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Existence of equilibria via Ekeland's principle

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Abstract

In the literature, when dealing with equilibrium problems and the existence of their solutions, the most used assumptions are the convexity of the domain and the generalized convexity and monotonicity, together with some weak continuity assumptions, of the function. In this paper, we focus on conditions that do not involve any convexity concept, neither for the domain nor for the function involved. Starting from the well-known Ekeland's theorem for minimization problems, we find a suitable set of conditions on the function f that lead to an Ekeland's variational principle for equilibrium problems. Via the existence of ϵ-solutions, we are able to show existence of equilibria on general closed sets for equilibrium problems and systems of equilibrium problems.

Keywords: Equilibrium problem; System of equilibrium problems; Approximate solution; Ekeland's principle

1. Introduction

By an equilibrium problem we understand the problem of finding

$$\bar{x} \in D \quad \text{such that } f(\bar{x}, y) \geqslant 0,\ \forall y \in D, \tag{EP}$$

where D is a given set, and $f : D \times D \to \mathbb{R}$ is a given function.

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This problem was considered in the past with the aim of extending results concerning particular problems like optimization problems, complementarity problems, fixed point problems and variational inequalities (see [6] for a survey).

More recently, inspired by the study of systems of vector variational inequalities, Ansari et al. [1] introduced and investigated systems of equilibrium problems.

Let m be a positive integer. By a system of equilibrium problems we understand the problem of finding $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m) \in D$ such that

$$f_i(\bar{x}, y_i) \geqslant 0, \quad \forall i \in I,\ \forall y_i \in D_i, \tag{SEP}$$

where $f_i : D \times D_i \to R$, $D = \prod_1^m D_i$, and D_i is a given set.

In literature, the convexity and closure of the set D and the generalized convexity and monotonicity, together with some weak continuity assumptions on f, were the most used assumptions in dealing with equilibrium problems (see, for instance, [3,5]). Similar assumptions can be found in the study of solutions of systems of equilibrium problems.

More recently a few authors have looked for methods aimed at finding approximate solutions; most of the algorithms developed for solving (EP) can be derived from equivalent formulations of the equilibrium problem itself. Along this line, Cohen [7] developed such methods for solving VI and optimization problems, and Mastroeni [10] generalized them focusing the attention on fixed-point formulations of (EP).

In this paper, we show the existence of approximated equilibria for (EP) and (SEP) on both compact and noncompact sets. Starting from the well-known Ekeland's theorem for minimization problems, we find a suitable set of conditions on the functions that do not involve convexity and lead to an Ekeland's variational principle for equilibrium and system of equilibrium problems. Via the existence of approximate solutions, we are able to show the existence of equilibria on general closed sets. Our setting is an Euclidean space, even though the results could be extended in reflexive Banach spaces, by adapting the assumptions in a standard way.

2. Ekeland's principle for equilibrium problems

The Ekeland's variational principle has been widely used in nonlinear analysis since it entails the existence of approximate solutions of a minimization problem for lower semicontinuous functions on a complete metric space (see, for instance, [2]). Since minimization problems are particular cases of equilibrium problems, where $f(x, y) = g(y) - g(x)$, one is interested in extending Ekeland's theorem to the setting of an equilibrium problem. We start with formulating this general result, involving a bifunction f.

Let $D \subseteq X$ be a closed set, where X is an Euclidean space, and $f : D \times D \to \mathbb{R}$.

Theorem 2.1. *Assume that the following assumptions are satisfied*:

(i) $f(x, \cdot)$ *is lower bounded and lower semicontinuous, for every* $x \in D$;
(ii) $f(t, t) = 0$, *for every* $t \in D$;
(iii) $f(z, x) \leqslant f(z, y) + f(y, x)$, *for every* $x, y, z \in D$.

Then, for every $\epsilon > 0$ *and for every* $x_0 \in D$, *there exists* $\bar{x} \in D$ *such that*

$$\begin{cases} f(x_0, \bar{x}) + \epsilon\|x_0 - \bar{x}\| \leqslant 0, \\ f(\bar{x}, x) + \epsilon\|\bar{x} - x\| > 0, \quad \forall x \in D,\ x \neq \bar{x}. \end{cases} \tag{2.1}$$

Proof. Without loss of generality, we can restrict the proof to the case $\epsilon = 1$. Denote by $F(x)$ the set

$$F(x) := \bigl\{y \in D\colon\ f(x, y) + \|y - x\| \leqslant 0\bigr\}.$$

By (i), $F(x)$ is closed, for every $x \in D$; by (ii), $x \in F(x)$, hence $F(x)$ is nonempty for every $x \in D$. Assume $y \in F(x)$, i.e., $f(x, y) + \|y - x\| \leqslant 0$, and let $z \in F(y)$ (i.e., $f(y, z) + \|y - z\| \leqslant 0$). Adding both sides of the inequalities, we get, by (iii),

$$0 \geqslant f(x, y) + \|y - x\| + f(y, z) + \|y - z\| \geqslant f(x, z) + \|z - x\|,$$

that is, $z \in F(x)$. Therefore $y \in F(x)$ implies $F(y) \subseteq F(x)$.

Define

$$v(x) := \inf_{z \in F(x)} f(x, z).$$

For every $z \in F(x)$,

$$\|x - z\| \leqslant -f(x, z) \leqslant \sup_{z \in F(x)} \bigl(-f(x, z)\bigr) = -\inf_{z \in F(x)} f(x, z) = -v(x),$$

that is,

$$\|x - z\| \leqslant -v(x), \quad \forall z \in F(x).$$

In particular, if $x_1, x_2 \in F(x)$,

$$\|x_1 - x_2\| \leqslant \|x - x_1\| + \|x - x_2\| \leqslant -v(x) - v(x) = -2v(x),$$

implying that

$$\operatorname{diam}\bigl(F(x)\bigr) \leqslant -2v(x), \quad \forall x \in D.$$

Fix $x_0 \in D$; $x_1 \in F(x_0)$ exists such that

$$f(x_0, x_1) \leqslant v(x_0) + 2^{-1}.$$

Denote by x_2 any point in $F(x_1)$ such that

$$f(x_1, x_2) \leqslant v(x_1) + 2^{-2}.$$

Proceeding in this way, we define a sequence $\{x_n\}$ of points of D such that $x_{n+1} \in F(x_n)$ and

$$f(x_n, x_{n+1}) \leqslant v(x_n) + 2^{-(n+1)}.$$

Notice that

$$\begin{aligned} v(x_{n+1}) &= \inf_{y \in F(x_{n+1})} f(x_{n+1}, y) \geqslant \inf_{y \in F(x_n)} f(x_{n+1}, y) \\ &\geqslant \inf_{y \in F(x_n)} \bigl(f(x_n, y) - f(x_n, x_{n+1})\bigr)\Bigl(\inf_{y \in F(x_n)} f(x_n, y)\Bigr) - f(x_n, x_{n+1}) \\ &= v(x_n) - f(x_n, x_{n+1}). \end{aligned}$$

Therefore,

$$v(x_{n+1}) \geqslant v(x_n) - f(x_n, x_{n+1}),$$

and

$$-v(x_n) \leqslant -f(x_n, x_{n+1}) + 2^{-(n+1)} \leqslant \big(v(x_{n+1}) - v(x_n)\big) + 2^{-(n+1)},$$

that entails

$$0 \leqslant v(x_{n+1}) + 2^{-(n+1)}.$$

It follows that

$$\operatorname{diam}\big(F(x_n)\big) \leqslant -2v(x_n) \leqslant 2 \cdot 2^{-n} \to 0, \quad n \to \infty.$$

The sets $\{F(x_n)\}$ being closed and $F(x_{n+1}) \subseteq F(x_n)$, we have that

$$\bigcap_n F(x_n) = \{\bar{x}\}.$$

Since $\bar{x} \in F(x_0)$, then

$$f(x_0, \bar{x}) + \|\bar{x} - x_0\| \leqslant 0.$$

Moreover, $\bar{x}$ belongs to all $F(x_n)$, and, since $F(\bar{x}) \subseteq F(x_n)$, for every n, we get that

$$F(\bar{x}) = \{\bar{x}\}.$$

It follows that $x \notin F(\bar{x})$ whenever $x \neq \bar{x}$, implying that

$$f(\bar{x}, x) + \|x - \bar{x}\| > 0.$$

This completes the proof. $\square$

Remark 2.1. Any function $f(x, y) = g(y) - g(x)$ trivially satisfies (iii), but there are other functions, not of this form, that fall into the framework of Theorem 2.1. Take, for instance, the function

$$f(x, y) = \begin{cases} e^{-\|x-y\|} + 1 + g(y) - g(x), & x \neq y, \\ 0, & x = y, \end{cases}$$

where g is a lower bounded and lower semicontinuous function.

Remark 2.2. Condition (iii) of Theorem 2.1 implies the cyclic monotonicity of $-f$, that extends a similar definition given for mappings in the framework of variational inequalities (see [11]): for every $x_1, x_2, \ldots, x_n \in D$ we have

$$\sum_{i=1}^{n} f(x_i, x_{i+1}) \geqslant 0, \tag{2.2}$$

where $x_{n+1} = x_1$. Indeed, taking $x = z$ in (iii), by (ii) we get $f(x, y) + f(y, x) \geqslant 0$, and (2.2) holds for $n = 2$. By induction, assuming that (2.2) holds for n, from (iii) the following inequalities hold:

$$\sum_{i=1}^{n+1} f(x_i, x_{i+1}) = \sum_{i=1}^{n-1} f(x_i, x_{i+1}) + f(x_n, x_{n+1}) + f(x_{n+1}, x_1)$$
$$\geqslant \sum_{i=1}^{n-1} f(x_i, x_{i+1}) + f(x_n, x_1) \geqslant 0.$$

In particular, the cyclic monotonicity of $-f$ implies the monotonicity of $-f$.

Let now m be a positive integer, and $I = \{1, 2, \ldots, m\}$. Consider the functions $f_i : D \times D_i \to R$, $i \in I$, where $D = \prod_{i\in I} D_i$, and $D_i \subset X_i$ is a closed subset of the Euclidean space X_i. An element of the set $D^i = \prod_{j\neq i} D_i$ will be represented by x^i; therefore, $x \in D$ can be written as $x = (x^i, x_i) \in D^i \times D_i$. If $x \in \prod X_i$, the symbol $|||x|||$ will denote the Chebyshev norm of x, i.e., $|||x||| = \max_i \|x_i\|_i$ and we shall consider the Euclidean space $\prod X_i$ endowed with this norm.

The following result is an extension of Theorem 2.1.

Theorem 2.2. *Assume that*

(i) $f_i(x, \cdot)$: $D_i \to R$ *is lower bounded and lower semicontinuous for every* $i \in I$;
(ii) $f_i(x, x_i) = 0$ *for every* $i \in I$ *and every* $x = (x_1, \ldots, x_m) \in D$;
(iii) $f_i(z, x_i) \leqslant f_i(z, y_i) + f_i(y, x_i)$, *for every* $x, y, z \in D$, *where* $y = (y^i, y_i)$, *and for every* $i \in I$.

Then for every $\varepsilon > 0$ *and for every* $x^0 = (x_1^0, \ldots, x_m^0) \in D$ *there exists* $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m) \in D$ *such that for each* $i \in I$ *one has*

$$f_i(x^0, \bar{x}_i) + \varepsilon \left\| x_i^0 - \bar{x}_i \right\|_i \leqslant 0 \tag{2.3}$$

and

$$f_i(\bar{x}, x_i) + \varepsilon \|\bar{x}_i - x_i\|_i > 0, \quad \forall x_i \in D_i,\ x_i \neq \bar{x}_i. \tag{2.4}$$

Proof. As before, we restrict the proof to the case $\varepsilon = 1$. Let $i \in I$ be arbitrarily fixed. Denote for every $x \in D$,

$$F_i(x) := \bigl\{ y_i \in D_i\colon\ f_i(x, y_i) + \|x_i - y_i\|_i \leqslant 0 \bigr\}.$$

These sets are closed and nonempty (for every $x = (x_1, \ldots, x_m) \in D$ we have $x_i \in F_i(x)$). Define for each $x \in D$,

$$v_i(x) := \inf_{z_i \in F_i(x)} f_i(x, z_i).$$

In a similar way as in the proof of Theorem 2.1 one can show that $\operatorname{diam}(F_i(x)) \leqslant -2v_i(x)$ for every $x \in D$ and $i \in I$.

Fix now $x^0 \in D$ and select for each $i \in I$ an element $x_i^1 \in F_i(x^0)$ such that

$$f_i\bigl(x^0, x_i^1\bigr) \leqslant v_i(x^0) + 2^{-1}.$$

Put $x^1 := (x_1^1, \dots, x_m^1) \in D$ and select for each $i \in I$ an element $x_i^2 \in F_i(x^1)$ such that

$$f_i\big(x^1, x_i^2\big) \leqslant v_i(x^1) + 2^{-2}.$$

Put $x^2 := (x_1^2, \dots, x_m^2) \in D$. Continuing this process we define a sequence $\{x^n\}$ in D such that $x_i^{n+1} \in F_i(x^n)$ for each $i \in I$ and $n \in N$ and

$$f_i\big(x^n, x_i^{n+1}\big) \leqslant v_i(x^n) + 2^{-(n+1)}.$$

Using the same argument as in the proof of Theorem 2.1, one can show that

$$\operatorname{diam}\big(F_i(x^n)\big) \leqslant -2v_i(x^n) \leqslant 2 \cdot 2^{-n} \to 0, \quad n \to \infty,$$

for each $i \in I$.

Now define for each $x \in D$ the sets

$$F(x) := F_1(x) \times \cdots \times F_m(x) \subseteq D.$$

The sets $F(x)$ are closed and using (iii) it is immediate to check that for each $y \in F(x)$ it follows that $F(y) \subseteq F(x)$. Therefore, we also have $F(x^{n+1}) \subseteq F(x^n)$ for each $n = 0, 1, \dots$. On the other hand, for each $y, z \in F(x^n)$ we have

$$|||y - z||| = \max_{i \in I} \|y_i - z_i\|_i \leqslant \max_{i \in I} \operatorname{diam}\big(F_i(x^n)\big) \to 0,$$

thus, $\operatorname{diam}(F(x^n)) \to 0$ as $n \to \infty$. In conclusion we have

$$\bigcap_{n=0}^{\infty} F(x^n) = \{\bar{x}\}, \quad \bar{x} \in D.$$

Since $\bar{x} \in F(x^0)$, i.e., $\bar{x}_i \in F_i(x^0)$ $(i \in I)$ we obtain

$$f_i(x^0, \bar{x}_i) + \big\|x_i^0 - \bar{x}_i\big\|_i \leqslant 0, \quad \forall i \in I,$$

and so, (2.3) holds. Moreover, $\bar{x} \in F(x^n)$ implies $F(\bar{x}) \subseteq F(x^n)$ for all $n = 0, 1, \dots$, therefore,

$$F(\bar{x}) = \{\bar{x}\}$$

implying

$$F_i(\bar{x}) = \{\bar{x}_i\}, \quad \forall i \in I.$$

Now for every $x_i \in D_i$ with $x_i \neq \bar{x}_i$ we have by the previous relation that $x_i \notin F_i(\bar{x})$ and so

$$f_i(\bar{x}, x_i) + \|\bar{x}_i - x_i\|_i > 0.$$

Thus (2.4) holds too, and this completes the proof. □

3. New existence results for equilibria on compact sets

In literature, the condition of proper quasimonotonicity is frequently used when dealing with (EP) on convex sets. Recall that a function $f : D \times D \to \mathbb{R}$ is *properly quasimonotone* on $D \times D$ if, for every finite set A of the convex set D, and for every $x \in \mathrm{co}(A)$, the following inequality is satisfied:

$$\max_{y \in A} f(x, y) \geqslant 0.$$

The following result provides a sufficient condition to solve (EP) on compact, convex sets.

Proposition 3.1 (see, for instance, [3]). *Let D be a compact, convex set, and let $f : D \times D \to \mathbb{R}$ be a properly quasimonotone and upper semicontinuous function in its first variable. Then the solution set of* (EP) *is nonempty.*

The next result shows that under suitable assumptions on f, proper quasimonotonicity is necessary and sufficient for solvability of (EP). Indeed, the following result holds.

Theorem 3.1 (see [3]). *Let D be a compact, convex set, and let $f : D \times D \to \mathbb{R}$ be an upper semicontinuous and quasiconvex function in its first variable*; *then the following conditions are equivalent*:

(i) *f is properly quasimonotone*;
(ii) *for any finite set $A \subseteq D$ there exists $\bar{x} \in \mathrm{co}(A)$ such that $\bar{x}$ is a solution of* (EP);
(iii) (EP) *has a solution on every compact convex subset of D.*

In this section, using Theorems 2.1 and 2.2, we are able to show the nonemptiness of the solution set of (EP) and (SEP), without any convexity requirement. To this purpose, we introduce a definition of approximate equilibrium point, for both cases (see [9] for a definition of approximate equilibrium for functions defined on product spaces). We start our analysis with (EP).

Definition 3.1. Given $f : D \times D \to \mathbb{R}$, and $\epsilon > 0$, $\bar{x}$ is said to be an ϵ-equilibrium point of f if

$$f(\bar{x}, y) \geqslant -\epsilon \|\bar{x} - y\|, \quad \forall y \in D. \tag{3.1}$$

The ϵ-equilibrium point is strict, if in (3.1) the inequality is strict for all $y \neq \bar{x}$.

Notice that the second relation of (2.1) gives the existence of a strict ϵ-equilibrium point, for every $\epsilon > 0$. Moreover, by (ii) and (iii) of Theorem 2.1 it follows by the first relation of (2.1) that

$$f(\bar{x}, x_0) \geqslant \epsilon \|\bar{x} - x_0\|,$$

"localizing," in a certain sense, the position of the $\bar{x}$.

We will show, using Theorem 2.1, that a set of conditions different and not comparable to Proposition 3.1, can be considered to ensure the nonemptiness of the solution set of (EP).

Proposition 3.2. *Let D be a compact* (*not necessarily convex*) *subset of an Euclidean space, and $f : D \times D \to \mathbb{R}$ be a function satisfying the assumptions*:

(i) $f(x,\cdot)$ *is lower semicontinuous, for every* $x \in D$;
(ii) $f(t,t) = 0$, *for every* $t \in D$;
(iii) $f(z,x) \leqslant f(z,y) + f(y,x)$, *for every* $x, y, z \in D$;
(iv) $f(\cdot, y)$ *is upper semicontinuous, for every* $y \in D$.

Then, the set of solutions of (EP) *is nonempty.*

Proof. For each $n \in \mathbf{N}$, let $x_n \in D$ a $1/n$-equilibrium point (such point exists by Theorem 2.1), i.e.,

$$f(x_n, y) \geqslant -\frac{1}{n}\|x_n - y\|, \quad \forall y \in D.$$

Since D is compact, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \bar{x}$ as $n \to \infty$. Then, by (iv),

$$f(\bar{x}, y) \geqslant \limsup_{k\to\infty}\left(f(x_{n_k}, y) + \frac{1}{n_k}\|x_{n_k} - y\|\right), \quad \forall y \in D,$$

thereby proving that $\bar{x}$ is a solution of (EP). □

Remark 3.1. Although Propositions 3.1 and 3.2 provide sufficient conditions for the existence of solutions of (EP), they are not comparable, i.e., none of them can be deduced from the other. While condition (iv) of Proposition 3.2 appears explicitly among the assumptions of Proposition 3.1 as well, in Proposition 3.2 no convexity of the set is required. However, lower semicontinuity with respect to the second variable of f is assumed only in Proposition 3.2. Observe also that a slightly weaker form of condition (ii) of Proposition 3.2 holds implicitly at Proposition 3.1: taking the set $A := \{x\}$ for arbitrary $x \in D$, by proper quasimonotonicity of f is immediate that $f(x,x) \geqslant 0$. What is interesting to remark is the difference between the main assumptions of these propositions: proper quasimonotonicity in case of Proposition 3.1, and condition (iii) in case of Proposition 3.2. As the following two simple examples show, these assumptions are not related. The function $f : [-1,1] \times [-1,1] \to \mathbb{R}$ defined as $f(x,y) = x^2 - y^2$ satisfies the assumptions of Proposition 3.2, but is not properly quasimonotone (take, for instance, $A = \{-1, 1\}$ and $x = 0$). On the other hand, the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined as $f(x,y) = xy - x^2$, is quasiconvex in its second variable and $f(t,t) = 0$, therefore is properly quasimonotone (see [3, Proposition 1.1]), but it does not satisfy condition (iii) of Proposition 3.2.

Another fact which can be observed is the difference between the proofs of these two propositions. While proving Proposition 3.1 one needs to apply “heavy” results of functional analysis like Ky Fan’s lemma [8], whose proof is based on the Brouwer’s fixed point theorem, the proof of Proposition 3.2 has been performed using only elementary results. Indeed, the proof of Theorem 2.1 (similarly to the proof of Ekeland’s principle) uses only the simple fact that the intersection of a sequence of descending closed sets in a complete metric space is a singleton, provided the sequence of their diameters converges to zero. On the other hand, as seen, Proposition 3.2 is a simple consequence of Theorem 2.1.

In this way, by means of Proposition 3.2 we have provided an existence result concerning (EP), whose proof is elementary.

Let us now consider the following definition of ϵ-equilibrium point for systems of equilibrium problems.

Definition 3.2. Let D_i, $i \in I$, be subsets of certain Euclidean spaces and put $D = \prod_{i\in I} D_i$. Given $f_i : D \times D_i \to \mathbb{R}$, $i \in I$ and $\epsilon > 0$, $\bar{x} \in D$ is said to be an ϵ-equilibrium point of $\{f_1, f_2, \ldots, f_m\}$ if

$$f_i(\bar{x}, y_i) \geqslant -\epsilon \|\bar{x}_i - y_i\|_i, \quad \forall y_i \in D_i,\ \forall i \in I. \tag{3.2}$$

The following result is an extension of Proposition 3.2, and it can be proved in a similar way.

Proposition 3.3. *Assume that, for every $i \in I$, D_i is compact and $f_i : D \times D_i \to \mathbb{R}$ is a function satisfying the assumptions*:

(i) $f_i(x, \cdot)$ *is lower semicontinuous, for every* $x \in D$;
(ii) $f_i(x, x_i) = 0$, *for every* $x = (x^i, x_i) \in D$;
(iii) $f_i(z, x_i) \leqslant f_i(z, y_i) + f_i(y, x_i)$, *for every* $x, y, z \in D$, *where* $y = (y^i, y_i)$;
(iv) $f_i(\cdot, y_i)$ *is upper semicontinuous, for every* $y_i \in D_i$.

Then, the set of solutions of (SEP) *is nonempty.*

4. Equilibria on noncompact sets

We consider now the case of a noncompact set D, first in the case of (EP), then for (SEP). The study of the existence of solutions of the equilibrium problems on unbounded domains usually involves the same sufficient assumptions as for bounded domains together with a coercivity condition. Bianchi and Pini [4] found coercivity conditions as weak as possible, exploiting the generalized monotonicity properties of the function f defining the equilibrium problem.

Let D be a closed subset of X, not necessarily convex, not necessarily compact, and $f : D \times D \to \mathbb{R}$ be a given function.

Consider the following coercivity condition (see [4]):

$$\exists r > 0\colon\ \forall x \in D \setminus K_r,\ \exists y \in D,\ \|y\| < \|x\|\colon\ f(x, y) \leqslant 0, \tag{C$_1$}$$

where $K_r := \{x \in D\colon\ \|x\| \leqslant r\}$.

We now show that within the framework of Proposition 3.2, condition (C_1) guarantees the existence of solutions of (EP) without supposing compactness of D.

Theorem 4.1. *Suppose that*

(i) $f(x, \cdot)$ *is lower bounded and lower semicontinuous for every* $x \in D$;
(ii) $f(t, t) = 0$ *for every* $t \in D$;
(iii) $f(z, x) \leqslant f(z, y) + f(y, x)$ *for every* $x, y, z \in D$;
(iv) $f(\cdot, y)$ *is upper semicontinuous for every* $y \in D$.

If (C_1) *holds, then* (EP) *has a solution.*

Proof. We may suppose without loss of generality that K_r is nonempty. For each $x \in D$ consider the nonempty set

$$S(x) := \{y \in D\colon \|y\| \leqslant \|x\|\colon f(x, y) \leqslant 0\}.$$

Observe that for every $x, y \in D$, $y \in S(x)$ implies $S(y) \subseteq S(x)$. Indeed, for $z \in S(y)$ we have $\|z\| \leqslant \|y\| \leqslant \|x\|$ and by (iii) $f(x,z) \leqslant f(x,y) + f(y,z) \leqslant 0$. On the other hand, since $K_{\|x\|}$ is compact, by (i) we obtain that $S(x) \subseteq K_{\|x\|}$ is a compact set for every $x \in D$. Furthermore, by Proposition 3.2, there exists an element $x_r \in K_r$ such that

$$f(x_r, y) \geqslant 0, \quad \forall y \in K_r. \tag{4.1}$$

Suppose that there exists $x \in D$ with $f(x_r, x) < 0$ and put

$$a := \min_{y \in S(x)} \|y\|$$

(the minimum is achieved since $S(x)$ is nonempty, compact and the norm is continuous). We distinguish two cases.

Case 1: $a \leqslant r$. Let $y_0 \in S(x)$ such that $\|y_0\| = a \leqslant r$. Then we have $f(x, y_0) \leqslant 0$. Since $f(x_r, x) < 0$, it follows by (iii) that

$$f(x_r, y_0) \leqslant f(x_r, x) + f(x, y_0) < 0,$$

contradicting (4.1).

Case 2: $a > r$. Let again $y_0 \in S(x)$ such that $\|y_0\| = a > r$. Then, by (C_1) we can choose an element $y_1 \in D$ with $\|y_1\| < \|y_0\| = a$ such that $f(y_0, y_1) \leqslant 0$. Thus, $y_1 \in S(y_0) \subseteq S(x)$ contradicting

$$\|y_1\| < a = \min_{y \in S(x)} \|y\|.$$

Therefore, there is no $x \in D$ such that $f(x_r, x) < 0$, i.e., x_r is a solution of (EP) (on D). This completes the proof. □

Next we consider (SEP) for noncompact setting. Let us consider the following coercivity condition:

$$\begin{aligned} &\exists r > 0\colon \forall x \in D \text{ such that } \|x_i\|_i > r \text{ for some } i \in I, \\ &\exists y_i \in D_i,\ \|y_i\|_i < \|x_i\|_i \text{ and } f_i(x, y_i) \leqslant 0. \end{aligned} \tag{CS_1}$$

We have the following result.

Theorem 4.2. *Suppose that, for every* $i \in I$,

(i) $f_i(x, \cdot)$ *is lower bounded and lower semicontinuous for every* $x \in D$;
(ii) $f_i(x, x_i) = 0$ *for every* $x = (x^i, x_i) \in D$;
(iii) $f_i(z, x_i) \leqslant f_i(z, y_i) + f_i(y, x_i)$ *for every* $x, y, z \in D$, *where* $y = (y^i, y_i)$;
(iv) $f_i(\cdot, y_i)$ *is upper semicontinuous for every* $y_i \in D_i$.

If (CS$_1$) *holds, then* (SEP) *has a solution.*

Proof. For each $x \in D$ and every $i \in I$ consider the set

$$S_i(x) := \{y_i \in D_i,\ \|y_i\|_i \leqslant \|x_i\|_i,\ f_i(x, y_i) \leqslant 0\}.$$

Observe that, by (iii), for every x and $y = (y^i, y_i) \in D$, $y_i \in S_i(x)$ implies $S_i(y) \subseteq S_i(x)$. On the other hand, since the set $\{y_i \in D_i\colon \|y_i\|_i \leqslant r\} = K_i(r)$ is a compact subset of D_i, by (i) we obtain that $S_i(x)$ is a nonempty compact set for every $x \in D$. Furthermore, by Proposition 3.3, there exists an element $x_r \in \prod_i K_i(r)$ (observe, we may suppose that $K_i(r) \neq \emptyset$ for all $i \in I$) such that

$$f_i(x_r, y_i) \geqslant 0, \quad \forall y_i \in K_i(r),\ \forall i \in I. \tag{4.2}$$

Suppose that x_r is not a solution of (SEP). In this case, there exists $j \in I$ and $z_j \in D_j$ with $f_j(x_r, z_j) < 0$. Let $z^j \in D^j$ be arbitrary and put $z = (z^j, z_j) \in D$. Define

$$a_j := \min_{y_j \in S_j(z)} \|y_j\|_j.$$

We distinguish two cases.

Case 1: $a_j \leqslant r$. Let $\bar{y}_j(z) \in S_j(z)$ such that $\|\bar{y}_j(z)\|_j = a_j \leqslant r$. Then we have $f_j(z, \bar{y}_j(z)) \leqslant 0$. Since $f_j(x_r, z_j) < 0$, it follows by (iii) that

$$f_j\big(x_r, \bar{y}_j(z)\big) \leqslant f(x_r, z_j) + f\big(z, \bar{y}_j(z)\big) < 0,$$

contradicting (4.2).

Case 2: $a_j > r$. Let again $\bar{y}_j(z) \in S_j(z)$ such that $\|\bar{y}_j(z)\|_j = a_j > r$. Let $\bar{y}^j \in D^j$ be arbitrary and put $\bar{y}(z) = (\bar{y}^j, \bar{y}_j(z)) \in D$. Then, by (CS$_1$) we can choose an element $y_j \in D_j$ with $\|y_j\|_j < \|\bar{y}_j(z)\|_j = a_j$ such that $f_j(\bar{y}(z), y_j) \leqslant 0$. Clearly, $y_j \in S_j(\bar{y}(z)) \subseteq S_j(z)$, a contradiction since $\bar{y}_j(z)$ has minimal norm in $S_j(z)$. This completes the proof. □

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