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J. Math. Anal. Appl. 302 (2005) 509–520

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces[☆]

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Received 23 April 2004

Available online 6 October 2004

Submitted by T.D. Benavides

Abstract

The iteration scheme for families of nonexpansive mappings, essentially due to Halpern [Bull. Amer. Math. Soc. 73 (1967) 957–961], is established in a Banach space. The main theorem extends a recent result of O'Hara et al. [Nonlinear Anal. 54 (2003) 1417–1426] to a Banach space setting. For the same iteration scheme, with finitely many mappings, a complementary result to a result of Jung and Kim [Bull. Korean Math. Soc. 34 (1997) 93–102] (also Bauschke [J. Math. Anal. Appl. 202 (1996) 150–159]) is obtained by imposing other condition on the sequence of parameters. Our results also improve results in [C. R. Acad. Sci. Sér A–B Paris 284 (1977) 1357–1359; J. Math. Anal. Appl. 211 (1997) 71–83; Arch. Math. 59 (1992) 486–491] in framework of a Hilbert space.
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Keywords: Nonexpansive mapping; Common fixed points; Iteration scheme; Sunny and nonexpansive retraction

1. Introduction

Let C be a nonempty closed convex subset of a Banach space E and let T_1, \dots, T_N be nonexpansive mappings from C into itself (recall that a mapping $T : C \rightarrow C$ is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$).

[☆] This paper was supported by the Dong-A University Research Fund in 2002.
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We consider the iteration scheme: for N , nonexpansive mappings T_1, T_2, \dots, T_N and $a, x_0 \in C$,

$$x_{n+1} = \lambda_{n+1}a + (1 - \lambda_{n+1})T_{n+1}x_n, \quad n \geq 0. \quad (1)$$

In 1967, Halpern [9] firstly introduced the iteration scheme (1) for $a = 0$, $N = 1$ (that is, he considered only one mapping T); see also Browder [3]. He pointed out that the conditions $\lim_{n \rightarrow \infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$ are necessary in the sense that, if the iteration scheme (1) converges to a fixed point of T , then these conditions must be satisfied. Ten years later, Lions [12] investigated the general case in Hilbert space under the conditions $\lim_{n \rightarrow \infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\lim_{n \rightarrow \infty} (\lambda_n - \lambda_{n+1})/\lambda_{n+1}^2 = 0$ on the parameters. However, Lions' conditions on the parameters were more restrictive and did not include the natural candidate $\lambda_n = 1/n + 1$. In 1980, Reich [16] gave the iteration scheme (1) for $N = 1$ in the case when E is uniformly smooth and $\lambda_n = n^{-a}$ with $0 < a < 1$.

In 1992, Wittmann [20] studied the iteration scheme (1) for $N = 1$ in the case when E is a Hilbert space and $\{\lambda_n\}$ satisfies

$$0 \leq \lambda_n \leq 1, \quad \lim_{n \rightarrow \infty} \lambda_n = 0, \quad \sum_{n=1}^{\infty} \lambda_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$$

In 1994, Reich [17] obtained a strong convergence of the iterates (1) for $N = 1$ with two necessary and decreasing conditions on parameters for convergence in the case when E is uniformly smooth with a weakly continuous duality mapping. In 1996, Bauschke [2] improves results of Wittmann to finitely many mappings with additional condition on the parameters $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}| < \infty$, where $T_n =: T_{n \bmod N}$, $N > 1$. He also provided an algorithmic proof which has been used successfully, with modifications, by many authors [5,13,18,21,22]. In 1997, Jung and Kim [10] extended Bauschke's result to a Banach space and Shioji and Takahashi [19] improved Wittmann's result to a Banach space. Shimizu and Takahashi [18], in 1997, dealt with the above iteration scheme with the necessary conditions on the parameters and some additional conditions imposed on the mappings in a Hilbert space.

Very recently, O'Hara et al. [13] generalized the result of Shimizu and Takahashi [18] and proved a result of Bauschke [1] by imposing a new condition on the parameters, $\lim_{n \rightarrow \infty} \lambda_n/\lambda_{n+N} = 1$, in the framework of a Hilbert space, which is not comparable with Bauschke's condition $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}| < \infty$.

In this paper, we establish the strong convergence of the iteration scheme $\{x_n\}$ defined by (1) for infinitely many nonexpansive mappings in a uniformly smooth Banach space with a weakly sequentially continuous duality mapping. The results extend results of O'Hara et al. [13] to a Banach space setting. Then we obtain a complementary result to a result of Jung and Kim [10] (also Bauschke [2]) for the same iteration scheme, with finitely many mappings. Our main results also improve and unify results in [12,18,20] in Hilbert spaces.

2. Preliminaries and lemmas

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f \rangle$. When $\{x_n\}$ is a sequence in E , then $x_n \rightarrow x$

(respectively $x_n \rightharpoonup x$, $x_n \xrightarrow{*} x$) will denote strong (respectively weak, weak*) convergence of the sequence $\{x_n\}$ to x .

The norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2)$$

exists for each x, y in its unit sphere $U = \{x \in E: \|x\| = 1\}$. It is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*) if the limit in (2) is attained uniformly for $(x, y) \in U \times U$.

The (normalized) *duality* mapping J from E into the family of nonempty (by Hahn–Banach theorem) weak-star compact subsets of its dual E^* is defined by

$$J(x) = \{f \in E^*: \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$$

for each $x \in E$. It is single valued if and only if E is smooth. It is also well known that if E has a uniformly Fréchet differentiable norm, J is uniformly continuous on bounded subsets of E (cf. [4,6]). Suppose that J is single valued. Then J is said to be *weakly sequentially continuous* if for each $\{x_n\} \in E$ with $x_n \rightharpoonup x$, $J(x_n) \xrightarrow{*} J(x)$.

We need the following lemma for the proof of our main results, which was given in Jung and Morales [11]. It is actually Lemma 1 of Petryshyn [15] (also see Asplund [1]).

Lemma 1. *Let X be a real Banach space and let J be the normalized duality mapping. Then for any given $x, y \in X$, we have*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle \quad (3)$$

for all $j(x + y) \in J(x + y)$.

A Banach space E is said to satisfy *Opial's condition* [14] if for any sequence $\{x_n\}$ in E , $x_n \rightharpoonup x$ implies

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$. We know that if E admits a weakly sequentially continuous duality mapping, then E satisfies Opial's condition; see [8].

Recall that a mapping T defined on a subset C of a Banach space E (and taking values in E) is said to be *demiclosed* if for any sequence $\{u_n\}$ in C the following implication holds:

$$u_n \rightharpoonup u \quad \text{and} \quad \lim_{n \rightarrow \infty} \|Tu_n - w\| = 0$$

implies

$$u \in C \quad \text{and} \quad Tu = w.$$

The following lemma can be found in [7, p. 108].

Lemma 2. *Let E be a reflexive Banach space which satisfies Opial's condition, let C be a nonempty closed convex subset of E , and suppose $T: C \rightarrow E$ is nonexpansive. Then the mapping $I - T$ is demiclosed on C , where I is the identity mapping.*

Let C be a nonempty closed convex subset of E . A mapping Q of C into C is said to be a *retraction* if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for every $z \in R(Q)$, where $R(Q)$ is range of Q . Let D be a subset of C and let Q be a mapping of C into D . Then Q is said to be *sunny* if each point on the ray $\{Qx + t(x - Qx) : t > 0\}$ is mapped by Q back onto Qx , in other words,

$$Q(Qx + t(x - Qx)) = Qx$$

for all $t \geq 0$ and $x \in C$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D ; for more details, see [6].

The following lemma is well known (cf. [6, p. 48]).

Lemma 3. *Let C be a nonempty closed convex subset of a smooth Banach space E , D a subset of C , $J : E \rightarrow E^*$ the duality mapping of E , and $Q : C \rightarrow D$ a retraction. Then the following are equivalent:*

- (a) $\langle x - Qx, J(y - Qx) \rangle \leq 0$ for all $x \in C$ and $y \in D$;
- (b) $\|Qz - Qw\|^2 \leq \langle z - w, J(Qz - Qw) \rangle$ for all z and w in C ;
- (c) Q is both sunny and nonexpansive.

Finally, we need the following lemma.

Lemma 4 (Xu [21]). *Let $\{\lambda_n\}$ be a sequence in $(0, 1)$ that satisfies $\lim_{n \rightarrow \infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Let $\{a_n\}$ be a sequence of nonnegative real numbers that satisfies any one of the following conditions:*

- (a) *For all $\varepsilon > 0$, there exists an integer $N \geq 1$ such that for all $n \geq N$,*

$$a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n \varepsilon;$$

- (b) $a_{n+1} \leq (1 - \lambda_n)a_n + \mu_n$, $n \geq 0$, where $\mu_n \geq 0$ satisfies $\lim_{n \rightarrow \infty} \mu_n / \lambda_n = 0$;
- (c) $a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n c_n$, where $\limsup_{n \rightarrow \infty} c_n \leq 0$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

3. Main results

First, we study the strong convergence result in a Banach space which generalizes Theorem 3.3 of O'Hara et al. [13].

Theorem 5. *Let E be a uniformly smooth Banach space with a weakly sequentially continuous duality mapping $J : E \rightarrow E^*$, C a nonempty closed convex subset of E , and $T_n : C \rightarrow C$ ($n = 1, 2, 3, \dots$) nonexpansive mappings such that*

$$F := \bigcap_{n=1}^{\infty} \text{Fix}(T_n) \neq \emptyset.$$

Assume that $V_1, \dots, V_N : C \rightarrow C$ are nonexpansive mappings with the property: for all $k = 1, 2, \dots, N$ and for any bounded subset \tilde{C} of C , there holds

$$\lim_{n \rightarrow \infty} \sup_{x \in \tilde{C}} \|T_n x - V_k(T_n x)\| = 0. \quad (4)$$

Let $\{\lambda_n\}$ be a sequence in $(0, 1)$ which satisfies $\lim_{n \rightarrow \infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. For any a and x_0 in C , define

$$x_{n+1} = \lambda_{n+1}a + (1 - \lambda_{n+1})T_{n+1}x_n, \quad n \geq 0.$$

Then the sequence $\{x_n\}$ converges strongly to $Q_{F(V)}a$, where Q is a sunny nonexpansive retraction of C onto $F(V) := \bigcap_{k=1}^N \text{Fix}(V_k)$.

Proof. First, we note that assumption (4) implies that $\bigcap_{k=1}^N \text{Fix}(V_k) \supset F$. Note that $\{x_n\}$ is bounded since $F \neq \emptyset$. In fact, by induction, we show that $\|x_n - z\| \leq \max\{\|x_0 - z\|, \|a - z\|\}$ for all $n \geq 0$ and all $z \in F$. The result is clearly true for $n = 0$. Suppose the result is true for n . Let $z \in F$. Then by the nonexpansivity of T_{n+1} ,

$$\begin{aligned} \|x_{n+1} - z\| &= \|\lambda_{n+1}a + (1 - \lambda_{n+1})T_{n+1}x_n - z\| \\ &\leq \lambda_{n+1}\|a - z\| + (1 - \lambda_{n+1})\|T_{n+1}x_n - z\| \\ &\leq \lambda_{n+1}\|a - z\| + (1 - \lambda_{n+1})\|x_n - z\| \\ &\leq \lambda_{n+1}\max\{\|x_0 - z\|, \|a - z\|\} \\ &\quad + (1 - \lambda_{n+1})\max\{\|x_0 - z\|, \|a - z\|\} \\ &= \max\{\|x_0 - z\|, \|a - z\|\}. \end{aligned}$$

Moreover, since for all $n \geq 0$ and for any $z \in F$,

$$\begin{aligned} \|T_{n+1}x_n\| &\leq \|T_{n+1}x_n - z\| + \|z\| \leq \|x_n - z\| + \|z\| \\ &\leq \max\{\|x_0 - z\|, \|a - z\|\} + \|z\|, \end{aligned}$$

it follows that $\{T_{n+1}x_n\}$ is bounded. Since

$$\|x_{n+1} - T_{n+1}x_n\| = \lambda_{n+1}\|a - T_{n+1}x_n\| \leq \lambda_{n+1}(\|a\| + \|T_{n+1}x_n\|) \leq \lambda_{n+1}M$$

for some M , we also have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - T_{n+1}x_n\| = 0. \quad (5)$$

Let a subsequence $\{T_{n_j+1}x_{n_j}\}$ of $\{T_{n+1}x_n\}$ be such that

$$\begin{aligned} &\lim_{j \rightarrow \infty} \langle a - Q_{F(V)}a, J(x_{n_j+1} - Q_{F(V)}a) \rangle \\ &= \limsup_{n \rightarrow \infty} \langle a - Q_{F(V)}a, J(x_{n+1} - Q_{F(V)}a) \rangle \end{aligned}$$

and $T_{n_j+1}x_{n_j} \rightharpoonup p$ for some $p \in C$. By assumption, we have for any $k = 1, 2, \dots, N$ and for $\tilde{C} = \{x_n\}$,

$$\begin{aligned}
0 &= \lim_{n \rightarrow \infty} \sup_{x \in \tilde{C}} \|T_{n+1}x - V_k(T_{n+1}x)\| \geq \lim_{n \rightarrow \infty} \sup \|T_{n+1}x_n - V_k(T_{n+1}x_n)\| \\
&\geq \lim_{j \rightarrow \infty} \sup \|T_{n_j+1}x_{n_j} - V_k(T_{n_j+1}x_{n_j})\|,
\end{aligned}$$

and so

$$\lim_{j \rightarrow \infty} \|T_{n_j+1}x_{n_j} - V_k(T_{n_j+1}x_{n_j})\| = 0 \quad \text{for all } k = 1, 2, \dots, N.$$

Thus, by Lemma 2, we have $p \in \text{Fix}(V_k)$ for $k = 1, 2, \dots, N$, that is, $p \in \bigcap_{k=1}^N \text{Fix}(V_k)$.

On the other hand, since E is uniformly smooth, F is a sunny nonexpansive retract of C (cf. [6, p. 49]). Thus, by weakly sequentially continuity of duality mapping J and Lemma 3, we have

$$\begin{aligned}
&\lim_{n \rightarrow \infty} \sup \langle a - Q_{F(V)}a, J(T_{n+1}x_n - Q_{F(V)}a) \rangle \\
&= \lim_{j \rightarrow \infty} \langle a - Q_{F(V)}a, J(T_{n_j+1}x_{n_j} - Q_{F(V)}a) \rangle \\
&= \langle a - Q_{F(V)}a, J(p - Q_{F(V)}a) \rangle \leq 0.
\end{aligned} \tag{6}$$

This together with (5) implies that

$$\lim_{n \rightarrow \infty} \sup \langle a - Q_{F(V)}a, J(x_{n+1} - Q_{F(V)}a) \rangle \leq 0. \tag{7}$$

Since $(1 - \lambda_{n+1})(T_{n+1}x_n - Q_{F(V)}a) = (x_{n+1} - Q_{F(V)}a) - \lambda_{n+1}(a - Q_{F(V)}a)$, by using the inequality (3) in Lemma 1, we have

$$\begin{aligned}
\|x_{n+1} - Q_{F(V)}a\|^2 &= \|(1 - \lambda_{n+1})(T_{n+1}x_n - Q_{F(V)}a) + \lambda_{n+1}(a - Q_{F(V)}a)\|^2 \\
&\leq (1 - \lambda_{n+1})^2 \|T_{n+1}x_n - Q_{F(V)}a\|^2 \\
&\quad + 2\lambda_{n+1} \langle a - Q_{F(V)}a, J(x_{n+1} - Q_{F(V)}a) \rangle \\
&\leq (1 - \lambda_{n+1}) \|x_n - Q_{F(V)}a\|^2 \\
&\quad + 2\lambda_{n+1} \langle a - Q_{F(V)}a, J(x_{n+1} - Q_{F(V)}a) \rangle.
\end{aligned} \tag{8}$$

Now, let $\varepsilon > 0$ be arbitrary. Then by (7), there exists N_ε such that

$$\langle a - Q_{F(V)}a, J(x_{n+1} - Q_{F(V)}a) \rangle \leq \frac{\varepsilon}{2} \quad \text{for all } n \geq N_\varepsilon.$$

Thus, from (8), we have

$$\|x_{n+1} - Q_{F(V)}a\|^2 \leq (1 - \lambda_{n+1}) \|x_n - Q_{F(V)}a\|^2 + \lambda_{n+1} \varepsilon. \tag{9}$$

Putting $a_n = \|x_n - Q_{F(V)}a\|^2$, we have from (9),

$$a_{n+1} \leq (1 - \lambda_{n+1})a_n + \lambda_{n+1} \varepsilon.$$

It follows from Lemma 4 that $a_n \rightarrow 0$ and hence $\{x_n\}$ converges strongly to $Q_{F(V)}a$. This completes the proof. \square

As a direct consequence, we have the following

Corollary 6 (O'Hara et al. [13, Theorem 3.3]). *Let H be a Hilbert space, C a nonempty closed convex subset of H , and $T_n: C \rightarrow C$ ($n = 1, 2, 3, \dots$) nonexpansive mappings such that*

$$F := \bigcap_{n=1}^{\infty} \text{Fix}(T_n) \neq \emptyset.$$

Assume that $V_1, \dots, V_N: C \rightarrow C$ are nonexpansive mappings with the property: for all $k = 1, 2, \dots, N$ and for any bounded subset \tilde{C} of C , there holds

$$\lim_{n \rightarrow \infty} \sup_{x \in \tilde{C}} \|T_n x - V_k(T_n x)\| = 0.$$

Let $\{\lambda_n\}$ be a sequence in $(0, 1)$ which satisfies $\lim_{n \rightarrow \infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. For any a and x_0 in C , define

$$x_{n+1} = \lambda_{n+1}a + (1 - \lambda_{n+1})T_{n+1}x_n, \quad n \geq 0.$$

Then the sequence $\{x_n\}$ converges strongly to $P_{F(V)}a$, where P is the nearest point projection of C onto $F(V) := \bigcap_{k=1}^N \text{Fix}(V_k)$.

Proof. Note that the nearest point projection P of C onto F is a sunny nonexpansive retraction. Thus the result follows from Theorem 5. \square

As in [13], by using Theorem 5 together with Lemmas 3.1 and 3.2 of [13] (Lemma 1 of [18]), we can also obtain the following result.

Corollary 7 (O'Hara et al. [13, Corollary 3.4]). *Let E be a Banach space, C a nonempty closed convex subset of E , and $T, S: C \rightarrow C$ nonexpansive mappings with fixed points.*

(a) *Set $T_n(x) = \frac{1}{n} \sum_{j=0}^{n-1} T^j x$ for $n \geq 1$ and $x \in C$. For $x_0, a \in C$, define*

$$x_{n+1} = \lambda_{n+1}a + (1 - \lambda_{n+1})T_{n+1}x_n, \quad n \geq 0.$$

If E is a uniformly convex and uniformly smooth Banach space with a weakly sequentially continuous duality mapping, then the sequence $\{x_n\}$ converges strongly to $Q_F a$, where Q is a sunny nonexpansive retraction of C onto $F := \text{Fix}(T)$.

(b) *Set $T_n(x) = \frac{2}{n(n+1)} \sum_{k=0}^{n-1} \sum_{i+j=k} S^i T^j(x)$ for $n \geq 1$ and $x \in C$, define*

$$x_{n+1} = \lambda_{n+1}a + (1 - \lambda_{n+1})T_{n+1}x_n, \quad n \geq 0.$$

Suppose that $ST = TS$ and $\text{Fix}(S) \cap \text{Fix}(T) \neq \emptyset$. If E is a Hilbert space H , then the sequence $\{x_n\}$ converges strongly to $P_{F(ST)}a$, where P is the nearest point projection of C onto $F(ST) := \text{Fix}(S) \cap \text{Fix}(T)$.

Remark 8. (1) Corollary 7(a) extends Corollary 3.4(a) in [13] to a Banach space setting.
(2) Theorem 1 of Shimizu and Takahashi [18] is just Corollary 7(b).

Now we consider the results developed by Bauschke [2] (also Jung and Kim [10]), in which he defined the following control conditions on the parameters $\{\lambda_n\}$:

- (B1) $\lim_{n \rightarrow \infty} \lambda_n = 0$;
 (B2) $\sum_{n=1}^{\infty} \lambda_n = \infty$; equivalently $\prod_{n=1}^{\infty} (1 - \lambda_n) = 0$;
 (B3) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}| < \infty$.

We will replace (B3) by the condition

$$(N3) \quad \lim_{n \rightarrow \infty} \frac{\lambda_n}{\lambda_{n+N}} = 1.$$

This condition also improves Lions' condition [12],

$$(L3) \quad \lim_{n \rightarrow \infty} \frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}^2} = 0.$$

Remark 9. Both (N3) and (B3) cover the natural candidate of $\lambda_n = \frac{1}{n+1}$, but (L3) does not. However, (B3) and (N3) are independent of each other. For more details, see [21].

We will give a complementary result to Theorem 1 of Jung and Kim [10] (also Theorem 3.1 of Bauschke [2]) with condition (B3) replaced by condition (N3).

We consider N mappings T_1, T_2, \dots, T_N . For $n > N$, set $T_n := T_{n \bmod N}$, where $n \bmod N$ is defined as follows: if $n = kN + l$, $0 \leq l < N$, then

$$n \bmod N := \begin{cases} l & \text{if } l \neq 0, \\ N & \text{if } l = 0. \end{cases}$$

Theorem 10. Let E be a uniformly smooth Banach space with a weakly sequentially continuous duality mapping $J : E \rightarrow E^*$ and C a nonempty closed convex subset of E . Let T_1, \dots, T_N be nonexpansive mappings from C into itself with $F := \bigcap_{i=1}^N \text{Fix}(T_i)$ nonempty and

$$F = \text{Fix}(T_N \dots T_1) = \text{Fix}(T_1 T_N \dots T_3 T_2) = \dots = \text{Fix}(T_{N-1} T_{N-2} \dots T_1 T_N).$$

Let $\{\lambda_n\}$ be a sequence in $(0, 1)$ which satisfies

- (N1) $\lim_{n \rightarrow \infty} \lambda_n = 0$;
 (N2) $\sum_{n=1}^{\infty} \lambda_n = \infty$;
 (N3) $\lim_{n \rightarrow \infty} \frac{\lambda_n}{\lambda_{n+N}} = 1$.

For any a and x_0 in C , define

$$x_{n+1} = \lambda_{n+1}a + (1 - \lambda_{n+1})T_{n+1}x_n, \quad n \geq 0.$$

Then the sequence $\{x_n\}$ converges strongly to $Q_F a$, where Q is a sunny nonexpansive retraction of C onto F .

Proof. We follow the same idea as in the proof of Theorem 1 in [10]. So we just sketch it. As in proof of Theorem 5, we can obtain the following facts and so the proofs are omitted:

- (1) $\|x_n - z\| \leq \max\{\|x_0 - z\|, \|a - z\|\}$ for all $n \geq 0$ and for all $z \in F$;
- (2) $\{x_n\}$ is bounded;
- (3) $\{T_{n+1}x_n\}$ is bounded;
- (4) $x_{n+1} - T_{n+1}x_n \rightarrow 0$.

Since (N3) is different from the condition (A3) in [2] (that is, (B3) above), we give the details of proof for $x_{n+N} - x_n \rightarrow 0$ as in [13]. By (3) above, there exists a constant $L > 0$ such that for all $n \geq 1$,

$$\|z - T_{n+1}x_n\| \leq L.$$

Since for all $n \geq 1$, $T_{n+N} = T_n$, we have

$$\begin{aligned} \|x_{n+N} - x_n\| &= \|(\lambda_{n+N} - \lambda_n)(z - T_{n+N}x_{n+N-1}) \\ &\quad + (1 - \lambda_{n+N})(T_nx_{n+N-1} - T_nx_{n-1})\| \\ &\leq L|\lambda_{n+N} - \lambda_n| + (1 - \lambda_{n+N})\|x_{n+N-1} - x_{n-1}\| \\ &= (1 - \lambda_{n+N})\|x_{n+N-1} - x_{n-1}\| + \lambda_{n+N}L \left| 1 - \frac{\lambda_n}{\lambda_{n+N}} \right|. \end{aligned}$$

By (N3), we have $\lim_{n \rightarrow \infty} L \left| 1 - \frac{\lambda_n}{\lambda_{n+N}} \right| = 0$, and so by Lemma 4,

$$x_{n+N} - x_n \rightarrow 0.$$

By the proof in [2], we also have

$$x_n - T_{n+N} \dots T_{n+1}x_n \rightarrow 0. \quad (10)$$

Finally we prove the strong convergence of $\{x_n\}$. Let a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ be such that

$$\lim_{j \rightarrow \infty} \langle a - Q_Fa, J(x_{n_j+1} - Q_Fa) \rangle = \limsup_{n \rightarrow \infty} \langle a - Q_Fa, J(x_{n+1} - Q_Fa) \rangle.$$

We assume (after passing to another subsequence if necessary) that $n_j + 1 \bmod N = i$ for some $i \in \{1, \dots, N\}$ and that $x_{n_j+1} \rightharpoonup x$. From (10), it follows that $\lim_{j \rightarrow \infty} \|x_{n_j+1} - T_{i+N} \dots T_{i+1}x_{n_j+1}\| = 0$. Hence, by Lemma 2, we have $x \in \text{Fix}(T_{i+N} \dots T_{i+1}) = F$.

On the other hand, since E is uniformly smooth, F is a sunny nonexpansive retract of C (cf. [6, p. 49]). Thus, by weakly sequentially continuity of duality mapping J and Lemma 3, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle a - Q_Fa, J(x_{n+1} - Q_Fa) \rangle &= \lim_{j \rightarrow \infty} \langle a - Q_Fa, J(x_{n_j+1} - Q_Fa) \rangle \\ &= \langle a - Q_Fa, J(x - Q_Fa) \rangle \leq 0. \end{aligned} \quad (11)$$

Since $(1 - \lambda_{n+1})(T_{n+1}x_n - Q_Fa) = (x_{n+1} - Q_Fa) - \lambda_{n+1}(a - Q_Fa)$, by Lemma 1, we have

$$\begin{aligned} \|x_{n+1} - Q_Fa\|^2 &\leq (1 - \lambda_{n+1})^2 \|T_{n+1}x_n - Q_Fa\|^2 \\ &\quad + 2\lambda_{n+1} \langle a - Q_Fa, J(x_{n+1} - Q_Fa) \rangle \\ &\leq (1 - \lambda_{n+1}) \|x_n - Q_Fa\|^2 \\ &\quad + 2\lambda_{n+1} \langle a - Q_Fa, J(x_{n+1} - Q_Fa) \rangle. \end{aligned} \quad (12)$$

Now, let $\varepsilon > 0$ be arbitrary. Then by (11), there exists N_ε such that

$$\langle a - Q_F a, J(x_{n+1} - Q_F a) \rangle \leq \frac{\varepsilon}{2} \quad \text{for all } n \geq N_\varepsilon.$$

Thus, from (12), we have

$$\|x_{n+1} - Q_F a\|^2 \leq (1 - \lambda_{n+1})\|x_n - Q_F a\|^2 + \lambda_{n+1}\varepsilon.$$

Thus, it follows from Lemma 4 that $\{x_n\}$ converges strongly to $Q_F a$. This completes the proof. \square

As an immediate consequence, we have the following

Corollary 11 (O'Hara et al. [13, Theorem 4.1]). *Let H be a Hilbert space, C a nonempty closed convex subset of H , and T_1, \dots, T_N nonexpansive mappings from C into itself with $F := \bigcap_{i=1}^N \text{Fix}(T_i)$ nonempty and*

$$F = \text{Fix}(T_N \dots T_1) = \text{Fix}(T_1 T_N \dots T_3 T_2) = \dots = \text{Fix}(T_{N-1} T_{N-2} \dots T_1 T_N).$$

Let $\{\lambda_n\}$ be a sequence in $(0, 1)$ which satisfies (N1)–(N3) in Theorem 2. For any a and x_0 in C , define

$$x_{n+1} = \lambda_{n+1}a + (1 - \lambda_{n+1})T_{n+1}x_n, \quad n \geq 0.$$

Then the sequence $\{x_n\}$ converges strongly to $P_F a$, where P is the nearest point projection of C onto F .

The following is a complementary result of the result of Wittmann [20].

Corollary 12. *Let H be a Hilbert space, C a nonempty closed convex subset of H , and T a nonexpansive mapping from C into itself with $\text{Fix}(T) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence in $(0, 1)$ which satisfies (N1)–(N3) in Theorem 10. For any a and x_0 in C , define (with $N = 1$)*

$$x_{n+1} = \lambda_{n+1}a + (1 - \lambda_{n+1})Tx_n, \quad n \geq 0.$$

Then the sequence $\{x_n\}$ converges strongly to $P_F a$, where P is the nearest point projection of C onto F .

Let D be a subset of a Banach space E . Recall that a mapping $T : D \rightarrow E$ is said to be firmly nonexpansive if for each x and y in D , the convex function $\phi : [0, 1] \rightarrow [0, \infty)$ defined by

$$\phi(s) = \|(1-s)x + sTx - ((1-s)y + sTy)\|$$

is nonincreasing. Since ϕ is convex, it is easy to check that a mapping $T : D \rightarrow E$ is firmly nonexpansive if and only if

$$\|Tx - Ty\| \leq \|(1-t)(x - y) + t(Tx - Ty)\|$$

for each x and y in D and $t \in [0, 1]$. It is clear that every firmly nonexpansive mapping is nonexpansive (cf. [6,7]).

The following result extends a Lions-type iteration scheme [12] with the condition (N3) to a Banach space setting.

Corollary 13. *Let E be a uniformly smooth Banach space with a weakly sequentially continuous duality mapping $J : E \rightarrow E^*$ and C a nonempty closed convex subset of E . Let T_1, \dots, T_N be firmly nonexpansive mappings from C into itself with $F := \bigcap_{i=1}^N \text{Fix}(T_i)$ nonempty and*

$$F = \text{Fix}(T_N \dots T_1) = \text{Fix}(T_1 T_N \dots T_3 T_2) = \dots = \text{Fix}(T_{N-1} T_{N-2} \dots T_1 T_N).$$

Let $\{\lambda_n\}$ be a sequence in $[0, 1)$ which satisfies (N1)–(N3) in Theorem 10. For any a and x_0 in C , define

$$x_{n+i} = \lambda_{n+1}a + (1 - \lambda_{n+1})T_{n+1}x_n, \quad n \geq 0.$$

Then the sequence $\{x_n\}$ converges strongly to $Q_F a$, where Q is a sunny nonexpansive retraction of C onto F .

Remark 14. (1) In Hilbert space, Lions [12, Théorème 4] had used

$$(L1) \quad \lim_{n \rightarrow \infty} \lambda_n = 0;$$

$$(L2) \quad \sum_{k=1}^{\infty} \lambda_{kN+i} = \infty \text{ for all } i = 0, \dots, N-1, \text{ which is more restrictive than (N2); and}$$

$$(L3)' \quad \lim_{k \rightarrow \infty} \frac{\sum_{i=1}^N |\lambda_{kN+i} - \lambda_{(k-1)N+i}|}{(\sum_{i=1}^N \lambda_{kN+i})^2} = 0 \text{ in place of (B3).}$$

(2) In general, (B3) and (L3)' are independent, even when $N = 1$. For more details, see [2].

Acknowledgment

The author thanks the referee for his/her valuable comments and suggestions for improving this paper.

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