

Nontrivial solutions of singular sublinear Sturm–Liouville problems [☆]

Jingxian Sun ^a, Guowei Zhang ^{b,*}

^a *Department of Mathematics, Xuzhou Normal University, Xuzhou 221116, Jiangsu, People's Republic of China*

^b *Department of Mathematics, Northeastern University, Shenyang 110004, Liaoning, People's Republic of China*

Received 29 August 2005

Available online 4 April 2006

Submitted by P.G.L. Leach

Abstract

The singular sublinear Sturm–Liouville problems

$$\begin{cases} -(L\varphi)(x) = h(x)f(\varphi(x)), & 0 < x < 1, \\ R_1(\varphi) = \alpha_1\varphi(0) + \beta_1\varphi'(0) = 0, & R_2(\varphi) = \alpha_2\varphi(1) + \beta_2\varphi'(1) = 0, \end{cases}$$

are considered under some conditions concerning the first eigenvalues corresponding to the relevant linear operators, where $(L\varphi)(x) = (p(x)\varphi'(x))' + q(x)\varphi(x)$ and $h(x)$ is allowed to be singular at both $x = 0$ and $x = 1$. In particular, f is not necessary to be nonnegative. The existence results of nontrivial solutions and positive solutions are given by means of the topological degree theory.

© 2006 Elsevier Inc. All rights reserved.

Keywords: Second-order singular equation; Two-point boundary value problem; Nontrivial solution; Positive solution

1. Introduction and preliminaries

Many authors are interested in the existence of positive solutions for second-order two-point boundary value problem (see [1–9] and references therein). In most work mentioned, they study the existence of positive solutions for second-order two-point boundary value problem by the method of upper and lower solutions, Schauder's fixed point theorem or the fixed point index in

[☆] The project supported by the National Natural Science Foundation of PR China (10371066, 10371013).

^{*} Corresponding author.

E-mail addresses: jxsun7083@sohu.com (J. Sun), gwzhangneum@sina.com (G. Zhang).

cone under some different conditions in which f is nonnegative. In this paper, we consider the following singular sublinear Sturm–Liouville problems:

$$\begin{cases} -(L\varphi)(x) = h(x)f(\varphi(x)), & 0 < x < 1, \\ R_1(\varphi) = \alpha_1\varphi(0) + \beta_1\varphi'(0) = 0, & R_2(\varphi) = \alpha_2\varphi(1) + \beta_2\varphi'(1) = 0, \end{cases} \quad (1.1)$$

where $(L\varphi)(x) = (p(x)\varphi'(x))' + q(x)\varphi(x)$ and $h(x)$ is allowed to be singular at $x = 0$ and $x = 1$. In particular, f is not necessary to be nonnegative, to our knowledge, for which case under superlinear and sublinear assumptions there are not many references [10,11].

The present paper is motivated by [10,11] in which the nonsingular case has been considered under the conditions concerning the first eigenvalue of the associated linear eigenvalue problem that cannot be improved again. Since $h(x)$ is allowed to be singular at $x = 0$ and $x = 1$, the method from [11] fails because for the singular case, inequality (16) in the proof of Theorem 1 in [11] is no longer true and the constant C in inequality (22) may not be finite. In particular, the integrations in the proof of Theorem 1 in [11] may not be convergent if $f(x, u)$ therein is singular at $x = 0$ and $x = 1$. For the singular case, we in this paper consider the right-hand side $h(x)f(u)$ of separating variables type instead of $f(x, u)$ and make the assumptions on $f(u)$ with respect to the first eigenvalue.

We obtain the existence results of nontrivial solutions, and the existence results of positive solutions for some cases, by means of the topological degree theory under some conditions on $f(u)$ concerning the first eigenvalue corresponding to the relevant linear operator. For the concepts and properties about the cone theory and the topological degree we refer to [12,13].

In this paper we suppose that

$$(H_1) \quad p(x) \in C^1[0, 1], \quad p(x) > 0, \quad q(x) \in C[0, 1], \quad q(x) \leq 0, \\ \alpha_1 \geq 0, \quad \beta_1 \leq 0, \quad \alpha_2 \geq 0, \quad \beta_2 \geq 0, \quad \alpha_1^2 + \beta_1^2 \neq 0, \quad \alpha_2^2 + \beta_2^2 \neq 0, \quad (1.2)$$

and the homogenous equation with respect to (1.1),

$$\begin{cases} -(L\varphi)(x) = 0, & 0 < x < 1, \\ R_1(\varphi) = R_2(\varphi) = 0, \end{cases} \quad (1.3)$$

has only the trivial solution. Let $k(x, y)$ be the Green's function with respect to (1.3), i.e.

$$k(x, y) = \begin{cases} \frac{1}{w}u(x)v(y), & 0 \leq x \leq y \leq 1, \\ \frac{1}{w}u(y)v(x), & 0 \leq y \leq x \leq 1, \end{cases} \quad (1.4)$$

where $u(x) \in C^2[0, 1]$ is an increasing function, $u(x) > 0$, $x \in (0, 1]$; $v(x) \in C^2[0, 1]$ is a decreasing function, $v(x) > 0$, $x \in [0, 1]$; w is a positive constant. It is easy to see that $k(x, y)$ is nonnegative and continuous over $[0, 1] \times [0, 1]$, $k(x, y) \leq k(y, y)$, $\forall x, y \in [0, 1]$.

$$(H_2) \quad h : (0, 1) \rightarrow [0, +\infty) \text{ is continuous, } h(x) \not\equiv 0 \text{ and}$$

$$\int_0^1 k(x, x)h(x) dx < +\infty. \quad (1.5)$$

$$(H_3) \quad f : (-\infty, +\infty) \rightarrow (-\infty, +\infty) \text{ is continuous.}$$

Remark 1. It is not difficult to verify (1.5) in (H_2) by Lemma 11 in [14] even if one does not know the explicit representation of the Green's function. Moreover, in assumption (H_3) it is not supposed that $f(u) \geq 0$, $\forall u \geq 0$.

In Banach space $C[0, 1]$ in which the norm is defined by $\|\varphi\| = \max_{0 \leq x \leq 1} |\varphi(x)|$, we set

$$P = \{\varphi \in C[0, 1] \mid \varphi(x) \geq 0, x \in [0, 1]\}, \quad (1.6)$$

then P is a positive cone in $C[0, 1]$. Denote by $B_r = \{\varphi \in C[0, 1] \mid \|\varphi\| < r\}$ ($r > 0$) the open ball of radius r and use θ to denote the zero function in $C[0, 1]$.

As is well known, the singular nonlinear Sturm–Liouville problems (1.1) can be converted into the equivalent Hammerstein nonlinear integral equation

$$\varphi(x) = \int_0^1 k(x, y)h(y)f(\varphi(y))dy, \quad x \in [0, 1]. \quad (1.7)$$

Let

$$(A\varphi)(x) = \int_0^1 k(x, y)h(y)f(\varphi(y))dy, \quad x \in [0, 1], \quad (1.8)$$

$$(T\varphi)(x) = \int_0^1 k(x, y)h(y)\varphi(y)dy, \quad x \in [0, 1]. \quad (1.9)$$

By the method similar to that in [2], we have

Lemma 1. Suppose that (H_1) – (H_3) are satisfied, then $A: C[0, 1] \rightarrow C[0, 1]$ is a completely continuous operator and $T: C[0, 1] \rightarrow C[0, 1]$ is a completely continuous linear operator; $T(P) \subset P$.

It is obvious that if the operator A has a fixed point φ , then φ is the solution of (1.1).

Lemma 2. Suppose that (H_1) , (H_2) are satisfied, then for the operator T , the spectral radius $r(T) \neq 0$ and T has positive eigenfunction corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$.

For the proof of Lemma 2, one may see [14]. We also need the following lemmas in [13].

Lemma 3. Let E be a Banach space, and P be a cone in E , and $\Omega(P)$ be a bounded open set in P . Suppose that $A: \overline{\Omega(P)} \rightarrow P$ is a completely continuous operator. If there exists $u_0 \in P \setminus \{\theta\}$ such that

$$u - Au \neq \mu u_0, \quad \forall u \in \partial\Omega(P), \mu \geq 0,$$

then the fixed point index $i(A, \Omega(P), P) = 0$.

Lemma 4. Let E be a Banach space, P a cone in E , and $\Omega(P)$ a bounded open set in P with $\theta \in \Omega(P)$. Suppose that $A: \overline{\Omega(P)} \rightarrow P$ is a completely continuous operator. If

$$Au \neq \mu u, \quad \forall u \in \partial\Omega(P), \mu \geq 1,$$

then the fixed point index $i(A, \Omega(P), P) = 1$.

2. Existence of nontrivial solutions

Theorem 1. Suppose that conditions (H_1) – (H_3) are satisfied. If there exists a constant $b \geq 0$ such that

$$f(u) \geq -b, \quad \forall u \in (-\infty, +\infty); \quad (2.1)$$

$$\liminf_{u \rightarrow 0} \frac{f(u)}{|u|} > \lambda_1; \quad (2.2)$$

$$\limsup_{u \rightarrow +\infty} \frac{f(u)}{u} < \lambda_1, \quad (2.3)$$

where λ_1 is the first eigenvalue of T defined by (1.9). Then the singular nonlinear Sturm–Liouville problem (1.1) has at least one nontrivial solution.

Proof. It follows from (2.2) that there exists $r_1 > 0$ such that

$$f(u) \geq \lambda_1 |u|, \quad \forall |u| \leq r_1. \quad (2.4)$$

For every $\varphi \in \bar{B}_{r_1}$, we have from (2.4) that

$$(A\varphi)(x) \geq \lambda_1 \int_0^1 k(x, y) h(y) |\varphi(y)| dy \geq 0, \quad x \in [0, 1],$$

and thus $A(\bar{B}_{r_1}) \subset P$. For any $\varphi \in \partial B_{r_1} \cap P$, it follows from (2.4) that

$$(A\varphi)(x) \geq \lambda_1 \int_0^1 k(x, y) h(y) \varphi(y) dy = \lambda_1 (T\varphi)(x), \quad x \in [0, 1]. \quad (2.5)$$

We may suppose that A has no fixed point on ∂B_{r_1} (otherwise, the proof completes). Let φ^* be the positive eigenfunction of T corresponding to λ_1 , thus $\varphi^* = \lambda_1 T\varphi^*$. Now we show that

$$\varphi - A\varphi \neq \mu\varphi^*, \quad \forall \varphi \in \partial B_{r_1} \cap P, \quad \mu \geq 0. \quad (2.6)$$

If otherwise, there exist $\varphi_1 \in \partial B_{r_1} \cap P$ and $\tau_0 \geq 0$ such that $\varphi_1 - A\varphi_1 = \tau_0\varphi^*$. Hence $\tau_0 > 0$ and

$$\varphi_1 = A\varphi_1 + \tau_0\varphi^* \geq \tau_0\varphi^*.$$

Put

$$\tau^* = \sup\{\tau \mid \varphi_1 \geq \tau\varphi^*\}. \quad (2.7)$$

It is easy to see that $\tau^* \geq \tau_0 > 0$ and $\varphi_1 \geq \tau^*\varphi^*$. We have from $T(P) \subset P$ that

$$\lambda_1 T\varphi_1 \geq \tau^* \lambda_1 T\varphi^* = \tau^*\varphi^*.$$

Therefore by (2.5),

$$\varphi_1 = A\varphi_1 + \tau_0\varphi^* \geq \lambda_1 T\varphi_1 + \tau_0\varphi^* \geq \tau^*\varphi^* + \tau_0\varphi^*,$$

which contradicts the definition of τ^* . Hence (2.6) is true. Since $A(\bar{B}_{r_1}) \subset P$, we have from the permanence property of fixed point index and Lemma 3 that

$$\deg(I - A, B_{r_1}, \theta) = i(A, B_{r_1} \cap P, P) = 0, \quad (2.8)$$

where \deg denotes the topological degree.

Letting $\tilde{\varphi}(x) = b \int_0^1 k(x, y)h(y)dy$. Obviously, $\tilde{\varphi} \in P$. It easy to see from (2.1) that $A : C[0, 1] \rightarrow P - \tilde{\varphi}$. Define $\tilde{A}\varphi = A(\varphi - \tilde{\varphi}) + \tilde{\varphi}$, $\varphi \in C[0, 1]$, then $\tilde{A} : C[0, 1] \rightarrow P$.

It follows from (2.3) that there exist $r_2 > r_1 + \|\tilde{\varphi}\|$ and $0 < \sigma < 1$ such that

$$f(u) \leq \sigma \lambda_1 u, \quad \forall u \geq r_2. \quad (2.9)$$

Let $T_1\varphi = \sigma \lambda_1 T\varphi$, $\varphi \in C[0, 1]$. Then $T_1 : C[0, 1] \rightarrow C[0, 1]$ is a bounded linear operator and $T_1(P) \subset P$. Let

$$M = 2 \max \left\{ \sup_{\varphi \in \tilde{B}_{r_2}} \int_0^1 k(y, y)h(y)|f(\varphi(y))|dy, 2\|\tilde{\varphi}\| \right\}. \quad (2.10)$$

It is clear that $M < +\infty$. Let

$$W = \{\varphi \in P \mid \varphi = \mu \tilde{A}\varphi, 0 \leq \mu \leq 1\}. \quad (2.11)$$

In the following, we prove that W is bounded.

For any $\varphi \in W$, set $\tilde{\psi}(x) = \min\{\varphi(x) - \tilde{\varphi}(x), r_2\}$ and denote

$$e(\varphi) = \{x \in [0, 1] \mid \varphi(x) - \tilde{\varphi}(x) > r_2\}.$$

When $\varphi(x) - \tilde{\varphi}(x) < 0$, $\tilde{\psi}(x) = \varphi(x) - \tilde{\varphi}(x) \geq \varphi(x) - r_2 \geq -r_2$, and so $\|\tilde{\psi}\| \leq r_2$. Thus for $\varphi \in W$, we have from (2.9)

$$\begin{aligned} \varphi(x) &= \mu(\tilde{A}\varphi)(x) \leq \int_0^1 k(x, y)h(y)f(\varphi(y) - \tilde{\varphi}(y))dy + \tilde{\varphi}(x) \\ &= \int_{e(\varphi)} k(x, y)h(y)f(\varphi(y) - \tilde{\varphi}(y))dy \\ &\quad + \int_{[0, 1] \setminus e(\varphi)} k(x, y)h(y)f(\varphi(y) - \tilde{\varphi}(y))dy + \tilde{\varphi}(x) \\ &\leq \sigma \lambda_1 \int_0^1 k(x, y)h(y)\varphi(y)dy + \int_0^1 k(x, y)h(y)f(\tilde{\psi}(y))dy + 2\tilde{\varphi}(x) \\ &\leq \sigma \lambda_1 \int_0^1 k(x, y)h(y)\varphi(y)dy + M = (T_1\varphi)(x) + M, \end{aligned}$$

where M is defined as (2.10). Thus $((I - T_1)\varphi)(x) \leq M$, $x \in [0, 1]$.

Since λ_1 is the first eigenvalue of T and $0 < \sigma < 1$, the first eigenvalue of T_1 , $(r(T_1))^{-1} > 1$. Therefore, the inverse operator $(I - T_1)^{-1}$ exists and

$$(I - T_1)^{-1} = I + T_1 + T_1^2 + \cdots + T_1^n + \cdots. \quad (2.12)$$

It follows from $T_1(P) \subset P$ that $(I - T_1)^{-1}(P) \subset P$. So we have $\varphi(x) \leq (I - T_1)^{-1}M$, $x \in [0, 1]$ and W is bounded.

Select $r_3 > \max\{r_2, \sup W + \|\tilde{\varphi}\|\}$ and thus \tilde{A} has no fixed point on ∂B_{r_3} . In fact, if there exists $\varphi_1 \in \partial B_{r_3}$ such that $\tilde{A}\varphi_1 = \varphi_1$, then $\varphi_1 \in W$ and $\|\varphi_1\| = r_3 > \sup W$, which is a contradiction.

Then we have from the permanence property and the homotopy invariance property of fixed point index that

$$\deg(I - \tilde{A}, B_{r_3}, \theta) = i(\tilde{A}, B_{r_3} \cap P, P) = i(\theta, B_{r_3} \cap P, P) = 1. \quad (2.13)$$

Set the completely continuous homotopy $H(t, \varphi) = A(\varphi - t\tilde{\varphi}) + t\tilde{\varphi}$, $(t, \varphi) \in [0, 1] \times \bar{B}_{r_3}$. If there exists $(t_0, \varphi_2) \in [0, 1] \times \partial B_{r_3}$ such that $H(t_0, \varphi_2) = \varphi_2$, then $A(\varphi_2 - t_0\tilde{\varphi}) = \varphi_2 - t_0\tilde{\varphi}$ and $\tilde{A}(\varphi_2 - t_0\tilde{\varphi} + \tilde{\varphi}) = \varphi_2 - t_0\tilde{\varphi} + \tilde{\varphi}$. Thus $\varphi_2 - t_0\tilde{\varphi} + \tilde{\varphi} \in W$ and

$$\|\varphi_2 - t_0\tilde{\varphi} + \tilde{\varphi}\| \geq \|\varphi_2\| - (1 - t_0)\|\tilde{\varphi}\| \geq r_3 - \|\tilde{\varphi}\| > \sup W,$$

a contradiction! From the homotopy invariance of topological degree and (2.13) we have

$$\begin{aligned} \deg(I - A, B_{r_3}, \theta) &= \deg(I - H(0, \cdot), B_{r_3}, \theta) \\ &= \deg(I - H(1, \cdot), B_{r_3}, \theta) = \deg(I - \tilde{A}, B_{r_3}, \theta) = 1. \end{aligned} \quad (2.14)$$

By (2.8) and (2.14) we have that

$$\deg(I - A, B_{r_3} \setminus \bar{B}_{r_1}, \theta) = \deg(I - A, B_{r_3}, \theta) - \deg(I - A, B_{r_1}, \theta) = 1,$$

which implies that A has at least one fixed point on $B_{r_3} \setminus \bar{B}_{r_1}$. This means that the singular nonlinear Sturm–Liouville problem (1.1) has at least one nontrivial solution. \square

Corollary 1. Suppose that conditions (H_1) – (H_3) are satisfied. If there exists a constant $b^* \geq 0$ such that

$$f(u) \geq -\frac{b^*}{M^*}, \quad \forall u \geq -b^*, \quad (2.15)$$

where $M^* = \max_{x \in [0, 1]} \int_0^1 k(x, y)h(y) dy$, in addition, (2.2) and (2.3) hold, then the singular nonlinear Sturm–Liouville problem (1.1) has at least one nontrivial solution.

Proof. Denote

$$f_1(u) = \begin{cases} f(u), & u \geq -b^*, \\ f(-b^*), & u < -b^*. \end{cases} \quad (2.16)$$

Define

$$(A_1\varphi)(x) = \int_0^1 k(x, y)h(y)f_1(\varphi(y)) dy, \quad x \in [0, 1]. \quad (2.17)$$

It follows from Theorem 1 that A_1 has at least one nonzero fixed point $\tilde{\varphi}$. Then

$$\tilde{\varphi}(x) = \int_0^1 k(x, y)h(y)f_1(\tilde{\varphi}(y)) dy \geq -\frac{b^*}{M^*} \int_0^1 k(x, y)h(y) dy \geq -b^*. \quad (2.18)$$

From (2.16) we have that $f_1(\tilde{\varphi}(x)) = f(\tilde{\varphi}(x))$, $x \in [0, 1]$, then

$$\tilde{\varphi}(x) = \int_0^1 k(x, y)h(y)f_1(\tilde{\varphi}(y)) dy = \int_0^1 k(x, y)h(y)f(\tilde{\varphi}(y)) dy.$$

Thus $\tilde{\varphi}$ is the nontrivial solution of the singular nonlinear Sturm–Liouville problem (1.1). \square

3. Existence of positive solutions

In this section we are concerned with the existence of positive solutions. Moreover, it is different from last section that f has no lower bound.

Theorem 2. Suppose that conditions (H_1) – (H_3) are satisfied. If

$$uf(u) \geq 0, \quad \forall u \in (-\infty, +\infty); \quad (3.1)$$

$$\liminf_{u \rightarrow 0} \frac{f(u)}{u} > \lambda_1; \quad (3.2)$$

$$\limsup_{|u| \rightarrow +\infty} \frac{f(u)}{u} < \lambda_1, \quad (3.3)$$

where λ_1 is the first eigenvalue of T defined by (1.9), then the singular nonlinear Sturm–Liouville problem (1.1) has at least one positive solution and one negative solution.

Proof. From (3.1) we have that $A(P) \subset P$. Similar to the proof of Theorem 1 in which $b = 0$, we have by Lemmas 3 and 4 that there exist $0 < r_1 < r_2$ such that

$$i(A, B_{r_1} \cap P, \theta) = 1, \quad i(A, B_{r_2} \cap P, \theta) = 0. \quad (3.4)$$

Then

$$i(A, (B_{r_2} \cap P) \setminus (\bar{B}_{r_1} \cap P), \theta) = i(A, B_{r_2} \cap P, \theta) - i(A, B_{r_1} \cap P, \theta) = -1.$$

So A has a fixed point in $(B_{r_2} \cap P) \setminus (\bar{B}_{r_1} \cap P)$ and (1.1) has at least one positive solution.

Denote $f_2(u) = -f(-u)$, $\forall u \in (-\infty, +\infty)$ and define

$$(A_2\varphi)(x) = \int_0^1 k(x, y)h(y)f_2(\varphi(y))dy, \quad x \in [0, 1]. \quad (3.5)$$

Then $A_2(P) \subset P$ and A_2 has a fixed point $\tilde{\psi} \in P \setminus \{\theta\}$, i.e. $A_2\tilde{\psi} = \tilde{\psi}$.

Since $f_2(\tilde{\psi}(x)) = -f(-\tilde{\psi}(x))$, $\forall x \in [0, 1]$, we have

$$-\tilde{\psi}(x) = \int_0^1 k(x, y)h(y)f(-\tilde{\psi}(y))dy = (A(-\tilde{\psi}))(x), \quad x \in [0, 1]. \quad (3.6)$$

So $-\tilde{\psi}$ is the negative solution of (1.1). \square

In the following we consider the singular nonlinear Sturm–Liouville problems:

$$\begin{cases} -(L\varphi)(x) = h(x)f(\varphi(x)), & 0 < x < 1, \\ \varphi(0) = \varphi(1) = 0. \end{cases} \quad (3.7)$$

Theorem 3. Suppose that conditions (H_1) – (H_3) are satisfied. If

$$f(0) = 0, \quad q(x) < 0, \quad \forall x \in [0, 1], \quad (3.8)$$

where $q(x)$ is as in (H_1) , in addition, (2.3) and

$$\liminf_{u \rightarrow 0^+} \frac{f(u)}{u} > \lambda_1 \quad (3.9)$$

hold, then (3.7) has at least one positive solution.

Proof. Denote

$$f_3(u) = \begin{cases} f(u), & u \geq 0, \\ 0, & u < 0. \end{cases} \quad (3.10)$$

Define

$$(A_3\varphi)(x) = \int_0^1 k(x, y)h(y)f_3(\varphi(y))dy, \quad x \in [0, 1]. \quad (3.11)$$

It is easy to see from (3.9) that f_3 is bounded below. Thus it follows from Theorem 1 that A_3 has a fixed point $\varphi_0 \neq \theta$. We only need to show $\varphi_0(x) \geq 0$, $x \in [0, 1]$.

If otherwise, $\varphi_0(x)$ achieves the minimum at $x_0 \in (0, 1)$ and

$$\varphi_0(x_0) < 0, \quad \varphi'_0(x_0) = 0, \quad \varphi''_0(x_0) \geq 0.$$

Hence

$$(L\varphi_0)(x_0) = p(x_0)\varphi''_0(x_0) + p'(x_0)\varphi'_0(x_0) + q(x_0)\varphi_0(x_0) > 0. \quad (3.12)$$

However, we also have

$$-(L\varphi_0)(x_0) = f_3(\varphi_0(x_0)) = 0,$$

which contradicts (3.12). \square

Remark 2. In Corollary 1, Theorems 2 and 3, f is not required to be bounded below, and in particular, the existence of positive solutions is obtained in Theorem 3 though A may be not a cone mapping.

4. Nonsingular case

In this section we consider the nonlinear Sturm–Liouville problem

$$\begin{cases} -(L\varphi)(x) = f(x, \varphi(x)), & 0 < x < 1, \\ R_1(\varphi) = \alpha_1\varphi(0) + \beta_1\varphi'(0) = 0, & R_2(\varphi) = \alpha_2\varphi(1) + \beta_2\varphi'(1) = 0. \end{cases} \quad (4.1)$$

Theorem 4. Suppose that condition (H_1) is satisfied, and $f(x, u)$ is continuous on $[0, 1] \times (-\infty, +\infty)$. If there exists a constant $b \geq 0$ such that

$$f(x, u) \geq -b, \quad \forall x \in [0, 1], u \in (-\infty, +\infty); \quad (4.2)$$

$$\liminf_{u \rightarrow 0} \frac{f(x, u)}{|u|} > \lambda_1, \quad \text{uniformly on } x \in [0, 1]; \quad (4.3)$$

$$\limsup_{u \rightarrow +\infty} \frac{f(x, u)}{u} < \lambda_1, \quad \text{uniformly on } x \in [0, 1], \quad (4.4)$$

where λ_1 is the first eigenvalue of T defined by (1.9) in which setting $h(y) \equiv 1$, then the nonlinear Sturm–Liouville problem (4.1) has at least one nontrivial solution.

The proof of Theorem 4 is similar to that in Theorem 1.

Theorem 5. Suppose that condition (H_1) is satisfied, and $f(x, u)$ is continuous on $[0, 1] \times (-\infty, +\infty)$. If there exists $M_1 \geq 0$ such that

$$uf(x, u) \geq -M_1 u^2, \quad 0 \leq x \leq 1, \quad -\infty < u < +\infty; \quad (4.5)$$

$$\liminf_{u \rightarrow 0} \frac{f(x, u)}{u} > \lambda_1, \quad \text{uniformly on } x \in [0, 1]; \quad (4.6)$$

$$\limsup_{|u| \rightarrow +\infty} \frac{f(x, u)}{u} < \lambda_1, \quad \text{uniformly on } x \in [0, 1], \quad (4.7)$$

where λ_1 is the first eigenvalue of T defined by (1.9) in which setting $h(y) \equiv 1$, then the nonlinear Sturm–Liouville problem (4.1) has at least one positive solution and one negative solution.

Proof. Let $f_4(x, u) = f(x, u) + M_1 u$, $0 \leq x \leq 1$, $-\infty < u < +\infty$ and denote $L_1 \varphi = L\varphi - M_1 \varphi$, then (4.1) is equivalent to

$$\begin{cases} -(L_1 \varphi)(x) = f_4(x, \varphi(x)), & 0 < x < 1, \\ R_1(\varphi) = R_2(\varphi) = 0. \end{cases} \quad (4.8)$$

$-M_1$ is not an eigenvalue of T since the eigenvalues of T are positive. The homogenous equation with respect to (4.8)

$$\begin{cases} -(L_1 \varphi)(x) = 0, & 0 < x < 1, \\ R_1(\varphi) = R_2(\varphi) = 0, \end{cases} \quad (4.9)$$

has only the trivial solution. Then the Green's function with respect to (4.9), $k_1(x, y)$ exists, and possesses the same properties as $k(x, y)$ above. It is well known that (4.8) is equivalent to the nonlinear integral equation of Hammerstein type

$$\varphi(x) = \int_0^1 k_1(x, y) f_4(x, \varphi(y)) dy = (A_4 \varphi)(x). \quad (4.10)$$

Obviously, $A_4 : C[0, 1] \rightarrow C[0, 1]$ is a completely continuous operator. By (4.5) we have that

$$uf_4(x, u) \geq 0, \quad 0 \leq x \leq 1, \quad -\infty < u < +\infty. \quad (4.11)$$

Define

$$(T_1 \varphi)(x) = \int_0^1 k_1(x, y) \varphi(y) dy. \quad (4.12)$$

Clearly, $T_1 : C[0, 1] \rightarrow C[0, 1]$ is completely continuous and $T_1(P) \subset P$. Let $\tilde{\lambda}_1 = \lambda_1 + M_1$. Since λ_1 is the first eigenvalue of T , we have that $\tilde{\lambda}_1$ is the first eigenvalue of T_1 . By (4.6) and (4.7),

$$\limsup_{u \rightarrow +\infty} \frac{f_4(x, u)}{u} > \tilde{\lambda}_1, \quad \text{uniformly on } x \in [0, 1]; \quad (4.13)$$

$$\liminf_{u \rightarrow 0} \frac{f_4(x, u)}{u} < \tilde{\lambda}_1, \quad \text{uniformly on } x \in [0, 1]. \quad (4.14)$$

Similar to the proof of Theorem 2, we have that (4.8) (i.e. (4.1)) has at least one positive solution and one negative solution. \square

There are the analogies of Corollary 1 and Theorem 3 for the nonsingular case.

5. Examples

Example 1. Let $h(x) = x^{p-1}(1-x)^{q-1}$, where $p, q > 0$. It is clear that $h(x)$ is singular at both $x = 0$ and $x = 1$ for $0 < p, q < 1$, and satisfies (H₂) by the convergence of Euler's integral.

Let $f(u) = \frac{1-u^2}{1+u^2}$. It is easy to see that $f(u)$ is bounded below and sign-changing for $u \geq 0$. In addition, $\lim_{u \rightarrow 0} \frac{f(u)}{|u|} = +\infty$ and $\lim_{u \rightarrow +\infty} \frac{f(u)}{u} = 0 < \lambda_1$. Thus by Theorem 1 one can obtain the existence of nontrivial solution of (1.2).

Example 2. Let $h(x)$ be as in Example 1 and let $f(u) = \frac{1-u^2}{3M^*(1+u^2)}e^{1-u}$, where M^* is as in Corollary 1. Obviously, $f(u)$ is unbounded below. It is not difficult to show that for $b^* = 1$ (b^* is as in Corollary 1), $f(u) \geq -\frac{b^*}{M^*}$, $\forall u \geq -b^*$. Then Corollary 1 can be applied since $\lim_{u \rightarrow 0} \frac{f(u)}{|u|} = +\infty$ and $\lim_{u \rightarrow +\infty} \frac{f(u)}{u} = 0 < \lambda_1$.

Example 3. Let $h(x)$ be as in Example 1 and (3.8) holds. Let $f(u) = \sqrt[3]{u} - u$. It is easy to see that $f(u)$ is unbounded below and sign-changing for $u \geq 0$. In addition, $\lim_{u \rightarrow 0^+} \frac{f(u)}{u} = +\infty$ and $\lim_{u \rightarrow +\infty} \frac{f(u)}{u} = -1 < \lambda_1$. Thus one can apply Theorem 3 to obtain the existence of positive solution of (3.7).

Acknowledgment

The authors express their gratitude to the referee for his valuable comments and suggestions.

References

- [1] L.H. Erbe, Eigenvalue criteria for existence of positive solutions to nonlinear boundary value problems, *Math. Comput. Modelling* 32 (2000) 529–539.
- [2] L.H. Erbe, R.M. Mathsen, Positive solutions for singular nonlinear boundary value problems, *Nonlinear Anal.* 46 (2001) 979–986.
- [3] P. Habets, F. Zanolin, Upper and lower solutions for a generalized Emden–Fowler equation, *J. Math. Anal. Appl.* 181 (1994) 684–700.
- [4] Z. Liu, F. Li, Multiple positive solutions of nonlinear two-point boundary value problems, *J. Math. Anal. Appl.* 203 (1996) 610–625.
- [5] R. Ma, Positive solutions of singular second order boundary value problem, *Acta Math. Sinica* 41 (1998) 1225–1230 (in Chinese).
- [6] D. O'Regan, *Theory of Singular Boundary Value Problems*, World Scientific, Singapore, 1994.
- [7] J. Wang, On positive solutions of singular nonlinear two-point boundary value problems, *J. Differential Equations* 107 (1994) 163–174.
- [8] Z. Wei, Positive solutions of singular boundary value problems of negative exponent Emden–Fowler equations, *Acta Math. Sinica* 41 (1998) 655–662 (in Chinese).
- [9] Y. Zhang, Positive solutions of singular sublinear Emden–Fowler boundary value problems, *J. Math. Anal. Appl.* 185 (1994) 215–222.
- [10] Y. Li, Positive solutions of second-order boundary value problems with sign-changing nonlinear terms, *J. Math. Anal. Appl.* 282 (2003) 232–240.
- [11] Y. Li, On the existence and nonexistence of positive solutions for nonlinear Sturm–Liouville boundary value problems, *J. Math. Anal. Appl.* 304 (2005) 74–86.
- [12] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.
- [13] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.
- [14] J. Sun, G. Zhang, Nontrivial solutions of singular superlinear Sturm–Liouville problem, *J. Math. Anal. Appl.* 313 (2006) 518–536.