Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 322 (2006) 599–609
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Maximal ideals of disjointness preserving operators

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Received 23 March 2005
Available online 21 October 2005
Submitted by K. Jarosz

Abstract

Let L and M be vector lattices with M Dedekind complete, and let $\mathcal{L}_r(L, M)$ be the vector lattice of all regular operators from L into M. We introduce the notion of maximal order ideals of disjointness preserving operators in $\mathcal{L}_r(L, M)$ (briefly, maximal δ-ideals of $\mathcal{L}_r(L, M)$) as a generalization of the classical concept of orthomorphisms and we investigate some aspects of this 'new' structure. In this regard, various standard facts on orthomorphisms are extended to maximal δ-ideals. For instance, surprisingly enough, we prove that any maximal δ-ideal of $\mathcal{L}_r(L, M)$ is a vector lattice copy of M, when L, in addition, has an order unit. Moreover, we pay a special attention to maximal δ-ideals on continuous function spaces. As an application, we furnish a characterization of lattice bimorphisms on such spaces in terms of weigthed composition operators.

Keywords: Continuous functions spaces; Disjointness preserving; Lattice isomorphism; Maximal order ideal; Regular operator; Vector lattice

1. Introduction

Over the last decades the importance of disjointness preserving operators in the general theory of vector lattices and spaces of real-valued continuous functions has steadily grown. Several aspects of these operators provoked interest in the literature. In this regard, special attention has been paid to multiplicative characterizations [1,8,11,15,17], polar decompositions [4,10,13,18],

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1 Aknowledges support from the NSF Grant INT-0423522.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.09.038

and criterions for invertibility [5,6,9,16]. In this paper, we look at disjointness preserving operators from a different 'global' point of view. Indeed, we focus on the lattice structure of certain spaces of operators preserving disjointness rather that the behavior of the disjointness preserving operators themselves. To be more precise, let L and M be vector lattices with M Dedekind complete, and let $\mathcal{L}_r(L, M)$ be the Dedekind complete vector lattice of all regular operators from L into M. We call an order ideal $\mathcal{I}$ of $\mathcal{L}_r(L, M)$ a *δ-ideal* if all the elements of $\mathcal{I}$ are disjointness preserving operators. The δ-ideal $\mathcal{I}$ of $\mathcal{L}_r(L, M)$ is said to be *maximal* if $\mathcal{I}$ is a maximal element with respect to the inclusion in the set of all δ-ideals of $\mathcal{L}_r(L, M)$. Though the concept of maximal δ-ideals may seem a little far-fetched, we believe that such a structure offer an interesting prospect, in part because it generalizes the standard notion of orthomorphisms on Dedekind complete vector lattices (see Proposition 2.1 below). In fact, this amazing observation is the stimulus of our investigation. Indeed, it seems to be natural to ask what results on orthomorphisms can be extended to maximal δ-ideals. In this paper we are interested, among other facts, in the next classical result on orthomorphisms, essentially due to Zaanen in [22] (see also [2]). If the Dedekind complete vector lattice M has an order unit then M is lattice isomorphic to the order ideal $\mathcal{Z}(M)$ of central operators on M. In this paper we generalize the Zaanen's theorem in the following way. We prove that, when L additionally has an order unit, $\mathcal{L}_r(L, M)$ has a unique (up to a lattice isomorphism) maximal δ-ideal which is a vector lattice copy of M. In spite of that, a systematic study of maximal δ-ideals is provided with a special attention to maximal δ-ideals on continuous functions spaces. As an application, we extend the standard characterization of lattice homomorphisms on continuous functions spaces as weigthed composition operators to lattice bimorphisms on such spaces.

For notations, terminology, and concepts not explained in this paper the reader can consult the remarkable new book [2] by Abramovich and Aliprantis, the important memoirs [5] of Abramovich and Kitover, and the standard monograph [7] by Aliprantis and Burkinshaw.

2. Some preliminaries

In this paper, all operators are linear and all vector lattices (also called Riesz spaces) are nontrivial and Archimedean.

Throughout this work, L and M are vector lattices with M Dedekind complete. The Dedekind complete vector lattice of all regular operators from L into M is denoted by $\mathcal{L}_r(L, M)$, as usual.

The first paragraph of this section is devoted to some preliminaries on disjointness preserving operators. An operator $T \in \mathcal{L}_r(L, M)$ is said to be *disjointness preserving* (or *separating*) if $|T(f)| \wedge |T(g)| = 0$ for all $f, g \in L$ such that $|f| \wedge |g| = 0$. An operator $T \in \mathcal{L}_r(L, M)$ is disjointness preserving if and only if $f \wedge g = 0$ in L implies $|T(f)| \wedge |T(g)| = 0$ in M. A positive disjointness preserving operator is called a *lattice homomorphism*. Hence, $T \in \mathcal{L}_r(L, M)$ is a lattice homomorphism if and only if $T(|f|) = |T(f)|$ for all $f \in L$. A *lattice isomorphism* is a bijective lattice homomorphism. The inverse of a lattice isomorphism is a lattice isomorphism as well. If $T \in \mathcal{L}_r(L, M)$ is disjointness preserving then its absolute value $|T|$ in $\mathcal{L}_r(L, M)$ is a lattice homomorphism given by

$$|T|\big(|f|\big) = \big|T(f)\big| = \big|T\big(|f|\big)\big|, \quad \text{for all } f \in L.$$

Conversely, if the absolute value $|T|$ of $T \in \mathcal{L}_r(L, M)$ is a lattice homomorphism then T preserves disjointness. More informations on disjointness preserving operators can be found in [5,7,18].

The next lines deal with the notion of orthomorphisms on Dedekind complete vector lattices. An operator T on the Dedekind complete vector lattice M is refereed to as an *orthomorphism* whenever $|f| \wedge |g| = 0$ implies $|f| \wedge |T(g)| = 0$. Clearly, orthomorphisms are disjointness preserving operators. Let $\mathcal{L}_r(M)$ denote the Dedekind complete vector lattice of all regular operators on M. An operator $T \in \mathcal{L}_r(M)$ is said to be *central* whenever $|T| \leqslant aI$ for some real number a, where I is the identity operator of M. Central operators are orthomorphisms. The set of all central operators of M is denoted by $\mathcal{Z}(M)$. We point out that $\mathcal{Z}(M)$ is the principal order ideal of $\mathcal{L}_r(M)$ generated by I, while the set $\mathrm{Orth}(M)$ of all orthomorphisms on M is the principal band of $\mathcal{L}_r(M)$ generated by I. For more background on orthomorphisms the reader is referred to [2,7].

At this point, we focus on order ideals of $\mathcal{L}_r(L, M)$, the elements of which preserve disjointness. For the sake of simplicity, we call an order ideal $\mathcal{I}$ of $\mathcal{L}_r(L, M)$ a *δ-ideal* if all operators in $\mathcal{I}$ are disjointness preserving. Positive elements in a δ-ideal are lattice homomorphisms. We define the δ-ideal $\mathcal{I}$ of $\mathcal{L}_r(L, M)$ to be *maximal* if $\mathcal{I}$ is a maximal element with respect to the inclusion in the set of all δ-ideals of $\mathcal{L}_r(L, M)$. The Zorn lemma reveals easily that any δ-ideal is contained in a maximal δ-ideal. The band of all orthomorphisms on M turns out to be a maximal δ-ideal of $\mathcal{L}_r(M)$ as we can see next.

Proposition 2.1. *Let M be a Dedekind complete vector lattice. Then* $\mathrm{Orth}(M)$ *is a maximal δ-ideal of $\mathcal{L}_r(M)$.*

Proof. Since $\mathrm{Orth}(M)$ is a band of $\mathcal{L}_r(M)$ and orthomorphisms on M are disjointness preserving, $\mathrm{Orth}(M)$ is a δ-ideal. We claim that $\mathrm{Orth}(M)$ is maximal as a δ-ideal of $\mathcal{L}_r(M)$. To this end, let $\mathcal{J}$ be a δ-ideal of $\mathcal{L}_r(M)$ that contains $\mathrm{Orth}(M)$ and let T be a positive operator in $\mathcal{J}$. As the identity operator I of M is an orthomorphism, $I \in \mathcal{J}$ and then $I + T$ is a positive operator in $\mathcal{J}$. Accordingly, $I + T$ is a lattice homomorphism. From [3, Problem 3.3.1], it follows that T is an orthomorphism on M and then $\mathrm{Orth}(M) = \mathcal{J}$. This means that $\mathrm{Orth}(M)$ is a maximal δ-ideal of $\mathcal{L}_r(M)$, as required. □

In other words, the concept of maximal δ-ideals generalizes the notion of orthomorphisms. Therefore, we expect certain properties of orthomorphisms to extend to the more general setting of maximal δ-ideals. A first result in this direction is the following.

Proposition 2.2. *Let L and M be vector lattices with M Dedekind complete. Then any maximal δ-ideal of $\mathcal{L}_r(L, M)$ is a band.*

Proof. Let $\mathcal{I}$ be a maximal δ-ideal of $\mathcal{L}_r(L, M)$ and $\mathcal{I}^{\mathrm{dd}}$ be the band of $\mathcal{L}_r(L, M)$ generated by $\mathcal{I}$. We claim that $\mathcal{I}^{\mathrm{dd}}$ is a δ-ideal of $\mathcal{L}_r(L, M)$. To this end, it suffices to show that any positive operator T in $\mathcal{I}^{\mathrm{dd}}$ is a lattice homomorphism. By [2, Theorem 1.27], there exists a directed upward set $\{T_\lambda\colon \lambda \in \Lambda\}$ of positive operators in $\mathcal{I}$ such that $\sup\{T_\lambda\colon \lambda \in \Lambda\} = T$. Since $\mathcal{I}$ is a δ-ideal, T_λ is a lattice homomorphism for all $\lambda \in \Lambda$. It follows from [7, Theorem 1.14] that

$$T(f) = \sup\bigl\{T_\lambda(f)\colon \lambda \in \Lambda\bigr\}, \quad \text{for all positive } f \in L.$$

Moreover, $T_\lambda + T_\mu$ is a lattice homomorphism for all $\lambda, \mu \in \Lambda$, where we use again that $\mathcal{I}$ is a δ-ideal. Hence if $\lambda, \mu \in \Lambda$ and $f, g \in L$ with $f \wedge g = 0$ then

$$0 \leqslant T_\lambda(f) \wedge T_\mu(g) \leqslant (T_\lambda + T_\mu)(f) \wedge (T_\lambda + T_\mu)(g) = 0.$$

This yields that $T(f) \wedge T(g) = 0$ and the proposition follows. □

In spite of the preceding result, the important fact on orthomorphisms we wish to extend to maximal δ-ideals is that M and $\mathrm{Orth}(M)$ are lattice isomorphic, when M, in addition, has an order unit. Notice that $\mathrm{Orth}(M)$ and $\mathcal{Z}(M)$ coincide in that situation. Such an extension will arise from our next discussion.

3. Characterizations of maximal δ-ideals

Recall that L and M are vector lattices with M Dedekind complete and assume, from now on, L to have an order unit $u > 0$. For an arbitrary order ideal $\mathcal{I}$ of $\mathcal{L}_r(L, M)$, we define the map $\Pi_{\mathcal{I}} : \mathcal{I} \to M$ by

$$\Pi_{\mathcal{I}}(T) = T(u), \quad \text{for all } T \in \mathcal{I}.$$

It is readily verified that $\Pi_{\mathcal{I}}$ is a positive operator. This operator plays a key role in the context of our investigation. In this direction, we get the following descriptions of δ-ideals of $\mathcal{L}_r(L, M)$.

Theorem 3.1. *Let L be a vector lattice with an order unit $u > 0$, M be a Dedekind complete vector lattice, and $\mathcal{I}$ be an order ideal of $\mathcal{L}_r(L, M)$. Then the following are equivalent*:

(i) *$\Pi_{\mathcal{I}}$ is a lattice homomorphism.*
(ii) *$\Pi_{\mathcal{I}}$ is injective.*
(iii) *$\mathcal{I}$ is a δ-ideal of $\mathcal{L}_r(L, M)$.*

Proof. (i) $\Rightarrow$ (ii). Assume $\Pi_{\mathcal{I}}$ to be a lattice homomorphism and let T be an element of the kernel of $\Pi_{\mathcal{I}}$. Then

$$|T|(u) = \Pi_{\mathcal{I}}(|T|) = |\Pi_{\mathcal{I}}(T)| = 0.$$

But then $|T| = 0$ because u is an order unit in L. This yields that $T = 0$ so $\Pi_{\mathcal{I}}$ is one-to-one.

(ii) $\Rightarrow$ (iii). It suffices to show that T is a lattice homomorphism whenever $0 < T \in \mathcal{I}$. To do this, denote L^{ru} the relatively uniform completion of L (see [19, Definition 2.12]). Obviously, u is again an order unit in L^{ru}. Moreover, let $T^{\mathrm{ru}} : L^{\mathrm{ru}} \to M$ be the unique extension of T to L^{ru}, where we use [21, Theorem 3.3]. Hence T^{ru} is a positive operator. Put $v = T^{\mathrm{ru}}(u) = T(u)$ and observe that $v > 0$ in M. It is clear that T^{ru} maps L^{ru} into the principal order ideal M_v of M generated by v. Both L^{ru} and M_v are relatively uniformly complete (uniformly closed in [14]) with order units u and v, respectively. It follows from [18, Proposition 1.2.13] that L^{ru} and M_v are Banach lattices with u and v as units, respectively. Pick two positive operators $R, S : L^{\mathrm{ru}} \to M_v$ such that

$$R(u) = S(u) = v \quad \text{and} \quad T = aR + (1 - a)S$$

for some real number $a \in (0, 1)$. Of course R can be seen as an operator from L^{ru} into M. Thus we have $0 \leqslant R \leqslant a^{-1}T$ in $\mathcal{L}_r(L^{\mathrm{ru}}, M)$. Consider now the operator $R_L : L \to M$ defined by the restriction of R to L. Clearly, R_L is positive, $R_L(u) = v$, and the inequalities $0 \leqslant R_L \leqslant a^{-1}T$ hold in $\mathcal{L}_r(L, M)$. We derive that $R_L \in \mathcal{I}$ as $\mathcal{I}$ is an order ideal of $\mathcal{L}_r(L, M)$. Hence

$$\Pi_{\mathcal{I}}(R_L) = R_L(u) = v = T(u) = \Pi_{\mathcal{I}}(T).$$

Since $\Pi_{\mathcal{I}}$ is one-to-one, we obtain $R_L = T$. In other words, $T^{\mathrm{ru}}(f) = R(f)$ for all $f \in L$. By uniqueness of extensions to L^{ru}, we obtain $T^{\mathrm{ru}} = R = S$. Consequently, T^{ru} is an extremal point in the convex set of all Markov operators from L^{ru} into M_v (notice here that a *Markov operator* between the two Banach lattices L^{ru} into M_v is a positive operator $U : L^{\mathrm{ru}} \to M_v$ such that $U(u) = v$). So, by [20, Proposition 9.2], T^{ru} is a lattice homomorphism from L^{ru} into M_v and then into M. It follows that T is again a lattice homomorphism.

(iii) $\Rightarrow$ (i). If T is an operator in the δ-ideal $\mathcal{I}$ of $\mathcal{L}_r(L, M)$ then T is a disjointness preserving operator. So,

$$\left|\Pi_{\mathcal{I}}(T)\right| = \left|T(u)\right| = |T|(u) = \Pi_{\mathcal{I}}\big(|T|\big).$$

This means that $\Pi_{\mathcal{I}}$ is a lattice homomorphism and the proof of the theorem is complete. $\square$

In order to prove the central result of this paper, we need the next lemma, which deals with the range $\mathrm{Im}(\Pi_{\mathcal{I}})$, when $\mathcal{I}$ is a maximal δ-ideal of $\mathcal{L}_r(L, M)$.

Lemma 3.2. *Let L be a vector lattice with an order unit $u > 0$ and M be a Dedekind complete vector lattice. If $\mathcal{I}$ is a maximal δ-ideal of $\mathcal{L}_r(L, M)$ then* $\mathrm{Im}(\Pi_{\mathcal{I}})$ *is a band of M.*

Proof. Since $\mathcal{I}$ is a δ-ideal of $\mathcal{L}_r(L, M)$, Theorem 3.1 yields that $\Pi_{\mathcal{I}}$ is a lattice homomorphism. Thus $\mathrm{Im}(\Pi_{\mathcal{I}})$ is a vector sublattice of M. On the other hand, let $g \in M$ sand assume that the inequalities

$$0 \leqslant g \leqslant \Pi_{\mathcal{I}}(T) = T(u)$$

hold in M for some positive operator $T \in \mathcal{I}$. By [7, Theorem 8.15], there exists a positive operator $R \in \mathcal{Z}(M)$ such that $R(T(u)) = g$. As $0 \leqslant RT \leqslant T$ and $\mathcal{I}$ is an order ideal of $\mathcal{L}_r(L, M)$, we get $RT \in \mathcal{I}$. Therefore,

$$g = R\big(T(u)\big) = (RT)(u) = \Pi_{\mathcal{I}}(RT) \in \mathrm{Im}(\Pi_{\mathcal{I}}).$$

Consequently, $\mathrm{Im}(\Pi_{\mathcal{I}})$ is an order ideal of M.

At this point, let $\{g_\lambda\colon \lambda \in \Lambda\}$ be a directed upward set of positive elements in $\mathrm{Im}(\Pi_{\mathcal{I}})$ such that the supremum $\sup\{g_\lambda\colon \lambda \in \Lambda\} = g$ exists in M. Since u is an order unit in L, there exists a directed upward set $\{T_\lambda\colon \lambda \in \Lambda\}$ of positive operators in $\mathcal{I}$ such that $T_\lambda(u) = g_\lambda$ for all $\lambda \in \Lambda$. If $f \in L$ and a is a real number such that $0 \leqslant f \leqslant au$ then

$$0 \leqslant T_\lambda(f) \leqslant aT_\lambda(u) = ag_\lambda \leqslant g, \quad \text{for all } \lambda \in \Lambda.$$

It follows from [7, Theorem 1.14] that $\sup\{T_\lambda\colon \lambda \in \Lambda\} = T$ exists in $\mathcal{L}_r(L, M)$ and satisfies the equalities

$$g = \sup\big\{T_\lambda(u)\colon \lambda \in \Lambda\big\} = T(u).$$

According to Proposition 2.2, $\mathcal{I}$ is a band of $\mathcal{L}_r(L, M)$ and so $T \in \mathcal{I}$. This leads to

$$g = T(u) = \Pi_{\mathcal{I}}(T) \in \mathrm{Im}(\Pi_{\mathcal{I}}),$$

which implies that $\mathrm{Im}(\Pi_{\mathcal{I}})$ is a band of M and we are done. $\square$

We have gathered now all the ingredients for the proof of the main theorem of this paper, which describes maximal δ-ideals of $\mathcal{L}_r(L, M)$.

Theorem 3.3. *Let L be a vector lattice with an order unit $u > 0$, M be a Dedekind complete vector lattice, and $\mathcal{I}$ be an order ideal of $\mathcal{L}_r(L, M)$. Then the following are equivalent*:

(i) $\Pi_{\mathcal{I}}$ *is a lattice isomorphism.*
(ii) $\Pi_{\mathcal{I}}$ *is bijective.*
(iii) $\mathcal{I}$ *is a maximal* δ*-ideal of* $\mathcal{L}_r(L, M)$.

Proof. (i) $\Rightarrow$ (ii). Trivial.

(ii) $\Rightarrow$ (iii). By Theorem 3.1, $\mathcal{I}$ is a δ-ideal of $\mathcal{L}_r(L, M)$. Then we only have to prove that $\mathcal{I}$ is maximal. To do this, let $\mathcal{J}$ be a δ-ideal of $\mathcal{L}_r(L, M)$ such that $\mathcal{I} \subset \mathcal{J}$ and choose $T \in \mathcal{J}$. Since $\Pi_{\mathcal{I}}$ is surjective, there exists $R \in \mathcal{I}$ such that $\Pi_{\mathcal{I}}(R) = T(u)$. Therefore,

$$\Pi_{\mathcal{J}}(T) = T(u) = \Pi_{\mathcal{I}}(R) = R(u) = \Pi_{\mathcal{J}}(R).$$

Again by Theorem 3.1, $\Pi_{\mathcal{J}}$ is one-to-one so $T = R \in \mathcal{I}$. It follows that $\mathcal{I} = \mathcal{J}$. Consequently, $\mathcal{I}$ is a maximal δ-ideal of $\mathcal{L}_r(L, M)$.

(iii) $\Rightarrow$ (i). In view of Theorem 3.1, it suffices to show that $\Pi_{\mathcal{I}}$ is surjective. Hence, let P denote the band projection of M on the disjoint complement $(\mathrm{Im}(\Pi_{\mathcal{I}}))^{\mathrm{d}}$ of $\mathrm{Im}(\Pi_{\mathcal{I}})$ and let $0 \leqslant h \in M$. We define the lattice homomorphism T_h from the vector sublattice $\mathbb{R}u$ of L generated by u into M by putting $T_h(u) = h$. Since u is an order unit of L, the vector sublattice $\mathbb{R}u$ is majorizing in L. By [7, Theorem 7.17], T_h extends to a lattice homomorphism from L into M, denoted again by T_h. Hence, the operator PT_h is a lattice homomorphism and the inclusion

$$\mathrm{Im}(PT_h) \subset \big(\mathrm{Im}(\Pi_{\mathcal{I}})\big)^{\mathrm{d}} \tag{1}$$

holds. On the other hand, let $T \in \mathcal{I}$ and $f \in L$. Since $|f| \leqslant au$ for some real number a, we get

$$\big|T(f)\big| = |T|\big(|f|\big) \leqslant |\lambda T|(u) = \Pi_{\mathcal{I}}\big(|\lambda T|\big).$$

By Lemma 3.2, $\mathrm{Im}(\Pi_{\mathcal{I}})$ is an order ideal of M. This implies that

$$\big|T(f)\big| \in \mathrm{Im}(\Pi_{\mathcal{I}}). \tag{2}$$

Combining (1) and (2), we obtain

$$\big|T(f)\big| \wedge \big|(PT_h)(g)\big| = 0, \quad \text{for all } f, g \in L \text{ and } T \in \mathcal{I}.$$

It follows that the subset $\mathcal{A} = \mathcal{I} \cup \{PT_h\}$ of $\mathcal{L}_r(L, M)$ has the property that

$$\big(|f| \wedge |g| = 0\big) \quad \Rightarrow \quad \big(\big|R(f)\big| \wedge \big|S(g)\big| = 0, \text{ for all } R, S \in \mathcal{A}\big). \tag{3}$$

Consider at this point the order ideal $\mathcal{I}_{\mathcal{A}}$ of $\mathcal{L}_r(L, M)$ generated by $\mathcal{A}$. We claim that $\mathcal{I}_{\mathcal{A}}$ is a δ-ideal of $\mathcal{L}_r(L, M)$. To this end, let T be an operator in $\mathcal{I}_{\mathcal{A}}$, $T_1, \ldots, T_n$ be operators in $\mathcal{A}$, and $a_1, \ldots, a_n$ be real numbers in $(0, \infty)$ such that

$$0 \leqslant T \leqslant R = a_1|T_1| + \cdots + a_n|T_n|$$

(see [2, p. 20]). Let $f, g \in L$ such that $f \wedge g = 0$ and observe that

$$|T_i|(f) \wedge |T_j|(g) = \big|T_i(f)\big| \wedge \big|T_j(g)\big| = 0, \quad \text{for all } i, j \in \{1, \ldots, n\},$$

where we use (3). It follows that

$$0 \leqslant T(f) \wedge T(g) \leqslant R(f) \wedge R(g) = 0.$$

Therefore, T is a lattice homomorphism. We derive that $\mathcal{I}_{\mathcal{A}}$ is a δ-ideal of $\mathcal{L}_r(L, M)$, as required. Since $\mathcal{I}_{\mathcal{A}}$ contains the maximal δ-ideal $\mathcal{I}$ of $\mathcal{L}_r(L, M)$, we get $\mathcal{I}_{\mathcal{A}} = \mathcal{I}$ and then $PT_h \in \mathcal{I}$. Accordingly,

$$P(h) = P\big(T_h(u)\big) = (PT_h)(u) = \Pi_{\mathcal{I}}(PT_h) \in \mathrm{Im}(\Pi_{\mathcal{I}}).$$

The latter equalities together with (1) lead to $P(h)=0$. Hence, $P=0$ so $(\mathrm{Im}(\Pi_{\mathcal{I}}))^{\mathrm{d}}=0$. It follows that $(\mathrm{Im}(\Pi_{\mathcal{I}}))^{\mathrm{dd}}=M$, where $(\mathrm{Im}(\Pi_{\mathcal{I}}))^{\mathrm{dd}}$ denotes the band of M generated by $\mathrm{Im}(\Pi_{\mathcal{I}})$. Whence, using Lemma 3.2, we obtain $\mathrm{Im}(\Pi_{\mathcal{I}})=M$. In other words, $\Pi_{\mathcal{I}}$ is surjective, which is the desired result. This completes the proof of the theorem. $\square$

As a consequence, we get the following corollary.

Corollary 3.4. *Let M be a Dedekind complete vector lattice. If L is a vector lattice with an order unit, then $\mathcal{L}_r(L,M)$ has a unique* (*up to a lattice isomorphism*) *maximal δ-ideal, which is a vector lattice copy of M.*

Proof. Examining the proof of the arrow (iii) $\Rightarrow$ (i) in Theorem 3.3, we see that there exists a nontrivial lattice homomorphism $T:L\to M$. It is easily seen that the principal order ideal $\mathcal{I}_T$ of $\mathcal{L}_r(L,M)$ generated by T is a δ-ideal. In other words, the collection of all δ-ideals of $\mathcal{L}_r(L,M)$ is nonempty. A classical argument based on the Zorn lemma shows that $\mathcal{L}_r(L,M)$ has a maximal δ-ideal $\mathcal{I}$. If $\mathcal{J}$ is another maximal δ-ideal of $\mathcal{L}_r(L,M)$, then Theorem 3.3 yields that both $\mathcal{I}$ and $\mathcal{J}$ are vector lattice copies of M. The uniqueness follows straightforwardly. $\square$

4. Examples of maximal δ-ideals

In this section, we describe maximal δ-ideals on some continuous functions spaces. We start with some useful notations and facts. Let X be a completely regular topological space. The vector lattice of all continuous real-valued functions on X is denoted by $C(X)$. By $\mathbf{1}_X$ we mean the function in $C(X)$ defined by $\mathbf{1}_X(x)=1$ for all $x\in X$. Therefore, the vector lattice $C(X)$ has $\mathbf{1}_X$ as an order unit if X, in addition, is compact. Recall also that the vector lattice $C(X)$ is Dedekind complete whenever X additionally is extremally disconnected, that is, every open set has an open closure. The cozeroset of a function f in $C(X)$ is denoted by $\mathrm{coz}(f)$ and defined by

$$\mathrm{coz}(f)=\big\{x\in X\colon\ f(x)\neq 0\big\}.$$

For more background on continuous functions spaces we refer to the classical book [12] by Gillman and Jerison.

Let X and Y be completely regular topological spaces with X compact. In the next corollary we characterize lattice homomorphisms from $C(X)$ into an arbitrary vector sublattice F of $C(Y)$. The proof is a slight modification of the proof of the standard Theorem 7.22 [7] (see also [12, Theorem 10.8]).

Proposition 4.1. *Let X and Y be a completely regular topological spaces with X compact. Let F be a vector sublattice of $C(Y)$ and $T:C(X)\to F$ be a lattice homomorphism. Then there exist a unique positive function $w\in C(Y)$ and a function $\tau:Y\to X$, which is continuous on* $\mathrm{coz}(w)$, *such that*

$$T(f)(y)=w(y)f\big(\tau(y)\big),\quad \textit{for all } f\in C(X),\ y\in Y.$$

Proof. The case where T is zero being obvious, we assume T to be nonzero. Let $w=T(\mathbf{1}_X)$ and $y\in\mathrm{coz}(w)$, and define δ_y to be the Dirac measure at y. Hence the linear functional $\delta_y\circ T: C(X)\to\mathbb{R}$ is a nonzero lattice homomorphism. By [7, Theorem 7.21], there exists a unique real number $w_y\in(0,\infty)$ and a unique point $\alpha_y\in X$ such that

$$(\delta_y\circ T)(f)=w_y f(\alpha_y),\quad \text{for all } f\in C(X).$$

Observe that

$$w_y = (\delta_y \circ T)(\mathbf{1}_X) = w(y).$$

Moreover, a function $\alpha : \operatorname{coz}(w) \to X$ can be defined by $\alpha(y) = \alpha_y$ for all $y \in \operatorname{coz}(w)$. We get

$$T(f)(y) = w(y) f\big(\alpha(y)\big), \quad \text{for all } f \in C(X),\ y \in \operatorname{coz}(w).$$

Now, let (y_λ) be a net of elements of $\operatorname{coz}(w)$ and $y \in \operatorname{coz}(w)$ such that $y_\lambda \to y$. Since w is continuous, $w(y_\lambda) \to w(y)$. For the same reason, $T(f)(y_\lambda) \to T(f)(y)$ for each $f \in C(X)$. In summary, $f(\alpha(y_\lambda)) \to f(\alpha(y))$ holds for all $f \in C(X)$. It follows that $\alpha(y_\lambda) \to \alpha(y)$ and thus α is continuous on $\operatorname{coz}(w)$. Let τ be an arbitrary function from Y into X which extends α. Since $\operatorname{coz}(w)$ is an open subset of Y, τ is continuous on $\operatorname{coz}(w)$. Besides, if $w(y) = 0$ for some $y \in Y$ then $T(f)(y) = 0$ for all $f \in C(X)$. This follows straightforwardly from the inequality

$$\big|T(f)(y)\big| \leqslant M w(y),$$

where $M = \sup\{|f(x)|\colon x \in X\}$. Consequently,

$$T(f)(y) = w(y) f\big(\alpha(y)\big), \quad \text{for all } f \in C(X), y \in Y.$$

On the other hand, the above formula implies quickly that $w = T(\mathbf{1}_X)$, which yields the uniqueness of w. This completes the proof of the proposition. □

Let A be a nonempty subset of $C(Y)$. A function $f \in C(Y)$ is said to be *A-regular* if $fg = 0$ and $g \in A$ imply $g = 0$. In other words, the function $f \in C(Y)$ is A-regular if and only if

$$\operatorname{coz}(f) \cap \operatorname{coz}(g) \neq \emptyset, \quad \text{for all } g \in A,\ g \neq 0.$$

This notion turns out to be necessary for the following main theorem of this section.

Theorem 4.2. *Let X and Y be completely regular topological spaces with X compact. Let F be a Dedekind complete vector sublattice of $C(Y)$ and assume that F contains an F-regular function w_0. The following are equivalent for a nonempty subset $\mathcal{M}$ of $\mathcal{L}_r(C(X), F)$:*

(i) *$\mathcal{M}$ is a maximal δ-ideal of $\mathcal{L}_r(C(X), F)$.*
(ii) *There exists a function $\tau : Y \to X$, which is continuous on* $\operatorname{coz}(w_0)$*, such that $T \in \mathcal{M}$ if and only if there is $w \in F$ for which*

$$T(f)(y) = w(y) f\big(\tau(y)\big), \quad \textit{for all } f \in C(X),\ y \in Y.$$

Proof. (ii) ⇒ (i). The condition (ii) implies directly that $\mathcal{M}$ is a vector sublattice of $\mathcal{L}_r(C(X), F)$ and that all elements of $\mathcal{M}$ are disjointness preserving operators. We claim that$\mathcal{M}$ is a δ-ideal of $\mathcal{L}_r(C(X), F)$. To this end, let $R \in \mathcal{M}$ with $0 \leqslant T \leqslant R$. Hence, there exists $w \in F^+$ such that

$$0 \leqslant T(f)(y) \leqslant w(y) f\big(\tau(y)\big), \quad \text{for all } y \in Y,\ 0 \leqslant f \in C(X).$$

Thus

$$T(f)(y) = \mu(y) w(y) f\big(\tau(y)\big), \quad \text{for all } f \in C(X),\ y \in Y,$$

for some function $\mu : Y \to \mathbb{R}$. Since $\mu w = T(\mathbf{1}_X) \in F$, we get $T \in \mathcal{M}$. This shows that $\mathcal{M}$ is a δ-ideal of $\mathcal{L}_r(C(X), F)$, as required. Observe now that the operator $\Pi_{\mathcal{M}} : \mathcal{M} \to F$ defined by $\Pi_{\mathcal{M}}(T) = T(\mathbf{1}_X)$ for all $T \in \mathcal{M}$ is a lattice isomorphism. Consequently, the δ-ideal $\mathcal{M}$ of $\mathcal{L}_r(C(X), F)$ is maximal, where we use Theorem 3.3.

(i) $\Rightarrow$ (ii). Assume $\mathcal{M}$ to be a maximal δ-ideal of $\mathcal{L}_r(C(X), F)$. By Theorem 3.3, the operator $\Pi_{\mathcal{M}} : \mathcal{M} \to F$ defined by $\Pi_{\mathcal{M}}(T) = T(\mathbf{1}_X)$ for all $T \in \mathcal{M}$ is a lattice isomorphism. Therefore, there exists a unique positive operator $T_0 \in \mathcal{M}$ such that $T_0(\mathbf{1}_X) = |w_0|$. But then T_0 is a lattice homomorphism from $C(X)$ into F. Hence, in view of Proposition 4.1, there exists a function $\tau : Y \to X$, which is continuous on $\operatorname{coz}(w_0)$, such that

$$T_0(f)(y) = \left|w_0(y)\right| f\left(\tau(y)\right), \quad \text{for all } f \in C(X),\ y \in Y.$$

Consider the set $\mathcal{I}$ of all operators $T \in \mathcal{L}_r(C(X), F)$ for which there exists a function $w \in F$ such that

$$T(f)(y) = w(y) f\left(\tau(y)\right), \quad \text{for all } f \in C(X),\ y \in Y.$$

In particular, $T_0 \in \mathcal{I}$. By the implication (ii) $\Rightarrow$ (i), $\mathcal{I}$ is a maximal δ-ideal of $\mathcal{L}_r(C(X), F)$. Furthermore, pick $T \in \mathcal{I}$ such that $T \wedge T_0 = 0$, that is, $(T - T_0)^+ = T$. Since $T - T_0$ preserves disjointness, [18, Theorem 3.1.4] leads to

$$T(\mathbf{1}_X) = (T - T_0)^+(\mathbf{1}_X) = \left(T(\mathbf{1}_X) - T_0(\mathbf{1}_X)\right)^+.$$

Hence, if $w \in F$ verifies

$$T(f)(y) = w(y) f\left(\tau(y)\right), \quad \text{for all } f \in C(X),\ y \in Y,$$

then

$$w = T(\mathbf{1}_X) = \left(T(\mathbf{1}_X) - T_0(\mathbf{1}_X)\right)^+ = (w - w_0)^+,$$

so that $ww_0 = 0$. But then $w = 0$ because w_0 is F-regular, which means that $T = 0$. Accordingly, $\mathcal{I} \subset \{T_0\}^{\mathrm{dd}}$, where $\{T_0\}^{\mathrm{dd}}$ is the principal band of $\mathcal{L}_r(C(X), F)$ generated by T_0. It follows that $\mathcal{I} \subset \mathcal{M}$ as $\mathcal{M}$ is a band of $\mathcal{L}_r(C(X), F)$ containing T_0 (see Lemma 3.2). Since $\mathcal{I}$ is a maximal δ-ideal of $\mathcal{L}_r(C(X), C(Y))$ and $\mathcal{M}$ is a δ-ideal of $\mathcal{L}_r(C(X), C(Y))$, we get $\mathcal{I} = \mathcal{M}$ and we are done. $\square$

Since the function $w_0 = \mathbf{1}_Y$ is $C(Y)$-regular and $\operatorname{coz}(w_0) = Y$, we immediately get the following corollary.

Corollary 4.3. *Let X and Y be completely regular topological spaces with X compact and Y extremally disconnected. The following are equivalent for a nonempty subset $\mathcal{M}$ of $\mathcal{L}_r(C(X), C(Y))$:*

(i) *$\mathcal{M}$ is a maximal δ-ideal of $\mathcal{L}_r(C(X), C(Y))$.*
(ii) *There exists a continuous function $\tau : Y \to X$ such that $T \in \mathcal{M}$ if and only if there is $w \in C(Y)$ for which*

$$T(f)(y) = w(y) f\left(\tau(y)\right), \quad \textit{for all } f \in C(X),\ y \in Y.$$

Another particular setting seems to be of some independent interest. Let us denote the set $\{1, 2, \ldots\}$ of positive integers by $\mathbb{N}$ and let $\beta\mathbb{N}$ be the Stone–Čech compactification of $\mathbb{N}$ [12]. For any real number $p \in (0, \infty)$, define ℓ^p to be the set of all real sequences $(x_n)_1^\infty$ such that the series $\sum |x_n|^p$ converges. Clearly, ℓ^p is a Dedekind complete vector sublattice of $C(\mathbb{N})$ and possesses ℓ^p-regular elements (consider, for instance, $(n^{-2/p})_1^\infty$). These observations together with Theorem 4.2 lead to the following.

Corollary 4.4. *Let $p \in (0, \infty)$ an $\mathcal{M}$ be a nonempty subset of $\mathcal{L}_r(C(\beta\mathbb{N}), \ell^p)$. The following are equivalent*:

(i) *$\mathcal{M}$ is a maximal δ-ideal of $\mathcal{L}_r(C(\beta\mathbb{N}), \ell^p)$.*
(ii) *There exists a sequence $(\tau_n)_1^\infty$ of elements of $\beta\mathbb{N}$ such that $T \in \mathcal{M}$ if and only if there is $(w_n)_1^\infty \in \ell^p$ for which*

$$T(f) = \big(w_n f(\tau_n)\big)_1^\infty, \quad \textit{for all } f \in C(\beta\mathbb{N}).$$

The last result of this paper again is an application of Theorem 4.2 and more precisely of Corollary 4.3. The result in question deals with lattice bimorphisms on continuous functions spaces. Let X, Y and Z be completely regular topological spaces with X, Y compact and Z extremally disconnected, and $\Phi : C(X) \times C(Y) \to C(Z)$ be a bilinear map. We say that Φ is a *lattice bimorphism* if, for any positive functions $f_0 \in C(X)$ and $g_0 \in C(Y)$, the operators $\Phi(f_0, \cdot) : C(Y) \to C(Z)$ and $\Phi(\cdot, g_0) : C(X) \to C(Z)$ defined by

$$\Phi(f_0, \cdot)(g) = \Phi(f_0, g), \quad \text{for all } g \in C(Y),$$

and

$$\Phi(\cdot, g_0)(f) = \Phi(f, g_0), \quad \text{for all } f \in C(X),$$

are lattice homomorphisms. It is not hard to see that the bilinear map $\Phi : C(X) \times C(Y) \to C(Z)$ is a lattice bimorphism if and only if

$$\big|\Phi(f, g)\big| = \Phi\big(|f|, |g|\big), \quad \text{for all } f \in C(X),\ g \in C(Y).$$

In particular, any lattice bimorphism $\Phi : C(X) \times C(Y) \to C(Z)$ is *positive*, that is, $\Phi(f, g)$ is a positive function in $C(Z)$ whenever f and g are positive functions in $C(X)$ and $C(Y)$, respectively. We finish this work with the next 'bilinear' version of the standard Theorem 7.22 [7] (see also [2, Theorem 4.25]).

Corollary 4.5. *Let X, Y and Z be completely regular topological spaces with X, Y compact and Z extremally disconnected. Let $\Phi : C(X) \times C(Y) \to C(Z)$ be a bilinear map. Then Φ is a lattice bimorphism if and only if there exists a positive function $w \in C(Z)$, and continuous functions $\sigma : Z \to X$ and $\tau : Z \to Y$ such that*

$$\Phi(f, g)(z) = w(y) f\big(\sigma(z)\big) g\big(\tau(z)\big)$$

for all $(f, g) \in C(X) \times C(Y)$, $z \in Z$.

Proof. Put $\mathcal{A} = \{\Psi(f, .)\colon\ 0 \leqslant f \in C(X)\}$. It is readily checked that the order ideal $\mathcal{I}_{\mathcal{A}}$ of $\mathcal{L}_r(C(X), C(Z))$ is a δ-ideal. By Zorn lemma, $\mathcal{I}_{\mathcal{A}}$ is contained in some maximal δ-ideal $\mathcal{M}$ of $\mathcal{L}_r(C(X), C(Z))$. From Corollary 4.3, it follows that there exists a continuous function $\tau : Z \to X$ such that, for any positive function $f \in C(X)$, there is a positive function $w_f \in C(Z)$ satisfying

$$\Phi(f, g)(z) = w_f(z) g\big(\tau(z)\big), \quad \text{for all } g \in C(Y),\ z \in Z.$$

In particular, $w_f = \Phi(f, \mathbf{1}_Y)$. On the other hand, $\Phi(\cdot, \mathbf{1}_Y) : C(X) \to C(Z)$ is a lattice homomorphism. Accordingly, there exist a positive function $w \in C(Z)$ and a continuous function $\sigma : Z \to Y$ such that

$$\Phi(f, \mathbf{1}_X)(z) = w(z) f\big(\sigma(z)\big), \quad \text{for all } f \in C(X),\ z \in Z$$

(see Proposition 4.1). Consequently, if f is a positive function in $C(X)$, $g \in C(Y)$, and $z \in Z$, then

$$\Phi(f,g)(z) = w_f(z)g\bigl(\tau(z)\bigr) = \Phi(f,\mathbf{1}_Y)(z)g\bigl(\tau(z)\bigr) = w(z)f\bigl(\sigma(z)\bigr)g\bigl(\tau(z)\bigr).$$

Since $f = f^+ - f^-$ for all $f \in C(X)$, an easy argument based on bilinearity yields that

$$\Phi(f,g)(z) = w(z)f\bigl(\sigma(z)\bigr)g\bigl(\tau(z)\bigr),$$

for all $(f,g) \in C(X) \times C(Y)$ and $z \in Z$. This completes the proof of the corollary. □

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