l is $\frac{\pi}{2}$-sectorial. In addition, by a direct computation, one has

$$\begin{aligned} \lambda\hat{l}'(\lambda) &= -(\lambda^\alpha + \mu\lambda^\beta)^{-2}(\alpha\lambda^\alpha + \mu\beta\lambda^\beta) \\ \lambda^2\hat{l}''(\lambda) &= 2(\lambda^\alpha + \mu\lambda^\beta)^{-3}[\alpha\lambda^\alpha + \mu\beta\lambda^\beta]^2 \\ &\quad + (\lambda^\alpha + \mu\lambda^\beta)^{-2}[\alpha(1-\alpha)\lambda^\alpha + \mu\beta(1-\beta)\lambda^\beta]. \end{aligned}$$

Observing that, for every $\eta_1, \eta_2 \in (0,1)$ and $\text{Re}\lambda > 0$,

$$|\eta_1\lambda^\alpha + \eta_2\mu\lambda^\beta| \le |\lambda^\alpha + \mu\lambda^\beta|,$$

we have

$$|\lambda\hat{l}'(\lambda)| \le |\hat{l}(\lambda)|, \ \ |\lambda^2\hat{l}''(\lambda)| \le 3|\hat{l}(\lambda)|,$$

which ensures that l is 2-regular, and (K^*) is fulfilled. So the Hölder regularity of the mild solution follows from Theorem 4.6.

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