

Global well-posedness for the viscous Camassa–Holm equation

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Abstract

We consider the initial value problem (IVP) of the Camassa–Holm equation with viscosity. We established global solution for the IVP with $u_0 \in L^2(\mathbb{R})$. This result improves the previous results.

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1. Introduction

In this note, we are interested in the global well-posedness of the initial value problem (IVP) associated to the viscous version of the one-dimensional shallow water equation

$$\partial_t u + \frac{1}{2} \partial_x(u^2) + \frac{1}{2} \partial_x(1 - \partial_x^2)^{-1}(u_x^2) + \partial_x(1 - \partial_x^2)^{-1}(u^2) = 0, \quad (1.1)$$

where $u = u(x, t)$, $(x, t) \in \mathbb{R} \times \mathbb{R}$.

Equation (1.1) was derived by Camassa and Holm [1] as a nonlinear model for water wave motion in shallow channels by using an asymptotic expansion directly in the Hamiltonian for Euler's equations. Together with Hyman [2], they have studied some basic properties including some explicit formulas for special solutions. They also studied the nonlinear peakon–antipeakon interaction. The immediate discovery is that the equation is bi-Hamiltonian and completely integrable which possesses infinitely many conserved quantities. Although possess such properties, yet it exhibits remarkable finite-time breakdown (for large class of initial data) [3].

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In [7], Li and Olver established local well-posedness in Sobolev space H^s , $s > \frac{3}{2}$, for the following equation:

$$u_t - \nu u_{xxt} = \alpha u_x + \beta u_{xxx} + 3\gamma uu_x - \gamma \nu (uu_{xxx} + 2u_x u_{xx}), \quad (1.2)$$

where $\alpha, \beta, \gamma, \nu$ are constants, with $\nu > 0$, by regularizing the equation and obtained a solution as the limit of the solutions to the regularized problem. Equation (1.2) can be transformed to the Camassa–Holm equation by just performing the following transformation $u(x, t) \rightarrow -\frac{1}{\gamma}u(\frac{x-\frac{\beta t}{\nu}}{\sqrt{\nu}}, \frac{t}{\sqrt{\nu}})$. They also proved local existence of weak solution in $H^s(\mathbb{R})$ for $1 < s \leq \frac{3}{2}$, provided that $u_{0x} \in H^{s-1}$ is essentially bounded. Later, Constantin and Molinet [4] showed the weak results in $H^1(\mathbb{R})$. Based on the idea by Constantin and Escher [3], which established that odd initial data in H^3 has blow-up solution, they were able to provide several blow-up scenarios even for small data. In particular, they showed that development of singularity in finite time for solutions corresponding to data $u_0 \in H^s$, $2 \leq s < \infty$, in the H^q -norm, $\frac{3}{2} < q \leq s$, is inevitable. Later Constantin and Strauss [5] were able to sharpen the result to $s > \frac{3}{2}$. They also showed the stability of peakons under certain perturbations of initial data.

We should notice that for general data in H^s , $s < \frac{3}{2}$, Himonas and Misiolek [6] proved that the map data-solution fails to be uniformly continuous. In other words, if we require uniformly continuous for the data in the definition of local well-posedness, then the IVP will be ill-posed. Thus, under the condition above, it is unlikely to construct a strong solution with data in H^s , $s < \frac{3}{2}$.

In [8], Stanislavova and Stefanov showed the global existence of the solution for the viscous version of (1.1) with $u_0 \in H^1(\mathbb{R})$. Using the “frequency localized” version of (1.1) together with the result for the viscous version, they established existence of weak solution for (1.1) but for a certain data with slow energy transfer property. They were also concerned with the blow-up result.

Here, we intend to sharpen the global result to less regular initial data. We consider Eq. (1.1) with an additional viscosity term

$$\left. \begin{aligned} \partial_t u &= \varepsilon \partial_x^2 u - \frac{1}{2} \partial_x (1 - \partial_x^2)^{-1} ((\partial_x u)^2) - \partial_x (1 - \partial_x^2)^{-1} (u^2) - \frac{1}{2} \partial_x u^2, \\ u(x, 0) &= u_0(x), \end{aligned} \right\} \quad (1.3)$$

where $\varepsilon > 0$, $u = u(x, t)$, $(x, t) \in \mathbb{R}^2$. We intend to show global existence and uniqueness of (1.3) for $u_0 \in L^2(\mathbb{R})$. The main result is as follow.

Theorem 1. *For any $\varepsilon > 0$ and $u_0 \in L^2(\mathbb{R})$, the IVP (1.3) has solution in*

$$X = C([0, \infty) : L_x^2(\mathbb{R})) \cap C((0, \infty) : H_x^1(\mathbb{R})). \quad (1.4)$$

To show Theorem 1, we will first establish local well-posedness (critical version) for $u_0 \in L^2(\mathbb{R})$, i.e.,

Theorem 2. *For any $\varepsilon > 0$ and $u_0 \in L^2(\mathbb{R})$, there exist a $T = T(u_0, \varepsilon) > 0$ and a unique solution u_ε of (1.3) such that*

$$u_\varepsilon(x, t) \in C([0, T] : L_x^2(\mathbb{R})) \cap C((0, T] : H_x^1(\mathbb{R})).$$

Note that the local existence depends on u_0 and not on $\|u_0\|_{L_x^2}$. The proof of Theorem 1 is the combination of Theorem 2, the energy estimate and the usual extension theory. We believe that Theorem 2 is optimal in the way that the IVP (1.3) is not locally well-posed in $H^s(\mathbb{R})$ for $s < 0$.

Our result is sharper than those in [8] regarding the viscous version of the Camassa–Holm equation (1.3) in the sense that we merely need data in $L_x^2(\mathbb{R})$ with immediate effect in $H_x^1(\mathbb{R})$. Our result for global extension is just a simple argument using the energy dissipation in $H_x^1(\mathbb{R})$.

The rest of the paper is organized as follow: In Section 2, we will set up and introduce a useful estimate concerning the operator $\partial_x(1 - \partial_x^2)^{-1}$. In Section 3, we will consider some estimates for the nonlinear part of the equation. Together with the crucial estimate in Lemma 4, we also obtain some estimates for $\partial_x u$. In Section 4, we will prove Theorem 2 and show the local existence of (1.3) using the contraction argument. The last section is a sketch of the proof of Theorem 1.

2. Preliminaries

Consider the IVP

$$\begin{cases} \partial_t u = \varepsilon \partial_x^2 u + f(u, \partial_x u), & x, t \in \mathbb{R}, \\ u(x, 0) = u_0(x), \end{cases} \quad (2.1)$$

where $f(u, \partial_x u) = -\frac{1}{2}\partial_x(1 - \partial_x^2)^{-1}((\partial_x u)^2) - \partial_x(1 - \partial_x^2)^{-1}(u^2) - \frac{1}{2}\partial_x(u^2)$.

The integral equivalent form of the equation is as follow:

$$u(x, t) = e^{\varepsilon t \partial_x^2} u_0 + \int_0^t e^{\varepsilon(t-t') \partial_x^2} f(u, \partial_x u) dt', \quad (2.2)$$

where $e^{\varepsilon t \partial_x^2} u_0 = (e^{-4\pi^2 \varepsilon t \xi} \hat{u}_0(\xi))^\vee$.

We will list some lemmas needed in the proof of Theorem 2. First, we state the following lemma which consists of the crucial inequality involving the operator $\partial_x(1 - \partial_x^2)^{-1}$.

Lemma 1. For $g, h \in L^2(\mathbb{R})$,

$$\|\partial_x(1 - \partial_x^2)^{-1}(gh)\|_{L_x^2} \leq c \|g\|_{L_x^2} \|h\|_{L_x^2} \quad (2.3)$$

or more general

$$\| |\partial_x|^s (1 - \partial_x^2)^{-1}(gh) \|_{L_x^2} \leq c \|g\|_{L_x^2} \|h\|_{L_x^2} \quad (2.4)$$

for all $s < \frac{3}{2}$.

Proof. Let $\hat{g}(\xi)$ denotes the Fourier transform of $g(x)$. Note that for $g, h \in L^2(\mathbb{R})$, we have

$$\begin{aligned} |(\partial_x(1 - \partial_x^2)^{-1}(gh))^\wedge| &= |\xi(1 + 4\pi^2 \xi^2)^{-1}(\widehat{gh})| \\ &\leq |(1 + 4\pi^2 \xi^2)^{-1/2}(\widehat{gh})|. \end{aligned} \quad (2.5)$$

From Plancherel's identity, Young's inequality, the fact that $(1 + 4\pi^2 \xi^2)^{-1/2} \in L^2(\mathbb{R})$ and (2.5), we have

$$\begin{aligned}
\|\partial_x(1 - \partial_x^2)^{-1}(gh)\|_{L_x^2} &\leq \|(1 + 4\pi^2\xi^2)^{-1/2}(\widehat{gh})\|_{L_\xi^2} \\
&\leq \|(1 + 4\pi^2\xi^2)^{-1/2}\|_{L_\xi^2} \|\widehat{gh}\|_{L_\xi^\infty} \\
&\leq \|(1 + 4\pi^2\xi^2)^{-1/2}\|_{L_\xi^2} \int_{\mathbb{R}} |g(x - \xi)h(x)| dx \\
&\leq c\|g\|_{L_x^2}\|h\|_{L_x^2}.
\end{aligned}$$

(2.4) can be proved in a similar manner. $\square$

The next two lemmas are regarding the nonlinear part of (1.3).

Lemma 2.

$$\begin{aligned}
&\left\| \int_0^t e^{\varepsilon(t-t')\partial_x^2} \left(-\partial_x(1 - \partial_x^2)^{-1}(u^2) - \frac{1}{2}\partial_x(u^2) \right) (x, t') dt' \right\|_{L_T^\infty L_x^2} \\
&\leq C(\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2),
\end{aligned} \tag{2.6}$$

$$\left\| \int_0^t e^{\varepsilon(t-t')\partial_x^2} (-\partial_x(1 - \partial_x^2)^{-1}((\partial_x u)^2))(x, t') dt' \right\|_{L_T^\infty L_x^2} \leq C\|\partial_x u\|_{L_T^2 L_x^2}^2, \tag{2.7}$$

$$\|u(x, t) - e^{\varepsilon t\partial_x^2} u_0\|_{L_T^\infty L_x^2} \leq C(\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2). \tag{2.8}$$

Proof. By Lemma 1 and Sobolev Embedding Theorem $H_x^1(\mathbb{R}) \hookrightarrow L_x^\infty(\mathbb{R})$, we have

$$\begin{aligned}
&\left\| \int_0^t e^{\varepsilon(t-t')\partial_x^2} \left(-\partial_x(1 - \partial_x^2)^{-1}(u^2) - \frac{1}{2}\partial_x(u^2) \right) (x, t') dt' \right\|_{L_T^\infty L_x^2} \\
&\leq \sup_{t \in [0, T]} \int_0^t \|e^{\varepsilon(t-t')\partial_x^2} (-\partial_x(1 - \partial_x^2)^{-1}(u^2))(x, t')\|_{L_x^2} dt' \\
&\quad + \frac{1}{2} \sup_{t \in [0, T]} \int_0^t \|e^{\varepsilon(t-t')\partial_x^2} \partial_x(u^2)(x, t')\|_{L_x^2} dt' \\
&\leq \int_0^T \|-\partial_x(1 - \partial_x^2)^{-1}(u^2)\|_{L_x^2} dt + C \int_0^T \|u\partial_x u\|_{L_x^2} dt \\
&\leq \int_0^T \|u\|_{L_x^2}^2 dt + C \int_0^T \|u\|_{L_x^\infty} \|\partial_x u\|_{L_x^2} dt \\
&\leq \int_0^T \|u\|_{L_x^2}^2 dt + C \int_0^T \|u\|_{H_x^1} \|\partial_x u\|_{L_x^2} dt
\end{aligned}$$

$$\begin{aligned}
&\leq C \left(\int_0^T \|u\|_{L_x^2}^2 dt + \int_0^T \|\partial_x u\|_{L_x^2}^2 dt \right) \\
&\leq C (\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2).
\end{aligned}$$

Similarly, (2.7) is as follow:

$$\begin{aligned}
&\left\| \int_0^t e^{\varepsilon(t-t')\partial_x^2} (-\partial_x(1-\partial_x^2)^{-1}((\partial_x u)^2))(x, t') dt' \right\|_{L_T^\infty L_x^2} \\
&\leq \int_0^T \|\partial_x(1-\partial_x^2)^{-1}((\partial_x u)^2)\|_{L_x^2} dt \\
&\leq \int_0^T \|\partial_x u\|_{L_x^2}^2 dt = \|\partial_x u\|_{L_T^2 L_x^2}^2.
\end{aligned}$$

(2.8) is a consequence of (2.6) and (2.7). $\square$

Lemma 3.

$$\left\| \int_0^t e^{\varepsilon(t-t')\partial_x^2} f(x, t') dt' \right\|_{L_T^2 L_x^2} \leq CT^{1/2} (\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2). \quad (2.9)$$

Proof.

$$\begin{aligned}
&\left\| \int_0^t e^{\varepsilon(t-t')\partial_x^2} f(x, t') dt' \right\|_{L_T^2 L_x^2} \\
&\leq \left(\int_0^T \left\| \int_{\mathbb{R}} \left(\int_0^t e^{\varepsilon(t-t')\partial_x^2} f(x, t') dt' \right)^2 dx \right\|_{L_T^\infty} dt \right)^{1/2} \\
&\leq T^{1/2} \left\| \int_0^t e^{\varepsilon(t-t')\partial_x^2} f(x, t') dt' \right\|_{L_T^\infty L_x^2}.
\end{aligned}$$

The result follows from Lemma 2. $\square$

To investigate the estimates in H^1 , we need the following lemma.

Lemma 4. For any $u_0 \in L^2(\mathbb{R})$, $\varepsilon > 0$ and $\delta > 0$, there exists $T = T(u_0, \varepsilon) > 0$ such that

$$\|\partial_x e^{\varepsilon t \partial_x^2} u_0\|_{L_T^2 L_x^2} = \left(\int_0^T \int_{\mathbb{R}} |\partial_x e^{\varepsilon t \partial_x^2} u_0|^2 dx dt \right)^{1/2} \leq \delta. \quad (2.10)$$

Proof. Let $\{u_0^n\} \subset H^1(\mathbb{R})$ such that

$$u_0^n \xrightarrow{L^2} u_0, \quad \text{i.e.,} \quad \|u_0^n - u_0\|_{L_x^2} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Then

$$\begin{aligned} & \|\partial_x e^{\varepsilon t \partial_x^2} u_0\|_{L_T^2 L_x^2} \\ & \leq \|\partial_x e^{\varepsilon t \partial_x^2} (u_0^n - u_0)\|_{L_T^2 L_x^2} + \|\partial_x e^{\varepsilon t \partial_x^2} u_0^n\|_{L_T^2 L_x^2} \\ & = \left(\int_0^T \int_{\mathbb{R}} |\partial_x e^{\varepsilon t \partial_x^2} (u_0^n - u_0)|^2 dx dt \right)^{1/2} + \left(\int_0^T \int_{\mathbb{R}} |\partial_x e^{\varepsilon t \partial_x^2} u_0^n|^2 dx dt \right)^{1/2} \\ & \leq \frac{1}{(2\varepsilon)^{1/2}} \|u_0^n - u_0\|_{L_x^2} + \left(\int_0^T \|e^{\varepsilon t \partial_x^2} u_0^n\|_{L_x^2}^2 dt \right)^{1/2} \\ & \leq \frac{1}{(2\varepsilon)^{1/2}} \|u_0^n - u_0\|_{L_x^2} + T^{1/2} \|\partial_x u_0^n\|_{L_x^2}. \end{aligned} \quad (2.11)$$

The estimate for the second term on the right-hand side is due to the group property of $e^{\varepsilon t \partial_x^2}$ and the estimate for the first term is due to the following linear problem argument: Consider the linear equation

$$\partial_t w = \varepsilon \partial_x^2 w.$$

Perform the standard energy estimate: Multiply w to the equation and integrate with respect to x over $\mathbb{R}$,

$$\frac{d}{dt} \int_{\mathbb{R}} w^2 dx = 2\varepsilon \int_{\mathbb{R}} w (\partial_x^2 w) dx = -2\varepsilon \int_{\mathbb{R}} (\partial_x w)^2 dx.$$

Next, integrate the above equation with respect to t from 0 to T , we have

$$\int_{\mathbb{R}} (w(x, T))^2 dx - \int_{\mathbb{R}} (w(x, 0))^2 dx = -2\varepsilon \int_0^T \int_{\mathbb{R}} (\partial_x w)^2 dx dt.$$

Thus we have the following estimate:

$$\int_0^T \int_{\mathbb{R}} (\partial_x w)^2 dx dt \leq \frac{1}{2\varepsilon} \int_{\mathbb{R}} w_0^2 dx. \quad (2.12)$$

Continue our discussion from (2.11), given $\delta > 0$, we first fix $N \in \mathbb{N}$ such that for any $n \geq N$,

$$\frac{1}{(2\varepsilon)^{1/2}} \|u_0^n - u_0\|_{L_x^2} \leq \frac{\delta}{2},$$

then choose $T > 0$ such that for $n \geq N$,

$$T^{1/2} \|\partial_x u_0^n\|_{L_x^2} \leq \frac{\delta}{2}.$$

Thus we have

$$\|\partial_x e^{\varepsilon t \partial_x^2} u_0\|_{L_T^2 L_x^2} \leq \delta. \quad \square$$

Next we consider the nonlinear part of (2.2). When written as $v = u - e^{\varepsilon t \partial_x^2} u_0$, v is the solution of the IVP

$$\left. \begin{aligned} \partial_t v &= \varepsilon \partial_x^2 v + f(u, \partial_x u), \\ v(x, 0) &= 0, \end{aligned} \right\} \quad (2.13)$$

i.e.,

$$v(x, t) = \int_0^t e^{\varepsilon(t-t') \partial_x^2} f(u, \partial_x u)(x, t') dt'. \quad (2.14)$$

First, we have the following basic estimate.

Lemma 5.

$$\|f\|_{L_T^1 L_x^2} \leq C(\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2). \quad (2.15)$$

Proof.

$$\begin{aligned} \|f\|_{L_T^1 L_x^2} &= \left\| \frac{1}{2} \partial_x (1 - \partial_x^2)^{-1} ((\partial_x u)^2) + \partial_x (1 - \partial_x^2)^{-1} (u^2) + \frac{1}{2} \partial_x (u^2) \right\|_{L_T^1 L_x^2} \\ &\leq \int_0^T \left\| \frac{1}{2} \partial_x (1 - \partial_x^2)^{-1} ((\partial_x u)^2) + \partial_x (1 - \partial_x^2)^{-1} (u^2) + \frac{1}{2} \partial_x (u^2) \right\|_{L_x^2} dt \\ &\leq \frac{1}{2} \int_0^T \|\partial_x (1 - \partial_x^2)^{-1} ((\partial_x u)^2)\|_{L_x^2} dt + \int_0^T \|\partial_x (1 - \partial_x^2)^{-1} (u^2)\|_{L_x^2} dt \\ &\quad + \int_0^T \|u \partial_x u\|_{L_x^2} dt \\ &\leq \frac{1}{2} \int_0^T \|\partial_x u\|_{L_x^2}^2 dt + \int_0^T \|u\|_{L_x^2}^2 dt + \int_0^T \|u\|_{L_x^\infty} \|\partial_x u\|_{L_x^2} dt \\ &\leq \frac{1}{2} \int_0^T \|\partial_x u\|_{L_x^2}^2 dt + C \int_0^T \|u\|_{L_x^2}^2 dt + \int_0^T \|u\|_{H_x^1} \|\partial_x u\|_{L_x^2} dt \\ &\leq C(\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2). \quad \square \end{aligned} \quad (2.16)$$

Then we have some estimates for v .

Lemma 6.

$$\|v\|_{L_T^\infty L_x^2} \leq C(\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2), \quad (2.17)$$

$$\|\partial_x v\|_{L_T^\infty L_x^2} \leq C\varepsilon^{-1/2}(\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2)^2. \quad (2.18)$$

Proof. We have that

$$\begin{aligned} \|v\|_{L_T^\infty L_x^2} &= \left\| \int_0^t e^{\varepsilon(t-t')\partial_x^2} f(u, \partial_x u)(x, t') dt' \right\|_{L_T^\infty L_x^2} \\ &\leq \int_0^T \|f\|_{L_x^2} dt = \|f\|_{L_T^1 L_x^2}. \end{aligned} \quad (2.19)$$

By Lemma 5, (2.17) follows.

Next, we will perform the standard energy estimate on (2.13), i.e., multiply v to (2.13) and integrate with respect to x over $\mathbb{R}$. After integration by parts, we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \frac{v^2}{2} dx + \varepsilon \int_{\mathbb{R}} (\partial_x v)^2 dx &= \int_{\mathbb{R}} v f dx, \\ \frac{1}{2} \int_{\mathbb{R}} v^2(T) dx - \frac{1}{2} \int_{\mathbb{R}} v^2(0) dx + \varepsilon \int_0^T \int_{\mathbb{R}} (\partial_x v)^2 dx dt &= \int_0^T \int_{\mathbb{R}} v f dx dt \end{aligned} \quad (2.20)$$

for any $\varepsilon > 0$. This gives

$$\begin{aligned} \varepsilon \int_0^T \int_{\mathbb{R}} (\partial_x v)^2 dx dt &\leq \int_0^T \int_{\mathbb{R}} v f dx dt, \\ \|\partial_x v\|_{L_T^2 L_x^2}^2 &\leq \frac{1}{\varepsilon} \int_0^T \|v\|_{L_x^2} \|f\|_{L_x^2} dt \leq \frac{1}{\varepsilon} \|v\|_{L_T^\infty L_x^2} \|f\|_{L_T^1 L_x^2}. \end{aligned} \quad (2.21)$$

Together with (2.19), we have that

$$\|\partial_x v\|_{L_T^2 L_x^2} \leq \frac{1}{\varepsilon^{1/2}} \|f\|_{L_T^1 L_x^2}^2. \quad (2.22)$$

By Lemma 5, (2.18) follows. $\square$

We are now ready to consider an estimate for $\partial_x u$.

Lemma 7.

$$\|\partial_x u\|_{L_T^2 L_x^2} \leq \delta + \frac{C}{\varepsilon^{1/2}} (\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2). \quad (2.23)$$

Proof. From Lemmas 4–6, we have

$$\begin{aligned}
 \|\partial_x u\|_{L_T^2 L_x^2} &\leq \|\partial_x e^{\varepsilon t \partial_x^2} u_0\|_{L_T^2 L_x^2} + \left\| \partial_x \int_0^t e^{\varepsilon(t-t') \partial_x^2} f(x, t') dt' \right\|_{L_T^2 L_x^2} \\
 &\leq \delta + \left\| \partial_x \int_0^t e^{\varepsilon(t-t') \partial_x^2} f(x, t') dt' \right\|_{L_T^2 L_x^2} \\
 &\leq \delta + \frac{1}{\varepsilon^{1/2}} \|f\|_{L_T^1 L_x^2} \\
 &\leq \delta + \frac{C}{\varepsilon^{1/2}} (\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2). \quad \square
 \end{aligned}$$

3. Local result: Proof of Theorem 2

Let

$$\begin{aligned}
 X_a^T &= \{u \in C([0, T] : L^2(\mathbb{R})) \cap C((0, T) : H^1(\mathbb{R})) : \\
 \|u\| &= \|u - e^{\varepsilon t \partial_x^2} u_0\|_{L_T^\infty L_x^2} + \|\partial_x u\|_{L_T^2 L_x^2} + \|u\|_{L_T^2 L_x^2} \leq a\}
 \end{aligned} \quad (3.1)$$

and define the mapping $\Phi : X_a^T \rightarrow X_a^T$ by

$$\begin{aligned}
 \Phi(u) &= e^{\varepsilon t \partial_x^2} u_0 + \int_0^t e^{\varepsilon(t-t') \partial_x^2} \left(-\frac{1}{2} \partial_x (1 - \partial_x^2)^{-1} ((\partial_x u)^2) \right. \\
 &\quad \left. - \partial_x (1 - \partial_x^2)^{-1} (u^2) - \frac{1}{2} \partial_x (u^2) \right) (x, t') dt'.
 \end{aligned} \quad (3.2)$$

Theorem 3. For any $\varepsilon > 0$, there exist $T = T_\varepsilon > 0$ and $a > 0$ such that $\Phi(X_a^T) \subseteq X_a^T$. In addition, $\Phi : X_a^T \rightarrow X_a^T$ is a contraction mapping.

Proof. We first need to show that the map is well defined for some appropriate a and T . Let $u \in X_a^T$. We have

$$\|\Phi u\| = \|\Phi u - e^{\varepsilon t \partial_x^2} u_0\|_{L_T^\infty L_x^2} + \|\Phi u\|_{L_T^2 L_x^2} + \|\partial_x(\Phi u)\|_{L_T^2 L_x^2}. \quad (3.3)$$

Consider the terms in (3.3) one by one: From Lemma 2, the first term in (3.3) can be estimated as follow:

$$\begin{aligned}
 \|\Phi u - e^{\varepsilon t \partial_x^2} u_0\|_{L_T^\infty L_x^2} &= \left\| \int_0^t e^{\varepsilon(t-t') \partial_x^2} f(u, \partial_x u)(x, t') dt' \right\|_{L_T^\infty L_x^2} \\
 &\leq C (\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2) \\
 &\leq C \|u\|^2;
 \end{aligned} \quad (3.4)$$

from Lemma 3, the second term in (3.3) can be estimated as follow:

$$\begin{aligned}
\|\Phi u\|_{L_T^2 L_x^2} &= \left\| e^{\varepsilon t \partial_x^2} u_0 + \int_0^t e^{\varepsilon(t-t') \partial_x^2} f(u, \partial_x u)(x, t') dt' \right\|_{L_T^2 L_x^2} \\
&\leq \|e^{\varepsilon t \partial_x^2} u_0\|_{L_T^2 L_x^2} + \left\| \int_0^t e^{\varepsilon(t-t') \partial_x^2} f(u, \partial_x u)(x, t') dt' \right\|_{L_T^2 L_x^2} \\
&\leq T^{1/2} \|u_0\|_{L_x^2} + CT^{1/2} (\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2) \\
&\leq T^{1/2} \|u_0\|_{L_x^2} + CT^{1/2} \|u\|^2;
\end{aligned} \tag{3.5}$$

from Lemma 7, the third term in (3.3) can be estimated as follow:

$$\begin{aligned}
\|\partial_x(\Phi u)\|_{L_T^2 L_x^2} &= \left\| \partial_x e^{\varepsilon t \partial_x^2} u_0 + \partial_x \int_0^t e^{\varepsilon(t-t') \partial_x^2} f(u, \partial_x u)(x, t') dt' \right\|_{L_T^2 L_x^2} \\
&\leq \|\partial_x e^{\varepsilon t \partial_x^2} u_0\|_{L_T^2 L_x^2} + \left\| \partial_x \int_0^t e^{\varepsilon(t-t') \partial_x^2} f(u, \partial_x u)(x, t') dt' \right\|_{L_T^2 L_x^2} \\
&\leq \delta + \frac{C}{\varepsilon^{1/2}} (\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2) \\
&\leq \delta + \frac{C}{\varepsilon^{1/2}} \|u\|.
\end{aligned} \tag{3.6}$$

Combine (3.4)–(3.6), we have that

$$\begin{aligned}
\|\Phi u\| &= \|\Phi u - e^{\varepsilon t \partial_x^2} u_0\|_{L_T^\infty L_x^2} + \|\Phi u\|_{L_T^2 L_x^2} + \|\partial_x(\Phi u)\|_{L_T^2 L_x^2} \\
&\leq \delta + T^{1/2} \|u_0\|_{L_x^2} + C(1 + T^{1/2}) \left(\|u\|^2 + \frac{1}{\varepsilon^{1/2}} \|u\| \right) \\
&\leq \delta + T^{1/2} \|u_0\|_{L_x^2} + C(1 + T^{1/2}) \left(1 + \frac{1}{\varepsilon^{1/2}} \right) a^2.
\end{aligned}$$

With appropriate values of δ , a and T , we are able to have that

$$\|\Phi u\| \leq a,$$

i.e., $\Phi : X_a^T \rightarrow X_a^T$ is well defined.

With a similar argument, we can show that $\Phi : X_a^T \rightarrow X_a^T$ is a contraction mapping:

$$\|\Phi(u) - \Phi(v)\| \leq C' \|u - v\|, \tag{3.7}$$

where $C' = C'(T, a, \varepsilon, \|u\|_{L_T^2 L_x^2}, \|v\|_{L_T^2 L_x^2}, \|\partial_x u\|_{L_T^2 L_x^2}, \|\partial_x v\|_{L_T^2 L_x^2})$ can be chosen as $0 < C' < 1$ with appropriate values of T and a . $\square$

Theorem 2 is merely Theorem 3 with a standard uniqueness argument.

4. Proof of Theorem 1

To prove Theorem 1, we need only to establish some “*a priori*” estimates. To do so, we apply the operator $1 - \partial_x^2$ to (1.3):

$$(1 - \partial_x^2) \partial_t u - \varepsilon (1 - \partial_x^2) \partial_x^2 u = -\frac{1}{2} \partial_x ((\partial_x u)^2) - \partial_x (u^2) - (1 - \partial_x^2) \frac{1}{2} \partial_x (u^2). \tag{4.1}$$

Perform the standard energy estimate on (4.1): Multiply u to the equation and integrate with respect to x over $\mathbb{R}$, we have that

$$\begin{aligned}
 & \int u(1 - \partial_x^2) \partial_t u \, dx - \varepsilon \int u(1 - \partial_x^2) \partial_x^2 u \, dx \\
 &= -\frac{1}{2} \int u \partial_x ((\partial_x u)^2) \, dx - \int u \partial_x (u^2) \, dx - \int u(1 - \partial_x^2) \frac{1}{2} \partial_x (u^2) \, dx \\
 &\implies \frac{1}{2} \frac{d}{dt} \left(\int u^2 + (\partial_x u)^2 \, dx \right) + \varepsilon \int (\partial_x u)^2 + (\partial_x^2 u)^2 \, dx \\
 &\quad = - \int u \partial_x u \partial_x^2 u \, dx + \int \partial_x (u^3) \, dx + \int u \partial_x^2 (u \partial_x u) \, dx \\
 &\implies \frac{1}{2} \frac{d}{dt} \left(\int u^2 + (\partial_x u)^2 \, dx \right) \leq -\varepsilon \int (\partial_x u)^2 + (\partial_x^2 u)^2 \, dx \leq 0.
 \end{aligned} \tag{4.2}$$

Thus we have an a priori estimate. Together with the local theory and the standard extension argument, we have a global solution for (1.3).

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