

Persistence in a Periodic Competitor-Competitor-Mutualist Diffusion System¹

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In this paper we investigate the existence and the asymptotic behavior of periodic solutions for a periodic reaction-diffusion system of a competitor-competitor-mutualist model under Dirichlet boundary conditions. We shall prove that under certain conditions this system is persistent and under some other conditions it exhibits extinction or, more exactly, the species of mutualist extincts.

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1. INTRODUCTION

This paper is concerned with a reaction-diffusion competitor-competitor-mutualist system,

$$\begin{aligned} u_{1t} - d_1 \Delta u_1 &= g_1 u_1 \left(1 - \frac{u_1}{a_1} - \frac{a_2 u_2}{1 + a_3 u_3} \right) \\ u_{2t} - d_2 \Delta u_2 &= g_2 u_2 \left(1 - b_1 u_1 - \frac{u_2}{b_2} \right) \\ u_{3t} - d_3 \Delta u_3 &= g_3 u_3 \left(1 - \frac{u_3}{c_1 + c_2 u_1} \right), \end{aligned} \quad \text{in } \Omega \times \mathbf{R}^+ \quad (1.1)$$

with Dirichlet boundary conditions

$$u_i(x, t) = 0 \quad \text{on } \partial\Omega \times \mathbf{R}^+, \quad i = 1, 2, 3, \quad (1.2)$$

and the initial conditions

$$u_i(x, 0) = u_{i0}(x) \quad \text{on } \Omega, \quad i = 1, 2, 3, \quad (1.3)$$

where $u_1(x, t)$, $u_2(x, t)$, and $u_3(x, t)$ represent the populations of a mutualist-competitor, a competitor, and a mutualist, respectively, $\Omega \subset \mathbf{R}^N$ ($N \geq 1$) is a bounded domain with $C^{2+\alpha}$ -smooth ($0 < \alpha < 1$) boundary, $d_i = d_i(t)$ ($i = 1, 2, 3$) are strictly positive smooth T -periodic ($T > 0$) functions, and $a_i = a_i(x, t)$, $b_i = b_i(x, t)$, $c_i = c_i(x, t)$, $g_i = g_i(x, t)$ ($i = 1, 2, 3$) are positive smooth functions on $\bar{\Omega} \times \mathbf{R}$ which are T -periodic in t . The periodicity of coefficients models seasonal fluctuations. We are interested in the existence of positive T -periodic solutions of system (1.1)–(1.2) and the asymptotic behavior of positive solutions of system (1.1)–(1.3).

The O. D. E. problem associated with (1.1)–(1.3) was proposed and studied by Rai *et al.* in [11]. In [16] Zheng studied problem (1.1)–(1.3) as well as the Neumann problem in the case where all coefficients are constant. He proved the existence of semitrivial nonnegative equilibrium solutions and discussed the asymptotic stability of both such solutions and the trivial equilibrium solutions. For the system (1.1) with T -periodic coefficients, Tineo [13] studied the homogeneous Neumann problem and considered the asymptotic behavior of positive solutions. Recently, Du [6] investigated the existence of positive T -periodic solutions of the Dirichlet problem by using degree and bifurcation theories, and Pao [8] proved the existence of maximal and minimal T -periodic solutions of the Dirichlet problem by using the method of upper and lower solutions. However, these authors did not consider the asymptotic behavior of general positive solutions of system (1.1)–(1.3).

Asymptotic behavior of solutions of a population dynamical system is intimately connected, in a certain sense, to equilibrium solutions of the system. In recent developments in mathematical ecology, in addition to the classical Liapunov stability, some Lagrange stability concepts, such as persistence and extinction, have been drawing increasingly wide attention of investigators (cf. [7, 12]). Recall that a species with density $u(x, t)$ ($x \in \Omega$, $t > 0$) is said to be persistent if $u(x, t) > 0$ for all $x \in \Omega$, $t > 0$ and $\liminf_{t \rightarrow \infty} u(x, t) > 0$ for all $x \in \Omega$, and a system of several species is said to be persistent if all species are persistent; a species with density $u(x, t)$ ($x \in \Omega$, $t > 0$) is said to be extinct if $\lim_{t \rightarrow \infty} u(x, t) = 0$ for all $x \in \Omega$, and a system of several species is said to exhibit extinction if at least one species is extinct.

In this paper, following some ideas of Ahmad and Lazer [1], we study the asymptotic behavior of solutions of system (1.1)–(1.3). We shall prove that under certain conditions this system is persistent and under some other conditions it exhibits extinction or, more exactly, the species of mutualist extincts.

For $i = 1, 2, 3$, let $\lambda(d_i, g_i)$ be the principal eigenvalue of the periodic eigenvalue problem

$$\begin{aligned} \varphi_{it} - d_i \Delta \varphi_i - g_i \varphi_i &= \lambda_i \varphi_i && \text{in } \Omega \times \mathbf{R} \\ \varphi_i &= 0 && \text{on } \partial\Omega \times \mathbf{R} \\ \varphi_i(x, 0) &= \varphi_i(x, T) && \text{in } \Omega, \quad i = 1, 2, 3. \end{aligned} \quad (1.4)$$

Then $\lambda_{i1} \equiv \lambda(d_i, g_i)$ is real and its corresponding eigenfunction φ_i is either positive or negative in $\Omega \times \mathbf{R}$ (see, e.g., [3]). We always choose positive φ_i and normalize it so that $\max \varphi_i = 1$ in $\Omega \times \mathbf{R}$. And, for a given bounded function $f: \overline{\Omega} \times \mathbf{R} \rightarrow \mathbf{R}$, we denote by f_M (resp., f_L) the supremum (resp., infimum) of f on $\overline{\Omega} \times \mathbf{R}$.

Our main results are the following Theorems 1 and 2.

THEOREM 1 (Coexistence). *Assume that*

$$\lambda_{11} < -g_{1M} a_{2M} b_{2M} \quad (1.5)$$

$$\lambda_{21} < -g_{2M} a_{1M} b_{1M}. \quad (1.6)$$

Assume further that $\lambda_{31} < 0$. Then system (1.1)–(1.2) admits two positive T -periodic solutions (u_{1*}, u_2^*, u_{3*}) and (u_1^*, u_{2*}, u_3^*) such that for any positive T -periodic solution (u_{1T}, u_{2T}, u_{3T}) of system (1.1)–(1.2),

$$u_{i*} \leq u_{iT} \leq u_i^*, \quad \text{on } \overline{\Omega} \times \mathbf{R}, \quad i = 1, 2, 3. \quad (1.7)$$

Moreover, if (u_1, u_2, u_3) is a solution of system (1.1)–(1.3) and

$$u_{i0}(x) \geq 0 \quad (\neq 0) \quad \text{in } \Omega \quad \text{and} \quad u_{i0}|_{\partial\Omega} = 0, \quad i = 1, 2, 3,$$

then for any $\epsilon > 0$, there exists corresponding $t_\epsilon > 0$ such that

$$u_{i*}(x, t) - \epsilon \varphi_i < u_i(x, t) < u_i^*(x, t) + \epsilon \varphi_i \quad (i = 1, 2, 3) \quad (1.8)$$

for all $x \in \bar{\Omega}$ and $t > t_\epsilon$.

COROLLARY 1. *Under the assumptions of Theorem 1, any trivial and semitrivial equilibrium solutions of system (1.1)–(1.2) are unstable.*

THEOREM 2 (Extinction of u_3). *Assume that (1.5), (1.6) hold. Assume further that $\lambda_{31} \geq 0$. Then problem (1.1)–(1.2) has semitrivial equilibrium solutions $(u_{1*}, u_{2*}^*, 0)$ and $(u_1^*, u_{2*}, 0)$ such that*

$$0 < u_{i*} \leq u_i^* \quad \text{on } \Omega \times \mathbf{R}.$$

Moreover, if (u_1, u_2, u_3) is a solution of system (1.1)–(1.3), and

$$u_{i0}(x) \geq 0 \ (\neq 0) \quad \text{in } \Omega \quad \text{and} \quad u_{i0}|_{\partial\Omega} = 0, \quad i = 1, 2, 3,$$

then for any $\epsilon > 0$, there exists corresponding $t_\epsilon > 0$, such that

$$u_{i*}(x, t) - \epsilon \varphi_i < u_i(x, t) < u_i^*(x, t) + \epsilon \varphi_i \quad (i = 1, 2)$$

$$u_3(x, t) < \epsilon \varphi_3$$

for all $x \in \bar{\Omega}$ and $t > t_\epsilon$.

Remark 1. When $a_3 = 0$, the system (1.1) reduces to the classical Lotka–Volterra competition model which has been investigated by many authors (see [1, 8]).

2. PRELIMINARIES

For $0 < \alpha < 1$, denote

$$E = C^{2+\alpha, 1+\alpha/2}(\bar{\Omega} \times \mathbf{R}^+),$$

$$F = \{w \in C^{2+\alpha, 1+\alpha/2}(\bar{\Omega} \times \mathbf{R}): w = w(x, t)$$

$$\text{is } T\text{-periodic in } t \text{ and } w|_{\partial\Omega \times \mathbf{R}} = 0\}.$$

By a classical solution of system (1.1)–(1.3) we mean a vector function $(u_1, u_2, u_3): \bar{\Omega} \times \mathbf{R}^+ \rightarrow E^3$ satisfying (1.1), (1.2), and (1.3), and by a classical T -periodic solution of system (1.1)–(1.2) we mean a vector function $(u_1, u_2, u_3): \bar{\Omega} \times \mathbf{R} \rightarrow F^3$ satisfying (1.1) and (1.2).

Consider the logistic problem

$$u_t - d\Delta u = u(a - bu) \quad \text{in } \Omega \times \mathbf{R}^+ \quad (2.1)$$

$$u(x, t) = 0 \quad \text{on } \partial\Omega \times \mathbf{R}^+ \quad (2.2)$$

$$u(x, 0) = u_0(x) \quad \text{on } \Omega, \quad (2.3)$$

where d is a strictly positive smooth T -periodic function, and a and b are positive smooth T -periodic functions on $\bar{\Omega} \times \mathbf{R}$. The following result is more or less well known [1, 15], and we omit its proof.

LEMMA 1. (1) *If $\lambda(d, a) < 0$, then problem (2.1), (2.2) has exactly one positive classical T -periodic solution $\theta[d, a, b] \equiv \theta[a, b]$ such that*

$$0 < \theta[a, b] \leq (a/b)_M \quad \text{in } \Omega \times \mathbf{R}. \quad (2.4)$$

Furthermore, if $u(x, t)$ is a solution of (2.1)–(2.3), then

$$\lim_{t \rightarrow \infty} |\theta[a, b] - u(x, t)| = 0$$

uniformly for $x \in \bar{\Omega}$, where $u|_{\partial\Omega \times \mathbf{R}^+} = 0$ and $u(x, 0) \geq 0$ ($\neq 0$) in Ω .

(2) *The problem (2.1), (2.2) has no nontrivial solutions, if and only if $\lambda_1(d, -a) \geq 0$.*

To derive the global asymptotic behavior of solutions of (1.1)–(1.3), the following result will play an essential role [5, 14].

LEMMA 2. *If $u(x), v(x) \in C^1(\bar{\Omega})$, $u|_{\partial\Omega} = v|_{\partial\Omega} = 0$, $v(x) > 0$ for $x \in \Omega$ and $\frac{\partial v}{\partial n}|_{\partial\Omega} < 0$, where $\frac{\partial}{\partial n}$ represents the outward normal derivative on $\partial\Omega$, then there exists a positive constant K such that $u(x) \leq K v(x)$ for all $x \in \Omega$.*

LEMMA 3. *If $\lambda(d, a) < 0$, $a \leq A$, and $0 < B < b$, then $\theta[a, b] \leq \theta[A, B]$.*

Proof. By Lemma 2.2 of [3], we have $\lambda(d, A) < 0$. By Lemma 1, we can choose a number $\gamma > 0$ such that $(\frac{\partial}{\partial t} - d\Delta + \gamma)\theta[A, B] \geq 0$. Since $\theta[A, B] = 0$ on $\partial\Omega \times \mathbf{R}$, by the extended parabolic maximum principle [10], $\frac{\partial \theta[A, B]}{\partial n}|_{\partial\Omega \times \mathbf{R}} < 0$. Thus, by Lemma 2, there exists a sufficiently small $\delta > 0$ such that $\delta\phi(0, x) \leq \theta[A, B]|_{t=0}$ and $\delta b_M < -\lambda(d, a)$, where ϕ is the principal eigenfunction of $\partial_t - d\Delta - a$ with Dirichlet boundary condition for the space variable and T -periodic condition for the time variable satisfying $\max_{\bar{\Omega} \times \mathbf{R}} \phi(x, t) = 1$.

It is easy to verify that $\theta[A, B]$ and $\delta\phi$ are B -related upper and lower solutions [2, Remark 5.5] of the following problem:

$$\begin{aligned} u_t - d\Delta u &= u(a - bu) \\ u|_{\partial\Omega \times \mathbf{R}} &= 0, \quad u(x, t + T) = u(x, t). \end{aligned}$$

Thus by Lemma 1.2 of [2] we have $\delta\phi(x) \leq \theta[a, b] \leq \theta[A, B]$.

From Lemma 1 and Lemma 3 we can easily obtain the following result.

LEMMA 4. Assume that $\lambda_{31} < 0$ and

$$a_{2M}\theta_{2M}g_{1M} < -\lambda_{11}, \quad (2.5)$$

$$a_{1M}\theta_{1M}g_{2M} < -\lambda_{21}. \quad (2.6)$$

If (u_1, u_2, u_3) is a classical positive T -periodic solution of (1.1), (1.2), then for $(x, t) \in \Omega \times \mathbf{R}$,

$$\begin{aligned} 0 < \theta \left[d_1, g_1(1 - a_2\theta_2), \frac{g_1}{a_1} \right] &\leq u_1 \leq \theta \left[d_1, g_1, \frac{g_1}{a_1} \right] \equiv \theta_1 \\ 0 < \theta \left[d_2, g_2(1 - b_1\theta_1), \frac{g_2}{b_2} \right] &\leq u_2 \leq \theta \left[d_2, g_2, \frac{g_2}{b_2} \right] \equiv \theta_2 \\ 0 < \theta \left[d_3, g_3, \frac{g_3}{c_1} \right] &\leq u_3 \leq \theta \left[d_3, g_3, \frac{g_3}{(c_1 + c_2\theta_1)} \right] \equiv \theta_3. \end{aligned} \quad (2.7)$$

Remark. Assume that (1.5) and (1.6) hold. Then (2.5) and (2.6) hold.

By Theorem 5.3 of [14] or, equivalently, the main results of [9], we have the following existence-comparison result.

LEMMA 5. Assume that $U_i \geq 0$, $V_i \geq 0$ ($i = 1, 2, 3$) are smooth functions on $\overline{\Omega} \times [0, \infty)$ such that (U_1, V_2, U_3) satisfies the inequalities

$$\begin{aligned} U_{1t} &\geq d_1\Delta U_1 + g_1U_1 \left(1 - U_1/a_1 - \frac{a_2V_2}{1 + a_3U_3} \right) \\ V_{2t} &\leq d_2\Delta V_2 + g_2V_2 \left(1 - b_1U_1 - \frac{V_2}{b_2} \right) \\ U_{3t} &\geq d_3\Delta U_3 + g_3U_3 \left(1 - \frac{U_3}{c_1 + c_2U_1} \right) \end{aligned} \quad (2.8)$$

in $\Omega \times (0, \infty)$ and (V_1, U_2, V_3) satisfies the corresponding reversed inequalities. Assume further that $U_i = V_i = 0$ on $\partial\Omega \times (0, \infty)$ and $V_i(x, 0) \leq u_{i0}(x) \leq U_i(x, 0)$ on $\overline{\Omega}$. Then for any positive smooth initial functions $u_{i0} \geq 0$ ($\neq 0$) ($i = 1, 2, 3$), the problem (1.1)–(1.3) has exactly one classical solution (u_1, u_2, u_3) and

$$V_i \leq u_i \leq U_i \quad \text{in } \Omega \times (0, \infty). \quad (2.9)$$

(U_1, U_2, U_3) (resp. (V_1, V_2, V_3)) is called an upper (resp. a lower) solution of the problem (1.1)–(1.3).

3. PROOFS OF THEOREMS

Proof of Theorem 1. In this section, we always assume that $\delta > 0$, $r \geq 1$, and

$$\delta\varphi_i(x, 0) \leq \theta_i \mid_{t=0}, \quad (3.1)$$

$$\delta\varphi_1/a_1 + a_2r\theta_2/(1 + a_3\delta\varphi_3) \leq -\lambda_{11}/g_1, \quad (3.2)$$

$$b_1r\theta_1 + \delta\varphi_2/b_2 \leq -\lambda_{21}/g_2, \quad (3.3)$$

$$\delta\varphi_3/(c_1 + c_2\delta\varphi_1) \leq -\lambda_{31}/g_3, \quad (3.4)$$

where φ_i is the principal eigenfunction of (1.4) satisfying $\max_{\bar{\Omega} \times \mathbf{R}} \varphi(x, t) = 1$. Such choices are obviously possible because $\lambda_{31} < 0$ and (1.5), (1.6) hold.

To prove Theorem 1, we proceed in several steps.

Step 1. First we consider the solution of system (1.1)–(1.3) with the following initial conditions:

$$\begin{aligned} u_{10}(x) &= r\theta \left[d_1, g_1, \frac{g_1}{a_1} \right] \mid_{t=0}, \\ u_{20}(x) &= \delta\varphi_2(x, 0), \\ u_{30}(x) &= r\theta \left[d_3, g_3, \frac{g_3}{(c_1 + c_2\theta_1)} \right]. \end{aligned}$$

From (3.1)–(3.4) we know that $\delta(\varphi_1(t, x), \varphi_2(t, x), \varphi_3(t, x))$ (resp. $r(\theta_1, \theta_2, \theta_3)$) is a subsolution (resp. supersolution) of problem (1.1)–(1.3). Therefore, by Lemma 5, there exists a unique global classical solution (u_1, u_2, u_3) of problem (1.1)–(1.3) such that

$$\delta\phi_i(x, t) \leq u_i(x, t) \leq r\theta_i(x, t) \quad \text{on } \bar{\Omega} \times [0, \infty), \quad i = 1, 2, 3.$$

Let $u_{i1}(x, t) = u_i(x, t + T)$. Then

$$\begin{aligned} \delta\varphi_1(x, 0) &\leq u_{11}(x, 0) = u_1(x, T) \leq r\theta_1 \mid_{t=0} = u_1(x, 0), \\ \delta\varphi_2(x, 0) &= u_2(x, 0) \leq u_{21}(x, 0) = u_2(x, T) \leq r\theta_2 \mid_{t=0}, \\ \delta\varphi_3(x, 0) &\leq u_{31}(x, 0) = u_3(x, T) \leq r\theta_3 \mid_{t=0} = u_3(x, 0). \end{aligned}$$

Choose $(U_1, U_2, U_3) = (u_1, u_{21}, u_3)$, $(V_1, V_2, V_3) = (u_{11}, u_2, u_{31})$. By Lemma 5, we have

$$\begin{aligned} u_{11}(x, t) &\leq u_1(x, t), \\ u_2(x, t) &\leq u_{21}(x, t), \\ u_{31}(x, t) &\leq u_3(x, t). \end{aligned}$$

If for each integer n , we define $u_{in}(x, t) = u_i(x, t + nT)$ ($i = 1, 2, 3$), then a similar argument shows that

$$\begin{aligned}\delta\varphi_1(x, t) &\leq u_{1n}(x, t) \leq u_{1,n-1}(x, t) \leq r\theta_1 \\ \delta\varphi_2(x, t) &\leq u_{2,n-1}(x, t) \leq u_{2n}(x, t) \leq r\theta_2 \\ \delta\varphi_3(x, t) &\leq u_{3n}(x, t) \leq u_{3,n-1}(x, t) \leq r\theta_3\end{aligned}\quad (3.5)$$

on $\overline{\Omega} \times [0, +\infty)$. It follows from (3.5) that there exist functions u_1^* , u_{2*} , and u_3^* defined on $\overline{\Omega} \times [0, +\infty)$ such that

$$\lim_{n \rightarrow \infty} (u_{1n}, u_{2n}, u_{3n})(x, t) = (u_1^*, u_{2*}, u_3^*)(x, t) \quad (3.6)$$

$$\delta\varphi_1 \leq u_1^* \leq r\theta_1, \quad \delta\varphi_2 \leq u_{2*} \leq r\theta_2, \quad \delta\varphi_3 \leq u_3^* \leq r\theta_3. \quad (3.7)$$

Since d_i, a_i, b_i, c_i , and g_i are smooth T -periodic functions, we can show, by applying a bootstrap argument similar to that used in the proofs of Theorems 2.1, 4.1 of Ahmad and Lazer [1], that (u_1^*, u_{2*}, u_3^*) is a classical positive T -periodic solution of the problem (1), (2), and

$$\begin{aligned}\lim_{t \rightarrow \infty} (u_i(x, t) - u_i^*(x, t)) &= 0, & i = 1, 3 \\ \lim_{t \rightarrow \infty} (u_2(x, t) - u_{2*}(x, t)) &= 0\end{aligned}\quad (3.8)$$

uniformly for x in $\overline{\Omega}$.

Step 2. Next we consider the solution of (1.1)–(1.3) with the initial conditions

$$u_{10}(x) = \delta\varphi_1(x, 0), \quad u_{20}(x) = r\theta[d_2, g_2, g_2/b_2]|_{t=0}, \quad u_{30}(x) = \delta\varphi_3(x, 0).$$

Notice that $r(\theta_1, \theta_2, \theta_3)(\delta(\varphi_1, \varphi_2, \varphi_3))$ is a supersolution (subsolution) of (1.1)–(1.3). By Lemma 5, there exists a unique classical global solution (v_1, v_2, v_3) of (1.1)–(1.3) such that

$$\delta\varphi_i(x, t) \leq v_i(x, t) \leq r\theta_i(x, t) \quad \text{on } \Omega \times [0, +\infty), \quad i = 1, 2, 3.$$

A parallel argument shows that there exist smooth functions u_{1*} , u_2^* , and u_{3*} defined on $\overline{\Omega} \times \mathbf{R}$ such that (u_{1*}, u_2^*, u_{3*}) is a T -periodic solution of the problem (1.1), (1.2), and

$$\begin{aligned}\lim_{t \rightarrow \infty} (v_i(x, t) - u_{i*}(x, t)) &= 0, & i = 1, 3 \\ \lim_{t \rightarrow \infty} (v_2(x, t) - u_2^*(x, t)) &= 0\end{aligned}\quad (3.9)$$

uniformly for x in $\overline{\Omega}$.

Since $u_i(x, 0) \geq v_i(x, 0)$ ($i = 1, 3$) and $u_2(x, 0) \leq v_2(x, 0)$, by Lemma 5 we have

$$\begin{aligned} u_i(x, t) &\geq v_i(x, t), & i = 1, 2, \\ u_2(x, t) &\leq v_2(x, t) \end{aligned} \quad (3.10)$$

on $\bar{\Omega} \times [0, +\infty)$. Furthermore, we see from (3.8) and (3.9) that

$$u_{i*}(x, t) \leq u_i^*(x, t) \quad (3.11)$$

for all $(x, t) \in \bar{\Omega} \times \mathbf{R}$.

Let (u_1, u_2, u_3) be a classical positive T -periodic solution of (1.1), (1.2). Then, by (1.5), (1.6) and Lemma 4, we have

$$u_{i*}(x, t) \leq u_i(x, t) \leq u_i^*(x, t) \quad \text{on } \bar{\Omega} \times \mathbf{R}, \quad i = 1, 2, 3. \quad (3.12)$$

Hence, all sufficiently small values of $\delta > 0$ and r satisfying (3.1)–(3.4) give the same T -periodic solutions (u_{1*}, u_2^*, u_{3*}) and (u_1^*, u_{2*}, u_3^*) by Steps 1 and 2.

Step 3. Now let us consider the solution of (1.1)–(1.3) with general initial conditions, i.e., $(u_{10}(x), u_{20}(x), u_{30}(x))$ are nonnegative smooth functions on $\bar{\Omega}$ such that $u_{i0}(x) \not\equiv 0$ in Ω and $u = v = 0$ on $\partial\Omega$. Let p_1, p_2 , and p_3 be the global solutions of the problems

$$\begin{aligned} u_t - d_1 \Delta u &= g_1 u(1 - u/a_1) \\ u|_{\partial\Omega \times \mathbf{R}^+} &= 0 \end{aligned} \quad (3.13)$$

$$\begin{aligned} u(x, 0) &= u_{10}(x), \\ v_t - d_2 \Delta v &= g_2 v(1 - v/b_2) \\ v|_{\partial\Omega \times \mathbf{R}^+} &= 0 \\ v(x, 0) &= u_{20}(x), \end{aligned} \quad (3.14)$$

and

$$\begin{aligned} w_t - d_3 \Delta w &= g_3 w[1 - w/(c_1 + c_2 p_1)] \\ w|_{\partial\Omega \times \mathbf{R}^+} &= 0 \\ w(x, 0) &= u_{30}(x), \end{aligned} \quad (3.15)$$

respectively. Then, by the parabolic minimum principle [10], we have $p_i(x, t) > 0$ for $(x, t) \in \Omega \times [0, +\infty)$, so that (p_1, p_2, p_3) (resp. $(0, 0, 0)$) is a supersolution (resp. subsolution) of problem (1.1)–(1.3). Therefore, there exists a unique global classical solution (u_1, u_2, u_3) such that

$$0 \leq u_i \leq p_i \quad \text{on } \bar{\Omega} \times [0, \infty), \quad i = 1, 2, 3. \quad (3.16)$$

Since $u_{1t} - d_1 u_1 + (p_{1M}/a_{1L} + a_{1M} p_{2M}) u_1 \geq 0$, by the extended parabolic minimum principle [10], we have, for each integer $m > 0$, $u_1(x, mT) > 0$ in Ω , and $\partial u_1 / \partial n < 0$ on $\partial\Omega \times \mathbf{R}^+$. By Lemma 2, for each $m > 0$, there exists a sufficiently small $\delta_0 > 0$ such that $\delta_0 \varphi_1(x, mT) \leq u_1(x, mT)$ in Ω . Similarly, there exists a sufficiently small δ (still denoted by δ_0), such that $\delta_0 \varphi_2(x, mT) \leq u_2(x, mT)$, $\delta_0 \varphi_3(mT, x) \leq u_3(mT, x)$ in Ω . To proceed further we need two claims.

Claim 1. For any $r > 1$, there exists $M \in \mathbf{N}$, such that $p_1(mT, x) \leq r\theta(0, x)$ on $\bar{\Omega}$ if $m > M$.

Proof. By Lemma 1 (or an argument similar to that used in the proof of Theorem 2.1 in [1]), we have

$$p_1(x, mT) \rightarrow \theta(x, 0) \quad \text{in } C^{2+\alpha}(\bar{\Omega}) \quad (m \rightarrow \infty), \quad (3.17)$$

so that

$$\frac{\partial p_1(x, mT)}{\partial n} \rightarrow \frac{\partial}{\partial n} \theta(x, 0) \quad \text{in } C(\partial\Omega) \quad (m \rightarrow \infty). \quad (3.18)$$

By (3.17), (3.18) and the facts that $\frac{\partial}{\partial n} \theta(x, 0)|_{\partial\Omega} < 0$, $\frac{\partial}{\partial n} p_1(x, mT) < 0$, we see that for any $r > 1$ there exists $M_1 \in \mathbf{N}$ such that if $m > M$ then $\frac{\partial}{\partial n} (r\theta(x, 0) - p_1(x, mT))|_{\partial\Omega} < 0$. Since $r\theta(x, 0) - p_1(x, mT) = 0$ on $\partial\Omega$, there exists a domain $\Omega_\epsilon \subset \subset \Omega$ such that $r\theta(x, 0) - p_1(x, mT) > 0$ for all $x \in (\Omega \setminus \Omega_\epsilon) \cup \partial\Omega_\epsilon$. Therefore, we need only prove that Claim 1 is true for $x \in \Omega_\epsilon$. If the conclusion is false, then there exists $r_0 > 1$, for each $n \in \mathbf{N}$, and there exist $k_n (> n)$ and $x_{k_n} \in \Omega_\epsilon$ such that

$$p_1(x, k_n T) > r_0 \theta_1(x, 0). \quad (3.19)$$

Since there exists a subsequence $\{x_{k'_n}\} \subset \{x_{k_n}\}$ such that $x_{k'_n} \rightarrow x_0$, by (3.19), we have $\theta_1(x_0, 0) \geq r_0 \theta_1(x_0, 0)$. This is impossible. Hence, there exists M_2 such that $p_1(x, mT) < r\theta_1(x, 0)$ in Ω_ϵ if $m > M_2$, and so the proof is complete.

Consider the auxiliary system

$$\begin{aligned} u_{1t} &= d_1 \Delta u_1 + g_1 u_1 (1 - u_1/a_1) \\ u_{3t} &= d_3 \Delta u_3 + g_3 u_3 \left(1 - \frac{u_3}{c_1 + c_2 u_1}\right) \\ u_1 &= u_3 = 0 \quad \text{on } \partial\Omega \times \mathbf{R}^+ \\ u_i(0, x) &= u_{i0}(x) \quad \text{in } \Omega, \quad i = 1, 3. \end{aligned} \quad (3.20)$$

If $u_{i0} (\geq 0, \neq 0, i = 1, 3)$ are sufficiently smooth functions, $u_{i0}(x) = 0$ for $x \in \Omega$ and

$$\lambda_{11}(d_1, g_1) < 0, \quad \lambda_{31}(d_3, g_3) < 0, \quad (3.21)$$

then, by Lemma 1 and arguments similar to those in the proof of Corollary 3.4 in [13], the problem (3.20) has exactly one solution (u_1, u_3) and

$$u_i(x, t) - \theta_i(x, t) \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad i = 1, 3, \quad (3.22)$$

uniformly on $\bar{\Omega}$.

Analogously, we have Claim 2.

Claim 2. For any $r > 1$, there exists $M \in \mathbb{N}$ such that $p_i(x, mT) \leq r\theta_i(x, 0)$ on $\bar{\Omega}$ if $m > M$.

The proof of this claim is similar to that of Claim 1.

By the above claims we know that, for any $r > 1$ satisfying (3.2) and (3.3), there exist a sufficiently large M and a sufficiently small $\delta > 0$ which satisfies (3.1)–(3.4) such that

$$\delta\varphi_i(x, 0) \leq u_i(x, mT) \leq p_i(x, mT) \leq r\theta_i(x, 0). \quad (3.23)$$

Let $(\underline{u}_1, \underline{u}_2, \underline{u}_3)$ (resp. $(\bar{u}_1, \bar{u}_2, \bar{u}_3)$) be the unique solution of (1.1), (1.2) with the function

$$\begin{aligned} (\underline{u}_1(x, mT), \bar{u}_2(x, mT), \underline{u}_3(x, mT)) &= (\delta\varphi_1(x, 0), r\theta_2(x, 0), \delta\varphi_3(x, 0)) \\ (\text{resp. } (\bar{u}_1(x, mT), \underline{u}_2(x, mT), \bar{u}_3(x, mT))) &= (r\theta_1(x, 0), \delta\varphi_2(x, 0), r\theta_3(x, 0)). \end{aligned}$$

By Lemma 5, we have

$$\delta\varphi_i(x, t) \leq \underline{u}_i(x, t) \leq u_i(x, t) \leq \bar{u}_i(x, t) \leq r\theta_i(x, t) \quad (3.24)$$

on $\bar{\Omega} \times [mT, \infty)$. By Steps 1 and 2, we still have

$$\lim_{t \rightarrow \infty} [\underline{u}_i(x, t) - u_{i*}(x, t)] = 0 = \lim_{t \rightarrow \infty} [\bar{u}_i(x, t) - u_i^*(x, t)] \quad (3.25)$$

uniformly for x in $\bar{\Omega}$. The foregoing conclusions show that the limits (3.25) hold in $C^{2,\alpha}(\Omega)$. Hence, for any $\epsilon > 0$, there exists corresponding $t_\epsilon > 0$ such that

$$u_{i*}(x, t) - \epsilon\varphi_i < u_i(x, t) < u_i^*(x, t) + \epsilon\varphi_i \quad (3.26)$$

on $\bar{\Omega} \times (t_\epsilon, \infty)$, and the proof of Theorem 1 is completed.

Proof of Theorem 2. Using the method of upper and lower solutions and bootstrap arguments, Theorem 2 can be proved in the same way as Theorem 1.

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