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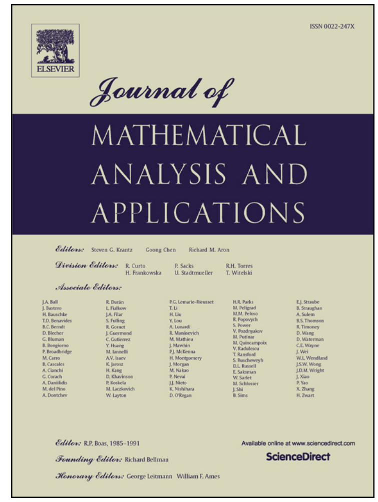


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# Systems with nonlocal vs. local diffusions and free boundaries[1]

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**Abstract**

We study a class of free boundary problems of ecological models with nonlocal and local diffusions, which are natural extensions of free boundary problems of reaction diffusion systems in there local diffusions are used to describe the population dispersal, with the free boundary representing the spreading front of the species. We first prove the existence, uniqueness and regularity of global solution. For the classical competition, prey-predator and mutualist models, we show that a spreading-vanishing dichotomy holds, and establish the criteria of spreading and vanishing, and long time behavior of the solution.

**Keywords**: Nonlocal-local diffusions; Free boundaries; Existence-uniqueness; Spreading-vanishing; Long-time behavior.

**AMS Subject Classification (2000)**: 35K57, 35R09, 35R20, 35R35, 92D25

## 1 Introduction

It is well known that random dispersal or local diffusion describes the movements of organisms between adjacent spatial locations. It has been increasingly recognized the movements and interactions of some organisms can occur between non-adjacent spatial locations. The evolution of nonlocal diffusion has attracted a lot of attentions for both theoretically and empirically; please refer to, for example, [1]-[3] and references therein. An extensively used nonlocal diffusion operator to replace the local diffusion term $d\Delta u$ (the Laplacian operator in $\mathbb{R}^N$) is given by

$$d(J*u-u)(t,x):=d\left(\int_{\mathbb{R}^N}J(x-y)u(t,y)dy-u(t,x)\right).$$

To describe the spatial spreading of species in the nonlocal diffusion processes, recently, Cao et al. ([4]) studied the following free boundary problem of Fisher-KPP nonlocal diffusion model:

$$\begin{cases} u_t=d\displaystyle\int_{g(t)}^{h(t)}J(x-y)u(t,y)\mathrm{d}y-du+f(t,x,u), & t>0,\ g(t)<x<h(t),\\ u(t,g(t))=u(t,h(t))=0, & t>0,\\ h'(t)=\mu\displaystyle\int_{g(t)}^{h(t)}\int_{h(t)}^{\infty}J(x-y)u(t,x)\mathrm{d}y\mathrm{d}x, & t>0,\\ g'(t)=-\mu\displaystyle\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)}J(x-y)u(t,x)\mathrm{d}y\mathrm{d}x, & t>0,\\ u(0,x)=u_0(x),\ h(0)=-g(0)=h_0, & |x|\le h_0,\end{cases}\tag{1.1}$$

where $x=g(t)$ and $x=h(t)$ are the moving boundaries to be determined together with $u(t,x)$, which is always assumed to be identically 0 for $x\in\mathbb{R}\setminus[g(t),h(t)]$; $d$, $\mu$ and $h_0$ are positive constants. The kernel function $J:\mathbb{R}\to\mathbb{R}$ is continuous and satisfies

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**(J)** $J(0) > 0,\ J(x) \geq 0,\ \int_{\mathbb{R}} J(x)\mathrm{d}x = 1,\ \ J$ is symmetric, $\sup J < \infty$.

The reaction function $f(t, x, u)$ satisfies

**(f)** $f(t, x, 0) \equiv 0$, there exists $K_0 > 0$ such that $f(t, x, u) < 0$ for $u \geq K_0$, $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, and $f$ is locally Lipschitz continuous in $u \in \mathbb{R}^+$, i.e., for any $A > 0$ there exists $L(A) > 0$ such that

$$|f(t, x, u_1) - f(t, x, u_2)| \leq L(A)|u_1 - u_2|, \quad \forall\ u_1, u_2 \in [0, A],\ (t, x) \in \mathbb{R}^+ \times \mathbb{R}.$$

It was shown in [4] that the problem (1.1) has a unique global solution. Furthermore, the spreading-vanishing dichotomy about free boundary problems of local diffusive logistic equation ([5]) holds true for the nonlocal diffusive problem (1.1) when $f(t, x, u) = f(u)$.

Motivated by the works of [4] and [6, 7, 8, 9, 10, 11] (two species local diffusion systems with common free boundary), the authors of [12] studied a class of free boundary systems with nonlocal diffusions and common free boundaries. They proved that such a nonlocal diffusion problem has a unique global solution, and for models with Lotka-Volterra type competition or predator-prey growth terms, they shown that a spreading-vanishing dichotomy holds, and obtained criteria for spreading and vanishing. Moreover, for the weak competition case and weak predation case, they determined the long-time asymptotic limit of the solution when spreading happens.

Kao et al. [13] studied the dynamics of a competitive model in which one diffusion is local and the other one is nonlocal.

Inspired by the above cited papers, recently, Wang and Wang [14] investigated free boundary problems with nonlocal and local diffusions and common free boundaries.

Free boundary problems of two species reaction diffusion systems, in which one species distributes in the whole space and the other one exists initially in a interval and invades into the new environment with double free boundaries, had been extensively studied. For example, [10, 11, 15, 16, 17] for the competition model, [18, 19] for the predator-prey model, and [7] for the Beddington-DeAngelis predator-prey model with nonlinear prey-taxis.

Motivated by the above mentioned works, in this paper we deal with the following free boundary problems with nonlocal and local diffusions:

$$\begin{cases} u_t = d_1(J * u - u) + f_1(t, x, u, v), & t > 0,\ x \in \mathbb{R}, \\ v_t = d_2 v_{xx} + f_2(t, x, u, v), & t > 0,\ g(t) < x < h(t), \\ v = 0,\ \ g'(t) = -\mu v_x, & t \geq 0,\ x = g(t), \\ v = 0,\ \ h'(t) = -\mu v_x, & t \geq 0,\ x = h(t), \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ v(0, x) = v_0(x), & x \in [-h_0, h_0], \\ -g(0) = h(0) = h_0 > 0, \end{cases} \tag{1.2}$$

where $[-h_0, h_0]$ represents the initial population range of the species $v(t, x)$; $x = g(t)$ and $x = h(t)$ are the free boundaries to be determined by $v(t, x)$; $d_i$ and $\mu$ are positive constants.

Denote by $C^{1-}(\mathbb{R})$ the space of global Lipschitz continuous functions in $\mathbb{R}$. We assume that the initial functions $u_0, v_0$ satisfy

$$\begin{cases} u_0 \in C^{1-}(\mathbb{R}) \cap C_b(\mathbb{R}), \quad v_0 \in W_p^2((-h_0, h_0)), \\ v_0(\pm h_0) = 0,\ v_0 > 0\ \text{ in }\ (-h_0, h_0), \quad u_0 > 0 \text{ in } \ \mathbb{R} \end{cases} \tag{1.3}$$

with $p > 3$, and denote by $L_0$ the Lipschitz constant of $u_0$.

**(J1)** The condition **(J)** holds, $J$ is compactly supported, $J \in C^{1-}(\mathbb{R})$, and denote by $L(J)$ the Lipschitz constant of $J$.

The growth terms $f_i : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ are assumed to be continuous and satisfy

**(f1)** $f_1(t, x, 0, v) = f_2(t, x, u, 0) = 0$, $f_i$ is locally Lipschitz continuous in $u, v \in \mathbb{R}^+$, i.e., for any given $A_1, A_2 > 0$, there exists $L(A_1, A_2) > 0$ such that

$$|f_i(t, x, u_1, v_1) - f_i(t, x, u_2, v_2)| \leq L(A_1, A_2)(|u_1 - u_2| + |v_1 - v_2|), \;\; i = 1, 2$$

for all $u_1, u_2 \in [0, A_1]$, $v_1, v_2 \in [0, A_2]$ and all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$;

**(f2)** There exists $k_0 > 0$ such that for all $v \geq 0$ and $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, there holds $f_1(t, x, u, v) < 0$ when $u > k_0$;

**(f3)** For the given $A > 0$, there exists $\Theta(A) > 0$ such that $f_2(t, x, u, v) < 0$ for $0 \leq u \leq A$, $v \geq \Theta(A)$ and $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$;

**(f4)** $f_i$ is Lipschitz continuous in $x \in \mathbb{R}$, i.e., for any $A_1, A_2 > 0$, there exists $L^*(A_1, A_2) > 0$ such that

$$|f_i(t, x_1, u, v) - f_i(t, x_2, u, v)| \leq L^*(A_1, A_2)|x_1 - x_2|, \quad i = 1, 2$$

for all $u \in [0, A_1]$, $v \in [0, A_2]$ and all $(t, x_1, x_2) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$.

The condition **(f1)** implies

$$|f_1(t, x, u, v)| \leq L(A_1, A_2)u, \quad |f_2(t, x, u, v)| \leq L(A_1, A_2)v$$

for all $u \in [0, A_1]$, $v \in [0, A_2]$ and all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$.

It is easily seen that the conditions **(f1)**–**(f4)** hold for the classical competition, prey-predator and mutualist models

$$\textit{Competition Model}: \quad f_1 = u(a - u - bv), \quad f_2 = v(1 - v - cu), \tag{1.4}$$

$$\textit{Prey-predator Model}: \quad f_1 = u(a - u - bv), \quad f_2 = v(1 - v + cu), \tag{1.5}$$

$$\textit{Mutualist Model}: \quad f_1 = r_1 u \left(a - u - \frac{u}{1 + bv}\right), \quad f_2 = r_2 v \left(1 - v - \frac{v}{1 + cu}\right), \tag{1.6}$$

where parameters are positive constants.

Except where otherwise stated, we always assume that **(f1)-(f4)** hold, the kernel function $J$ satisfies **(J1)** and $u_0$, $v_0$ satisfy the condition (1.3) throughout this paper. Sometimes, we simply write $\|(\phi, \varphi)\|$ as $\|\phi, \varphi\|$.

The organization of this paper is as follows. In Section 2 we prove that the problem (1.2) has a unique global solution. For later discussion, we give some preliminary results in section 3. Section 4 is devoted to the long time behaviors of global solution, and Section 5 deals with conditions for spreading and vanishing. In the last section we shall give some estimates of spreading speeds.

## 2 Global existence and uniqueness of solution of (1.2)

In this section we prove the global existence and uniqueness of the solution to problem (1.2). For convenience, we first introduce some notations. Let $k_0, \Theta(\cdot)$ be given in **(f2)**, **(f3)**. Denote

$$\begin{aligned} A_1 &= \max\left\{\|u_0\|_\infty,\, k_0\right\},\ A_2 = \max\left\{\|v_0\|_\infty,\, \Theta(A_1)\right\},\ L = L(A_1, A_2), \\ L^* &= L^*(A_1, A_2), \quad A_3 = 2A_2 \max\left\{\sqrt{\frac{L}{2d_2}},\ \frac{4\|v_0\|_{C^1([-h_0,h_0])}}{3A_2}\right\}, \\ \Pi_T &= [0,T]\times\mathbb{R}, \quad \Delta_T = [0,T]\times[-1,1] \quad \text{with} \ \ T>0. \end{aligned}$$

For the given $h_0, T > 0$, define

$$\begin{aligned} \mathbb{H}^T &= \left\{h\in C^1([0,T])\ :\ h(0)=h_0,\ 0<h'(t)\le \mu A_3\right\}, \\ \mathbb{G}^T &= \left\{g\in C^1([0,T])\ :-g\in\mathbb{H}^T\right\}. \end{aligned}$$

And for $g\in\mathbb{G}^T$, $h\in\mathbb{H}^T$, define

$$\begin{aligned} D^T_{g,h} &= \left\{(t,x)\in\mathbb{R}^2:\ 0<t\le T,\ g(t)<x<h(t)\right\}, \\ \mathbb{X}^T_1 &= X^T_{u_0} = \left\{\varphi\in C(\Pi_T):\ 0\le\varphi\le A_1,\ \varphi\big|_{t=0}=u_0\right\}, \\ \mathbb{X}^T_2 &= \mathbb{X}^T_{v_0,g,h} = \left\{\varphi\in C(\overline{D}^T_{g,h}):\ 0\le\varphi\le A_2,\ \varphi\big|_{t=0}=v_0,\ \varphi\big|_{x=g(t),h(t)}=0\right\}, \\ \mathbb{X}^T &= \mathbb{X}^T_1\times\mathbb{X}^T_2 \end{aligned}$$

The following theorem is the main result of this section.

**Theorem 2.1.** *The problem* (1.2) *has a unique local solution* $(u, v, g, h)$ *defined on* $[0, T]$ *for some* $0 < T < \infty$. *Moreover,* $(g, h) \in \mathbb{G}^T \times \mathbb{H}^T$, $(u, v) \in \mathbb{X}^T$, $v \in W^{1,2}_p(D^T_{g,h})$ *and*

$$(u,v,g,h)\in C^{1,1-}(\Pi_T)\times W^{1,2}_p(D^T_{g,h})\times[C^1([0,T])]^2, \tag{2.1}$$

$$0<u\le A_1 \quad \text{in} \ \ \Pi_T, \quad 0<v\le A_2 \quad \text{in} \ \ D^T_{g,h}, \tag{2.2}$$

$$0<-v_x(t,h(t)),\ v_x(t,g(t))\le A_3, \quad 0<t\le T, \tag{2.3}$$

*where* $u \in C^{1,1-}(\Pi_T)$ *means that* $u$ *is differentiable continuously in* $t \in [0, T]$ *and is Lipschitz continuous in* $x \in \mathbb{R}$.

*If we further assume that*

**(f5)** *For any given* $\tau,\ l,\ A_1,\ A_2 > 0$, *there exists* $\tilde{L}(\tau, l, A_1, A_2)$ *such that*

$$\|f_2(\cdot,x,u,v)\|_{C^{\frac{\alpha}{2}}([0,\tau])}\le\tilde{L}(\tau,l,A_1,A_2) \tag{2.4}$$

*for all* $x\in[-l,l],\ u\in[0,A_1],\ v\in[0,A_2]$.

*Then the solution* $(u, v, g, h)$ *of* (1.2) *exists globally. Moreover, for any given* $T > 0$, (2.1)-(2.3) *hold true, and*

$$v\in C^{1+\alpha/2,\,2+\alpha}((0,T]\times[g(t),h(t)]), \quad g,\ h\in C^{1+\alpha/2}([0,T]). \tag{2.5}$$

To prove Theorem 2.1, we first give some Lemmas which are crucial in the proof of Theorem 2.1.

**Lemma 2.2** (Maximum Principle)**.** *Assume that $J$ satisfies* **(J)***, $u$ and $u_t$ are continuous and $u$ is bounded in $\Pi_T$. If $u$ satisfies, for some $c(t,x) \in L^\infty(\Pi_T)$*

$$\begin{cases} u_t \geq d(J * u - u) + c(t,x)u, & (t,x) \in (0,T] \times \mathbb{R}, \\ u(0,x) \geq 0, & x \in \mathbb{R}. \end{cases}$$

*Then $u \geq 0$ on $\Pi_T$. Moreover, $u > 0$ in $(0,T] \times \mathbb{R}$ provided $u(0,x) \not\equiv 0$ in $\mathbb{R}$.*

*Proof.* This lemma may be known, but we can't find references, so we give it's proof. The idea of this proof comes from [20, Lemma 2.3] and [21, Lemma 3.3]. As $u$ is continuous and bounded, the function $f(t) := \inf_{x\in\mathbb{R}} u(t,x)$ is continuous in $[0,T]$. Set $p(t,x) = d - c(t,x)$ and $g(t) = e^{-2Kt}f(t)$ where $K := \|p(t,x)\|_\infty + d$. Suppose on the contrary and due to $g(0) \geq 0$, there exist $\varepsilon > 0$, $0 < T_0 \leq T$ such that $g(T_0) = \inf_{[0,T]} g(t) = -\varepsilon$, $g(t) > -\varepsilon$ for $0 \leq t < T_0$, and there exists $(t_*, x_*) \in (0,T_0) \times \mathbb{R}$ such that $u(t_*,x_*) < -\frac{7}{8}\varepsilon e^{2Kt_*}$. Therefore,

$$u > -\varepsilon e^{2Kt} \quad \text{in} \quad [0,T_0) \times \mathbb{R}, \quad u(t_*,x_*) < -\frac{7}{8}\varepsilon e^{2Kt_*}.$$

Let $z(x)$ be continuous in $\mathbb{R}$ and satisfy $\min_{\mathbb{R}} z = z(x_*) = 1$, $\sup_{\mathbb{R}} z = z(\pm\infty) = 3$. Set

$$w_\sigma(t,x) = -\varepsilon(3/4 + \sigma z(x))e^{2Kt}, \quad \text{with } \sigma \in [0,1].$$

Obviously, $w_\sigma$ is bounded and continuous in $[0,1] \times [0,T] \times \mathbb{R}$. Notice that, when $\sigma \leq 1/8$,

$$\inf_{\Pi_{T_0}} (u - w_\sigma) \leq u(t_*,x_*) - w_\sigma(t_*,x_*) < -\frac{7}{8}\varepsilon e^{2Kt_*} + \varepsilon\left(\frac{3}{4} + \frac{1}{8}\right)e^{2Kt_*} = 0;$$

when $\sigma > 1/4$, for all $(t,x) \in \Pi_{T_0}$,

$$u(t,x) - w_\sigma(t,x) \geq -\varepsilon e^{2Kt} + \varepsilon(3/4 + \sigma z(x))e^{2Kt} \geq \varepsilon e^{2Kt}(3/4 + \sigma - 1) > \varepsilon(\sigma - 1/4).$$

One can find a $\sigma_* \in (1/8, 1/4]$ such that $\inf_{\Pi_{T_0}}(u - w_{\sigma_*}) = 0$. As $w_{\sigma_*}(t,\pm\infty) \leq -\frac{9}{8}\varepsilon e^{2Kt} < u(t,\pm\infty)$ for $t \in [0,T_0]$, and $u(0,x) \geq 0 > -3\varepsilon/4 > w_{\sigma_*}(0,x)$ for $x \in \mathbb{R}$. There exists $(t_0,x_0) \in (0,T_0] \times \mathbb{R}$ such that

$$u(t_0,x_0) - w_{\sigma_*}(t_0,x_0) = 0 = \inf_{\Pi_{T_0}}(u - w_{\sigma_*}),$$

which implies $u_t(t_0,x_0) - w_{\sigma_* t}(t_0,x_0) \leq 0$. Recall $\sigma_* \in (1/8, 1/4]$, $1 \leq z(x) \leq 3$, we have

$$\begin{aligned} 0 &\geq u_t(t_0,x_0) - w_{\sigma_* t}(t_0,x_0) \\ &\geq d(J * u)(t_0,x_0) - p(t_0,x_0)u(t_0,x_0) + 2K\varepsilon(3/4 + \sigma_* z(x_0))e^{2Kt_0} \\ &> -d\varepsilon e^{-2Kt_0} - \|p(t,x)\|_\infty |u(t_0,x_0)| + \frac{7}{4}K\varepsilon e^{2Kt_0} \\ &> -d\varepsilon e^{2Kt_0} - \frac{3}{2}\varepsilon \|p(t,x)\|_\infty e^{2Kt_0} + \frac{7}{4}K\varepsilon e^{2Kt_0} \\ &= \varepsilon e^{2Kt_0}(7K/4 - d - 3\|p(t,x)\|_\infty/2) > 0 \end{aligned}$$

as $K = \|p(t,x)\|_\infty + d$. This derives a contradiction. So $u \geq 0$ on $\Pi_T$.

If $u(0,x) \not\equiv 0$ in $\mathbb{R}$, then for any given $N > 0$ big enough such that $u(0,x) \not\equiv 0$ in $[-N,N]$, we have

$$\begin{cases} u_t \geq d\displaystyle\int_{-N}^{N} J(x,y)u(t,y)\mathrm{d}y - du + c(t,x)u, & (t,x) \in (0,T] \times [-N,N], \\ u(0,x) \geq, \not\equiv 0, & x \in [-N,N]. \end{cases}$$

It follows by [4, Lemma 3.3] that $u(t,x)>0$ in $(0,T]\times[-N,N]$. The arbitrariness of $N$ implies $u>0$ in $(0,T]\times\mathbb{R}$. □

**Lemma 2.3.** *Assume that $J$ satisfies* **(J)**. *Consider the following Cauchy problem*

$$\begin{cases} z_t = d(J*z-z)+f(t,x,z), & (t,x)\in(0,T]\times\mathbb{R},\\ z(0,x)=z_0(x)>0, & x\in\mathbb{R}, \end{cases} \tag{2.6}$$

*where $z_0\in C_b(\mathbb{R})$, $f:\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}$ is continuous. If $f$ satisfies the condition* **(f)**, *then* (2.6) *has a unique solution $z\in C_b(\Pi_T)$ and $0<z\le K_1:=\max\{K_0,\|z_0\|_\infty\}$.*

*Proof.* This lemma may be known. For the convenience to readers we shall give its proof. Define $A:=\|z_0\|_\infty+1$, $0<T_0\le\min\big\{1,T,\frac{1}{(2d+L(A))A}\big\}$, and

$$\tilde f(t,x,z)=0 \ \text{ when } z\le 0,\quad \tilde f(t,x,z)=f(t,x,z) \ \text{ when } z>0.$$

Then $\tilde f(t,x,z)$ is still continuous. Clearly, the problem

$$\begin{cases} z_t = d(J*z-z)+\tilde f(t,x,z), & (t,x)\in(0,T_0]\times\mathbb{R},\\ z(0,x)=z_0(x)>0, & x\in\mathbb{R} \end{cases} \tag{2.7}$$

is equivalent to the following integral equation

$$z(t,x)=z_0(x)+\int_0^t\big[d(J*z-z)(s,x)+\tilde f(s,x,z(s,x))\big]\mathrm{d}s.$$

Let us define

$$z_n(t,x)=z_0(x)+\int_0^t\big[d(J*z_{n-1}-z_{n-1})(s,x)+\tilde f(s,x,z_{n-1}(s,x))\big]\mathrm{d}s,\ \ n\ge1,$$

and $z_0(t,x):=z_0(x)$. The direct calculation gives

$$\|z_n-z_0\|_{C_b(\Pi_{T_0})}\le 1,\quad \|z_{n+1}-z_n\|_{C_b(\Pi_{T_0})}\le T_0(2d+L(A))\|z_{n-1}-z_n\|_{C_b(\Pi_{T_0})}.$$

Due to our choice of $T_0$, we have $q:=T_0(2d+L(A))<1$. Thus

$$\|z_{n+1}-z_n\|_{C_b(\Pi_{T_0})}\le q^n\|z_1-z_0\|_{C_b(\Pi_{T_0})}$$

inductively. This shows that $\{z_n\}$ is a Cauchy sequence of $C_b(\Pi_{T_0})$. Thanks to the completeness of $C_b(\Pi_{T_0})$, it follows that $z_n\to z$ in $C_b(\Pi_{T_0})$ and $\|z-z_0\|_{C_b(\Pi_{T_0})}\le1$. It is easy to see that $z$ is the unique solution of (2.7). Due to $f(t,x,z)\ge -L(A)\mathrm{Sgn}(z)z$, we have $z>0$ in $\Pi_{T_0}$ by Lemma 2.2. Hence $z(t,x)$ is the unique solution of (2.6) and $0<z\le K_1$ in $\Pi_{T_0}$ by the comparison principle.

According to the above arguments we can regard $z(T_0,x)$ as the initial data to extend the unique solution of (2.6) to some $[0,T_1]$ with $T_1>T_0$ and it is easy to see that $T_1-T_0$ depends only on $K_1$, $f$ and $d$. Furthermore, the extended solution $z$ still satisfies $0<z\le K_1$ in $[0,T_1]\times\mathbb{R}$. By repeating this extension process finitely many times, the solution can be uniquely extended to $T$ and satisfies $0<z\le K_1$ in $[0,T]\times\mathbb{R}$. □

*Proof of Theorem 2.1. Step 1*: Given $T > 0$, we say $u \in C_x^{1-}(\Pi_T)$ if there is a constant $L(u, T)$ such that

$$|u(t, x_1) - u(t, x_2)| \le L(u, T)|x_1 - x_2|, \ \ \forall\, x_1, x_2 \in \mathbb{R},\ t \in [0, T].$$

For $s > 0$ we define

$$\mathbb{X}_{u_0}^s = \{\phi \in C_b(\Pi_s) : \phi(0, x) = u_0(x),\ 0 \le \phi \le A_1\}\,.$$

Choose $u \in \mathbb{X}_{u_0}^1 \cap C_x^{1-}(\Pi_1)$ and consider the following problem

$$\begin{cases} v_t = d_2 v_{xx} + f_2(t, x, u(t, x), v), & 0 < t \le 1,\ g(t) < x < h(t), \\ v(t, g(t)) = v(t, h(t)) = 0, & 0 \le t \le 1, \\ h'(t) = -\mu v_x(t, h(t)), & 0 \le t \le 1, \\ g'(t) = -\mu v_x(t, g(t)), & 0 \le t \le 1, \\ v(0, x) = v_0(x),\ h(0) = -g(0) = h_0 > 0, & |x| \le h_0. \end{cases} \tag{2.8}$$

Due to the properties of $f_2$ and $u$, using the similar arguments in the proof of [22, Theorem 1.1] we can show that (2.8) has a unique solution $(v, g, h) \in C^{\frac{1+\alpha}{2}, 1+\alpha}(\overline{D}_{g,h}^T) \times [C^{1+\alpha/2}([0, T])]^2$ for some $0 < T \le 1$, and $(v, g, h)$ satisfies

$$0 < v \le A_2 \ \text{ in } \ D^T; \quad 0 < -v_x(t, g(t)),\ v_x(t, h(t)) \le A_3 \ \text{ in } \ (0, T], \tag{2.9}$$

where $T$ depends only on $v_0$, $h_0$, $\alpha$, $A_1$, $A_2$, $f_2$ and the Lipschitz constant $L(u, 1)$ of $u$. Put $v(t, x) = 0$ when $x \in \mathbb{R} \setminus (g(t), h(t))$ and consider the following problem

$$\begin{cases} \tilde{u}_t = d(J * \tilde{u} - \tilde{u}) + f_1(t, x, \tilde{u}, v(t, x)), & (t, x) \in (0, T] \times \mathbb{R}, \\ \tilde{u}(0, x) = u_0(x) > 0, & x \in \mathbb{R}. \end{cases} \tag{2.10}$$

It is easy to verify that the function $f(t, x, \tilde{u}) := f_1(t, x, \tilde{u}, v(t, x))$ satisfies the condition **(f)**. By virtue of Lemma 2.3 we can see that (2.10) has a unique solution $\tilde{u} \in \mathbb{X}_{u_0}^T$.

In the following it will be shown that $\tilde{u} \in C_x^{1-}(\Pi_T)$. We straighten the boundaries and define $w(t, y) = u(t, x(t, y))$ and $z(t, y) = v(t, x(t, y))$, where

$$x(t, y) = \frac{(h(t) - g(t))y + h(t) + g(t)}{2}.$$

Then $z$ satisfies

$$\begin{cases} z_t = d_2 \xi(t) z_{yy} + \zeta(t, y) z_y + f_2(t, x(t, y), w, z), & 0 < t \le T,\ |y| < 1, \\ z(t, -1) = z(t, 1) = 0, & 0 \le t \le T, \\ z(0, y) = v_0(h_0 y) =: z_0(y), & |y| \le 1 \end{cases} \tag{2.11}$$

with

$$\xi(t) = \frac{4}{(h(t) - g(t))^2}, \quad \zeta(t, y) = \frac{h'(t) + g'(t)}{h(t) - g(t)} + \frac{(h'(t) - g'(t))y}{h(t) - g(t)}.$$

Due to (2.9) we have

$$\|\xi\|_{L^\infty([0,T])} \le 1/h_0^2, \quad \|\zeta\|_{L^\infty(\Delta_T)} \le 2\mu A_3/h_0, \quad \|f_2\|_{L^\infty(\Delta_T)} \le C_0, \tag{2.12}$$

where $C_0$ depends only on $A_1, A_2$. By the parabolic $L^p$ theory, $z \in W_p^{1,2}(\Delta_T)$ and

$$\|z\|_{W_p^{1,2}(\Delta_T)} \le C, \tag{2.13}$$

where $C$ depends only on $h_0$, $v_0$, $A_1$, $A_2$, $A_3$ and $\mu$. Using the arguments in the proof of [22, Theorem 1.1] we can obtain

$$[z,\, z_y]_{C^{\alpha/2,\alpha}(\Delta_T)} \le C_1, \tag{2.14}$$

where $C_1$ is independent of $T^{-1}$. This implies

$$\|z_y\|_{C(\Delta_T)} \le \|z_0'(y)\|_{C([-1,1])} + C_1 T^{\alpha/2} \le \|z_0'(y)\|_{C([-1,1])} + C_1 := C_2. \tag{2.15}$$

Thanks to (2.9) and $v_x(t,x) = z_y(t,y)\frac{2}{h(t)-g(t)}$, it yields

$$\|v_x\|_{C(\overline{D}_{g,h}^T)} \le C_2/h_0 =: \bar{C}. \tag{2.16}$$

For $(t,x), (t,\bar{x}) \in \Pi_T$, we set $q(t,x,\bar{x}) = \tilde{u}(t,x) - \tilde{u}(t,\bar{x})$ and $\Omega(x,\bar{x}) = \mathrm{supp}J(x-\cdot) \cup \mathrm{supp}J(\bar{x}-\cdot)$. Noticing $J$ is compactly supported. Then

$$\begin{aligned}
|q(t,x,\bar{x})| \le\ & |q(0,x,\bar{x})| + \int_0^t \Big( d_1 \int_{\Omega(x,\bar{x})} |J(x-\rho) - J(\bar{x}-\rho)| \tilde{u}(s,\rho)\mathrm{d}\rho \\
& + d_1 |q(s,x,\bar{x})| + |f_1(s,x,\tilde{u}(s,x),v(s,x)) - f_1(s,\bar{x},\tilde{u}(s,\bar{x}),v(s,\bar{x}))| \Big)\mathrm{d}s \\
\le\ & \big[L_0 + \big(2d_1 A_1 L(J)|\mathrm{supp}J| + L^* + L\bar{C}\big)T\big]|x-\bar{x}| + \int_0^t (d_1 + L)|q(s,x,\bar{x})|\mathrm{d}s.
\end{aligned}$$

Thanks to Gronwall's inequality, it derives that

$$|\tilde{u}(t,x) - \tilde{u}(t,\bar{x})| \le \bar{L}(T)|x-\bar{x}|, \tag{2.17}$$

where

$$\bar{L}(T) := \big[L_0 + \big(2d_1 A_1 L(J)|\mathrm{supp}J| + L^* + L\bar{C}\big)T\big] e^{(d_1+L)T}.$$

This shows that $\tilde{u} \in C_x^{1-}(\Pi_T)$ and $L(u,T) = \bar{L}(T) \le \bar{L}(1) := \bar{L}$.

Now we define the mapping $\Gamma(u) = \tilde{u}$ and

$$\mathbb{Y}_{u_0}^T = \left\{\phi \in C(\Pi_T) : \phi(0,x) = u_0(x),\ 0 \le \phi \le A_1,\ |\phi(t,x) - \phi(t,y)| \le \bar{L}|x-y|\right\}.$$

Clearly, $\mathbb{Y}_{u_0}^T$ is complete with the metric $d(\phi_1,\phi_2) = \sup_{\Pi_T} |\phi_1 - \phi_2|$ and $\Gamma$ maps $\mathbb{Y}_{u_0}^T$ into itself. We shall prove that $\Gamma$ is a contraction mapping in $\mathbb{Y}_{u_0}^T$ provided $T$ sufficiently small (depends only on $v_0$, $h_0$, $\alpha$, $A_1$, $A_2$, $f_2$ and $\bar{L}$).

*Step 2*: Let $u_1, u_2 \in \mathbb{Y}_{u_0}^T$, $\Gamma(u_i) = \tilde{u}_i$, we shall show that

$$\|\tilde{u}\|_{C_b(\Pi_T)} \le \frac{1}{3}\|u\|_{C_b(\Pi_T)} \ \text{ if } \ 0 < T \ll 1. \tag{2.18}$$

Let $(v_i, g_i, h_i)$ be the unique solution of (2.8) with $u$ replaced by $u_i$. Set $u = u_1 - u_2$, $\tilde{u} = \tilde{u}_1 - \tilde{u}_2$, $v = v_1 - v_2$ and $\Omega_T = D_{g_1,h_1}^T \cup D_{g_2,h_2}^T$. Then

$$|\tilde{u}(t,x)| = \left| \int_0^t d_1 (J * \tilde{u} - \tilde{u})(s,x) + f_1(s,x,\tilde{u}_1,v_1) - f_1(s,x,\tilde{u}_2,v_2)\mathrm{d}s \right|$$

$$\le (2d_1+L)T\|\tilde{u}\|_{C(\Pi_T)} + LT\|v_1 - v_2\|_{C(\bar{\Omega}_T)}, \quad \forall\ (t,x)\in\Pi_T. \tag{2.19}$$

The main content of this step is to estimate $\|v_1 - v_2\|_{C(\bar{\Omega}_T)}$. To this aim, we first gives some useful estimates. Let $x_i(t,y)$, $\xi_i(t)$ and $\zeta_i(t,y)$ be as $x(t,y)$, $\xi(t)$ and $\zeta(t,y)$ with $(g(t),h(t))$ replaced by $(g_i(t),h_i(t))$, and let $z_i(t,y)=v_i(t,x_i(t,y))$. Then $z_i$, $v_i$ satisfy (2.15) and (2.16), respectively. Set $\xi=\xi_1-\xi_2$, $\zeta=\zeta_1-\zeta_2$, and $g=g_1-g_2$, $h=h_1-h_2$. Then

$$\|\xi\|_{L^\infty((0,T))} \le \frac{h_0+\mu A_3}{h_0^4}\|g,\,h\|_{C([0,T])}, \quad \|\zeta\|_{L^\infty(\Delta_T)} \le \frac{h_0+\mu A_3}{h_0^2}\|g,\,h\|_{C^1([0,T])}.$$

Define

$$f_2^i(t,y,u,v)=f_2(t,x_i(t,y),u,v), \quad w_i(t,y)=u_i(t,x_i(t,y)), \quad z_i(t,y)=v_i(t,x_i(t,y)),$$

and set $z=z_1-z_2$, $w=w_1-w_2$. It then follows from (2.11) that

$$\begin{cases} z_t - d_2\xi_1 z_{yy} - \zeta_1 z_y - az = d_2\xi z_{2,yy} + \zeta z_{2,y} + bw + c, & 0<t\le T,\ |y|<1,\\ z(t,\pm1)=0, & 0\le t\le T,\\ z(0,y)=0, & |y|\le 1, \end{cases} \tag{2.20}$$

where

$$\begin{aligned} a=a(t,y) &= \int_0^1 f^1_{2,v}(t,y,w_1,z_2+(z_1-z_2)\tau)\mathrm{d}\tau,\\ b=b(t,y) &= \int_0^1 f^2_{2,u}(t,y,w_2+(w_1-w_2)\tau,z_2)\mathrm{d}\tau,\\ c=c(t,y) &= f_2^1(t,y,w_1,z_2)-f_2^2(t,y,w_1,z_2). \end{aligned}$$

It is easy to see that

$$\|a,\,b\|_{L^\infty(\Delta_T)}\le L, \quad \|c\|_{L^\infty(\Delta_T)} \le L^*\|g,\,h\|_{C([0,T])}.$$

Due to (2.12), (2.13), applying the parabolic $L^p$ theory to (2.20) we can obtain

$$\|z\|_{W_p^{1,2}(\Delta_T)} \le C_3\big(\|g,\,h\|_{C^1([0,T])} + \|w\|_{C(\Delta_T)}\big),$$

where $C_3$ depends on $h_0,\mu,A_i$. The same as (2.14), we have

$$[z]_{C^{\alpha/2,\alpha}(\Delta_T)} + [z_y]_{C^{\alpha/2,\alpha}(\Delta_T)} \le C_4\big(\|g,\,h\|_{C^1([0,T])} + \|w\|_{C(\Delta_T)}\big), \tag{2.21}$$

where $C_4>0$ is independent of $T^{-1}$. When $(t,y)\in\Delta_T$, we have

$$\begin{aligned} |w_1(t,y)-w_2(t,y)| = \ & |u_1(t,x_1(t,y))-u_2(t,x_2(t,y))| \le |u_1(t,x_1(t,y))-u_2(t,x_1(t,y))|\\ & +|u_2(t,x_1(t,y))-u_2(t,x_2(t,y))| \le \|u\|_{C_b(\Pi_T)} + \bar{L}|x_1(t,y)-x_2(t,y)|\\ & \le C_5(\|u\|_{C_b(\Pi_T)} + \|g,h\|_{C([0,T])}), \end{aligned}$$

where $C_5$ depends only on $h_0,\bar{L}$. Therefore, $\|w\|_{C(\Delta_T)} \le C_5\big(\|u\|_{C_b(\Pi_T)} + \|g,h\|_{C([0,T])}\big)$. This combined with (2.21) asserts

$$[z]_{C^{\alpha/2,\alpha}(\Delta_T)} + [z_y]_{C^{\alpha/2,\alpha}(\Delta_T)} \le C_6\big(\|g,\,h\|_{C^1([0,T])} + \|u\|_{C_b(\Pi_T)}\big). \tag{2.22}$$

Notice $z_y(0,1)=0$. The above estimate implies

$$|z_y(t,1)|_{C([0,T])}\le C_6T^{\alpha/2}\big(\|g,\,h\|_{C^1([0,T])}+\|u\|_{C_b(\Pi_T)}\big). \tag{2.23}$$

As $h(0)=g(0)=0$, it is easy to see that

$$|h(t)|\le T\|h'\|_{C([0,T])},\quad |g(t)|\le T\|g'\|_{C([0,T])}. \tag{2.24}$$

Making use of (2.15) and (2.23) we have

$$\begin{aligned}
|h_1'(t)-h_2'(t)| &= \mu|v_{1,x}(t,h_1(t))-v_{2,x}(t,h_2(t))|\\
&= \mu\left|\frac{2[z_{1,y}(t,1)-z_{2,y}(t,1)]}{h_1(t)-g_1(t)}+2z_{2,y}(t,1)\frac{g(t)-h(t)}{[h_1(t)-g_1(t)][h_2(t)-g_2(t)]}\right|\\
&\le \mu\frac{1}{h_0}|z_y(t,1)|+2\mu|z_{2,y}(t,1)|\frac{|h(t)|+|g(t)|}{4h_0^2}\\
&\le C_7T^{\alpha/2}\big(\|g,\,h\|_{C^1([0,T])}+\|u\|_{C_b(\Pi_T)}\big).
\end{aligned}\tag{2.25}$$

Therefore, by use of (2.24),

$$\|h'\|_{C([0,T])}\le C_8T^{\alpha/2}\big(\|g',\,h'\|_{C([0,T])}+\|u\|_{C_b(\Pi_T)}\big).$$

Similarly, we have

$$\|g'\|_{C([0,T])}\le C_8T^{\alpha/2}\big(\|g',\,h'\|_{C([0,T])}+\|u\|_{C_b(\Pi_T)}\big).$$

Consequently, $\|g',\,h'\|_{C([0,T])}\le\|u\|_{C_b(\Pi_T)}$ provided $T$ small enough. Recalling (2.24) we get

$$\|g,\,h\|_{C^1([0,T])}\le C_9\|u\|_{C_b(\Pi_T)}. \tag{2.26}$$

Moreover, as $z(0,y)=0$, we have $|z(t,y)|=|z(t,y)-z(0,y)|\le t^{\alpha/2}[z]_{C^{\alpha/2,\alpha}(\Delta_T)}$ for all $(t,y)\in\Delta_T$. This combined with (2.22) allows us to derive

$$\|z\|_{C(\Delta_T)}\le T^{\alpha/2}[z]_{C^{\alpha/2,\alpha}(\Delta_T)}\le C_6T^{\alpha/2}\big(\|g,\,h\|_{C^1([0,T])}+\|u\|_{C_b(\Pi_T)}\big). \tag{2.27}$$

Now we estimate $\|v_1-v_2\|_{C(\bar\Omega_T)}$. Fixed $(t,x)\in\overline{\Omega}_T$, let

$$y_i(t,x)=\frac{2x-g_i(t)-h_i(t)}{h_i(t)-g_i(t)}.$$

Case 1: $x\in[g_1(t),h_1(t)]\cap[g_2(t),h_2(t)]$. Using (2.27), (2.15) and (2.26), respectively, we have

$$\begin{aligned}
|v_1(t,x)-v_2(t,x)| &= |z_1(t,y_1)-z_2(t,y_2)|\\
&\le |z_1(t,y_1)-z_2(t,y_1)|+|z_2(t,y_1)-z_2(t,y_2)|\\
&\le \|z\|_{C(\Delta_T)}+\|z_{2,y}\|_{C(\Delta_T)}|y_1-y_2|\\
&\le \|z\|_{C(\Delta_T)}+\frac{h_0+\mu A_3}{h_0^2}\|z_{2,y}\|_{C(\Delta_T)}\|g',h'\|_{C([0,T])}\\
&\le C_6T^{\alpha/2}\big(\|g,\,h\|_{C^1([0,T])}+\|u\|_{C_b(\Pi_T)}\big)+\frac{C_2(h_0+\mu A_3)}{h_0^2}\|g',h'\|_{C([0,T])}\\
&\le C_{10}\|u\|_{C_b(\Pi_T)}.
\end{aligned}\tag{2.28}$$

Case 2: $x \in [g_1(t), h_1(t)] \setminus [g_2(t), h_2(t)]$. In this case $v_2(t,x) = 0$. Without loss of generality, we may think of $x \in [g_1(t), g_2(t))$ and $g_2(t) \le h_1(t)$. Take advantage of (2.16) and (2.28), it yields

$$\begin{aligned} |v_1(t,x) - v_2(t,x)| &= |v_1(t,x) - v_2(t, g_2(t))| \\ &\le |v_1(t,x) - v_1(t, g_2(t))| + |v_1(t, g_2(t)) - v_2(t, g_2(t))| \\ &\le \|v_{1,x}\|_{C(\overline{D}^T_{g_1,h_1})} |g_1(t) - g_2(t)| + C_{10}\|u\|_{C_b(\Pi_T)} \\ &\le C_{11}\|u\|_{C_b(\Pi_T)}. \end{aligned}$$

Case 3: $x \in [g_2(t), h_2(t)] \setminus [g_1(t), h_1(t)]$. Similar to Case 2, we still have $|v_1(t,x) - v_2(t,x)| \le C_{12}\|u\|_{C_b(\Pi_T)}$.

In a word, $\|v_1 - v_2\|_{C(\bar\Omega_T)} \le C_{13}\|u\|_{C_b(\Pi_T)}$. Substitute this estimate into (2.19) derives (2.18).

*Step 3*: The estimate (2.18) shows that $\Gamma$ is a contraction mapping in $\mathbb{Y}^T_{u_0}$. Thus, $\Gamma$ has a unique fixed point $u$ in $\mathbb{Y}^T_{u_0}$. Let $(v, g, h)$ be the unique solution of (2.8). Then $(u, v, g, h)$ is a solution of (1.2) and it is the unique one provided $u \in \mathbb{Y}^T_{u_0}$.

To prove the uniqueness of solution of (1.2), we need to prove that $u \in \mathbb{Y}^T_{u_0}$ for any solution $(u, v, g, h)$ of (1.2). Firstly, by virtue of Lemma 2.2 and the parabolic maximum principle for the strong solution we can see $0 < u \le A_1$ in $[0,T] \times \mathbb{R}$ and $0 < v \le A_2$ in $[0,T] \times (g(t), h(t))$. From the above analysis we can see that $u$ satisfies (2.17). Thus $u \in \mathbb{Y}^T_{u_0}$ and uniqueness is proved.

*Step 4*: Global existence and uniqueness. Assume that (2.4) holds. We have known that (1.2) admits a unique solution $(u, v, g, h)$ in some interval $(0,T]$, and $u \in C^{1,1-}(\Pi_T)$, $w \in C^{1,1-}(\Delta_T)$, $g', h' \in C^{\alpha/2}([0,T])$. Thus $\xi \in C^{\alpha/2}([0,T])$, $\zeta \in C^{\alpha/2,\alpha}(\Delta_T)$. Set $F_2(t,y,z) = f_2^*(t, y, w(t,y), z)$. Then $F_2(\cdot,\cdot,z) \in C^{\alpha/2,\alpha}(\Delta_T)$. By the interior Schauder theory we have $z \in C^{1+\alpha/2,2+\alpha}([\varepsilon,T] \times [-1,1])$ for any $0 < \varepsilon < T$, which implies $v(T,x) \in C^2([g(T), h(T)])$. Moreover since $u(T,x)$ satisfies $0 < u \le A_1$ and is Lipschitz continuous in $x \in \mathbb{R}$, we can take $(u(T,x), v(T,x))$ as an initial function and $[g(T), h(T)]$ as the initial habitat of $v$ and then use the above Steps 1-3 to extend the solution from $t = T$ to some $T' > T$. Assume that $(0, T_0)$ is the maximal existence interval of $(u, v, g, h)$ obtained by such extension process. We shall prove that $T_0 = \infty$. Assume on the contrary that $T_0 < \infty$.

Evidently, we have

$$\begin{gathered} 0 < u \le A_1 \ \text{ in } \ [0, T_0) \times \mathbb{R}; \ \ 0 < v \le A_2 \ \text{ in } (0, T_0) \times (g(t), h(t)); \\ 0 < -g'(t), \, h'(t) \le \mu A_3 \ \text{ in } \ (0, T_0). \end{gathered}$$

Set $\Lambda = \{h_0, \ \mu, \ \|v_0\|_{W^2_p(-h_0,h_0)}, \ A_i, \ i = 1,2,3\}$. For any $0 < T < T_0$, applying $L^p$ theory to (2.11) we obtain $\|z\|_{W^{1,2}_p(\Delta_T)} \le C_{14}(\Lambda, T_0)$. This implies that $z \in W^{1,2}_p(\Delta_{T_0})$ and

$$\|z\|_{W^{1,2}_p(\Delta_{T_0})} + \|z\|_{C^{(1+\alpha)/2,1+\alpha}(\Delta_{T_0})} \le C_{15}(\Lambda, T_0),$$

This inequalities combined with the definition of $h'$ and $g'$ yields that $g, h \in C^{1+\alpha/2}([0, T_0])$ and

$$\|g, h\|_{C^{1+\alpha/2}([0,T_0])} \le C_{16}(\Lambda, T_0). \tag{2.29}$$

Moreover, it follows from the above arguments that $v \in C^{(1+\alpha)/2,1+\alpha}(\overline{D}^{T_0}_{g,h})$. These facts show that the first differential equation holds for $0 < t \le T_0$ and $u \in C^{1,1-}(\Pi_{T_0})$. Same as above, we have

$z \in C^{1+\alpha/2,2+\alpha}([\varepsilon, T_0] \times [-1,1])$ and $\|z\|_{C^{1+\alpha/2,\,2+\alpha}([\varepsilon,T_0]\times[-1,1])} \le C_{17}(\varepsilon, \Lambda, T_0)$ for any $0 < \varepsilon < T_0$. Therefore, $v \in C^{1+\alpha/2,2+\alpha}((0, T_0] \times [g(t), h(t)])$ and

$$\|v\|_{C^{1+\alpha/2,2+\alpha}([\varepsilon,T_0]\times[g(t),h(t)])} \le C_{18}(\varepsilon, \Lambda, T_0). \tag{2.30}$$

The above analysis implies that the solution $(u, v, g, h)$ exists on $[0, T_0]$. We choose $t_n \in (\varepsilon, T_0)$ with $t_n \nearrow T_0$, and regard $t_n$ and $(u(t_n, x), v(t_n, x), g(t_n), h(t_n))$ as the initial time and initial datum. According to the arguments of Steps 1-3, we can find a constant $0 < T \ll 1$, which depends only on $\{g(t_n),\ h(t_n),\ g'(t_n),\ h'(t_n),\ \|v(t_n,\cdot)\|_{W_p^2(g(t_n),h(t_n))},\ L(u, T_0),\ \mu,\ A_i,\ i = 1, 2, 3\}$, such that the problem (1.2) has a unique solution $(u_n, v_n, g_n, h_n)$ in $[t_n, t_n + T]$. Due to the uniqueness of solution, $(u, v, g, h) = (u_n, v_n, g_n, h_n)$ for $t_n \le t < \min\{t_n + T, T_0\}$. This indicates that the solution $(u, v, g, h)$ can be extended uniquely to $[0, t_n + T)$. Thanks to (2.29) and (2.30), we can choose $T > 0$ independent of $n$. Hence, $t_n + T > T_0$ when $n$ is large enough. This contradicts with the definition of $T_0$. So $T_0 = \infty$.

It follows from the above arguments that $(g, h) \in \mathbb{G}^T \times \mathbb{H}^T$, $(u, v) \in \mathbb{X}^T$, and $(u, v, g, h)$ satisfies (2.2), (2.3) and (2.5). The proof is end. □

Since $g'(t) < 0$, $h'(t) > 0$, we have the limits $\lim_{t\to\infty} g(t) = g_\infty \ge -\infty$, $\lim_{t\to\infty} h(t) = h_\infty \le \infty$. If $h_\infty - g_\infty = \infty$ we call that $v$ spreading, if $g_\infty > -\infty$ and $h_\infty < \infty$ we call that $v$ vanishing.

## 3 Preliminaries

To establish the long time behaviors of $(u, v)$ and conditions for spreading and vanishing, in this section we will present some preliminaries. The first one focus on the comparison principle.

**Lemma 3.1.** *Let $(f_1, f_2)$ satisfy (1.4) and $(u, v, g, h)$ be the unique solution of (1.2). Let $T > 0$, $\bar g$, $\bar h \in C^1([0, T])$, $\bar u \in C_b(\Pi_T)$, $\bar u_t \in C(\Pi_T)$, $\bar v \in C^{1,2}(D^T_{\bar g,\bar h}) \cap C(\overline{D}^T_{\bar g,\bar h})$ and satisfy*

$$\begin{cases} \bar u_t \ge d_1(J * \bar u - \bar u) + \bar u(a - \bar u), & (t,x) \in (0,T] \times \mathbb{R}, \\ \bar v_t \ge d_2 \bar v_{xx} + \bar v(1 - \bar v + c\bar u), & (t,x) \in D^T_{\bar g,\bar h}, \\ \bar v(t, \bar g(t)) \ge 0, \quad \bar v(t, \bar h(t)) \ge 0, & t \in [0,T], \\ \bar h'(t) \ge -\mu \bar v_x(t, \bar h(t)),\ \bar g'(t) \le -\mu \bar v_x(t, \bar g(t)), & t \in [0,T], \\ \bar u(0,x) \ge u_0(x),\ x \in \mathbb{R};\ \bar v(0,x) \ge v_0(x), & x \in [-h_0, h_0], \\ \bar h(0) \ge h_0, \quad \bar g(0) \le -h_0. \end{cases} \tag{3.1}$$

*Then we have*

$$u \le \bar u \quad \text{in } \Pi_T; \quad g \ge \bar g,\ h \le \bar h \quad \text{in } [0,T]; \quad v \le \bar v \quad \text{in } D^T_{\bar g,\bar h}.$$

Lemma 3.1 can be proved by using Lemma 2.2 and the similar argument as that in proofs of [8, Lemma 4.1] and [18, Lemma 3.1]. We omit the details here.

The following lemma will play an important role in the study of long time behaviors of $(u, v)$ when $v$ vanishes and conditions for spreading and vanishing.

**Lemma 3.2.** ([23, Proposition 2]) *Let $d$, $C$, $\mu$ and $\eta_0$ be positive constants, $\eta \in C^1([0,\infty))$, $w \in W_p^{1,2}((0,T)\times(0,\eta(t)))$ and $w_0 \in W_p^2((0,\eta_0))$ for some $p > 1$ and any $T > 0$, and $w_x \in C([0,\infty)\times(0,\eta(t)])$. If $(w,\eta)$ satisfies*

$$\begin{cases} w_t - dw_{xx} \geq -Cw, & t > 0, \quad 0 < x < \eta(t), \\ w \geq 0, & t > 0, \quad x = 0, \\ w = 0, \quad \eta'(t) \geq -\mu w_x, & t > 0, \quad x = \eta(t), \\ w(0,x) = w_0(x) \geq, \not\equiv 0, & x \in (0,\eta_0), \\ \eta(0) = \eta_0, \end{cases}$$

*and* $\lim\limits_{t\to\infty} \eta(t) = \eta_\infty < \infty$, $\lim\limits_{t\to\infty} \eta'(t) = 0$,

$$\|w(t,\cdot)\|_{C^1([0,\,\eta(t)])} \leq M, \ \ \forall\, t \geq 1$$

*for some constant $M > 0$. Then* $\lim\limits_{t\to\infty} \max\limits_{0\leq x\leq\eta(t)} w(t,x) = 0$.

To study the long time behaviors of $(u,v)$ when $v$ spreading, we shall prove a lemma regarding the estimate of solution of nonlocal diffusion inequality.

**Lemma 3.3.** *Let $K$, $d$ and $\theta$ be positive constants, $w$ be a non-negative continuous function satisfying $w(t,x) \leq K$ for $t \geq 0$ and $x \in \mathbb{R}$. Assume that $u$ satisfies*

$$\begin{cases} u_t = d(J*u-u) + u(\theta - u - w), & (t,x) \in (0,T]\times\mathbb{R}, \\ u(0,x) = u_0(x) \geq, \not\equiv 0 & x \in \mathbb{R}, \end{cases}$$

*where $J$ satisfies* **(J)** *and $u_0(x) \in C_b(\mathbb{R})$. Then the following conclusions hold:*

(i) *If for some constant $m \in [0,K]$,*

$$\liminf_{t\to\infty} w(t,x) \geq m \ \ \text{locally uniformly in } \mathbb{R}, \tag{3.2}$$

*then*

$$\limsup_{t\to\infty} u(t,x) \leq (\theta-m)_+ \ \ \text{locally uniformly in } \mathbb{R},$$

*where $(\theta-m)_+$ is the positive part of $\theta - m$.*

(ii) *If $\theta > M$ and*

$$\limsup_{t\to\infty} w(t,x) \leq M \ \ \text{locally uniformly in } \mathbb{R} \tag{3.3}$$

*for some constant $M$, then*

$$\liminf_{t\to\infty} u(t,x) \geq \theta - M \ \ \text{locally uniformly in } \mathbb{R}.$$

*Proof.* Since we can prove this lemma by applying the methods used in the proof of [12, Lemma 3.14] with some modifications, we just give the outline.

(i) For any integer $n \geq 1$, it follows from (3.2) that there exists $T_n \nearrow \infty$ such that

$$w(t,x) \geq m - 1/n \ \ \text{for } t \geq T_n \ \text{ and } x \in [-n-1, n+1].$$

For any given small $\varepsilon > 0$, define

$$\sigma_n = \begin{cases} \theta - m + 1/n, & \theta - m > 0, \\ \varepsilon + 1/n, & \theta - m \le 0, \end{cases}$$

and

$$a_n(x) = \begin{cases} \sigma_n, & |x| < n, \\ \sigma_n + 2(\theta + K + 1 - \sigma_n)(|x| - n), & n \le |x| \le n + 1/2, \\ \theta + K + 1, & |x| > n + 1/2. \end{cases}$$

Clearly $a_n \in C(\mathbb{R})$, $a - w(t,x) \le a_n(x)$ for $t > T_n$ and $x \in \mathbb{R}$. Moreover, $a_n(x)$ is nonincreasing in $n$, $\sigma_n \le a_n(x) \le \theta + K + 1$ and

$$\lim_{n\to\infty} a_n(x) = \sigma_\infty := \begin{cases} \theta - m, & \theta - m > 0, \\ \varepsilon, & \theta - m \le 0. \end{cases}$$

Let $K_1 := \max\{\theta + K + 1, \|u_0\|_\infty\}$. It follows from Lemma 2.2 that

$$0 \le u(t,x) \le K_1 \quad \text{for} \quad t \ge 0, \quad x \in \mathbb{R}.$$

Let $z_1$ be the unique solution of

$$\begin{cases} z_t = d(J * z - z) + z[a_1(x) - z], & t > T_1,\ x \in \mathbb{R}, \\ z(T_1, x) = K_1, & x \in \mathbb{R}. \end{cases} \tag{3.4}$$

Making use of Lemma 2.2, we have

$$\sigma_\infty \le z_1(t,x) \le K_1, \quad z_1(t,x) \ge u(t,x), \quad (t,x) \in [T_1, \infty) \times \mathbb{R}.$$

For $n \ge 2$, let $z_n$ be the unique solution of

$$\begin{cases} z_t = d(J * z - z) + z[a_n(x) - z], & t > T_n,\ x \in \mathbb{R}, \\ z(T_n, x) = z_{n-1}(T_n, x), & x \in \mathbb{R}. \end{cases} \tag{3.5}$$

By virtue of Lemma 2.2 again, it follows that

$$\sigma_\infty \le z_n(t,x) \le K_1, \quad z_n(t,x) \ge u(t,x), \quad (t,x) \in [T_n, \infty) \times \mathbb{R}.$$

Due to $a_n(x) \ge \sigma_\infty > 0$ for $x \in \mathbb{R}$ and [12, Proposition 3.12], we can see that (3.4) and (3.5) admit a unique positive steady state $\widetilde{z}_n \in C(\mathbb{R})$:

$$d \int_{\mathbb{R}} J(x-y)\widetilde{z}_n(y)dy - d\widetilde{z}_n + \widetilde{z}_n(a_n(x) - \widetilde{z}_n) = 0,\ x \in \mathbb{R}, \tag{3.6}$$

and

$$\lim_{t\to\infty} z_n(t,x) = \widetilde{z}_n(x) \ \text{ locally uniformly in } \mathbb{R}. \tag{3.7}$$

Obviously, we have $\sigma_\infty \le \widetilde{z}_n(x) \le K_1$ for $x \in \mathbb{R}$. Moreover, it follows from the monotonicity of $a_n(x)$ in $n$ that $z_{n+1}(t,x) \le z_n(t,x)$ for $t \ge T_{n+1}$ and $x \in \mathbb{R}$. Then $\widetilde{z}_{n+1}(x) \le \widetilde{z}_n(x)$ for $x \in \mathbb{R}$. Therefore, there exists $\widetilde{z}_\infty(x)$ such that

$$\lim_{n\to\infty} \widetilde{z}_n(x) = \widetilde{z}_\infty(x) \ \text{ for every } x \in \mathbb{R},$$

where $\widetilde{z}_\infty(x)$ satisfies $\sigma_\infty \le \widetilde{z}_\infty(x) \le K_1$ in $\mathbb{R}$. Then similar to the arguments in the proof of [12, Lemma 3.14], we can obtain the desired result.

(ii) Due to the equations of $u$, it is easy to prove conclusion (ii). So the details are omitted here. □

**Proposition 3.4.** ([18, Proposition B.1 and B.2]) *Let $d$, $\beta$, $\zeta$ be fixed positive constants, and $k$ be a fixed non-negative constant. For any given $\varepsilon, N > 0$, there exist $T_\varepsilon > 0$ and $l_\varepsilon > \max\left\{N, \frac{\pi}{2}\sqrt{d/\beta}\right\}$, such that when the continuous and non-negative function $z$ satisfies*

$$\begin{cases} z_t - dz_{xx} \ge \ (\le)\, z(\beta - \zeta z), & t > 0, \quad -l_\varepsilon < x < l_\varepsilon, \\ z(0,x) > 0, & \qquad\quad\ -l_\varepsilon < x < l_\varepsilon, \end{cases}$$

*and for $t > 0$, $z(t, \pm l_\varepsilon) \ge \ (\le)\, k$ if $k > 0$, while $z(t, \pm l_\varepsilon) \ge \ (=)\, 0$ if $k = 0$, we must have*

$$z(t,x) > \beta/\zeta - \varepsilon \ \big(z(t,x) < \beta/\zeta + \varepsilon\big), \quad \forall\ t \ge T_\varepsilon, \quad x \in [-N, N].$$

*This implies*

$$\liminf_{t\to\infty} z(t,x) \ge \beta/\zeta - \varepsilon \ \ \left(\limsup_{t\to\infty} z(t,x) < \beta/\zeta + \varepsilon\right) \quad \textit{uniformly on } [-N,N].$$

# 4 Longtime behavior of $(u, v)$

## 4.1 Vanishing case $(h_\infty - g_\infty < \infty)$

Firstly, we shall use Lemma 3.2 to deduce $\lim_{t\to\infty} \max_{g(t)\le x\le h(t)} v(t,x) = 0$. To this purpose, we first provide an estimate for the solution component $v$.

**Lemma 4.1.** *Let $(u, v, g, h)$ be the unique global solution of (1.2) and $h_\infty - g_\infty < \infty$. Then there exists a constant $C > 0$ such that*

$$\|v(t,\cdot)\|_{C^1([g(t),h(t)])} \le C, \quad \forall\ t \ge 1; \qquad \lim_{t\to\infty} g'(t) = \lim_{t\to\infty} h'(t) = 0.$$

This lemma can be proved by the similar arguments to those of [26, Theorem 2.1] and [27, Theorem 2.2], we omit the details here.

**Theorem 4.2.** *Let $f_1, f_2$ satisfy one of (1.4), (1.5), (1.6). If $h_\infty - g_\infty < \infty$, then the solution $(u, v, g, h)$ of (1.2) satisfies $\lim_{t\to\infty} \|v(t,\cdot)\|_{C([g(t),h(t)])} = 0$, and*

$$\begin{aligned} &\lim_{t\to\infty} u(t,x) = a \ \text{ locally uniformly in } \mathbb{R} \ \text{ if } f_1, f_2 \text{ satisfy (1.4) } \textit{or } (1.5), \\ &\lim_{t\to\infty} u(t,x) = a/2 \ \text{ locally uniformly in } \mathbb{R} \ \text{ if } f_1, f_2 \text{ satisfy (1.6).} \end{aligned}$$

*Proof.* As $0 < u \le A_1$, $0 < v \le A_2$, we have $f_2(t,x,u,v) \ge -Lv$. Then $\lim_{t\to\infty} \|v(t,\cdot)\|_{C([g(t),h(t)])} = 0$ is deduced by Lemmas 3.2 and 4.1 immediately.

We only prove $\lim_{t\to\infty} u(t,x) = a$ locally uniformly in $\mathbb{R}$ when $f_1, f_2$ satisfy (1.4). Firstly, by the comparison principle, $\limsup_{t\to\infty} u(t,x) \le a$ uniformly in $\mathbb{R}$. Secondly, as $\lim_{t\to\infty} \|v(t,\cdot)\|_{C([g(t),h(t)])} = 0$ and $v \equiv 0$ for $x \in \mathbb{R}/(g(t),h(t))$, it follows that $\lim_{t\to\infty} v(t,x) = 0$ uniformly in $\mathbb{R}$. This combined with Lemma 3.3 arrives at $\liminf_{t\to\infty} u \ge a$ locally uniformly in $\mathbb{R}$. The proof is finished. □

## 4.2 Spreading case ($h_\infty - g_\infty = \infty$)

### 4.2.1 The competition model

In this part we always suppose that $f_1, f_2$ satisfy (1.4).

**Theorem 4.3.** *Assume $ac < 1$. Then $h_\infty - g_\infty = \infty$ if and only if $h_\infty = \infty$ and $g_\infty = -\infty$.*

The proof of this lemma is similar to those of [18, Proposition 4.1] and [23, Proposition 3], and we omit the details.

**Theorem 4.4.** *Assume $h_\infty - g_\infty = \infty$. For the weak competition case: $ac < 1 < a/b$ we have*

$$\lim_{t\to\infty} u = (a-b)/(1-bc), \quad \lim_{t\to\infty} v = (1-ac)/(1-bc)$$

*locally uniformly in $\mathbb{R}$.*

*Proof.* **Step 1**. It is easy to show that

$$\limsup_{t\to\infty} u \le a =: \bar{u}_1 \ \text{ uniformly in } \ \mathbb{R} \tag{4.1}$$

by the comparison principle. For any given $N > 0$ and $0 < \varepsilon, \sigma \ll 1$, let $l_\varepsilon$ be determined in Proposition 3.4 with $d = d_2$, $\beta = 1 - c(\bar{u}_1 + \sigma)$ and $\zeta = 1$. In view of $h_\infty = \infty$, $g_\infty = -\infty$ and (4.1), there exists $T_1 > 0$ such that

$$u(t,x) \le \bar{u}_1 + \sigma, \ g(t) < -l_\varepsilon, \ h(t) > l_\varepsilon, \quad \forall\, t \ge T_1, \ x \in [-l_\varepsilon, l_\varepsilon].$$

Hence, $v$ satisfies

$$\begin{cases} v_t \ge d_2 v_{xx} + v(1 - c(\underline{u}_1 + \sigma) - v), & t > T_1, \ x \in [-l_\varepsilon, l_\varepsilon], \\ v(t, \pm l_\varepsilon) \ge 0 & t \ge T_1. \end{cases}$$

As $v(T_1, x) > 0$ in $[-l_\varepsilon, l_\varepsilon]$, it deduces by Proposition 3.4 that $\liminf_{t\to\infty} v \ge 1 - c(\bar{u}_1 + \sigma) + \varepsilon$ uniformly in $[-N, N]$. The arbitrariness of $\varepsilon$, $\sigma$ and $N$ assert

$$\liminf_{t\to\infty} v \ge 1 - c\bar{u}_1 =: \underline{v}_1 \ \text{ locally uniformly in } \ \mathbb{R}.$$

It follows from $ac < 1 < a/b$ that $a - b\underline{v}_1 > 0$. Making use of Lemma 3.3, we have

$$\limsup_{t\to\infty} u \le a - b\underline{v}_1 =: \bar{u}_2 \ \text{ locally uniformly in } \ \mathbb{R}.$$

Clearly, $1 - c\bar{u}_2 > 0$. By using Proposition 3.4 again, we see that

$$\liminf_{t\to\infty} v \geq 1 - c\bar{u}_2 =: \underline{v}_2 \text{ locally uniformly in } \mathbb{R}.$$

The assumption $ac < 1 < a/b$ implies that $a - b\underline{v}_2 > 0$. Similarly, we have

$$\limsup_{t\to\infty} u \leq a - b\underline{v}_2 =: \bar{u}_3 \text{ locally uniformly in } \mathbb{R}.$$

**Step 2**. We can repeat the above procedure and get two sequences $\{\bar{u}_k\}$ and $\{\underline{v}_k\}$ such that

$$\limsup_{t\to\infty} u \leq \bar{u}_k, \quad \liminf_{t\to\infty} v \geq \underline{v}_k \quad \text{locally uniformly in } \mathbb{R}.$$

Using the inductive method we have the following expressions:

$$\bar{u}_{k+1} = (a-b)(1+q+q^2+\cdots+q^{k-1}) + aq^k, \quad \underline{v}_k = 1 - c\bar{u}_k, \quad k \geq 1,$$

where $q = bc < 1$. Thus, $\bar{u}_k \to (a-b)/(1-bc)$ and $\underline{v}_k \to (1-ac)/(1-bc)$ as $k \to \infty$. Thus we have

$$\limsup_{t\to\infty} u \leq (a-b)/(1-bc), \quad \liminf_{t\to\infty} v \geq (1-ac)/(1-bc) \text{ locally uniformly in } \mathbb{R}.$$

**Step 3**. It is easy to see that $\limsup_{t\to\infty} v \leq 1 =: \bar{v}_1$ uniformly in $\mathbb{R}$. Similar to Step 1-2, by virtue of Lemma 3.3 and Proposition 3.4 we can find two sequences $\{\underline{u}_k\}$ and $\{\bar{v}_k\}$ such that

$$\liminf_{t\to\infty} u \geq \underline{u}_k, \quad \limsup_{t\to\infty} v \leq \bar{v}_k \text{ locally uniformly in } \mathbb{R},$$

and

$$\bar{v}_k = (1-ac)(1+q+q^2+\cdots+q^{k-2}) + q^{k-1}, \quad \underline{u}_k = a - b\bar{v}_k,$$

where $q = bc < 1$ and $k \geq 2$. Take $k \to \infty$ and we have

$$\liminf_{t\to\infty} u \geq (a-b)/(1-bc), \quad \limsup_{t\to\infty} v \leq (1-ac)/(1-bc) \text{ locally uniformly in } \mathbb{R}.$$

Combining with our early conclusion, we complete the proof. □

**Theorem 4.5.** *Assume $h_\infty = -g_\infty = \infty$. For the strong competition case: $ac \geq 1 > b/a$ we have*

$$\lim_{t\to\infty} u(t,x) = a, \quad \lim_{t\to\infty} v(t,x) = 0 \quad \text{locally uniformly in } \mathbb{R}.$$

*Proof.* The idea of this proof comes from [9, Theorem 2.4]. It follows from the comparison principle that $\limsup\limits_{t\to\infty} v \leq 1 =: \bar{v}_1$ uniformly in $\mathbb{R}$. Applying Lemma 3.3, similar to the above we can get

$$\liminf_{t\to\infty} u \geq a - b\bar{v}_1 = a - b =: \underline{u}_1 \text{ locally uniformly in } \mathbb{R}.$$

If $1 - c\underline{u}_1 \leq 0$, then $\underline{u}_1 - 1/c \geq 0$. For any given $N > 0$, $\sigma > \underline{u}_1 - 1/c$ and $0 < \varepsilon \ll 1$, using Proposition 3.4 with $d = d_2$, $\beta = 1 - c(\underline{u}_1 - \sigma)$ and $\zeta = 1$, we can deduce that, similar to the above in the proof of Theorem 4.4,

$$\limsup_{t\to\infty} v < 1 - c(\underline{u}_1 - \sigma) + \varepsilon \text{ uniformly in } [-N, N].$$

By the arbitrariness of $\varepsilon$, $\sigma$, $N$ and $v \geq 0$, it yields that

$$\lim_{t\to\infty} v = 0 \text{ locally uniformly in } \mathbb{R}.$$

If $1 - c\underline{u}_1 > 0$, apply Proposition 3.4 and Lemma 3.3 to $v$ and $u$, respectively, it follows that

$$\limsup_{t\to\infty} v \leq 1 - c\underline{u}_1 =: \bar{v}_2, \quad \liminf_{t\to\infty} u \geq a - b\bar{v}_2 =: \underline{u}_2 \text{ locally uniformly in } \mathbb{R}.$$

Similar to the above, it can be shown that if $1 - c\underline{u}_2 \leq 0$ then

$$\lim_{t\to\infty} v = 0 \text{ locally uniformly in } \mathbb{R},$$

if $1 - c\underline{u}_2 > 0$ then

$$\limsup_{t\to\infty} v \leq 1 - c\underline{u}_2 := \bar{v}_3 \text{ locally uniformly in } \mathbb{R}.$$

Repeating this process we know that if there exists a first $k \geq 1$ such that $c\underline{u}_k \geq 1$, then $\lim_{t\to\infty} v = 0$ locally uniformly in $\mathbb{R}$. Similar to the proof of Theorem 4.2, $\lim_{t\to\infty} u = a$ locally uniformly in $\mathbb{R}$.

If $c\underline{u}_k < 1$ for all $k \geq 1$, then

$$\liminf_{t\to\infty} u \geq \underline{u}_k, \quad \limsup_{t\to\infty} v \leq \bar{v}_k \text{ locally uniformly in } \mathbb{R}.$$

The inductive method shows

$$c\underline{u}_k = ac(1 + q + q^2 + \cdots + q^{k-1}) - (q + q^2 + \cdots + q^k),$$

where $q = bc$. Since $0 < c\underline{u}_k < 1$ for $k \geq 1$, we can see that $bc < 1$. Moreover,

$$\bar{v}_k = (1 - ac)\frac{1 - q^{k-1}}{1 - q} + q^{k-1} > 0, \quad \forall\ k \geq 1.$$

As $ac \geq 1$ and $\bar{v}_k > 0$, it must be derived that $ac = 1$ by taking $k \to \infty$, and hence $\lim_{t\to\infty} \bar{v}_k = 0$. Recall $v \geq 0$, we have $\lim_{t\to\infty} v = 0$ and then $\lim_{t\to\infty} u = a$ locally uniformly in $\mathbb{R}$. □

Similarly, the following conclusion can be proved by use of Lemma 3.3 and Proposition 3.4.

**Theorem 4.6.** *If $ac < 1 \leq b/a$ and $h_\infty - g_\infty = \infty$, then*

$$\lim_{t\to\infty} u = 0, \quad \lim_{t\to\infty} v = 1 \text{ locally uniformly in } \mathbb{R}.$$

### 4.2.2 The predator-prey model

In what follows we always suppose that $f_1, f_2$ satisfy (1.5).

**Theorem 4.7.** *$h_\infty - g_\infty = \infty$ if and only if $h_\infty = \infty$ and $g_\infty = -\infty$.*

The proof of Theorem 4.7 is similar to that of [18, Proposition 4.1], and we omit the details here.

**Theorem 4.8.** *Assume that $h_\infty - g_\infty = \infty$. For the weakly hunting case: $a > b$, $bc < 1$ we have*

$$\lim_{t\to\infty} u(t,x) = (a-b)/(1+bc), \quad \lim_{t\to\infty} v(t,x) = (1+ac)/(1+bc) \text{ locally uniformly in } \mathbb{R}.$$

*Proof.* The idea of this proof comes from [18, Theorem 4.3]. We just give the outline and omit the details.

**Step 1**. It can be derived from Proposition 3.4 that $\liminf_{t\to\infty} v \geq 1 =: \underline{v}_1$ locally uniformly in $\mathbb{R}$. Taking advantage of $a - b\underline{v}_1 > 0$ and Lemma 3.3 we have $\limsup_{t\to\infty} u \leq a - b\underline{v}_1 =: \bar{u}_1$ locally uniformly in $\mathbb{R}$. Then making use of Proposition 3.4 again one have $\limsup_{t\to\infty} v \leq 1 + c\bar{u}_1 =: \bar{v}_1$ locally uniformly in $\mathbb{R}$. The assumption $bc < 1 < a/b$ implies that $a - b\bar{v}_1 > 0$. By means of Lemma 3.3 repeatedly, we have $\liminf_{t\to\infty} u \geq a - b\bar{v}_1 =: \underline{u}_1$ locally uniformly in $\mathbb{R}$. Similarly it follows from Proposition 3.4 that $\liminf_{t\to\infty} v \geq 1 + c\underline{u}_1 =: \underline{v}_2$ locally uniformly in $\mathbb{R}$.

**Step 2**. Repeating the above procedure, we can obtain four sequences $\{\underline{v}_k\}$, $\{\bar{u}_k\}$, $\{\bar{v}_k\}$ and $\{\underline{u}_k\}$ such that

$$\underline{u}_k \leq \liminf_{t\to\infty} u \leq \limsup_{t\to\infty} u \leq \bar{u}_k, \quad \underline{v}_k \leq \liminf_{t\to\infty} v \leq \limsup_{t\to\infty} v \leq \bar{v}_k$$

locally uniformly in $\mathbb{R}$. By direct calculations we have

$$\lim_{k\to\infty} \underline{u}_k = \lim_{k\to\infty} \bar{u}_k = (a-b)/(1+bc), \quad \lim_{k\to\infty} \underline{v}_k = \lim_{k\to\infty} \bar{v}_k = (1+ac)/(1+bc).$$

Thus the proof is completed. □

**Theorem 4.9.** *Assume $h_\infty - g_\infty = \infty$. For the strongly hunting case: $b \geq a$ we have*

$$\lim_{t\to\infty} u(t,x) = 0, \quad \lim_{t\to\infty} v(t,x) = 1 \ \text{ locally uniformly in } \ \mathbb{R}.$$

The proof of this theorem is similar to that of Theorem 4.5, we omit the details here.

### 4.2.3 The mutualist model

In what follows we always suppose that $f_1, f_2$ satisfy (1.6).

**Lemma 4.10.** *$h_\infty - g_\infty = \infty$ if and only if $h_\infty = \infty$ and $g_\infty = -\infty$.*

*Proof.* Since $r_2 v\left(1 - v - \frac{v}{1+cu}\right) \geq r_2 v\,(1-2v)$, this lemma can be proved by similar arguments with Theorem 4.7. We omit the details here. □

Let $(u^*, v^*)$ is the unique positive solution of the problem:

$$\begin{cases} a - u - \dfrac{u}{1+bv} = 0, \\ 1 - v - \dfrac{v}{1+cu} = 0. \end{cases}$$

**Theorem 4.11.** *Assume $h_\infty - g_\infty = \infty$. Then we have*

$$\lim_{t\to\infty} u(t,x) = u^*, \quad \lim_{t\to\infty} v(t,x) = v^* \ \text{ locally uniformly in } \ \mathbb{R}.$$

*Proof.* **Step 1**. Since this proof is similar to the proof of [28, Theorem 4.3], we omit the details and give the sketch. Firstly, it is easily seen that $\limsup_{t\to\infty} u(t,x) \leq a =: \bar{u}_1$ uniformly in $\mathbb{R}$. Then any $0 < \varepsilon \ll 1$ there exists $T_\varepsilon > 0$ such that $u \leq \bar{u}_1 + \varepsilon$ for $t > T_\varepsilon$ and $x \in \mathbb{R}$. Hence $v$ satisfies

$$v_t \leq d_2 v_{xx} + r_2 v\Big(1 - v - \frac{v}{1 + c(\bar{u}_1 + \varepsilon)}\Big), \quad t > T_\varepsilon, \ x \in (g(t), \ h(t)).$$

It follows from comparison principle and the arbitrariness of $\varepsilon$ that

$$\limsup_{t\to\infty} v(t,x) \le \frac{1}{(1+c\bar{u}_1)^{-1}+1} =: \bar{v}_1 \ \text{ uniformly in } \mathbb{R}.$$

Repeating the above procedure we have

$$\limsup_{t\to\infty} u(t,x) \le \frac{a}{(1+b\bar{v}_k)^{-1}+1} =: \bar{u}_{k+1} \ \text{ uniformly in } \mathbb{R}$$

and

$$\limsup_{t\to\infty} v(t,x) \le \frac{1}{(1+c\bar{u}_k)^{-1}+1} =: \bar{v}_k \ \text{ uniformly in } \mathbb{R}.$$

**Step 2**. Moreover, Since $u$ satisfies

$$u_t \ge d_1(J*u-u)+r_1u(a-2u), \quad t>0, \ x\in\mathbb{R},$$

it follows from Lemma 3.3 that $\liminf_{t\to\infty} u(t,x) \ge a/2 =: \underline{u}_1$ locally uniformly in $\mathbb{R}$. Making use of Proposition 3.4, we have

$$\liminf_{t\to\infty} v(t,x) \ge \frac{1}{(1+c\underline{u}_1)^{-1}+1} =: \underline{v}_1 \ \text{ locally uniformly in } \mathbb{R}.$$

Then similar to Lemma 3.3 (ii) we have

$$\liminf_{t\to\infty} u(t,x) \ge \frac{a}{(1+b\underline{v}_1)^{-1}+1} =: \underline{u}_2 \ \text{ locally uniformly in } \mathbb{R}.$$

Repeating the above procedure arrives at

$$\liminf_{t\to\infty} u(t,x) \ge \frac{a}{(1+b\underline{v}_k)^{-1}+1} =: \underline{u}_{k+1} \ \text{ locally uniformly in } \mathbb{R},$$

and

$$\liminf_{t\to\infty} v(t,x) \ge \frac{1}{(1+c\underline{u}_k)^{-1}+1} =: \underline{v}_k \ \text{ locally uniformly in } \mathbb{R}.$$

**Step 3**. From the above arguments we have

$$\underline{u}_k \le \liminf_{t\to\infty} u(t,x) \le \limsup_{t\to\infty} u(t,x) \le \bar{u}_k \ \text{ locally uniformly in } \mathbb{R},$$

and

$$\underline{v}_k \le \liminf_{t\to\infty} v(t,x) \le \limsup_{t\to\infty} v(t,x) \le \bar{v}_k \ \text{ locally uniformly in } \mathbb{R}.$$

By the similar arguments with [28, Theorem 4.3], we have

$$\lim_{k\to\infty} \underline{u}_k = \lim_{k\to\infty} \bar{u}_k = u^*, \quad \lim_{k\to\infty} \underline{v}_k = \lim_{k\to\infty} \bar{v}_k = v^*.$$

The proof is finished. □

# 5 The criteria governing spreading and vanishing

To study the criteria governing spreading and vanishing, we first give one abstract lemma to affirm that the habitat can be large provided that the moving parameter of free boundary is large enough.

**Lemma 5.1.** ([23, Lemma 4.2]) *Let $C$ be a positive constant. For any given positive constants $r_0, H$, and any function $w_0 \in W_p^2((-r_0, r_0))$ with $p > 1$, $w_0(\pm r_0) = 0$ and $w_0 > 0$ in $(-r_0, r_0)$, there exists $\mu^0 > 0$ such that when $\mu \geq \mu^0$ and $(w, l, r)$ satisfies*

$$\begin{cases} w_t - w_{xx} \geq -Cw, & t > 0,\ l(t) < x < r(t), \\ w = 0,\ l'(t) \leq -\mu w_x, & t \geq 0,\ x = l(t), \\ w = 0,\ r'(t) \geq -\mu w_x, & t \geq 0,\ x = r(t), \\ w(0, x) = w_0(x), & -r_0 \leq x \leq r_0, \\ r(0) = -l(0) = r_0, \end{cases}$$

*we must have $\lim_{t\to\infty} l(t) \leq -H$, $\lim_{t\to\infty} r(t) \geq H$.*

## 5.1 The competition model

In this subsection we always suppose that $f_1, f_2$ satisfy (1.4) and $ac < 1$.

**Lemma 5.2.** *If $h_\infty - g_\infty < \infty$, then $h_\infty - g_\infty \leq \pi\sqrt{d_2/(1 - ac)} =: \Lambda_c$.*

Since the proof is similar to [18, Theorem 5.1] and [23, Theorem 4.1], we omit the details.

From Lemma 5.2 and $g'(t) < 0, h'(t) > 0$ for $t > 0$, we have

**Corollary 5.3.** *If $h_0 \geq \Lambda_c/2$, then spreading happens, i.e., $h_\infty - g_\infty = +\infty$.*

**Lemma 5.4.** *If $h_0 < \Lambda_c/2$, there exists $\mu^0 > 0$ such that $h_\infty = \infty$ and $g_\infty = -\infty$ when $\mu \geq \mu^0$.*

Since $1 - v - cu$ is bounded and $v \geq 0$, we can prove Lemma 5.4 by using Lemma 5.1 and Lemma 5.2, and the details are omitted here.

**Lemma 5.5.** *Define $\Lambda_c^* := \pi\sqrt{d_2}$. If $h_0 < \Lambda_c^*/2$, then there exists $\mu_0 > 0$ such that $h_\infty - g_\infty < \infty$ when $0 < \mu \leq \mu_0$.*

*Proof.* Clearly, $(v, g, h)$ is a lower solution of the problem

$$\begin{cases} \bar{v}_t - d_2\bar{v}_{xx} = \bar{v}(1 - \bar{v}), & t > 0,\ \bar{g}(t) < x < \bar{h}(t), \\ \bar{v}(t, x) = 0, & t > 0,\ x = \bar{g}(t),\ \bar{h}(t), \\ \bar{g}'(t) = -\mu\bar{v}_x(t, \bar{g}(t)), & t > 0, \\ \bar{h}'(t) = -\mu\bar{v}_x(t, \bar{h}(t)), & t > 0, \\ \bar{v}(0, x) = v_0(x), & x \in [-h_0, h_0], \\ \bar{h}(0) = -\bar{g}(0) = h_0. \end{cases} \tag{5.1}$$

Thus $g(t) \geq \bar{g}(t)$, $h(t) \leq \bar{h}(t)$ for all $t \geq 0$. Since $h_0 < \Lambda_c^*/2$, it follows from [5, Lemma 5.10] that there exists $\mu_0 > 0$ such that $\bar{h}_\infty - \bar{g}_\infty \leq \pi\sqrt{d_2}$ when $0 < \mu \leq \mu_0$. Hence $h_\infty - g_\infty < \infty$. $\square$

**Lemma 5.6.** *Suppose that* $h_0 < \Lambda_c^*/2$, *then there exist* $\mu^* \geq \mu_* > 0$, *such that* $h_\infty = \infty$ *and* $g_\infty = -\infty$ *if* $\mu > \mu^*$, *and* $h_\infty - g_\infty < \infty$ *if* $\mu \leq \mu_*$ *or* $\mu = \mu^*$.

*Proof.* Taking advantage of Lemmas 5.4 and 5.5, we can prove this conclusion by the same manner as that of [18, Theorem 5.2]. The details are omitted here. □

From the above discussion we immediately obtain the following spreading-vanishing dichotomy and criteria for spreading and vanishing.

**Theorem 5.7.** (*The competition model*) *Let* $(u, v, g, h)$ *be the unique solution of* (1.2) *with* $f_1, f_2$ *satisfy* (1.4). *Then the following alternative holds:*

*Either*

(i) *Spreading:* $h_\infty - g_\infty = \infty$. *If we further assume that* $a > b$ *and* $ac < 1$, *then we have* $h_\infty - g_\infty = \infty$ *if and only if* $h_\infty = -g_\infty = \infty$, *and*

$$\lim_{t\to\infty} u = (a-b)/(1-bc), \quad \lim_{t\to\infty} v = (1-ac)/(1-bc) \text{ locally uniformly in } \mathbb{R}.$$

*or*

(ii) *Vanishing:* $h_\infty - g_\infty < \infty$ *and*

$$\lim_{t\to\infty} u(t,x) = a \text{ locally uniformly in } \mathbb{R}, \quad \lim_{t\to\infty} \|v(t,\cdot)\|_{C([g(t),h(t)])} = 0.$$

*Moreover,*

(iii) *If* $h_0 \geq \Lambda_c/2$, *then* $h_\infty = -g_\infty = \infty$ *for all* $\mu > 0$.

(iv) *If* $h_0 < \Lambda_c/2$, *then there exists* $\mu^0 > 0$ *such that* $h_\infty = -g_\infty = \infty$ *when* $\mu \geq \mu^0$.

(v) *If* $h_0 < \Lambda_c^*/2$, *then there exist* $\mu^* \geq \mu_* > 0$ *such that* $h_\infty = -g_\infty = \infty$ *if* $\mu > \mu^*$, *and* $h_\infty - g_\infty < \infty$ *if* $\mu \leq \mu_*$ *or* $\mu = \mu^*$.

## 5.2 The predator-prey model

In this part we always suppose that $f_1, f_2$ satisfy (1.5).

**Lemma 5.8.** *If* $h_\infty - g_\infty < \infty$, *then* $h_\infty - g_\infty \leq \pi\sqrt{d_2/(1+ac)} =: \Lambda_p$.

Since the proof is similar to [18, Theorem 5.1] and [23, Theorem 4.1], we omit the details.

**Corollary 5.9.** *If* $h_0 \geq \Lambda_p/2$, *then spreading happens, i.e.,* $h_\infty - g_\infty = \infty$.

**Lemma 5.10.** *If* $h_0 < \Lambda_p/2$, *then there exists* $\bar{\mu} > 0$ *such that* $h_\infty - g_\infty = \infty$ *when* $\mu \geq \bar{\mu}$.

The proof of Lemma 5.10 is similar to that of Lemma 5.4. So we omit the details.

**Lemma 5.11.** *If* $h_0 < \Lambda_p/2$, *then there exists* $\underline{\mu} > 0$ *such that* $h_\infty - g_\infty < \infty$ *when* $0 < \mu \leq \underline{\mu}$.

*Proof.* We can prove this conclusion by follow the proof of [18, Lemma 5.2]. For the convenience of readers, we shall give the outline here.

Define functions

$$\bar{u}(t) = ae^{at}\left(e^{at} - 1 + \frac{a}{\|u_0\|_\infty}\right)^{-1},$$

and

$$
\begin{aligned}
f(t) &= M \exp\left\{\int_0^t \left[1 + c\bar{u}(s) - d_2\left(\frac{\pi}{2\zeta}\right)^2\right] \mathrm{d}s\right\},\\
\eta(t) &= \left(h_0^2(1+\delta)^2 + \mu\pi\int_0^t f(s)\,\mathrm{d}s\right)^{1/2}, \quad w(y) = \cos\frac{\pi y}{2},\\
\bar{v}(t,x) &= f(t)w(\frac{x}{\eta(t)}), \quad t \geq 0,\ -\eta(t) \leq x \leq \eta(t),
\end{aligned}
$$

where $\zeta = \frac{1}{2}h_0 + \frac{1}{4}\Lambda_p$, $\delta$ is a fixed positive constant such that $\zeta > h_0(1+\delta)$ and $M$ is a positive constant to be determined later. Direct calculation shows that

$$
\frac{f'(t)}{f(t)} = 1 + c\bar{u}(t) - d_2\left(\frac{\pi}{2\zeta}\right)^2 \quad \text{for } \ t > 0.
$$

Since $\zeta < \Lambda_p/2$ and $\lim\limits_{t\to\infty}\bar{u}(t) = a$, we have $1 + c\bar{u}(t) - d_2\left(\pi/(2\zeta)\right)^2 < 0$ when $t$ is large enough. Thus the integration $\int_0^\infty f(t)\,\mathrm{d}t$ is convergent. Define

$$
\underline{\mu} = \frac{\zeta^2 - h_0^2(1+\delta)^2}{\pi\int_0^\infty f(t)\,\mathrm{d}t}.
$$

To apply Lemma 3.1, we need to verify that $(\bar{u}, \bar{v}, -\eta(t), \eta(t))$ satisfies all the inequalities of (3.2) when $0 < \mu \leq \underline{\mu}$. Clearly, $\bar{u}$ satisfies

$$
\begin{cases}
\bar{u}_t \geq d_1(J * \bar{u} - \bar{u}) + \bar{u}(a - \bar{u}), & t > 0,\ x \in \mathbb{R},\\
\bar{u}(0,x) \geq u_0(x), & x \in \mathbb{R},
\end{cases}
$$

and $\eta(t) > 0$ for $t \geq 0$. Moreover, since $\zeta \geq \eta(t)$ for $t \geq 0$, it follows that

$$
\begin{aligned}
\bar{v}_t - d_2\bar{v}_{xx} - \bar{v}(1 - \bar{v} + c\bar{u}) &= f'w - fw'\frac{x\eta'}{\eta^2} + d_2(\frac{\pi}{2\eta})^2 fw - fw(1 - fw + c\bar{u})\\
&\geq fw\left[\frac{f'}{f} + d_2(\frac{\pi}{2\eta})^2 - 1 - c\bar{u}\right]\\
&= \frac{d_2\pi^2}{4}fw(\eta^{-2} - \zeta^{-2}) \geq 0.
\end{aligned}
$$

for $t > 0$ and $-\eta(t) < x < \eta(t)$. Recalling the definition of $\eta$ and $\bar{v}$, it is easily seen that

$$
-\eta'(t) = -\mu\bar{v}_x(t, -\eta(t)), \quad \eta'(t) = -\mu\bar{v}_x(t, \eta(t)).
$$

Choose $M > 0$ such that $v_0(x) \leq M\cos\frac{\pi x}{2h_0(1+\delta)}$ for $x \in [-h_0, h_0]$. Consequently, it follows from the above analysis and Lemma 3.1 that

$$
g_\infty \geq -\lim_{t\to\infty}\eta(t) > -\zeta, \quad h_\infty \leq \lim_{t\to\infty}\eta(t) < \zeta.
$$

This completes the proof. □

Similar to the proof of Lemma 5.6, it is easy to prove the following lemma.

**Lemma 5.12.** *Suppose that $h_0 < \Lambda_p/2$, then there exist $\mu^* \geq \mu_* > 0$, such that $h_\infty = \infty$ and $g_\infty = -\infty$ if $\mu > \mu^*$, and $h_\infty - g_\infty < \infty$ if $\mu \leq \mu_*$ or $\mu = \mu^*$.*

**Theorem 5.13.** (*The predator-prey model*) *Let* $(u, v, g, h)$ *be the unique solution of* (1.2) *in there* $f_1, f_2$ *satisfy* (1.5). *Then the following alternative holds:*

*Either*

(i) *Spreading:* $h_\infty = -g_\infty = \infty$. *If we further assume that* $a > b$ *and* $bc < 1$, *then we have*

$$\lim_{t\to\infty} u(t,x) = (a-b)/(1+bc), \quad \lim_{t\to\infty} v(t,x) = (1+ac)/(1+bc) \ \text{ locally uniformly in } \ \mathbb{R}.$$

*or*

(ii) *Vanishing:* $h_\infty - g_\infty < \infty$ *and*

$$\lim_{t\to\infty} u(t,x) = a \ \text{ locally uniformly in } \ \mathbb{R}, \quad \lim_{t\to\infty} \|v(t,\cdot)\|_{C([g(t),h(t)])} = 0.$$

*Moreover,*

(iii) *If* $h_0 \geq \frac{1}{2}\Lambda_p$, *then* $h_\infty = -g_\infty = \infty$ *for all* $\mu > 0$.

(iv) *If* $h_0 < \frac{1}{2}\Lambda_p$, *then there exist* $\mu^* \geq \mu_* > 0$ *such that* $h_\infty = -g_\infty = \infty$ *if* $\mu > \mu^*$, *and* $h_\infty - g_\infty < \infty$ *if* $\mu \leq \mu_*$ *or* $\mu = \mu^*$.

### 5.3 The mutualist model

In this part we suppose that $f_1, f_2$ satisfy (1.6). Taking advantage of comparison principle and results of logistic equation ([5]), we can easily obtain the criteria governing spreading and vanishing. Combining with our early results about long-time behaviors of $(u, v)$, we have the following theorem.

**Theorem 5.14.** (*The mutualist model*) *Let* $(u, v, g, h)$ *be the unique solution of* (1.2) *in there* $f_1, f_2$ *satisfy* (1.6). *Then the following alternative holds:*

*Either*

(i) *Spreading:* $h_\infty = -g_\infty = \infty$ *and*

$$\lim_{t\to\infty} u(t,x) = u^*, \quad \lim_{t\to\infty} v(t,x) = v^* \ \text{ locally uniformly in } \ \mathbb{R}.$$

*or*

(ii) *Vanishing:* $h_\infty - g_\infty < \infty$ *and*

$$\lim_{t\to\infty} u(t,x) = a/2 \ \text{ locally uniformly in } \ \mathbb{R}, \quad \lim_{t\to\infty} \|v(t,\cdot)\|_{C([g(t),h(t)])} = 0.$$

*Moreover,*

(iii) *If* $h_0 \geq \pi\sqrt{d_2/r_2} =: \Lambda_m/2$, *then* $h_\infty - g_\infty = \infty$ *for all* $\mu > 0$;

(iv) *If* $h_0 < \Lambda_m/2$, *then there exist* $\mu^* \geq \mu_* > 0$ *such that* $h_\infty = -g_\infty = \infty$ *if* $\mu > \mu^*$, *and* $h_\infty - g_\infty < \infty$ *if* $\mu \leq \mu_*$ *or* $\mu = \mu^*$.

## 6 Estimates of spreading speeds

In this section we give some rough estimates on the spreading speed of $g(t)$ and $h(t)$ when spreading happens. We first recall one proposition whose proof is given in [29].

**Proposition 6.1.** *For any given positive constants $a$, $d$, $b$, and $k \in [0, 2\sqrt{ad})$, the problem*

$$-dU'' + kU' = U(a - bU) \quad \text{in } (0,\infty), \quad U(0) = 0.$$

*admits a unique positive solution $U = U_k = U_{a,d,k,b}$, and satisfies $U(x) \to a/b$ as $x \to \infty$. Moreover, for each $\mu > 0$, there exists a unique $k_0 = k_0(\mu, a, d, b) \in (0, 2\sqrt{ad})$ such that $\mu U'_{k_0}(0) = k_0$. Furthermore, $k_0 = k_0(\mu, a, d, b)$ is strictly increasing in $\mu$ and $a$ and decreasing in $b$.*

Utilizing the function $k_0(\mu, a, d, b)$, we have the following estimates for spreading speed of $g(t)$ and $h(t)$. Since the proof is similar, we just give the proof of Theorem 6.2. For simplicity, we denote $k_0(\mu, a, d, 1) = k_0(\mu, a, d)$.

**Theorem 6.2.** (*The competition model*) *Suppose that $f_1, f_2$ satisfy* (1.4) *and $ac < 1$. If $h_\infty - g_\infty = \infty$, then*

$$-k_0(\mu, 1, d_2) \le \liminf_{t\to\infty} \frac{g(t)}{t} \le \limsup_{t\to\infty} \frac{g(t)}{t} \le -k_0(\mu, 1 - ac, d_2),$$

*and*

$$k_0(\mu, 1 - ac, d_2) \le \liminf_{t\to\infty} \frac{h(t)}{t} \le \limsup_{t\to\infty} \frac{h(t)}{t} \le k_0(\mu, 1, d_2).$$

*Proof.* Since the proof is similar to [29, Theorem 5.2], we just give the sketch. Clearly, the triplet $(v, g, h)$ is a lower solution of the problem (5.1). It then follows that $\bar{g}(t) \le g(t) \to -\infty$ and $\bar{h}(t) \ge h(t) \to \infty$ as $t \to \infty$. Thanks to [5, Theorem 5.12], we have

$$\lim_{t\to\infty} \frac{\bar{g}(t)}{t} = -k_0(\mu, 1, d_2) \quad \lim_{t\to\infty} \frac{\bar{h}(t)}{t} = k_0(\mu, 1, d_2).$$

Thus

$$\liminf_{t\to\infty} \frac{g(t)}{t} \ge -k_0(\mu, 1, d_2) \quad \limsup_{t\to\infty} \frac{h(t)}{t} \le k_0(\mu, 1, d_2).$$

Since $\limsup_{t\to\infty} u \le a$ uniformly in $\mathbb{R}$, for any $0 < \varepsilon \ll 1$, there exists $T_\varepsilon$ such that $u(t,x) \le a + \varepsilon$ for $t \ge T_\varepsilon$, $x \in \mathbb{R}$ and $h(T_\varepsilon) - g(T_\varepsilon) > \pi\sqrt{d_2/\big(1 - c(a+\varepsilon)\big)}$. The comparison principle implies that $(v, g, h)$ is a upper solution to the following problem

$$\begin{cases} \underline{v}_t - d_2\underline{v}_{xx} = \underline{v}(1 - c(a+\varepsilon) - \underline{v}), & t > T_\varepsilon, \ \ \underline{g}(t) < x < \underline{h}(t), \\ \underline{v}(t,x) = 0, & t > T_\varepsilon, \ \ x = \underline{g}(t), \ \underline{h}(t), \\ \underline{g}'(t) = -\mu\underline{v}_x(t, \underline{g}(t)), & t > T_\varepsilon, \\ \underline{h}'(t) = -\mu\underline{v}_x(t, \underline{h}(t)), & t > T_\varepsilon, \\ \underline{h}(T_\varepsilon) = h(T_\varepsilon), \ \ \underline{g}(T_\varepsilon) = g(T_\varepsilon), & \\ \underline{v}(T_\varepsilon, x) = v(T_\varepsilon, x), & g(T_\varepsilon) \le x \le h(T_\varepsilon). \end{cases}$$

It follows by [5] that $\underline{h}_\infty = -\underline{g}_\infty = \infty$ since $h(T_\varepsilon) - g(T_\varepsilon) > \pi\sqrt{d_2/\big(1 - c(a+\varepsilon)\big)}$. Moreover, making use of [5, Theorem 5.12] yields that

$$\lim_{t\to\infty} \frac{\underline{g}(t)}{t} = -k_0(\mu, 1 - c(a+\varepsilon), d_2), \quad \lim_{t\to\infty} \frac{\underline{h}(t)}{t} = k_0(\mu, 1 - c(a+\varepsilon), d_2) =: k_0^*(\varepsilon).$$

By virtue of $\underline{g}(t) \ge g(t)$ and $\underline{h}(t) \le h(t)$ for $t \ge T_\varepsilon$ and the arbitrariness of $\varepsilon$, we have

$$\limsup_{t\to\infty} \frac{g(t)}{t} \le -k_0(\mu, 1 - ac, d_2), \quad \liminf_{t\to\infty} \frac{h(t)}{t} \ge k_0(\mu, 1 - ac, d_2).$$

The proof is completed. □

**Theorem 6.3.** (*The predator-prey model*) *Suppose that $f_1, f_2$ satisfy* (1.5)*. If $h_\infty - g_\infty = \infty$, then*

$$-k_0(\mu, 1+ac, d_2) \le \liminf_{t\to\infty} \frac{g(t)}{t} \le \limsup_{t\to\infty} \frac{g(t)}{t} \le -k_0(\mu, 1, d_2),$$

*and*

$$k_0(\mu, 1, d_2) \le \liminf_{t\to\infty} \frac{h(t)}{t} \le \limsup_{t\to\infty} \frac{h(t)}{t} \le k_0(\mu, 1+ac, d_2).$$

**Theorem 6.4.** (*The mutualist model*) *Suppose that $f_1, f_2$ satisfy* (1.6)*. If $h_\infty - g_\infty = \infty$, then*

$$-k_0\left(\mu, r_2, d_2, r_2(2+cu^*)/(1+cu^*)\right) \le \liminf_{t\to\infty} \frac{g(t)}{t} \le \limsup_{t\to\infty} \frac{g(t)}{t} \le -k_0(\mu, r_2, d_2, 2r_2),$$

*and*

$$k_0(\mu, r_2, d_2, 2r_2) \le \liminf_{t\to\infty} \frac{h(t)}{t} \le \limsup_{t\to\infty} \frac{h(t)}{t} \le k_0\left(\mu, r_2, d_2, r_2(2+cu^*)/(1+cu^*)\right).$$

## References


[1] R. Natan, E. Klein, J. J. Robledo-Arnuncio and E. Revilla, *Dispersal kernels: Review*, in *Dispersal Ecology and Evolution*, J. Clobert, M. Baguette, T. G. Benton and J. M. Bullock, eds., Oxford University Press, Oxford, UK, 2012, 187-210.

[2] H. Berestycki, J. Coville and H. Vo, *On the definition and the properties of the principal eigenvalue of some nonlocal operators*, J. Funct. Anal., **271** (2016), 2701-2751.

[3] H. Berestycki, J. Coville and H. Vo, *Persistence criteria for populations with non-local dispersion*, J. Math. Biol., **72** (2016), 1693-1745.

[4] J. F. Cao, Y. H. Du, F. Li and W. T. Li, *The dynamics of a Fisher-KPP nonlocal diffusion model with free boundaries*, J. Functional Analysis, 2019. https://doi.org/10.1016/j.jfa.2019.02.013.

[5] Y. H. Du and Z. G. Lin, *Spreading-Vanishing dichotomy in the diffusive logistic model with a free boundary*, SIAM J. Math. Anal., **42** (2010), 377-405.

[6] J. Guo, C. H. Wu, *On a free boundary problem for a two-species weak competition system*, J. Dynam. Differential Equations **24** (2012), 873-895.

[7] J. P. Wang and M. X. Wang, *The diffusive Beddington-DeAngelis predator-prey model with nonlinear prey-taxis and free boundary*, Math. Meth. Appl. Sci. **41** (2018), 6741-6762.

[8] M. X. Wang, *On some free boundary problems of the prey-predator model*, J. Differential Equations, **256** (2014), 3365-3394.

[9] M. X. Wang and J. F. Zhao, *Free boundary problems for a Lotka-Volterra competition system*, J. Dyn. Diff. Equat., **26** (2014), 655-672.

[10] M. X. Wang and Y. Zhang, *The time-periodic diffusive competition models with a free boundary and sign-changing growth rates*, Z. Angew. Math. Phys. **67** (5) (2016), 132.

[11] Y. G. Zhao and M. X. Wang, *Free boundary problems for the diffusive competition system in higher dimension with sign-changing coefficients*, IMA J. Appl. Math., **81** (2016), 255-280.

[12] Y. H. Du, M. X. Wang and M. Zhao, *Two species nonlocal diffusion systems with free boundaries*, Preprint.

[13] C. Y. Kao, Y. Lou and W. X. Shen, *Random dispersal vs. non-local dispersal*, Discrete Contin. Dyn. Syst., **26** (2010), 551-596.

[14] J. P. Wang and M. X. Wang, *Free boundary problems of ecological models with nonlocal and local diffusions*, Preprint.

[15] Y. H. Du and Z. G. Lin, *The diffusive competition model with a free boundary: Invasion of a superior or inferior competitor*, Discrete Cont. Dyn. Syst.-B, **19** (2014), 3105-3132.

[16] Y. H. Du, M. X. Wang and M. L. Zhou, *Semi-wave and spreading speed for the diffusive competition model with a free boundary*, J. Math. Pures Appl., **107** (2017), 253-287.

[17] L. Zhou, S. Zhang and Z. H. Liu, *An evolutional free-boundary problem of a reaction-diffusion-advection system*, Proc. Royal Soc. Edinburgh A, **147** (2017), 615-648.

[18] M. X. Wang and J. F. Zhao, *A free boundary problem for the predator-prey model with double free boundaries*, J. Dyn. Diff. Equat., **29** (3) (2017), 957-979.

[19] J. F. Zhao and M. X. Wang, *A free boundary problem of a predator-prey model with higher dimension and heterogeneous environment*, Nonlinear Anal.: RWA, **16** (2014), 250-263.

[20] F. X. Chen, *Almost periodic traveling waves of nonlocal evolution equations*, Nonlinear Anal. TMA, **50** (2002), 807-838.

[21] X. Chen, J. S. Guo, *Existence and asymptotic stability of travelling waves of discrete quasilinear monostable equations*, J. Differential Equations, **184** (2002), 549-569.

[22] M. X. Wang, *Existence and uniqueness of solutions of free boundary problems in heterogeneous environments*. Discrete Cont. Dyn. Syst. B., **24** (2) (2019), 415-421.

[23] M. X. Wang and Q. Y. Zhang, *Dynamics for the diffusive leslie-gower model with double free boundaries*, Discrete Cont. Dyn. Syst., **38** (5) (2018), 2591-2607.

[24] P. Bates and G. Y. Zhao, *Existence, uniqueness and stability of the stationary solution to a nonlocal evolution equation arising in population dispersal*, J. Math. Anal. Appl., **332** (2007), 428-440.

[25] J. Coville, *On a simple criterion for the existence of a principal eigenfunction of some nonlocal operators*, J. Differential Equations, **249** (2010), 2921-2953.

[26] M. X. Wang, *A diffusive logistic equation with a free boundary and sign-changing coefficient in time-periodic environment*, J. Funct. Anal., **270** (2016), 483-508.

[27] M. X. Wang and Y. Zhang, *Dynamics for a diffusive prey-predator model with different free boundaries*, J. Differental Equatons, **264** (2018), 3527-3558.

[28] Q. Y. Zhang, M. X. Wang, *Dynamics for the diffusive mutualist model with advection and different free boundaries*, J. Math. Anal. Appl., **474** (2019), 1512-1535.

[29] G. Bunting, Y. H. Du and K. Krakowski, *Spreading speed revisited: Analysis of a free boundary model*, Networks and Heterogeneous Media, **42** (7) (2012), 583-603.