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On projective Banach lattices of the form $C(K)$ and $FBL[E]$ ☆

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ABSTRACT

We show that if a Banach lattice is projective, then every bounded sequence that can be mapped by a homomorphism onto the basis of c_0 must contain an ℓ_1-subsequence. As a consequence, if Banach lattices ℓ_p or $FBL[E]$ are projective, then $p = 1$ or E has the Schur property, respectively. On the other hand, we show that $C(K)$ is projective whenever K is an absolute neighbourhood retract, answering a question by de Pagter and Wickstead.

1. Introduction

In this paper we continue the program proposed by B. de Pagter and A. W. Wickstead [10] of studying the projective Banach lattices.

Definition 1.1. Let $\lambda > 1$ be a real number. A Banach lattice P is λ-*projective* if whenever X is a Banach lattice, J a closed ideal in X and $Q\colon X \longrightarrow X/J$ the quotient map, then for every Banach lattice homomorphism $T\colon P \longrightarrow X/J$, there is a Banach lattice homomorphism $\hat{T}\colon P \longrightarrow X$ such that $T = Q \circ \hat{T}$ and $\|\hat{T}\| \leq \lambda \|T\|$.

A Banach lattice is called *projective* in [10] if it is $(1+\varepsilon)$-projective for every $\varepsilon > 0$. For a more intuitive terminology, and by analogy to similar notions in Banach spaces, we will call this 1^+-*projective* instead of just projective. Note that if P is λ-projective, then P is μ-projective for every $\mu \geq \lambda$. We will call a Banach

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lattice ∞-*projective* if it is λ-projective for some $\lambda > 1$. It is clear that, in the case of ∞-projective, Q can be taken any surjective Banach lattice homomorphism.

The notion of free Banach lattice was also introduced in [10]. If A is a set with no extra structure, the free Banach lattice generated by A, denoted by $FBL(A)$, is a Banach lattice together with a bounded map $u\colon A \longrightarrow FBL(A)$ having the following universal property: for every Banach lattice Y and every bounded map $v\colon A \longrightarrow Y$ there is a unique Banach lattice homomorphism $S\colon FBL(A) \longrightarrow Y$ such that $S \circ u = v$ and $\|S\| = \sup\{\|v(a)\| : a \in A\}$. The same idea is applied by A. Avilés, J. Rodríguez and P. Tradacete to define the concept of the free Banach lattice generated by a Banach space E, $FBL[E]$. This is a Banach lattice together with a bounded operator $u\colon E \longrightarrow FBL[E]$ such that for every Banach lattice Y and every bounded operator $T\colon E \longrightarrow Y$ there is a unique Banach lattice homomorphism $S\colon FBL[E] \longrightarrow Y$ such that $S \circ u = T$ and $\|S\| = \|T\|$.

In [4] and [10] it is shown that both objects exist and are unique up to Banach lattices isometries. A more explicit description of these spaces is given in [4] as follows:

Let A be a non-empty set. For $x \in A$, let $\delta_x\colon [-1,1]^A \longrightarrow \mathbb{R}$ be the evaluation function given by $\delta_x(x^*) = x^*(x)$ for every $x^* \in [-1,1]^A$, and for every $f\colon [-1,1]^A \longrightarrow \mathbb{R}$ define

$$\|f\| = \sup\left\{\sum_{i=1}^n |f(x_i^*)| : n \in \mathbb{N},\, x_1^*, \dots, x_n^* \in [-1,1]^A,\ \sup_{x\in A}\sum_{i=1}^n |x_i^*(x)| \le 1\right\}.$$

The Banach lattice $FBL(A)$ is the Banach lattice generated by the evaluation functions δ_x inside the Banach lattice of all functions $f\colon [-1,1]^A \longrightarrow \mathbb{R}$ with finite norm. The natural identification of A inside $FBL(A)$ is given by the map $u\colon A \longrightarrow FBL(A)$ where $u(x) = \delta_x$. Since every function in $FBL(A)$ is a uniform limit of such functions, they are all continuous (with respect to the product topology) and positively homogeneous, i.e. they commute with multiplication by positive scalars.

Now, let E be a Banach space. For a function $f\colon E^* \longrightarrow \mathbb{R}$ consider the norm

$$\|f\|_{FBL[E]} = \sup\left\{\sum_{i=1}^n |f(x_i^*)| : n \in \mathbb{N},\, x_1^*, \dots, x_n^* \in E^*,\ \sup_{x\in B_E}\sum_{i=1}^n |x_i^*(x)| \le 1\right\}.$$

The Banach lattice $FBL[E]$ is the closure of the vector lattice in $\mathbb{R}^{E^*}$ generated by the evaluations $\delta_x\colon x^* \mapsto x^*(x)$ with $x \in E$. These evaluations form the natural copy of E inside $FBL[E]$. All the functions in $FBL[E]$ are positively homogeneous and *weak**-continuous when restricted to the closed unit ball B_{E^*}. An alternative approach to the construction of $FBL[E]$ has been given in [11].

In our previous work [3] we answered a question by B. de Pagter and A. W. Wickstead by showing that c_0 is not a projective Banach lattice. In Section 2 we exploit some of the ideas of that paper further, so that we are able to show that projective Banach lattices enjoy the following property:

Theorem 1.2. *Let $(u_i)_{i\in\mathbb{N}}$ be a bounded sequence of vectors in an ∞-projective Banach lattice X. Suppose that there exists a Banach lattice homomorphism $T\colon X \longrightarrow c_0$ such that $T(u_i) = e_i$ for every $i \in \mathbb{N}$, where $(e_i)_{i\in\mathbb{N}}$ is the canonical basis of c_0. Then there is a subsequence $(u_{i_k})_{k\in\mathbb{N}}$ equivalent to the canonical basis of ℓ_1.*

From this theorem we can deduce for example that no Banach lattice ℓ_p is λ-projective, for any $\lambda > 0$ or $p \neq 1$. The prototype of 1^+-projective Banach lattice is $FBL(A) = FBL[\ell_1(A)]$ (see [4, Corollary 2.8] and [10, Proposition 10.2]), so it would be natural to wonder whether $FBL[E]$ might be 1^+-projective as well for other Banach spaces E. We show that, for this to happen, the structure of E must be very close to that of $\ell_1(A)$:

Theorem 1.3. *Let E be a Banach space. If $FBL[E]$ is ∞-projective, then E has the Schur property (i.e. every weakly convergent sequence in E converges in norm).*

Moreover, at the end of Section 2 we provide a counterexample which shows that, in the category of nonseparable Banach spaces, the converse of this result does not hold. We still do not know if there exists a separable Banach space E which has the Schur property and such that $FBL[E]$ is not ∞-projective.

Section 3 is devoted to the study of projectivity on Banach lattices of the form $C(K)$ of continuous functions on a compact space with the supremum norm. It was shown by de B. Pagter and A. W. Wickstead [10] that if $C(K)$ is 1^+-projective then K is an absolute neighbourhood retract, and the converse holds true if K is a compact subset of $\mathbb{R}^n$. They asked whether the converse holds for general K [10, Question 12.12]. We solve their problem in the positive:

Theorem 1.4. *If K is a compact Hausdorff topological space, then $C(K)$ is 1^+-projective if, and only if, K is an absolute neighbourhood retract.*

2. Schur property in Banach spaces with projective free Banach lattice

As a preparation towards Theorem 1.2 we provide a criterion to obtain ℓ_1-subsequences in the free Banach lattice $FBL(L)$. We denote the index set L instead of A for convenience in latter application.

Lemma 2.1. *Let L be an infinite set, $(x_n^*)_{n\in\mathbb{N}}$ a sequence in $[-1,1]^L$ and $(f_n)_{n\in\mathbb{N}}$ a sequence in $FBL(L)$ with the following properties:*

(1) *$(f_n)_{n\in\mathbb{N}}$ converges pointwise to 0, i.e. $\lim_{n\to\infty} f_n(x^*) = 0$ for every $x^* \in [-1,1]^L$;*
(2) *$f_n(x_n^*) = 1$ for every $n \in \mathbb{N}$;*
(3) *For every finite set $F \subset L$ there is a natural number n such that $x_n^*|_F = 0$, i.e. the restriction of x_n^* to F is null.*

Then, for every $\varepsilon > 0$ there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that for every $l \in \mathbb{N}$ and for every $\lambda_1, \dots, \lambda_l \in \mathbb{R}$,

$$\left\| \sum_{k=1}^{l} \lambda_k f_{n_k} \right\| \geq (1-\varepsilon) \sum_{k=1}^{l} |\lambda_k|.$$

Proof. Fix $\varepsilon > 0$ and $(\varepsilon_{ij})_{i,j=1}^{\infty}$ a family of positive real numbers such that $\varepsilon = \sum_{i,j=1}^{\infty} \varepsilon_{ij}$ and $\varepsilon_{ij} = \varepsilon_{ji}$ for every i, j.

We are going to define a subsequence $(f_{m_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ as follows:

Let $m_1 := 1$. Since the elements of $FBL(L)$ are continuous with respect to the product topology, there is a neighbourhood U_{m_1} of $x_{m_1}^*$ such that $f_{m_1}(x^*) \in [1-\varepsilon_{11}, 1+\varepsilon_{11}]$ whenever $x^* \in U_{m_1}$. In particular, there is a finite set $F_{m_1} \subset L$ such that $f_{m_1}(x^*) \in [1-\varepsilon_{11}, 1+\varepsilon_{11}]$ whenever $x^*|_{F_{m_1}} = x_{m_1}^*|_{F_{m_1}}$.

By property (3), there exists $m_2 \in \mathbb{N}$ such that $x_{m_2}^*|_{F_{m_1}} = 0$. Since f_{m_2} is continuous, there exists a finite set $F_{m_2} \supset F_{m_1}$ such that $f_{m_2}(x^*) \in [1-\varepsilon_{22}, 1+\varepsilon_{22}]$ whenever $x^*|_{F_{m_2}} = x_{m_2}^*|_{F_{m_2}}$.

Suppose that we have $f_{m_1}, \dots, f_{m_{k-1}}$ for some $k \geq 2$, and $F_{m_1}, \dots, F_{m_{k-1}}$ finite subsets of L such that $F_{m_1} \subset \cdots \subset F_{m_{k-1}}$, $x_{m_i}^*|_{F_{m_{i-1}}} = 0$ for every $i = 2, \dots, k-1$ and $f_{m_i}(x^*) \in [1-\varepsilon_{ii}, 1+\varepsilon_{ii}]$ whenever $x^*|_{F_{m_i}} = x_{m_i}^*|_{F_{m_i}}$.

Property (3) guarantees the existence of a number $m_k \in \mathbb{N}$ such that $x_{m_k}^*|_{F_{m_{k-1}}} = 0$. It follows from property (2) that there is a finite set $F_{m_k} \subset L$, with $F_{m_{k-1}} \subset F_{m_k}$, such that $f_{m_k}(x^*) \in [1-\varepsilon_{kk}, 1+\varepsilon_{kk}]$ whenever $x^*|_{F_{m_k}} = x_{m_k}^*|_{F_{m_k}}$.

For each $k \in \mathbb{N}$ define $y_{m_k}^* \colon L \longrightarrow [-1,1]$ such that $y_{m_k}^*|_{F_{m_k}} = x_{m_k}^*|_{F_{m_k}}$ and $y_{m_k}^*(x) = 0$ whenever $x \in L \setminus F_{m_k}$. Notice that $f_{m_k}(y_{m_k}^*) \in [1-\varepsilon_{kk}, 1+\varepsilon_{kk}]$ for every $k \in \mathbb{N}$. On the other hand, if $m_k < m_{k'}$ and

$y^*_{m_k}(x) \neq 0$ then $x \in F_{m_k}$ (by the definition of $y^*_{m_k}$) and therefore $x^*_{m_{k'}}(x) = 0$, so $y^*_{m_{k'}}(x) = 0$. It follows that $y^*_{m_k}$ and $y^*_{m_{k'}}$ have disjoint supports. In particular,

$$\sup_{x\in L} \sum_{k=1}^{l} \left| y^*_{m_k}(x) \right| \leq 1.$$

Let $\nu_1 := m_1 = 1$. Combining property (1) with the fact that the functions f_n are continuous in $[-1,1]^L$ and the functions $y^*_{m_n}$ converge to 0 in the product topology, we have that there exists $\nu_2 \in \mathbb{N}$ such that

$$|f_{m_n}(y^*_{m_{\nu_1}})| \leq \varepsilon_{12} \text{ and } |f_{m_{\nu_1}}(y^*_{m_n})| \leq \varepsilon_{21} = \varepsilon_{12} \text{ for every } n \geq \nu_2.$$

Again, using the above, there exists a natural number $\nu_3 \geq \nu_2$ such that

$$|f_{m_n}(y^*_{m_{\nu_1}})| \leq \varepsilon_{13},\ |f_{m_{\nu_1}}(y^*_{m_n})| \leq \varepsilon_{31} = \varepsilon_{13}$$

and

$$|f_{m_n}(y^*_{m_{\nu_2}})| \leq \varepsilon_{23},\ |f_{m_{\nu_2}}(y^*_{m_n})| \leq \varepsilon_{32} = \varepsilon_{23}$$

for every $n \geq \nu_3$.

Suppose that we have $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_p \in \mathbb{N}$ such that

$$|f_{m_n}(y^*_{m_{\nu_j}})| \leq \varepsilon_{jp} \text{ and } |f_{m_{\nu_j}}(y^*_{m_n})| \leq \varepsilon_{pj} = \varepsilon_{jp} \text{ for every } j < p \text{ and every } n \geq \nu_p.$$

Then, there exists a natural number $\nu_{p+1} \geq \nu_p$ such that

$$|f_{m_n}(y^*_{m_{\nu_j}})| \leq \varepsilon_{j(p+1)} \text{ and } |f_{m_{\nu_j}}(y^*_{m_n})| \leq \varepsilon_{(p+1)j} = \varepsilon_{j(p+1)} \text{ for every } j < p+1$$

and every $n \geq \nu_{p+1}$.

Since $f_{m_{\nu_i}}(y^*_{m_{\nu_i}}) \in [1-\varepsilon_{\nu_i\nu_i}, 1+\varepsilon_{\nu_i\nu_i}]$ for every i, we can write $f_{m_{\nu_i}}(y^*_{m_{\nu_i}}) = 1+\eta_{\nu_i\nu_i}$ with $|\eta_{\nu_i\nu_i}| \leq \varepsilon_{\nu_i\nu_i}$.

On the other hand, if $k \neq i$, we have that $f_{m_{\nu_k}}(y^*_{m_{\nu_i}}) \in [-\varepsilon_{ik}, \varepsilon_{ik}]$, and we will write $f_{m_{\nu_k}}(y^*_{m_{\nu_i}}) = \eta_{\nu_i\nu_k}$ with $|\eta_{\nu_i\nu_k}| \leq \varepsilon_{ik}$.

We take the subsequence $f_{n_k} := f_{m_{\nu_k}}$ for every $k \in \mathbb{N}$.

Now, let $\lambda_1, \ldots, \lambda_l \in \mathbb{R}$. We have that

$$\begin{aligned}
\left\| \sum_{k=1}^{l} \lambda_k f_{n_k} \right\| &= \sup \left\{ \sum_{i=1}^{q} \left| \sum_{k=1}^{l} \lambda_k f_{n_k}(z_i^*) \right| : q \in \mathbb{N},\ z_i^* \in [-1,1]^L,\ \sup_{x\in L} \sum_{i=1}^{q} |z_i^*(x)| \leq 1 \right\} \\
&\geq \sum_{i=1}^{l} \left| \sum_{k=1}^{l} \lambda_k f_{m_{\nu_k}}(y^*_{m_{\nu_i}}) \right| = \sum_{i=1}^{l} \left| \lambda_i f_{m_{\nu_i}}(y^*_{m_{\nu_i}}) + \sum_{k\neq i} \lambda_k f_{m_{\nu_k}}(y^*_{m_{\nu_i}}) \right| \\
&= \sum_{i=1}^{l} \left| \lambda_i (1+\eta_{\nu_i\nu_i}) + \sum_{k\neq i} \lambda_k \eta_{\nu_i\nu_k} \right| = \sum_{i=1}^{l} \left| \lambda_i + \sum_{k=1}^{l} \lambda_k \eta_{\nu_i\nu_k} \right| \\
&\geq \sum_{i=1}^{l} |\lambda_i| - \sum_{i=1}^{l} \sum_{k=1}^{l} |\lambda_k| \, |\eta_{\nu_i\nu_k}| = \sum_{i=1}^{l} |\lambda_i| - \sum_{k=1}^{l} |\lambda_k| \left(\sum_{i=1}^{l} |\eta_{\nu_i\nu_k}| \right) \\
&\geq \sum_{i=1}^{l} |\lambda_i| - \sum_{k=1}^{l} |\lambda_k| \left(\varepsilon_{\nu_k\nu_k} + \sum_{i\neq k} \varepsilon_{ik} \right) \geq (1-\varepsilon) \sum_{k=1}^{l} |\lambda_k| . \quad \square
\end{aligned}$$

We are ready to prove Theorem 1.2 from the introduction:

Proof of Theorem 1.2. Suppose that there is no subsequence equivalent to the canonical basis of ℓ_1. Then, by Rosenthal's ℓ_1-theorem [1, Theorem 10.2.1], the sequence $(u_i)_{i\in\mathbb{N}}$ has a weakly Cauchy subsequence $(u_{i_k})_{k\in\mathbb{N}}$. Thus, the sequence $(y_n)_{n\in\mathbb{N}}$, with $y_n = u_{i_{2n+1}} - u_{i_{2n}}$ for every $n \in \mathbb{N}$, is weakly null and bounded.

Let us denote $T(x) = (T(x)_j)_{j\in\mathbb{N}} \in c_0$ for $x \in X$. Let $\tilde{T}\colon X \longrightarrow c_0$ be the map given by $\tilde{T}(x) = (T(x)_{i_{2k+1}})_{k\in\mathbb{N}}$.

Let $L = \mathcal{P}_{fin}(\omega)\setminus\{\emptyset\}$ be the set of the finite parts of ω without the empty set, and let $\Phi\colon FBL(L) \longrightarrow c_0$ be the map given by

$$\Phi(f) = \left(f\left((\chi_A(\{1\}))_{A\in L}\right), f\left((\chi_A(\{2\}))_{A\in L}\right), \dots\right) = \left(f\left((\chi_A(\{n\}))_{A\in L}\right)\right)_{n\in\mathbb{N}}$$

for every $f\colon [-1,1]^L \longrightarrow \mathbb{R} \in FBL(L)$, where for $A \in L$, $\chi_A\colon L \longrightarrow [-1,1]$ is the map given by $\chi_A(B) = 1$ if $B \subset A$ and $\chi_A(B) = 0$ if $B \not\subset A$.

By [3, Lemma 2.2], Φ is a surjective Banach lattice homomorphism. Since X is ∞-projective, there exists a bounded Banach lattice homomorphism $\ddot{T}\colon X \longrightarrow FBL(L)$ such that $\Phi \circ \ddot{T} = \tilde{T}$. We are going to find now f_n and x_n^* for the application of Lemma 2.1.

Let $f_n := \ddot{T}(y_n)$ for every $n \in \mathbb{N}$. The sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to 0, since $(y_n)_{n\in\mathbb{N}}$ is weakly null. It follows from the equality $\Phi(f_n) = (\Phi \circ \ddot{T})(y_n) = \tilde{T}(y_n) = e_n$ and the definition of Φ that

$$f_n\left((\chi_A(\{n\}))_{A\in L}\right) = \Phi(f_n)_n = e_n(n) = 1$$

for every $n \in \mathbb{N}$. Set $x_n^* = (\chi_A(\{n\}))_{A\in L} \in [-1,1]^L$ for every $n \in \mathbb{N}$. Notice that if $F \subset L$ is finite, then $x_n^*(S) = 0$ whenever $n \notin \bigcup_{S\in F} S$, so condition (3) of Lemma 2.1 is also satisfied.

We can now apply Lemma 2.1, so for every $\varepsilon > 0$ there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that for every $l \in \mathbb{N}$ and for every $\lambda_1, \dots, \lambda_l \in \mathbb{R}$,

$$\left\|\sum_{k=1}^{l} \lambda_k f_{n_k}\right\| \geq (1-\varepsilon)\sum_{k=1}^{l} |\lambda_k|.$$

On the other hand, since $\ddot{T}$ and $(y_n)_{n\in\mathbb{N}}$ are bounded, there are two constants $C, M > 0$ such that

$$\left\|\sum_{k=1}^{l} \lambda_k f_{n_k}\right\| = \left\|\ddot{T}\left(\sum_{k=1}^{l} \lambda_k y_{n_k}\right)\right\| \leq C\left\|\sum_{k=1}^{l} \lambda_k y_{n_k}\right\| \leq CM\sum_{k=1}^{l} |\lambda_k|.$$

Thus,

$$(1-\varepsilon)\sum_{k=1}^{l} |\lambda_k| \leq \left\|\sum_{k=1}^{l} \lambda_k f_{n_k}\right\| \leq CM\sum_{k=1}^{l} |\lambda_k|,$$

so that $(f_{n_k})_{k\in\mathbb{N}}$ is equivalent to the canonical basis of ℓ_1, and in consequence, $(y_{n_k})_{k\in\mathbb{N}}$ is also equivalent to the canonical basis of ℓ_1, which is a contradiction. $\square$

Corollary 2.2. *The Banach lattices c_0 and l_p (for $2 \leq p < \infty$) are not ∞-projective.*

Proof. On the one hand, the canonical basis $(u_i)_{i\in\mathbb{N}}$ of c_0 does not have subsequences equivalent to the canonical basis of ℓ_1, and the identity map $T = id_{c_0}$ is a Banach lattice homomorphism such that $T(u_i) = e_i$ for every $i \in \mathbb{N}$, where $(e_i)_{i\in\mathbb{N}}$ is the canonical basis of c_0. On the other hand, the canonical basis $(u_i)_{i\in\mathbb{N}}$ of

l_p does not have subsequences equivalent to the canonical basis of ℓ_1, and the formal inclusion T of l_p into c_0 is a Banach lattice homomorphism such that $T(u_i) = e_i$ for every $i \in \mathbb{N}$, where $(e_i)_{i\in\mathbb{N}}$ is the canonical basis of c_0. $\square$

As we mentioned, the fact that c_0 is not ∞-projective was already proved in [3, Theorem 2.4]. The following is a corollary of Theorem 1.2 in the context of free Banach lattices $FBL[E]$:

Lemma 2.3. *Let E be a Banach space such that $FBL[E]$ is ∞-projective, and let $(u_i)_{i\in\mathbb{N}}$ be a bounded sequence of vectors in E. Suppose that there exists an operator $S\colon E \longrightarrow c_0$ such that $S(u_i) = e_i$ for every $i \in \mathbb{N}$, where $(e_i)_{i\in\mathbb{N}}$ is the canonical basis of c_0. Then there is a subsequence $(u_{i_k})_{k\in\mathbb{N}}$ equivalent to the canonical basis of ℓ_1.*

Proof. Let $\phi\colon E \longrightarrow FBL[E]$ be the inclusion of E into $FBL[E]$, and let $T\colon FBL[E] \longrightarrow c_0$ be the Banach lattice homomorphism given by the universal property of the free Banach lattice which extends the operator S.

The sequence $(\phi(u_i))_{i\in\mathbb{N}}$ is bounded in $FBL[E]$ and $T(\phi(u_i)) = S(u_i) = e_i$ for every $i \in \mathbb{N}$, so that applying Theorem 1.2 we have that $(\phi(u_i))_{i\in\mathbb{N}}$ has a subsequence $(\phi(u_{i_k}))_{k\in\mathbb{N}}$ equivalent to the canonical basis of ℓ_1, which implies that $(u_{i_k})_{k\in\mathbb{N}}$ is a subsequence of $(u_i)_{i\in\mathbb{N}}$ equivalent to the canonical basis of ℓ_1. $\square$

We pass now to the proof of Theorem 1.3, which states that E has the Schur property when $FBL[E]$ is ∞-projective. Lemmas 2.4, 2.5 and 2.6 are necessary only to deal with the case when E is nonseparable. The reader interested in the separable case may skip those lemmas and just apply Sobczyk's extension theorem [1, Theorem 2.5.8] where appropriate.

Lemma 2.4. *Let E be a Banach space. If $FBL[E]$ is ∞-projective, then E is isomorphic to a subspace of $C([-1,1]^\Gamma)$ for some set Γ.*

Proof. Let Γ be a dense subset of the unit ball B_E of E. Let B_{E^*} be the closed unit ball of the dual space E^*, endowed with the weak* topology. We have a surjective Banach lattice homomorphism $P\colon C([-1,1]^\Gamma) \longrightarrow C(B_{E^*})$ given by $P(f)(x^*) = f((x^*(x))_{x\in\Gamma})$. This is just the composition operator with the continuous injection $x^* \mapsto (x^*(x))_{x\in\Gamma}$ from B_{E^*} into $[-1,1]^\Gamma$. Let $\iota\colon E \longrightarrow C(B_{E^*})$ be the canonical inclusion $\iota(x)(x^*) = x^*(x)$, and let $\hat{\iota}\colon FBL[E] \longrightarrow C(B_{E^*})$ be the Banach lattice homomorphism given by the universal property of the free Banach lattice. Since $FBL[E]$ is supposed to be ∞-projective, there exists $\hat{T}\colon FBL[E] \longrightarrow C([-1,1]^\Gamma)$ such that $P \circ \hat{T} = \hat{\iota}$. We take the restriction $T := \hat{T}|_E\colon E \longrightarrow C([-1,1]^\Gamma)$. Notice that $PTx = \iota x$, and therefore

$$\|Tx\| \geq \|PTx\| = \|\iota x\| = \|x\|$$

for every $x \in E$. This implies that T gives an isomorphism of E onto a subspace of $C([-1,1]^\Gamma)$. $\square$

The following fact is well known in the context of a more general theory about Valdivia compacta, Plichko spaces and projectional skeletons (cf. for instance [9]), but we provide a short proof for the reader's convenience:

Lemma 2.5. *For every set Γ, the Banach space $C([-1,1]^\Gamma)$ has the separable complementation property. That is, for every separable subspace $G \subset C([-1,1]^\Gamma)$ there exists a separable complemented subspace G_0 of $C([-1,1]^\Gamma)$ such that $G \subset G_0$.*

Proof. Let S be a countable dense subset of G. By Mibu's theorem [2, page 80, Theorem 4], for every $f \in S$ there exists a countable subset $\Gamma_f \subset \Gamma$ such that $f(x) = f(y)$ whenever $x|_{\Gamma_f} = y|_{\Gamma_f}$. The set $A = \bigcup_{f\in S} \Gamma_f$ is a countable set such that $f(x) = f(y)$ whenever $x|_A = y|_A$ and $f \in G$. The desired separable complemented subspace is

$$G_0 = \left\{ f \in C([-1,1]^\Gamma) : x|_A = y|_A \Rightarrow f(x) = f(y) \right\} \cong C([-1,1]^A).$$

The projection $P\colon C([-1,1]^\Gamma) \longrightarrow G_0$ is given by $P(f)(x) = f(\tilde{x})$ where $\tilde{x}_i = x_i$ if $i \in A$ and $\tilde{x}_i = 0$ if $i \notin A$. $\square$

Lemma 2.6. *Let E be a Banach space such that $FBL[E]$ is ∞-projective, and let $F \subset E$ be a separable subspace. Every operator $S_0\colon F \longrightarrow c_0$ can be extended to an operator $S\colon E \longrightarrow c_0$.*

Proof. By Lemma 2.4, there is an operator $T\colon E \longrightarrow C([-1,1]^\Gamma)$ that is an isomorphism onto its range, so that $G = T(F)$ is a separable subspace of $C([-1,1]^\Gamma)$. By Lemma 2.5, we can find a complemented separable subspace G_0 of $C([-1,1]^\Gamma)$ with $G \subset G_0$. Let $P\colon C([-1,1]^\Gamma) \longrightarrow G_0$ be the projection. If $S_0'\colon G_0 \longrightarrow c_0$ is the extension of S_0 given by the Sobczyk's theorem, then $S := S_0' \circ P \circ T\colon E \longrightarrow c_0$ is the desired operator. $\square$

Theorem 1.3 follows from the previous results:

Proof of Theorem 1.3. If E does not have the Schur property, then there is a weakly null sequence $(u_i)_{i\in\mathbb{N}}$ that does not converge to 0 in norm. By passing to a subsequence we may assume that 0 is not in the norm closure of $\{u_i\}_{i\in\mathbb{N}}$. By the theorem of Kadets and Pełczyński [1, Theorem 1.5.6], by passing to a further subsequence, we can suppose that $(u_i)_{i\in\mathbb{N}}$ is a basic sequence. We are going to see that there exists an operator $S\colon E \longrightarrow c_0$ such that $S(u_i) = e_i$ for every $i \in \mathbb{N}$, where $(e_i)_{i\in\mathbb{N}}$ is the canonical basis of c_0, and then by Lemma 2.3, this will mean that $(u_i)_{i\in\mathbb{N}}$ has a subsequence equivalent to the canonical basis of ℓ_1, a contradiction with the fact that it is weakly null.

Let $F = \overline{\text{span}}\,\{u_i : i \in \mathbb{N}\} \subset E$. For every $n \in \mathbb{N}$ let $u_n^*\colon F \longrightarrow \mathbb{R}$ be the n-th coordinate functional, given by $u_n^*(\sum_{i=1}^\infty \alpha_i u_i) = \alpha_n$, and let $S_0\colon F \longrightarrow \ell_\infty$ be the map given by $S_0(x) = (u_n^*(x))_{n\in\mathbb{N}}$ for every $x \in F$. Since the sequence $(u_n^*)_{n\in\mathbb{N}}$ is weak*-null, we have that $S_0(F) \subset c_0$. On the other hand, $S_0(u_i) = e_i$ for every $i \in \mathbb{N}$. Now, since F is separable and $FBL[E]$ is ∞-projective, applying Lemma 2.6 we can extend S_0 to an operator $S\colon E \longrightarrow c_0$ such that $S(u_i) = e_i$ for every $i \in \mathbb{N}$. $\square$

As a remark, along the first lines of the proof we justify that the Schur property is characterized by the property that every basic sequence contains a subsequence equivalent to the canonical basis of ℓ_1. We may refer to [8] for a study of this kind of fact in a more general context.

Finally, let us see that, in the category of nonseparable Banach spaces, the converse does not hold. By [7, Theorem 1, e) and f)], there exist a separable Banach space F and a bounded set Λ in F^* such that $E := \overline{\text{span}}(\Lambda)$ is nonseparable, has the Schur property and does not contain any copy of $\ell_1(\omega_1)$. Now, since for every set Γ the space $[-1,1]^\Gamma$ is a continuous image of $\{0,1\}^m$ for some infinite cardinal number m, by [6, Corollary 3] we have that E is not isomorphic to any subspace of $C([-1,1]^\Gamma)$ for any set Γ, and then, by Lemma 2.4, we have that $FBL[E]$ cannot be ∞-projective.

3. Projectivity of $C(K)$

In [10, Theorem 11.4] it is proved that for a compact subset K of $\mathbb{R}^n$, $C(K)$, with the supremum norm, is a 1^+-projective Banach lattice if, and only if, K is an absolute neighbourhood retract of $\mathbb{R}^n$. In this section we prove a similar result for K being a compact Hausdorff topological space not necessarily inside $\mathbb{R}^n$. We first recall some basic definitions and facts.

Definition 3.1. We say that a compact space K is an *absolute neighbourhood retract* (ANR) if whenever $i\colon K \longrightarrow X$ is a homeomorphism between K and a subspace of the compact space X, there exist an open set U and a continuous function $\phi\colon U \longrightarrow K$ such that $i(K) \subset U \subset X$ and $\phi(i(k)) = k$ for all $k \in K$.

Lemma 3.2. *In the situation of Definition 3.1, there exist a continuous function $u\colon X \longrightarrow [0,1]$ and a continuous function $\varphi\colon X \setminus u^{-1}(0) \longrightarrow K$ such that $u(i(k)) = 1$ and $\varphi(i(k)) = k$ for every $k \in K$.*

Proof. By the Urysohn's lemma, we can find a continuous function $u\colon X \longrightarrow [0,1]$ such that $u(i(k)) = 1$ for every $k \in K$, $u(x) = 0$ for every $x \in X \setminus U$, and $u(x) \in (0,1)$ for every $x \in U \setminus K$. Notice that $X \setminus u^{-1}(0) \subset U$, so the statement of the Lemma is satisfied. $\square$

Proposition 3.3. [5, Proposition 2.1] *Let P be a 1^+-projective Banach lattice, $\mathcal{I}$ an ideal of P and $T\colon P \longrightarrow P/\mathcal{I}$ the quotient map. The quotient $P/\mathcal{I}$ is 1^+-projective if, and only if, for every $\varepsilon > 0$ there exists a Banach lattice homomorphism $S_\varepsilon\colon P/\mathcal{I} \longrightarrow P$ such that $T \circ S_\varepsilon = id_{P/\mathcal{I}}$ and $\|S_\varepsilon\| \leq 1 + \varepsilon$.*

Lemma 3.4. *Let A be a set, $f\colon [-1,1]^A \longrightarrow \mathbb{R}$ a function, and $a \in A$. Then, the $FBL(A)$-norm of the pointwise product $f \cdot |\delta_a|$ is less than or equal to the supremum norm $\|f\|_\infty$.*

Proof.

$$
\begin{aligned}
\|f \cdot |\delta_a|\| &:= \sup\left\{\sum_{k=1}^m |f \cdot |\delta_a|(x_k^*)| : m \in \mathbb{N},\ x_k^* \in [-1,1]^A,\ \sup_{x \in A} \sum_{k=1}^m |x_k^*(x)| \leq 1\right\} \\
&= \sup\left\{\sum_{k=1}^m |f(x_k^*)| \cdot |\delta_a(x_k^*)| : m \in \mathbb{N},\ x_k^* \in [-1,1]^A,\ \sup_{x \in A} \sum_{k=1}^m |x_k^*(x)| \leq 1\right\} \\
&\leq \sup\left\{\sum_{k=1}^m |f(x_k^*)| \cdot |x_k^*(a)| : m \in \mathbb{N},\ x_k^* \in [-1,1]^A,\ \sum_{k=1}^m |x_k^*(a)| \leq 1\right\} \\
&\leq \sup\left\{\max\left\{|f(x_k^*)| : k = 1,\dots,m\right\} : m \in \mathbb{N},\ x_k^* \in [-1,1]^A,\ \sum_{k=1}^m |x_k^*(a)| \leq 1\right\} \\
&\leq \|f\|_\infty. \quad \square
\end{aligned}
$$

Proof of Theorem 1.4. In [10, Proposition 11.7] it is proved that if $C(K)$ is 1^+-projective, then K is an ANR.

For the converse, let $X := [-1,1]^{B_{C(K)}}$, where $B_{C(K)} = \{f \in C(K) : \|f\|_\infty \leq 1\}$ is the closed unit ball of the space of continuous functions. The map $i\colon K \longrightarrow X$ given by $i(k) = (\gamma(k))_{\gamma \in B_{C(K)}}$ is an homeomorphism between K and $i(K)$. By Lemma 3.2 there exist a continuous function $u\colon X \longrightarrow [0,1]$ and a continuous function $\varphi\colon X \setminus u^{-1}(0) \longrightarrow K$ such that $u(i(k)) = 1$ and $\varphi(i(k)) = k$ for every $k \in K$.

By the universal property of the free Banach lattice, there is a Banach lattice homomorphism $T\colon FBL(B_{C(K)}) \longrightarrow C(K)$ that extends the inclusion $B_{C(K)} \hookrightarrow C(K)$. This is clearly a quotient map and its action is given by $Tf(k) = f(i(k))$ for every $f \in FBL(B_{C(K)})$, $k \in K$.

Since $FBL(B_{C(K)})$ is 1^+-projective ([10, Proposition 10.2]), by Proposition 3.3, it is enough to prove that there exists a Banach lattice homomorphism $S\colon C(K) \longrightarrow FBL(B_{C(K)})$ such that $T \circ S = id_{C(K)}$ and $\|S\| \leq 1$.

Let $\bar{1} \in B_{C(K)}$ be the constant function equal to 1, and let $v\colon \{x \in X : x_{\bar{1}} \neq 0\} \longrightarrow X$ be the map given by $v(x) = (v(x)_\gamma)_{\gamma \in B_{C(K)}}$, where

$$
v(x)_\gamma = \begin{cases} -1 & \text{if} \quad \frac{x_\gamma}{x_{\bar{1}}} < -1, \\ \frac{x_\gamma}{x_{\bar{1}}} & \text{if} \quad \frac{x_\gamma}{x_{\bar{1}}} \in [-1,1], \\ 1 & \text{if} \quad \frac{x_\gamma}{x_{\bar{1}}} > 1, \end{cases}
$$

for every $x = (x_\gamma)_{\gamma \in B_{C(K)}} \in X$ with $x_{\bar{1}} \neq 0$.

For a given $h \in C(K)$, define $f\colon X \longrightarrow \mathbb{R}$ by

$$f(x) := \begin{cases} (h \circ \varphi \circ v) \cdot (u \circ v)(x) & \text{if } x_{\bar{1}} \neq 0 \text{ and } u(v(x)) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Formally, we should call the function f_h as it depends on h. But we omit the subindex for a more friendly notation (in fact the subindex would always be "h" along the proof). Notice also that $x_{\bar{1}} \neq 0$ is required for x to be in the domain of v and $u(v(x)) \neq 0$ is required for $v(x)$ to be in the domain of φ.

The desired $S\colon C(K) \longrightarrow FBL(B_{C(K)})$ will be the map given by $Sh(x) = (f \cdot |\delta_{\bar{1}}|)(x)$ for every $h \in C(K)$, $x \in X$. The function Sh is a real-valued function on $X = [-1,1]^{B_{C(K)}}$, and we will need to prove that, in fact, $Sh \in FBL(B_{C(K)})$. Once that is proved, the rest of properties required for S are relatively easy to check: It is clear that S is a linear map, and it preserves the lattice operations $\wedge$ and $\vee$. The fact that $\|S\| \leq 1$ comes from Lemma 3.4:

$$\|Sh\| = \|f \cdot |\delta_{\bar{1}}|\| \leq \|f\|_\infty = \|(h \circ \varphi \circ v)(u \circ v))\|_\infty \leq \|h\|_\infty .$$

To see that $T \circ S = id_{C(K)}$, take $h \in C(K)$ and $k \in K$. Remember that $u(i(k)) = 1$ and $\varphi(i(k)) = k$ and notice that $i(k)_{\bar{1}} = 1$ and $v(i(k)) = i(k)$ for every $k \in K$, so

$$TSh(k) = Sh(i(k)) = (f \cdot |\delta_{\bar{1}}|)(i(k)) = h(\varphi(i(k))) \cdot u(i(k)) = h(k).$$

So we turn now to the remaining more delicate question whether $Sh \in FBL(B_{C(K)})$ for every $h \in C(K)$. Functions in the free Banach lattice must be continuous (in the product topology) and positively homogeneous (commute with multiplication by positive scalars). We check first that Sh has these two properties. Clearly, Sh is continuous on the open set $\{x \in X : x_{\bar{1}} \neq 0, u(v(x)) \neq 0\}$ because Sh is expressed there by the formula $(h \circ \varphi \circ v) \cdot (u \circ v) \cdot |\delta_{\bar{1}}|$. If $x_{\bar{1}} = 0$, then for every $\varepsilon > 0$ there is a neighbourhood W such that $|f(y)| \cdot |y_{\bar{1}}| \leq \|h\|_\infty \cdot \varepsilon$ for all $y \in W$, so Sh is also continuous at those x. If $x_{\bar{1}} \neq 0$ but $u(v(x)) = 0$, again, given $\varepsilon > 0$, we can find a neighbourhood W of x where $y_{\bar{1}} \neq 0$ and $|f(y)| \cdot |y_{\bar{1}}| \leq \|h\|_\infty \cdot \varepsilon$ for all $y \in W$. For positive homogeneity, on the one hand, if $x_{\bar{1}} \neq 0$, then $v(\lambda x) = v(x)$ for every $0 < \lambda \leq 1$ and $x \in X$, while $|\delta_{\bar{1}}|$ is positively homogeneous. If $x_{\bar{1}} = 0$, then for every $0 < \lambda \leq 1$ we have that $Sh(\lambda x) = 0 = \lambda Sh(x)$.

Being continuous and positively homogeneous is a sufficient condition to belong to $FBL(A)$ in the case when A is finite [10]. What we can deduce from this in the infinite case is that a function $g\colon [-1,1]^A \longrightarrow \mathbb{R}$ belongs to $FBL(A)$ provided that is continuous, positively homogeneous and depends on finitely many coordinates [3, Lemma 3.1]. Depending on finitely many coordinates means that there is a finite subset $A_0 \subset A$ such that $g(x) = g(y)$ whenever $x|_{A_0} = y|_{A_0}$. We will prove that Sh can be obtained as the limit, in the $FBL(B_{C(K)})$-norm, of a sequence of continuous and positively homogeneous functions that only depend on a finite number of coordinates from $[-1,1]^{B_{C(K)}}$. This proves that $Sh \in FBL(B_{C(K)})$ since it is a closed space.

Consider $L = \{x \in X : x_{\bar{1}} = 1\} \subset X$. Since the restriction $f|_L$ is a continuous function on the compact space L, by the Stone-Weierstrass' theorem, for every $n \in \mathbb{N}$ we can find a continuous function $f_n^+ \in C(L)$ that depends only on finitely many coordinates of the cube $[-1,1]^{B_{C(K)}}$ such that

$$\left\| f|_L - f_n^+ \right\|_\infty < \frac{1}{n}.$$

Define $f_n\colon X \longrightarrow \mathbb{R}$ by

$$f_n(x) := \begin{cases} f_n^+(v(x)) & \text{if } x_{\bar{1}} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $f_n(\lambda x) = f_n(x)$ for all $0 < \lambda \leq 1$ and $x \in X$, since $v(\lambda x) = v(x)$. Moreover, f_n depends on finitely many coordinates because f_n^+ does so, and each coordinate of v depends on two coordinates ($v(x)_\gamma$ only depends on x_γ and $x_{\bar{1}}$). On the other hand, $f_n \cdot |\delta_{\bar{1}}|$ is continuous in X. This is because $f_n \cdot |\delta_{\bar{1}}|$ is continuous in $\{x \in X : x_{\bar{1}} \neq 0\}$ clearly, and, if $x_{\bar{1}} = 0$, then for every $\varepsilon > 0$ there is a neighbourhood W such that $|f_n(y)| \cdot |y_{\bar{1}}| \leq \|f_n^+\|_\infty \cdot \varepsilon$ for all $y \in W$. Thus, the functions $f_n \cdot |\delta_{\bar{1}}|$ are all continuous, positively homogeneous and depend on finitely many coordinates. It follows from the aforementioned fact [3, Lemma 3.1] that $f_n \cdot |\delta_{\bar{1}}| \in FBL(B_{C(K)})$ for every $n \in \mathbb{N}$. It only remains to prove that $\|Sh - f_n \cdot |\delta_{\bar{1}}|\| \to 0$ when $n \to \infty$. For this, first notice that $v(v(x)) = v(x)$ for all $x \in X$ with $x_{\bar{1}} \neq 0$. This is just because $v(x)_{\bar{1}} = 1$. From this, it follows that $f(x) = f(v(x))$ for all x with $x_{\bar{1}} \neq 0$. This together with Lemma 3.4 gives:

$$\begin{aligned}
\|Sh - f_n \cdot |\delta_{\bar{1}}|\| &= \|f \cdot |\delta_{\bar{1}}| - f_n \cdot |\delta_{\bar{1}}|\| = \|(f - f_n) \cdot |\delta_{\bar{1}}|\| \\
&\leq \|f - f_n\|_\infty \\
&= \sup\{|f(x) - f_n(x)| : x \in X\} \\
&= \sup\{|f(x) - f_n(x)| : x \in X, x_{\bar{1}} \neq 0\} \\
&= \sup\left\{|f(x) - f_n^+(v(x))| : x \in X, x_{\bar{1}} \neq 0\right\} \\
&= \sup\left\{|f(v(x)) - f_n^+(v(x))| : x \in X, x_{\bar{1}} \neq 0\right\} \\
&\leq \sup\left\{|f(y) - f_n^+(y)| : y \in L\right\} = \|f|_L - f_n^+\|_\infty < \frac{1}{n}. \quad \square
\end{aligned}$$

4. Problems

Concerning the different variations of projectivity, it was already observed in [10] that if a Banach lattice P has the property that every homomorphism into a quotient $T\colon P \longrightarrow X/J$ can be lifted to a homomorphism $\hat{T}\colon P \longrightarrow X$, then P is λ-projective for some λ. It is obvious that the class of ∞-projective Banach lattices is closed under renorming but the 1^+-projective class is not. It was asked in [10] whether every ∞-projective Banach lattice is the renorming of a 1^+-projective Banach lattice. But, in fact, we do not know a single example that separates these two classes, even by renorming.

Problem 1. Find an equivalent norm on a 1^+-projective Banach lattice that makes it ∞-projective but not 1^+-projective.

A natural candidate would be $FBL[E]$ with E a suitable Banach space renorming of ℓ_1.

Theorems 1.2 and 1.3 suggest a large presence of the Banach space ℓ_1 inside projective Banach lattices. This does not exclude other subspaces ($C[0,1]$ is 1^+-projective and contains isometric copies of any separable Banach space) but we may at least ask:

Problem 2. If X is ∞-projective and infinite-dimensional, must X contain a Banach subspace isomorphic to ℓ_1?

We proved that E has the Schur property if $FBL[E]$ is ∞-projective. But the only positive case that we know is that $FBL[\ell_1(A)] = FBL(A)$ is 1^+-projective.

Problem 3. Is there a Banach space E with the Schur property, not isometric to $\ell_1(A)$, for which $FBL[E]$ is 1^+-projective? Is there a Banach space E with the Schur property, not isomorphic to $\ell_1(A)$, for which $FBL[E]$ is ∞-projective?

For a more complete picture, let us mention that other Banach lattices proved by B. de Pagter and A. W. Wickstead to be 1^+-projective include all finite-dimensional Banach lattices ([10, Theorem 11.1]), ℓ_1 and any countable ℓ_1-sum of separable 1^+-projective Banach lattices ([10, Theorem 11.11]).

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