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Positive solutions for multi-point boundary value problem on the half-line ☆

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Abstract

This paper presents a variety of existence results for nonlinear multi-point boundary value problem on the half-line. In particular, we consider

$$x''(t) - px'(t) - qx(t) + f\big(t, x(t)\big) = 0, \quad t \in [0, \infty),$$
$$\alpha x(0) - \beta x'(0) - \sum_{i=1}^{n} k_i x(\xi_i) = a, \qquad \lim_{t\to\infty} \frac{x(t)}{e^{rt}} = b, \quad r \in \left[0, \frac{p + \sqrt{p^2 + 4q}}{2}\right].$$

Existence results are established using fixed-point theorems on cone.

Keywords: Multi-point boundary value problem; Half-line; Positive solutions; Existence criteria

1. Introduction

This paper is concerned with the existence of solutions to multi-point boundary value problem

$$\begin{cases} x''(t) - px'(t) - qx(t) + f\big(t, x(t)\big) = 0, & t \in I = [0, \infty), \\ \alpha x(0) - \beta x'(0) - \sum_{i=1}^{n} k_i x(\xi_i) = a \geqslant 0, & \lim_{t\to\infty} \frac{x(t)}{e^{rt}} = b \geqslant 0, \end{cases} \tag{1.1}$$

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where $p, \alpha, \beta, k_i \geqslant 0$, $\alpha^2 + \beta^2 + \sum_{i=1}^{n} k_i^2 \neq 0$, $q > 0$, $0 \leqslant \xi_i < \infty$, $i = 1, 2, \ldots, n$, $r \in [0, \frac{p+\sqrt{p^2+4q}}{2}]$.

Definition 1.1. A function $x \in C(I, I)$ is said to be a positive solution of boundary value problem (1.1), if $x(t) \geqslant 0$, and x satisfies (1.1) for $t \in I$.

By employing fixed-point theorems on cones, the existence of positive solutions to (1.1) will be obtained.

The motivation for the present work stems from both practical and theoretical aspects. In fact, boundary value problems (BVPs) on the half-line occur naturally in the study of radially symmetric solutions of nonlinear elliptic equations, see [5,13], and various physical phenomena [3,11], such as unsteady flow of gas through a semi-infinite porous media, the theory of drain flows, plasma physics, in determining the electrical potential in an isolated neutral atom. In all these applications, it is frequent that only solutions that are positive are useful. Recently there have been many papers investigated the positive solutions of boundary value problem on the half-line, see [1,2,4,15–17]. Zima [17] studied the existence of at least one positive solution to boundary value problem on the half-line for the following second-order differential equation:

$$\begin{cases} x''(t) - k^2 x(t) + f\big(t, x(t)\big) = 0, & t \in [0, \infty), \\ x(0) = 0, \qquad \lim\limits_{t\to\infty} x(t) = 0, \end{cases} \tag{1.2}$$

where $k > 0$ and f is a continuous, nonnegative function. Yan [15] considered the following two-point boundary value problem:

$$\begin{cases} \dfrac{1}{p(t)}\big(p(t)x'(t)\big)' + f(t, x_t) = 0, & t \in [0, \infty), \\ x(0) = a \geqslant 0, \qquad \lim\limits_{t\to\infty} p(t)x'(t) = b \geqslant 0. \end{cases}$$

By the Leray–Schauder theorem, the existence of unbounded solutions was proved. Bai and Fang [4] applied the fixed-point theorem on cone to obtain the positive solutions for the following boundary value problem on infinite interval for second-order functional differential equations:

$$\begin{cases} x''(t) - px'(t) - qx(t) + f(t, x_t) = 0, & t \in [0, \infty), \\ \alpha x(0) - \beta x'(0) = \xi(t), \quad t \in [-\tau, 0], \qquad \lim\limits_{t\to\infty} x(t) = 0, \end{cases}$$

here $x_t \in C([-\tau, 0], I)$.

However, in [1,2,4,15–17], the authors only studied two-point boundary value problem on the half-line. By so far, very few results are obtained for multi-point boundary value problem on the half-line. It is well known that the study of multi-point BVPs is very important. For finite interval, there are many results, see [8–10,12]. So it is necessary to discuss the existence of the multi-point boundary value problem on the half-line. As a result, the goal of this paper is to fill the gap in this area. In particular, the corresponding Green's function and some useful inequalities are obtained in Section 3. Indeed our results extend and complement many results in the literature, see [1,2,4, 15–17] and references therein.

2. Preliminary

For convenience, let

$$r_1 = \frac{p + \sqrt{p^2 + 4q}}{2}, \qquad r_2 = \frac{p - \sqrt{p^2 + 4q}}{2}.$$

It is easy to know that $r_1 > 0$, $r_2 < 0$.

The following conditions will be used in this paper:

(H1) $f : [0, \infty) \times C([0, \infty), [0, \infty)) \to [b(r^2 - pr - q)e^{rt}, \infty)$ is continuous such that $f(t, x) \leqslant c(t) + d(t)x$ for $(t, x) \in [0, \infty) \times C([0, \infty), [0, \infty))$, and $c, d : [0, \infty) \to [0, \infty)$ are continuous functions,

$$c^* = \int_0^\infty c(s)\, ds < \infty, \qquad d^* = \int_0^\infty d(s)e^{r_1 s}\, ds < \infty.$$

(H2) $$\alpha - \beta r_1 - \sum_{i=1}^{n} k_i e^{r_1 \xi_i} > 0.$$

Let

$$C_l\big([0, \infty), R\big) = \Big\{x \in C\big([0, \infty), R\big) \colon \lim_{t\to\infty} x(t) \text{ exists}\Big\}.$$

From [10], we know that C_l is a Banach space with the norm $\|x\|_l = \sup_{t\in[0,+\infty)} |x(t)|$. Let

$$X = C_\infty\big([0, \infty), R\big) = \left\{x \in C\big([0, \infty), R\big) \colon \lim_{t\to\infty} \frac{x(t)}{e^{r_1 t}} \text{ exists}\right\}$$

with the norm $\|x\|_\infty = \sup_{t\in[0,\infty)} |e^{-r_1 t} x(t)|$.

Lemma 2.1. *$X = C_\infty$ is a Banach space.*

Proof. It is easy to see that C_∞ is a normed and linear space. If $\{x_m\} \subseteq C_\infty$ is a Cauchy sequence, then $\{y_m \mid y_m(t) = e^{-r_1 t} x_m(t)\} \subseteq C_l$ is a Cauchy sequence, too. So there exists a $y_0 \in C_l$ such that $\lim_{m\to\infty} \|y_m - y_0\|_l = 0$. Let $x_0(t) = e^{r_1 t} y_0(t)$. Then $x_0 \in C_\infty$ and $\lim_{m\to\infty} \|x_m - x_0\|_\infty = \lim_{m\to\infty} \|y_m - y_0\|_l = 0$. So C_∞ is a Banach space. $\square$

Lemma 2.2. [6,14] *Let $M \subseteq C_l([0, \infty), R)$, then M is precompact if the following conditions hold*:

(a) *M is bounded in C_l*;
(b) *the functions belonging to M are locally equicontinuous on any interval of* $[0, \infty)$;
(c) *the functions from M are equiconvergent, that is, given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|x(t) - x(\infty)| < \varepsilon$ for any $t \geqslant T(\varepsilon)$ and $x \in M$.*

Lemma 2.3. *Let $M \subseteq C_\infty([0, \infty), R)$. Then M is relatively compact in C_∞ if the following conditions hold*:

(i) *M is bounded in C_∞*;

(ii) *the functions belonging to* $\{y \mid y(t) = e^{-r_1 t}x(t),\ x \in M\}$ *are locally equicontinuous on* $[0, \infty)$;
(iii) *the functions from* $\{y \mid y(t) = e^{-r_1 t}x(t),\ x \in M\}$ *are equiconvergent, at* ∞.

Proof. Let $M' = \{y \mid y(t) = e^{-r_1 t}x(t)\}$. It is easy to know that $M' \subseteq C_l$ satisfies the conditions of Lemma 2.2. So there exists a sequence $\{y_n\} \subseteq M'$ and $y_0 \in C_l$ such that $\lim_{n\to\infty} \|y_n - y_0\|_l = 0$. Let $x_n(t) = e^{r_1 t} y_n(t)$, $n = 1, 2, \dots,$ and $x_0 = e^{r_1 t} y_0$. Obviously that $\{x_n\} \subseteq M$ and $x_0 \in C_\infty$ and $\lim_{n\to\infty} \|x_n - x_0\|_\infty = \lim_{n\to\infty} \|y_n - y_0\|_l = 0$. □

Remark 2.1. Lemma 2.3 is a simple improvement of Corduneanu theorem in [6].

Lemma 2.4. [7] *Let X be a bounded open set in real Banach space E, P be a cone of E, $\theta \in \Omega$ and $A : \overline{\Omega} \cap P \to P$ be completely continuous. Suppose*

$$\lambda Ax \neq x, \quad \forall x \in \partial\Omega \cap P,\ \lambda \in [0, 1],$$

then

$$i(A, \Omega \cap P, P) = 1.$$

Lemma 2.5 *(Krasnosel'skii fixed-point theorem). Let E be a Banach space and $K \subseteq E$ be a cone in E. Assume that Ω_1 and Ω_2 are two bounded open sets in E such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Let $F : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(1) $\|Fx\| \leqslant \|x\|$ *for* $x \in K \cap \partial\Omega_1$ *and* $\|Fx\| \geqslant \|x\|$ *for* $x \in K \cap \partial\Omega_2$ *or*
(2) $\|Fx\| \geqslant \|x\|$ *for* $x \in K \cap \partial\Omega_1$ *and* $\|Fx\| \leqslant \|x\|$ *for* $x \in K \cap \partial\Omega_2$

is satisfied. Then F has at least one fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Related lemmas

Lemma 3.1. *Assume that* (H2) *holds. Then $x \in C(I)$ is a solution of the BVP* (1.1) *if and only if $y = x - \bar{x} \in C(I)$ is a solution of the following BVP*:

$$\begin{cases} y'' - py' - qy + g(t, y) = 0, \quad t \in I, \\ \alpha y(0) - \beta y'(0) = \displaystyle\sum_{i=1}^{n} k_i y(\xi_i), \qquad \lim_{t\to\infty} \frac{y(t)}{e^{rt}} = 0, \end{cases} \tag{3.1}$$

where $g(t, y) = b(r^2 - pr - q)e^{rt} + f(t, y + \bar{x})$, $\bar{x}(t) = be^{rt} + \frac{a - b(\alpha - \beta r - \sum_{i=1}^n k_i e^{r\xi_i})}{\alpha - \beta r_2 - \sum_{i=1}^n k_i e^{r_2 \xi_i}} e^{r_2 t}$.

Proof. Clearly, $\bar{x}$ satisfies boundary condition of BVP (1.1). Let $y = x - \bar{x}$, then y is a solution of BVP (3.1) is equivalent to x is a solution of BVP (1.1). □

Now we consider the following boundary value problem for second-order differential equation on the half-line:

$$\begin{cases} x''(t) - px'(t) - qx(t) + \sigma(t) = 0, \quad t \in I, \\ \alpha x(0) - \beta x'(0) = \displaystyle\sum_{i=1}^{n} k_i x(\xi_i), \qquad \lim_{t\to\infty} \frac{x(t)}{e^{rt}} = 0, \end{cases} \tag{3.2}$$

where $\sigma \in C(I)$.

Lemma 3.2. *Let $\sigma \in C(I)$ and $0 \leqslant \int_0^\infty e^{-r_1 s}\sigma(s)\,ds < \infty$. Then $x \in C^2(I)$ is a solution of* (3.2) *if and only if $x \in C^1(I)$ is a solution of the following integral equation*:

$$x(t) = \int_0^\infty G(t,s)\sigma(s)\,ds, \quad t \in I, \tag{3.3}$$

where

$$G(t,s) = \frac{1}{\Delta} \times \begin{cases} e^{r_2 t}\big[(\alpha - \beta r_2)e^{-r_2 s} - (\alpha - \beta r_1)e^{-r_1 s}\big], \\ \quad s \leqslant t,\ s \leqslant \xi_1; \\ e^{r_1(t-s)}\left(\alpha - \beta r_2 - \sum_{i=1}^n k_i e^{r_2 \xi_i}\right) - e^{r_2 t}\left[(\alpha - \beta r_1)e^{-r_1 s} - \sum_{i=1}^n k_i e^{r_2(\xi_i - s)}\right], \\ \quad t \leqslant s \leqslant \xi_1; \\ e^{r_2 t}\left[e^{-r_2 s}\left(\alpha - \beta r_2 - \sum_{i=1}^j k_i e^{r_2 \xi_i}\right) - e^{-r_1 s}\left(\alpha - \beta r_1 - \sum_{i=1}^j k_i e^{r_1 \xi_i}\right)\right], \\ \quad \xi_j \leqslant s \leqslant \xi_{j+1},\ s \leqslant t,\ j = 1,2,\ldots,n-1; \\ e^{-r_1 s}\left[e^{r_1 t}\left(\alpha - \beta r_2 - \sum_{i=1}^n k_i e^{r_2 \xi_i}\right)\right. \\ \quad \left. - e^{r_2 t}\left(\alpha - \beta r_1 - \sum_{i=1}^j k_i e^{r_1 \xi_i} - \sum_{i=j+1}^n k_i e^{r_2(\xi_i - s) + r_1 s}\right)\right], \\ \quad \xi_j \leqslant s \leqslant \xi_{j+1},\ t \leqslant s,\ j = 1,2,\ldots,n-1; \\ e^{r_2 t}\left[e^{-r_2 s}\left(\alpha - \beta r_2 - \sum_{i=1}^n k_i e^{r_2 \xi_i}\right) - e^{-r_1 s}\left(\alpha - \beta r_1 \sum_{i=1}^n k_i e^{r_1 \xi_i}\right)\right], \\ \quad \xi_n \leqslant s \leqslant t; \\ e^{-r_1 s}\left[e^{r_1 t}\left(\alpha - \beta r_2 - \sum_{i=1}^n k_i e^{r_2 \xi_i}\right) - e^{r_2 t}\left(\alpha - \beta r_1 - \sum_{i=1}^n k_i e^{r_1 \xi_i}\right)\right], \\ \quad t \leqslant s,\ \xi_n \leqslant s, \end{cases}$$

$$\Delta = (r_1 - r_2)\left(\alpha - \beta r_2 - \sum_{i=1}^n k_i e^{r_2 \xi_i}\right).$$

Proof. It is easy to know that the general solution for the equation in boundary value problem (3.2) is as follows

$$x(t) = \frac{1}{r_1 - r_2}\left[A e^{r_1 t} + B e^{r_2 t} + \int_0^t \left(e^{r_2(t-s)} - e^{r_1(t-s)}\right)\sigma(s)\,ds\right], \quad t \in I, \tag{3.4}$$

where A, B are constants. Now we claim that

$$\lim_{t\to\infty} e^{(r_2-r_1)t}\int_0^t e^{-r_2 s}\sigma(s)\,ds = 0. \tag{3.5}$$

In fact, if $\int_0^\infty e^{-r_2 s}\sigma(s)\,ds < \infty$, (3.5) holds clearly; if $\int_0^\infty e^{-r_2 s}\sigma(s)\,ds = \infty$, by $-\infty < \int_0^\infty e^{-r_1 s}\sigma(s)\,ds < \infty$, we have that $\lim_{s\to\infty} e^{-r_1 s}\sigma(s)\,ds = 0$. Then by the rule of l'Hospital, we obtain

$$\lim_{t\to\infty} e^{(r_2-r_1)t}\int_0^t e^{-r_2 s}\sigma(s)\,ds = \lim_{t\to\infty}\frac{\int_0^t e^{-r_2 s}\sigma(s)\,ds}{e^{(r_1-r_2)t}} = \lim_{t\to\infty}\frac{e^{-r_2 t}\sigma(t)}{(r_1-r_2)e^{(r_1-r_2)t}} = 0. \tag{3.6}$$

Thus by (3.4), (3.5) and boundary condition we get

$$A = \int_0^\infty e^{-r_1 s}\sigma(s)\,ds,$$

$$B = \left(\alpha - \beta r_2 - \sum_{i=1}^n k_i e^{r_2\xi_i}\right)^{-1}\left\{\sum_{i=1}^n k_i\int_0^{\xi_i}\left(e^{r_2(\xi_i-s)} - e^{r_1(\xi_i-s)}\right)\sigma(s)\,ds - \left(\alpha - \beta r_1 - \sum_{i=1}^n k_i e^{r_1\xi_i}\right)\int_0^\infty e^{-r_1 s}\sigma(s)\,ds\right\}.$$

Substituting the expressions of A, B into (3.4), after tedious compute, (3.3) holds.

Conversely, if $x \in C^1(I)$ is a solution of (3.3), it is easy to obtain x satisfies boundary value problem (3.2). □

It is easy to prove that the following lemma holds, so we omit the proof here.

Lemma 3.3. *Assume that* (H2) *holds and $G(t,s)$ is given in Lemma* 3.2*, then we have*

(a) $G(t,s) \geqslant 0$, *for* $(t,s) \in I \times I$;
(b) $e^{-r_1 t}G(t,s) \leqslant e^{-r_1 s}G(s,s)$, $G(s,s) \leqslant \frac{\alpha-\beta r_2}{\Delta}$, *for* $(t,s) \in I \times I$, *where*

$$\Delta = (r_1 - r_2)\left(\alpha - \beta r_2 - \sum_{i=1}^n k_i e^{r_2\xi_i}\right) > 0;$$

(c) $G(t,s) \geqslant \gamma e^{-r_1 s}G(s,s)$, *for* $(t,s) \in [l_1,l_2] \times I$, *where* $\gamma = \min\{e^{r_2 l_2}, e^{r_1 l_1} - e^{r_2 l_1}\}$.

We denote

$$C = \left\{y \in X\colon\ y(t) \geqslant 0 \text{ on } [0,\infty) \text{ and } y(t) \geqslant \gamma\|y\|,\ \forall t \in [l_1,l_2]\right\}$$

and

$$B(0,R) = \left\{y \in X\colon\ \|y\|_\infty \leqslant R\right\}.$$

Let $F : C \cap B(0, R) \to X$ be defined by

$$F(x)(t) = \int_0^\infty G(t, s)g\bigl(s, x(s)\bigr)\, ds \quad \text{for } t \in [0, \infty), \tag{3.7}$$

where g is defined in Lemma 3.1.

Remark 3.1. It follows from Lemmas 3.1, 3.2 that $x \in C(I)$ is a fixed point of operator F if, and only if $x + \bar{x} \in C^2(I)$ is a solution of BVP (1.1) under the assumption that $0 \leqslant \int_0^\infty e^{-r_1 s} g(s, x(s))\, ds < \infty$.

Lemma 3.4. *Suppose that* (H1), (H2) *hold, then* $F : C \cap B(0, R) \to C$ *is continuous and compact.*

Proof. (1) We need to show that for all $x \in C \cap B(0, R)$, $F(x) \in C$.

By Lemma 3.3 it is clear that $F(x)(t) \geqslant 0$ for $t \in [0, \infty)$. Now by Lemma 3.3 implies that for $t \in [l_1, l_2]$,

$$\begin{aligned} F(x)(t) &= \int_0^\infty G(t, s)g\bigl(s, x(s)\bigr)\, ds \geqslant \min_{t \in [l_1, l_2]} \int_0^\infty G(t, s)g\bigl(s, x(s)\bigr)\, ds \\ &\geqslant \gamma \int_0^\infty e^{-r_1 s} G(s, s) g\bigl(s, x(s)\bigr)\, ds \\ &\geqslant \gamma \int_0^\infty e^{-r_1 y} G(y, s) g\bigl(s, x(s)\bigr)\, ds \quad \text{for any } y \in [0, \infty). \end{aligned}$$

As a result, we have $F(x)(t) \geqslant \gamma \|F(x)\|_\infty$ for any $t \in [l_1, l_2]$. Then $F(x) \in C$.

(2) We will show that $F : C \cap B(0, R) \to C$ is continuous.

In fact suppose $\{x_m\} \subseteq C \cap B(0, R)$, $x_0 \in C \cap B(0, R)$ and $\lim_{m \to \infty} x_m = x_0$. By (H1) and Lemma 3.3,

$$\begin{aligned} &\int_0^\infty \frac{G(t, s)}{e^{r_1 t}} g\bigl(s, x_m(s)\bigr)\, ds \\ &\quad = \int_0^\infty \frac{G(t, s)}{e^{r_1 t}} \bigl[b\bigl(r^2 - pr - q\bigr)e^{rt} + f\bigl(s, x_m(s) + \bar{x}\bigr)\bigr]\, ds \\ &\quad \leqslant \int_0^\infty \frac{G(s, s)}{e^{r_1 s}} \bigl[c(s) + d(s)\bar{x} + d(s)x_m(s)\bigr]\, ds \\ &\quad = \frac{\alpha - \beta r_2}{\Delta} \int_0^\infty e^{-r_1 s} \biggl[c(s) + bd(s)e^{rs} + d(s)e^{r_2 s} \frac{a - b(\alpha - \beta r - \sum_{i=1}^n k_i e^{r \xi_i})}{\alpha - \beta r_2 - \sum_{i=1}^n k_i e^{r_2 \xi_i}} \\ &\qquad + d(s)x_m(s)\biggr]\, ds \\ &\quad \leqslant \frac{\alpha - \beta r_2}{\Delta} (c^* + d^* \Gamma + d^* R), \end{aligned} \tag{3.8}$$

where

$$\Gamma = b + \frac{a - b(\alpha - \beta r - \sum_{i=1}^{n} k_i e^{r\xi_i})}{\alpha - \beta r_2 - \sum_{i=1}^{n} k_i e^{r_2 \xi_i}} \quad \text{and} \quad \lim_{m\to\infty} f\big(s, x_m(s)\big) = f\big(s, x_0(s)\big).$$

Therefore by Lebesgue's dominated convergence theorem one has

$$\begin{aligned}\lim_{m\to\infty} \big\|F(x_m) - F(x_0)\big\|_\infty &= \lim_{m\to\infty} \sup_{t\in[0,\infty)} e^{-r_1 t}\big|F(x_m)(t) - F(x_0)(t)\big| \\ &\leqslant \lim_{m\to\infty} \sup_{t\in[0,\infty)} \int_0^\infty e^{-r_1 t} G(t,s)\big|g\big(s, x_m(s)\big) - g\big(s, x_0(s)\big)\big|\, ds = 0.\end{aligned}$$

Thus $F : C \cap B(0, R) \to C$ is continuous.

(3) We will show that $F : C \cap B(0, R) \to C$ is compact.

First we will show that $F(C \cap B(0, R))$ is bounded. For all $x \in C \cap B(0, R)$, by (3.8)

$$\big\|F(x)\big\|_\infty = \sup_{t\in[0,\infty)} e^{-r_1 t}\left|\int_0^\infty G(t,s) g\big(s, x(s)\big)\, ds\right| \leqslant \frac{\alpha - \beta r_2}{\Delta}\big(c^* + d^*\Gamma + d^* R\big)$$

independent of b_n. So $F(C \cap B(0, R))$ is bounded.

Second it is easy to show that $\{e^{-r_1 t} F(C \cap B(0, R))\}$ is locally equicontinuous by (H1).

It remains to show that $\{e^{-r_1 t} F(C \cap B(0, R))\}$ is equiconvergent at ∞.

For $x \in C \cap B(0, R)$, it follows from the expression of Green's function $G(t, s)$ in Lemma 3.2 that

$$\lim_{t\to\infty} \frac{G(t,s) g(s, x(s))}{e^{r_1 t}} = 0 \tag{3.9}$$

for $s \in [0, \infty)$. So

$$\Big|e^{-r_1 t} F(x)(t) - \lim_{s\to\infty} e^{-r_1 s} F(x)(s)\Big| = \big|e^{-r_1 t} F(x)(t)\big| = \left|\int_0^\infty e^{-r_1 t} G(t,s) g\big(s, x(s)\big)\, ds\right|. \tag{3.10}$$

By (H1), Lemma 3.3 and (3.8) we have

$$\begin{aligned}\int_0^\infty \frac{G(t,s)}{e^{r_1 t}} g\big(s, x(s)\big)\, ds &\leqslant \int_0^\infty \frac{G(s,s)}{e^{r_1 s}}\big[c(s) + d(s)\bar{x} + d(s)x(s)\big]\, ds \\ &\leqslant \frac{\alpha - \beta r_2}{\Delta}\big(c^* + d^*\Gamma + d^* R\big).\end{aligned} \tag{3.11}$$

By (3.9)–(3.11) and Lebesgue's dominated convergence theorem we have

$$\begin{aligned}&\lim_{t\to\infty}\Big|e^{-r_1 t} F(x)(t) - \lim_{s\to\infty} e^{-r_1 s} F(x)(s)\Big| \\ &\quad = \lim_{t\to\infty}\left|\int_0^\infty e^{-r_1 t} G(t,s) g\big(s, x(s)\big)\, ds\right| = \int_0^\infty \lim_{t\to\infty} e^{-r_1 t} G(t,s) g\big(s, x(s)\big)\, ds = 0,\end{aligned}$$

which means that $\{e^{-r_1 t} F(C \cap B(0, R))\}$ is equiconvergent at ∞. By Lemma 2.3, $F(C \cap B(0, R))$ is relatively compact. $\square$

4. Existence results

Theorem 4.1. *Suppose that* (H1), (H2) *hold, and assume*

(H3) $\frac{(\alpha-\beta r_2)d^*}{\Delta} < 1$, *where* Δ *is given in Lemma* 3.2.

Then boundary value problem (1.1) *has a positive solution.*

Proof. Banach space X with the norm $\|\cdot\|_\infty$ has been defined in Section 2. The operator F is defined as in (3.7). We apply Lemma 2.4 to prove the existence of fixed point of F. Lemma 3.4 shows that F is completely continuous. By (H1), Lemma 3.3 and (3.8), for all $t \in I$, $\|x\|_\infty = R$ (R is a constant to be determined later),

$$\begin{aligned} \left|F(x)(t)\right|e^{-r_1 t} &= e^{-r_1 t}\lambda\left|\int_0^\infty G(t,s)g\big(s,x(s)\big)\,ds\right| \leqslant \left|\int_0^\infty e^{-r_1 s}G(s,s)g\big(s,x(s)\big)\,ds\right| \\ &\leqslant \frac{\alpha-\beta r_2}{\Delta}\left(c^* + d^*\Gamma + d^*R\right). \end{aligned}$$

So by (H3) fix $R > \frac{(\alpha-\beta r_2)(c^*+d^*\Gamma)}{\Delta - d^*(\alpha-\beta r_2)}$ and define

$$\Omega = \left\{x \in X\colon\ \|x\|_\infty \leqslant R\right\}.$$

Therefore, $\|F(x)\|_\infty \leqslant \|x\|_\infty$, i.e., $\lambda F(x) \neq x$ for $\lambda \in (0,1]$, $x \in C \cap \partial\Omega$. Applying Lemma 2.4, F has a fixed point x in $C \cap \Omega$. Furthermore, we have from (H1) and (3.8) that

$$\int_0^\infty e^{-r_1 s}g\big(s,x(s)\big)\,ds \leqslant c^* + d^*\Gamma + d^*R < \infty.$$

Therefore, Remark 3.1 implies that $x + \bar{x}$ is a solution of BVP (1.1). Condition (H1) implies that $\bar{x}(t) \geqslant 0$ for $t \in I$. So $x(t) + \bar{x}(t) \geqslant 0$, $t \in I$. $\square$

Theorem 4.2. *Suppose that* (H1), (H2) *hold, and assume there exists* $0 < r < R$ *such that*

(H4)
$$\int_0^\infty e^{-r_1 s}G(s,s)\left[c(s) + bd(s)e^{rs} + d(s)e^{r_2 s}\frac{a - b(\alpha - \beta r - \sum_{i=1}^n k_i e^{r\xi_i})}{\alpha - \beta r_2 - \sum_{i=1}^n k_i e^{r_2\xi_i}} + d(s)e^{r_1 s}R\right]ds < R;$$

(H5)
$$g(s,x) > \frac{r}{\gamma e^{-2r_1 l_2}\int_{l_1}^{l_2} G(s,s)\,ds},$$

for $(s,x) \in [l_1,l_2] \times [\gamma r, r]$, *where* γ, $G(s,s)$ *is given in Lemma* 3.3.

Then boundary value problem (1.1) *has a positive solution* y *with*

$$\sup_{t\in I}\left|e^{-r_1 t}\big(y(s) - \bar{x}(s)\big)\right| \leqslant R, \quad y(t) - \bar{x}(t) \geqslant \gamma r, \quad \text{for } t \in [l_1,l_2]. \tag{4.1}$$

Proof. Banach space X with the norm $\|\cdot\|_\infty$ have been defined in Section 2. The operator F is defined as in (3.7). We apply Lemma 2.5 to prove the existence of fixed point of F.

Define

$$\Omega_1 = \{x \in X:\ \|x\|_\infty \leqslant r\} \quad \text{and} \quad \Omega_2 = \{x \in X:\ \|x\|_\infty \leqslant R\}.$$

By Lemma 3.4 we know that F is completely continuous on $C \cap \overline{\Omega}_2$.

If $x \in C \cap \partial\Omega_2$, we have by Lemma 3.3, (H1), (H4) and (3.8)

$$\begin{aligned}
\left|F(x)(t)e^{-r_1 t}\right| &= \int_0^\infty e^{-r_1 t} G(t,s) g\big(s, x(s)\big)\, ds \\
&\leqslant \int_0^\infty e^{-r_1 s} G(s,s)\bigg[c(s) + bd(s)e^{rs} + d(s)e^{r_2 s}\frac{a - b(\alpha - \beta r - \sum_{i=1}^n k_i e^{r\xi_i})}{\alpha - \beta r_2 - \sum_{i=1}^n k_i e^{r_2 \xi_i}} \\
&\qquad + d(s)e^{r_1 s} R\bigg]\, ds \\
&\leqslant R,
\end{aligned}$$

which means that $\|F(x)\|_\infty \leqslant \|x\|_\infty$ for $x \in C \cap \partial\Omega_2$.

If $x \in C \cap \partial\Omega_1$, by Lemma 3.3 and (H5) we have

$$\begin{aligned}
e^{-r_1 t} F(x)(t) &\geqslant e^{-r_1 t}\int_0^\infty G(t,s) g\big(s, x(s)\big)\, ds \geqslant \min_{t\in[0,b_1]} e^{-r_1 t}\int_{l_1}^{l_2} G(t,s) g\big(s, x(s)\big)\, ds \\
&\geqslant \min_{t\in[0,b_1]} \gamma \int_{l_1}^{l_2} e^{-r_1(t+s)} G(s,s) g\big(s, x(s)\big)\, ds \\
&\geqslant \gamma e^{-2r_1 l_2}\int_{l_1}^{l_2} G(s,s)\, ds \min\big\{g(s,x):\ s \in [l_1, l_2],\ x \in [\gamma r, r]\big\} \\
&\geqslant r, \quad l_1 < l_2 = b_1,
\end{aligned}$$

i.e., $\|F(x)\|_\infty \geqslant \|x\|_\infty$ for $x \in C \cap \partial\Omega_1$. Now, we apply Lemma 2.5 to deduce that F has a fixed point $x \in C \cap (\overline{\Omega}_2 \setminus \Omega_1)$. Similar to Theorem 4.1 we have $0 \leqslant \int_0^\infty e^{-r_1 s} g(s, x(s))\, ds < \infty$. So Remark 3.1 implies that $x + \bar{x}$ is a solution of BVP (1.1). □

Corollary 4.3. *Suppose that* (H1)*,* (H2)*,* (H4) *hold and*

(H6) $$\lim_{r\to 0} \frac{r}{\inf_{(s,x)\in[l_1,l_2]\times[\gamma r, r]} g(s,x)} = 0.$$

Then boundary value problem (1.1) *has a positive solution.*

Proof. Suppose $\varepsilon^* = \gamma e^{-2r_1 l_2}\int_{l_1}^{l_2} G(s,s)\, ds$.

By (H6) there exists $0 < r < R$ such that

$$\frac{u}{\inf_{(s,x)\in[l_1,l_2]\times[\gamma u, u]} g(s,x)} < \varepsilon^*, \quad \forall u \leqslant r.$$

So (H5) holds. By Theorem 4.2, BVP (1.1) has a positive solution. □

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