

Unconditional martingale difference sequences in Banach spaces

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Abstract

For a Banach space Y , the question of whether $L^p(\mu, Y)$ has an unconditional basis if $1 < p < \infty$ and Y has unconditional basis, stood unsolved for a long time and was answered in the negative by Aldous. In this work we prove a weaker, positive result related to this question. We show that if (y_j) is a basis of Y and (d_i) is a martingale difference sequence spanning $L^p(\mu)$ then the sequence $(d_i \otimes y_j)$ is a basis of $L^p(\mu, Y)$ for $1 \leq p < \infty$. Moreover, if $1 < p < \infty$ and (y_j) is unconditional then $(d_i \otimes y_j)$ is strictly dominated by an unconditional tensor product basis. In addition, for $1 < p < \infty$, we show that if $(d_i) \subset L^p(\mu)$ is a martingale difference sequence then there exists a constant $K > 0$ so that

$$\left\| \sum_{i,j \in \mathbb{N}} (\alpha_{ij} y_j) d_i \right\|_{L^p(\mu, Y)} \leq K \left\| \sum_{i,j \in \mathbb{N}} \|\alpha_{ij} y_j\| d_i \right\|_{L^p(\mu)}$$

holds for every sequence $(y_j) \subset Y$ and every choice of finitely supported scalars (α_{ij}) .

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1. Introduction

Let (Ω, Σ, μ) denote a probability space. Then, for $1 \leq p < \infty$ and Y a Banach space, let $L^p(\mu, Y)$ denote the space of (classes of a.e. equal) Bochner p -integrable functions $f: \Omega \rightarrow Y$

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and denote the Bochner norm on $L^p(\mu, Y)$ by Δ_p , i.e., $\Delta_p(f) = (\int_{\Omega} \|f\|_Y^p d\mu)^{1/p}$. Here, we only consider Banach spaces over the real numbers.

A sequence (y_i) in a Banach space Y is called a *basis* if each element $y \in Y$ has a unique series expansion $y = \sum_{i=1}^{\infty} \alpha_i y_i$ where (α_i) is a sequence of scalars. A sequence $(u_j) \subset Y$ is called a *block basis* of (y_i) if $u_j = \sum_{i=n_j+1}^{n_{j+1}} \alpha_i y_i$ for each $j \in \mathbb{N}$ with (α_i) scalars and $n_1 < n_2 < \dots$ an increasing sequence of natural numbers. A basis $(y_i) \subset Y$ is called *unconditional* if the unique series expansion $\sum_{i=1}^{\infty} \alpha_i y_i$ of each element in Y converges unconditionally, i.e., $\sum_{i=1}^{\infty} \alpha_{\pi(i)} y_{\pi(i)}$ converges for every permutation π of the natural numbers. A sequence $(y_i) \subset Y$ is called a *basic sequence* if (y_i) is a basis of its closed linear span, which we denote by $[y_i]$. If (y_i) is an unconditional basis of $[y_i]$, then (y_i) is referred to as an *unconditional basic sequence*.

In [8, p. 114] it is asked whether $L^p(\mu, Y)$ has an unconditional basis if $1 < p < \infty$ and Y has unconditional basis. This question was answered in the negative by Aldous in [1], who showed that this cannot happen if Y is not super reflexive (see definition in [2, p. 225]), e.g., $Y = \ell^1$ or $Y = c_0$. In this work we show that if (y_j) is a basis of Y and (d_i) is a martingale difference sequence spanning $L^p(\mu)$ then the sequence $(d_i \otimes y_j)$ is a basis of $L^p(\mu, Y)$ for $1 \leq p < \infty$. Moreover, if $1 < p < \infty$ and (y_j) is unconditional then $(d_i \otimes y_j)$ is strictly dominated by an unconditional tensor product basis. In addition, for $1 < p < \infty$, we show that if $(d_i) \subset L^p(\mu)$ is a martingale difference sequence then there exists a constant $K > 0$ so that

$$\left\| \sum_{i,j \in \mathbb{N}} (\alpha_{ij} y_j) d_i \right\|_{L^p(\mu, Y)} \leq K \left\| \sum_{i,j \in \mathbb{N}} \|\alpha_{ij} y_j\| d_i \right\|_{L^p(\mu)}$$

holds for every sequence $(y_j) \subset Y$ and every choice of finitely supported scalars (α_{ij}) . Here, a sequence $(d_i) \subset L^p(\mu)$ is called a *martingale difference sequence (m.d.s.)* if

$$\mathbb{E}(d_{i+1} \mid \sigma(d_1, \dots, d_i)) = 0,$$

for each $i \in \mathbb{N}$, where $\mathbb{E}(\cdot \mid \sigma(d_1, \dots, d_i)) : L^p(\mu) \rightarrow L^p(\mu)$ denotes the conditional expectation operator relative to the smallest σ -algebra allowing d_1, \dots, d_i to be measurable. Note that every m.d.s. is a *monotone* basic sequence in $L^p(\mu)$, i.e., $\|\sum_{i=1}^n \alpha_i d_i\| \leq \|\sum_{i=1}^m \alpha_i d_i\|$ for all natural numbers $n < m$ and scalars $\alpha_1, \dots, \alpha_m$.

2. Preliminaries

We assume that the reader is familiar with the basic concepts and notation of Banach spaces, Banach lattices, vector-valued L^p -spaces and Riesz spaces as can be found in [2,8,13,16,17,19,22,23,25].

Let X and Y be Banach spaces. A sequence $(x_i) \subset X$ is said to *dominate* a sequence $(y_i) \subset Y$ provided for all sequences of scalars (α_i) we have

$$\sum_{i=1}^{\infty} \alpha_i x_i \text{ converges} \Rightarrow \sum_{i=1}^{\infty} \alpha_i y_i \text{ converges.}$$

In this case we shall use the notation $(x_i) \succcurlyeq (y_i)$. We shall say that (x_i) *strictly dominates* (y_i) if there exists a bounded linear mapping $T : [x_i] \rightarrow [y_i]$ such that $T x_i = y_i$ for each $i \in \mathbb{N}$. In this case we shall write $(x_i) \succ (y_i)$. The sequences (x_i) and (y_i) are said to be *equivalent* if $(x_i) \succcurlyeq (y_i) \succcurlyeq (x_i)$ and *strictly equivalent* if $(x_i) \succ (y_i) \succ (x_i)$. In these cases we shall use the notations $(x_i) \sim (y_i)$ and $(x_i) \approx (y_i)$, respectively. It is immediate that if $(x_i) \approx (y_i)$ then $[x_i]$ is isomorphic to $[y_i]$ under the bounded linear map that takes x_i to y_i for each $i \in \mathbb{N}$. Strict

domination clearly implies domination and strict equivalence clearly implies equivalence, but the reverse implications need not be true (cf. [23, p. 69]).

If X is a Banach space with basis (x_i) we define the *natural projections* associated with (x_i) to be the sequence (P_i) given by $P_i(\sum_{k=1}^{\infty} \alpha_k x_k) = \sum_{k=1}^i \alpha_k x_k$ for each $i \in \mathbb{N}$. It follows, by the principle of uniform boundedness, that (P_i) is a uniformly bounded collection of linear projections. The quantity $K = \sup_i \|P_i\| < \infty$ is known as the *basis constant*. A basis can be characterized as follows (cf. [16, Proposition 1.a.3]):

Proposition 2.1. *Let (x_i) be a sequence in a Banach space X . Then (x_i) is a basis of X if and only if the following three conditions hold:*

- (a) $x_i \neq 0$ for all $i \in \mathbb{N}$,
- (b) *there is a constant K so that, for every choice of scalars (α_i) and positive integers $n < m$, we have*

$$\left\| \sum_{i=1}^n \alpha_i x_i \right\| \leq K \left\| \sum_{i=1}^m \alpha_i x_i \right\|, \quad \text{and}$$

- (c) *the closed linear span of (x_i) is all of X .*

Throughout this work, we assume all our sequences to have no zero terms. Note that, if $K = 1$ in the above proposition, then (x_i) is a monotone basis. If $(x_i) \subset X$ is a basic sequence and $(y_i) \subset Y$ is an arbitrary sequence, then we have $(x_i) \succ (y_i)$ if and only if $(x_i) \succ (y_i)$. If, in addition, we have that (y_i) is also a basic sequence then $(x_i) \sim (y_i)$ if and only if $(x_i) \approx (y_i)$ (cf. [23, Chapter I, Theorem 8.1]).

An important example of a basis is the (generalized) *Haar system* (cf. [16, Definition 1.a.4] and [9, p. 54]), denoted by (χ_i) , which is a basis of $L^p(\mu)$ for $1 \leq p < \infty$ (cf. [16, p. 3]) and an unconditional basis of $L^p(\mu)$ for $1 < p < \infty$ (cf. [17, Theorem 2.c.5] and [3, Theorem 9]). Note that the Haar system is also an m.d.s. (cf. [17, pp. 151, 152]). The unconditionality of the Haar system was first proved by Paley in 1931 (cf. [20]).

The unconditional convergence of a series in a Banach space can be characterized as follows (cf. [16, Proposition 1.c.1]):

Proposition 2.2. *Let (x_i) be a sequence of vectors in a Banach space X . Then the following are equivalent:*

- (a) *The series $\sum_{i=1}^{\infty} x_{\pi(i)}$ converges for every permutation π of the natural numbers.*
- (b) *The series $\sum_{k=1}^{\infty} x_{n_k}$ converges for every choice of $n_1 < n_2 < n_3 < \dots$.*
- (c) *The series $\sum_{i=1}^{\infty} \theta_i x_i$ converges for every choice of signs (θ_i) .*

Suppose (x_i) is an unconditional basis of a Banach space X , then it follows that for $\sigma \subset \mathbb{N}$, the map $P_\sigma : X \rightarrow X$ defined by $P_\sigma(\sum_{i=1}^{\infty} \alpha_i x_i) = \sum_{i \in \sigma} \alpha_i x_i$ is a bounded linear projection. Similarly, for every choice of signs $\theta = (\theta_i)$, we have a bounded linear operator $M_\theta : X \rightarrow X$ given by $M_\theta(\sum_{i=1}^{\infty} \alpha_i x_i) = \sum_{i=1}^{\infty} \theta_i \alpha_i x_i$. Moreover, we have that $\sup_\sigma \|P_\sigma\|$ and $\sup_\theta \|M_\theta\|$ are finite and these quantities are related by the inequality $\sup_\sigma \|P_\sigma\| \leq \sup_\theta \|M_\theta\| \leq 2 \sup_\sigma \|P_\sigma\|$. The quantity $\sup_\theta \|M_\theta\|$ is known as the *unconditional constant* of the unconditional basis (x_i) and is always larger or equal to the basis constant.

3. Martingale difference sequences in a Banach space

We introduce abstract definitions for a filtration and a martingale in a Banach space. For the convenience of the reader, we first recall the classical definitions.

Let $1 \leq p < \infty$ and (Ω, Σ, μ) denote a probability space with Σ_1 a sub σ -algebra of Σ . The conditional expectation of $f \in L^p(\mu)$ relative to Σ_1 , denoted by $\mathbb{E}(f \mid \Sigma_1)$, is a Σ_1 -measurable element of $L^p(\mu)$ which is uniquely given by

$$\int_A \mathbb{E}(f \mid \Sigma_1) d\mu = \int_A f d\mu \quad \text{for all } A \in \Sigma_1. \quad (3.1)$$

It is well known that the map $f \rightarrow \mathbb{E}(f \mid \Sigma_1)$ is a linear contractive projection on $L^p(\mu)$.

A monotone increasing sequence (Σ_i) of sub σ -algebras of Σ is called a *filtration*. For a filtration (Σ_i) and $i \leq j$, it follows from (3.1) that

$$\mathbb{E}(\cdot \mid \Sigma_i) = \mathbb{E}(\mathbb{E}(\cdot \mid \Sigma_j) \mid \Sigma_i) = \mathbb{E}(\mathbb{E}(\cdot \mid \Sigma_i) \mid \Sigma_j),$$

which implies $\mathcal{R}(\mathbb{E}(\cdot \mid \Sigma_i)) \subseteq \mathcal{R}(\mathbb{E}(\cdot \mid \Sigma_j))$. Here, we use the notation $\mathcal{R}(T)$ to denote the range of a function T .

If (Σ_i) is a filtration, a sequence $(f_i) \subset L^p(\mu)$ is called a *martingale* relative to (Σ_i) if each f_i is Σ_i -measurable and

$$\mathbb{E}(f_j \mid \Sigma_i) = f_i \quad \text{for all } i \leq j.$$

A martingale (f_i) is *norm-convergent* if there exists $f \in L^p(\mu)$ such that $\|f - f_i\|_p \rightarrow 0$ as $i \rightarrow \infty$. From this point on we shall simply refer to a norm-convergent martingale as *convergent*.

Note that a filtration (Σ_i) corresponds to a uniformly bounded sequence of commuting projections with increasing range on $L^p(\mu)$. Using this observation, we generalize the notions of a filtration and a martingale to a Banach space.

Definition 3.1. Let X be a Banach space.

- (a) A sequence (T_i) of projections on X with the property that $T_{i \wedge j} = T_i T_j$, for each $i, j \in \mathbb{N}$, is called a *K-filtration* on X if $\sup_i \|T_i\| = K < \infty$. A 1-filtration will simply be referred to as a filtration.
- (b) If (T_i) is a *K-filtration* on X , then (f_i, T_i) is called a *K-martingale* on X if $T_i f_j = f_i$ for all $i \leq j$. A 1-martingale will simply be referred to as a martingale.

Examples of the above definition are furnished by [24] as well as further reading on the space of bounded martingales defined on a Banach lattice. These generalized notions are also studied in [5]. In [5, Proposition 3.2] it is shown that if (f_i, T_i) is a martingale in a Banach space, then $T_i f$ converges to f if and only if $f \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$. In [5, Corollary 3.2], it is deduced that (f_i, T_i) converges to an element f if and only if $f \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ and $f_i = T_i f$ for all $i \in \mathbb{N}$. A notational change in the proofs of these results shows that the same results hold for *K-filtrations* and *K-martingales*, namely:

Proposition 3.2. Let X be a Banach space and let (T_i) be a K -filtration on X . Then $f \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$ if and only if $\|T_i f - f\| \rightarrow 0$.

Corollary 3.3. Let X be a Banach space and let (f_i, T_i) be a K -martingale in X . Then (f_i, T_i) converges to f if and only if $f \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$ and $f_i = T_i f$ for all $i \in \mathbb{N}$.

We now recall the classical definition of a martingale difference sequence. Let $(d_i) \subset L^p(\mu)$ be a sequence and $\sigma(d_1, \dots, d_i)$ denote the smallest σ -algebra allowing d_1, \dots, d_i to be measurable. Then, as mentioned in the introduction, (d_i) is called a martingale difference sequence (m.d.s.) if

$$\mathbb{E}(d_{i+1} \mid \sigma(d_1, \dots, d_i)) = 0 \quad (3.2)$$

for each $i \in \mathbb{N}$. Notice that $(\sigma(d_1, \dots, d_i))_{i=1}^{\infty}$ is a filtration and

$$d_{i+1} \in \mathcal{R}(\mathbb{E}(\cdot \mid \sigma(d_1, \dots, d_{i+1})) - \mathbb{E}(\cdot \mid \sigma(d_1, \dots, d_i)))$$

for each $i \in \mathbb{N}$. Conversely, if (Σ_i) is a filtration and $(g_i) \subset L^p(\mu)$ is a sequence such that $g_{i+1} \in \mathcal{R}(\mathbb{E}(\cdot \mid \Sigma_{i+1}) - \mathbb{E}(\cdot \mid \Sigma_i))$ for each $i \in \mathbb{N}$, then it follows from (3.1) that

$$0 = \int_A \mathbb{E}(g_{i+1} \mid \Sigma_i) d\mu = \int_A \mathbb{E}(g_{i+1} \mid \sigma(g_1, \dots, g_i)) d\mu \quad \text{for all } A \in \sigma(g_1, \dots, g_i),$$

which implies (g_i) satisfies (3.2) and is, therefore, an m.d.s. Using this characterization, we introduce an abstract notion for an m.d.s. in a Banach space.

Let (T_i) be a K -filtration and $i < j$, then $T_j - T_i$ is a projection due to the fact that the T_i s commute. This justifies the following definition.

Definition 3.4. Let (T_i) be a K -filtration on a Banach space X . Then the *difference projections* (D_i) relative to (T_i) are given by $D_1 = T_1$ and $D_i = T_i - T_{i-1}$ for $i \geq 2$.

It is then clear that $T_i = \sum_{k=1}^i D_k$ for each $i \in \mathbb{N}$ and that $D_i D_j = 0$ whenever $i \neq j$.

Definition 3.5. Let (D_i) be the difference projections relative to a K -filtration (T_i) on a Banach space X . Then a sequence (d_i) is called a *K -martingale difference sequence (K -m.d.s.) relative to (T_i)* if $d_i \in \mathcal{R}(D_i)$ for each $i \in \mathbb{N}$. A 1-m.d.s. will simply be referred to as an m.d.s.

A sequence $(d_i) \subset L^p(\mu)$ obeying (3.2) will be called a *classical m.d.s.* and is clearly a special case of an m.d.s. in the above definition.

Notice that $D_i d_j = d_j$ whenever $i = j$ and $D_i d_j = 0$ whenever $i \neq j$. The sequence of partial sums $f_i = \sum_{k=1}^i d_k$, for each $i \in \mathbb{N}$, form a K -martingale with respect to (T_i) . Conversely, if (f_i, T_i) is a K -martingale then the sequence of differences, defined by $d_1 = f_1$ and $d_i = f_i - f_{i-1}$ for $i \geq 2$, form a K -m.d.s. relative to (T_i) .

If (D_i) is the sequence of difference projections relative to a K -filtration (T_i) on a Banach space X with $\bigcup_{i=1}^{\infty} \mathcal{R}(T_i) = X$ then, for each $x \in X$, Proposition 3.2 asserts that $(\sum_{k=1}^i D_k)x = T_i x \rightarrow x$ as $i \rightarrow \infty$. Thus $\sum_{k=1}^{\infty} D_k x = x$, which gives $X = \bigoplus_{i=1}^{\infty} \mathcal{R}(D_i)$.

Suppose that (d_i) is a K -m.d.s. in a Banach space X relative to (T_i) and that (α_i) is a sequence of scalars. Then the partial sums $f_i = \sum_{k=1}^i \alpha_k d_k$ form a K -martingale with respect to (T_i) . If $i < j$, then

$$\|f_i\| = \left\| \sum_{k=1}^i \alpha_k d_k \right\| = \left\| T_i \left(\sum_{k=1}^j \alpha_k d_k \right) \right\| \leq K \left\| \sum_{k=1}^j \alpha_k d_k \right\| = K \|f_j\|.$$

Hence, (d_i) is a basic sequence with basis constant K . On the other hand, if (x_i) is a basic sequence in a Banach space X with basis constant K , then (x_i) is a K -m.d.s. in $[x_i]$ relative to the associated natural projections (P_i) on $[x_i]$. In short, we have the following result:

Proposition 3.6. *Every K -m.d.s. in a Banach space is a basic sequence with basis constant K . Also, every basic sequence in a Banach space with basis constant K is a K -m.d.s. in its closed linear span.*

If (d_i) is a K -m.d.s. relative to (T_i) , it is easily observed that $[d_i] \subset \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ which is, in general, necessarily strict. Indeed, let (χ_i) denote the Haar system and consider the m.d.s. of Rademacher functions (r_i) in $L^p(\mu)$ for $1 \leq p < \infty$, where the Rademacher functions are defined by $r_1 = \chi_1$ and $r_i = \sum_{k=2^{(i-2)}+1}^{2^{(i-1)}} \chi_k$ for $i \geq 2$. It is clear that (r_i) is a block basis of the Haar system and so it follows that $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(\mathbb{E}(\cdot | \sigma(r_1, \dots, r_i)))} = L^p(\mu)$. On the other hand, it follows from Khintchine's inequality that $[r_i]$ is isomorphic to ℓ^2 (see [2, Chapter VI, §1, Proposition 1]), in which case, the inclusion $[r_i] \subset \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(\mathbb{E}(\cdot | \sigma(r_1, \dots, r_i)))}$ is certainly strict. The next result characterizes the situation.

Proposition 3.7. *Let (d_i) be a K -m.d.s. in a Banach space X relative to (T_i) . Then $[d_i] = \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ if and only if $\text{rank}(T_i) = i$ for each $i \in \mathbb{N}$.*

Proof. Suppose $[d_i] = \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ and let (D_i) be the difference projections relative to (T_i) , then it follows from the above discussion that $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)} = \bigoplus_{i=1}^{\infty} \mathcal{R}(D_i)$. On the other hand, for $f \in [d_i]$, we have a unique basis expansion so that $f = \sum_{i=1}^{\infty} \alpha_i d_i = \sum_{i=1}^{\infty} D_i f$. The uniqueness of both these expansions implies $\alpha_i d_i = D_i f$ for each $i \in \mathbb{N}$ so that $T_i f = \sum_{k=1}^i \alpha_k d_k$. Thus, (T_i) are just the natural projections associated to the basic sequence (d_i) . But then $\text{rank}(T_i) = i$ for each $i \in \mathbb{N}$.

Conversely, it is sufficient to show $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)} \subset [d_i]$, since the reverse inclusion is always true. Since the T_i s have increasing ranges and $\text{rank}(T_i) = i$ for each $i \in \mathbb{N}$, the difference projections (D_i) relative to (T_i) are all of rank one. As before, $f \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ has a unique expansion $f = \sum_{i=1}^{\infty} D_i f$. The fact that $\dim(\mathcal{R}(D_i)) = 1$ implies there exists a scalar α_i such that $D_i f = \alpha_i d_i$ for each $i \in \mathbb{N}$. It follows that $\sum_{i=1}^n D_i f = \sum_{i=1}^n \alpha_i d_i \in \text{span}(d_i)$ for each $n \in \mathbb{N}$, which completes the proof. \square

It is apparent from the above proof that, in the case where $[d_i] \subset \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$, the restriction $T_i|_{[d_i]}$ of T_i to $[d_i]$ is just the i th natural projection on $[d_i]$ associated to the basic sequence (d_i) for each $i \in \mathbb{N}$.

In view of the fact that every K -m.d.s. is a basic sequence, we formulate the analogous notion of an unconditional K -m.d.s.

Definition 3.8. Let (d_i) be a K -m.d.s. in a Banach space X . Then (d_i) is said to be *unconditional* if there exists a constant $M > 0$ such that for every choice of scalars (α_i) , signs (θ_i) and natural numbers n we have

$$\left\| \sum_{i=1}^n \theta_i \alpha_i d_i \right\| \leq M \left\| \sum_{i=1}^n \alpha_i d_i \right\|.$$

The smallest constant M for which the above inequality holds is called the *unconditional constant* of (d_i) .

Note that M in the above definition is never smaller than K . It is evident that if (d_i) is an unconditional K -m.d.s., then it forms an unconditional basis of $[d_i]$ with unconditional constant M .

4. The Banach lattice of unconditional m.d.s. multipliers

Definition 4.1. Let (x_i) be a sequence in a Banach space X such that $x_i \neq 0$ for each $i \in \mathbb{N}$. We define the normed linear space of sequences of coefficients of (x_i) to be

$$A^{(x_i)} = \left\{ (\alpha_i) \subset \mathbb{R} : \sum_{i=1}^{\infty} \alpha_i x_i \text{ converges in } X \right\}$$

endowed with the norm $\|\cdot\|_{A^{(x_i)}}$ defined by $\|(\alpha_i)\|_{A^{(x_i)}} = \sup_n \left\| \sum_{i=1}^n \alpha_i x_i \right\|$ for each $(\alpha_i) \in A^{(x_i)}$.

It is shown in [23, Chapter I, Proposition 3.1] that $A^{(x_i)}$ is a Banach space. Furthermore, it is shown in [23, Chapter I, Proposition 8.1] that the unit vectors $e_i = (\delta_{ik})_{k=1}^{\infty}$ ($i = 1, 2, \dots$) constitute a basis of $A^{(x_i)}$ such that $(e_i) \sim (x_i)$ and $(e_i) \succ (x_i)$.

Definition 4.2. If X is a Banach space and $(x_i) \subset X$ is a sequence, then the map from $A^{(x_i)}$ into X given by $(\alpha_i) \mapsto \sum_{i=1}^{\infty} \alpha_i x_i$ will be referred to as the *co-ordinate map*.

It is evident that the co-ordinate map for any sequence is linear and of norm one. If (x_i) is a basic sequence, [23, Chapter I, Theorem 8.1] implies that $(e_i) \approx (x_i)$. Thus, $A^{(x_i)}$ is isomorphic to $[x_i]$ under the co-ordinate map (also see [23, Chapter I, Proposition 3.2]). We shall mainly consider $A^{(d_i)}$ where (d_i) is a K -m.d.s. (and thus a basic sequence).

Definition 4.3. Let (d_i) be a K -m.d.s. in a Banach space. The order on $A^{(d_i)}$, defined by $A^{(d_i)} \ni (\alpha_i) \geq 0 \Leftrightarrow \alpha_i \geq 0$ for each $i \in \mathbb{N}$, is called the *sequential ordering* induced by (d_i) and the set $A_+^{(d_i)} := \{(\alpha_i) \in A^{(d_i)} : (\alpha_i) \geq 0\}$ is called the *positive cone* induced by (d_i) .

Evidently, $\lambda A^{(d_i)} \subset A^{(d_i)}$ where $\lambda \in \mathbb{R}_+$, $A^{(d_i)} + A^{(d_i)} \subset A^{(d_i)}$ and $A_+^{(d_i)} \cap (-A_+^{(d_i)}) = \{0\}$. Thus $(A^{(d_i)}, A_+^{(d_i)})$ is a partially ordered vector space.

Lemma 4.4. Suppose that (d_i) is an unconditional K -m.d.s., then $A^{(d_i)}$ is a Dedekind complete Riesz space under the sequential ordering. Furthermore, $A^{(d_i)}$ can be renormed so that it becomes a Dedekind complete Banach lattice with the unit vectors (e_i) as an unconditional basis.

Proof. Let $(\alpha_i) \in A^{(d_i)}$. The unconditional convergence of $\sum_{i=1}^{\infty} \alpha_i d_i \in [d_i]$ implies that the series $\sum_{i=1}^{\infty} |\alpha_i| d_i$, $\sum_{i=1}^{\infty} \alpha_i^+ d_i$ and $\sum_{i=1}^{\infty} \alpha_i^- d_i$ also converge in $[d_i]$, where $\alpha_i^+ := \max\{0, \alpha_i\}$

and $\alpha_i^- := \max\{0, -\alpha_i\}$. Hence $(|\alpha_i|), (\alpha_i^+), (\alpha_i^-) \in A^{(d_i)}$. It is clear from the definition of the sequential ordering that

$$|(\alpha_i)| = (\alpha_i) \vee (-\alpha_i) = (\alpha_i \vee (-\alpha_i)) = (|\alpha_i|) \in A^{(d_i)}.$$

Consequently, $A^{(d_i)}$ is a Riesz space. In addition, we have $(\alpha_i) = (\alpha_i^+) - (\alpha_i^-)$ and $(\alpha_i^+) \wedge (\alpha_i^-) = 0$, thus $(\alpha_i)^+ = (\alpha_i^+)$ and $(\alpha_i)^- = (\alpha_i^-)$. Moreover, the unconditional convergence of $\sum_{i=1}^{\infty} \alpha_i d_i \in [d_i]$ also implies the convergence of $\sum_{i=1}^{\infty} \gamma_i d_i$ whenever $|\gamma_i| \leq |\alpha_i|$ for each $i \in \mathbb{N}$ (see [16, Proposition 1.c.6]). Thus, $(\gamma_i) \in A^{(d_i)}$ provided that $|\gamma_i| \leq |\alpha_i|$ for each $i \in \mathbb{N}$. It is now evident from the Dedekind completeness of \mathbb{R} that $(A^{(d_i)}, A_+^{(d_i)})$ is a Dedekind complete Riesz space.

Since $(e_i) \approx (d_i)$ with (d_i) an unconditional basic sequence, it follows that (e_i) is an unconditional basis of $A^{(d_i)}$. It is an easy consequence of Hahn Banach (see [16, Proposition 1.c.7]) that, for all $(\alpha_i) \in A^{(d_i)}$ and $(\lambda_i) \in \ell^\infty$, we have $\|\sum_{i=1}^{\infty} \lambda_i \alpha_i e_i\| \leq M \|(\lambda_i)\|_\infty \|\sum_{i=1}^{\infty} \alpha_i e_i\|$ where M is the unconditional constant of (e_i) . Thus, if $(\alpha_i), (\beta_i) \in A^{(d_i)}$ with $|(\alpha_i)| \leq |(\beta_i)|$, it follows that $\|(\alpha_i/\beta_i)\|_\infty \leq 1$. Consequently,

$$\left\| \sum_{i=1}^{\infty} \alpha_i e_i \right\| = \left\| \sum_{i=1}^{\infty} (\alpha_i/\beta_i) \beta_i e_i \right\| \leq M \left\| \sum_{i=1}^{\infty} \beta_i e_i \right\|.$$

Thus, $|(\alpha_i)| \leq |(\beta_i)|$ implies $\|(\alpha_i)\| \leq M \|(\beta_i)\|$, i.e., $A^{(d_i)}$ is a partially ordered Banach space. Define $\|\cdot\|_0$ on $A^{(d_i)}$ by

$$\|(\alpha_i)\|_0 = \sup\{\|(\beta_i)\| : |(\beta_i)| \leq |(\alpha_i)|\},$$

for each $(\alpha_i) \in A^{(d_i)}$. Then it is readily verified that $\|\cdot\|_0$ is an equivalent norm on $A^{(d_i)}$ and $(A^{(d_i)}, A_+^{(d_i)}, \|\cdot\|_0)$ is a Dedekind complete Banach lattice. \square

If (d_i) is an unconditional K -m.d.s., we shall denote the Dedekind complete Banach lattice obtained by renorming $A^{(d_i)}$ again by $A^{(d_i)}$. Note that, after renorming, the unit vectors (e_i) are now an unconditional basis of $A^{(d_i)}$ with unconditional constant one and the co-ordinate map from $A^{(d_i)}$ onto $[d_i]$ is still of norm one. We refer to $A^{(d_i)}$ as *the Banach lattice of m.d.s. multipliers*. It should be clear that $A^{(d_i)}$ is isomorphic to $[d_i]$ but not necessarily Riesz isomorphic; in fact, $[d_i]$ need not even be a Riesz space.

Theorem 4.5. *Let (d_i) be a K -m.d.s. in a Banach space X , then (d_i) is unconditional if and only if $A^{(d_i)}$ is an order continuous Banach lattice.*

Proof. Suppose that (d_i) is an unconditional K -m.d.s. Then Lemma 4.4 asserts that $A^{(d_i)}$ is a Dedekind complete Banach lattice. It is now sufficient to show that every positive, order bounded, disjoint sequence in $A^{(d_i)}$ converges in norm to zero (cf. [25, Theorem 17.14]).

To this end, let $(x_k) \subset A^{(d_i)}$ be a positive disjoint sequence which is order bounded. By the Dedekind completeness of $A^{(d_i)}$ it follows that $\sup_k x_k \in A^{(d_i)}$. Let $s = \sup_k x_k$ and define the sequence of partial sums (s_j) by $s_j = \sum_{k=1}^j x_k$. Since (x_k) is disjoint, we have that $s_j = \bigvee_{k=1}^j x_k$ for each $j \in \mathbb{N}$ with $s_j \uparrow s$. We claim that $s_j \rightarrow s$ in norm. To see this, for each $j \in \mathbb{N}$ let

$$\sigma_j = \bigcup_{1 \leq k \leq j} \{i \in \mathbb{N} : x_k = (\alpha_i^{(k)}), \alpha_i^{(k)} \neq 0\}$$

and define the family of projections (P_{σ_j}) on $A^{(d_i)}$ by $P_{\sigma_j}(\gamma_i) = \sum_{i \in \sigma_j} \gamma_i e_i$ for each $(\gamma_i) \in A^{(d_i)}$. By Lemma 4.4 we have that (e_i) is an unconditional basis of $A^{(d_i)}$. Thus (P_{σ_j}) is a filtration. Now observe

$$x_j \wedge x_k = (\alpha_i^{(j)}) \wedge (\alpha_i^{(k)}) = (\alpha_i^{(j)} \wedge \alpha_i^{(k)}) = 0$$

for $j \neq k$. Hence $\alpha_i^{(j)} \wedge \alpha_i^{(k)} = 0$, giving either $\alpha_i^{(j)} = 0$ or $\alpha_i^{(k)} = 0$ for each $i \in \mathbb{N}$. Thus, the sequence $(\alpha_i^{(k)})_{k=1}^\infty$ has at most one nonzero element for each $i \in \mathbb{N}$. As a consequence, (s_j, P_{σ_j}) is a martingale and $P_{\sigma_j}(s) = s_j$ for each $j \in \mathbb{N}$ with $s \in \overline{\bigcup_{j=1}^\infty \mathcal{R}(P_{\sigma_j})}$. An appeal to Corollary 3.3 gives $s_j \rightarrow s$ in norm, which proves the claim. It is now evident that $\|x_{k+1}\| = \|s_{k+1} - s_k\| \rightarrow 0$ since (s_k) is a Cauchy sequence.

Conversely, suppose that $A^{(d_i)}$ is an order continuous Banach lattice under the sequential ordering. Note that the order continuity of the norm on $A^{(d_i)}$ implies that $A^{(d_i)}$ is Dedekind complete. Since $A^{(d_i)}$ is a Riesz space, we may decompose any element uniquely as the difference of two disjoint positive elements and so we need only consider positive elements.

To this end, let $f = \sum_{i=1}^\infty \alpha_i e_i \in A_+^{(d_i)}$ and let $(n_r)_{r=1}^\infty$ be a strictly increasing sequence of natural numbers. For each $k \in \mathbb{N}$ define $x_k = \sum_{r=1}^k \alpha_{n_r} e_{n_r}$. Then (x_k) is an increasing sequence which is bounded above by f . The Dedekind completeness of $A^{(d_i)}$ implies that $x_k = \sum_{r=1}^k \alpha_{n_r} e_{n_r} \uparrow \sum_{r=1}^\infty \alpha_{n_r} e_{n_r} := x \in S$. Now $(x - x_k) \downarrow 0$ and the order continuity of the norm implies $\|x - x_k\| \rightarrow 0$. Hence, the series $\sum_{r=1}^\infty \alpha_{n_r} e_{n_r}$ is summable, from which we deduce the unconditional summability of $f = \sum_{i=1}^\infty \alpha_i e_i$. Thus (e_i) is an unconditional basis of $A^{(d_i)}$ with $(e_i) \approx (d_i)$, which completes the proof. \square

As an immediate consequence of this result, we obtain a familiar characterization of an unconditional basis that can be found in [23, Chapter II, Proposition 16.2].

Corollary 4.6. *Let X be a Banach space and $(x_i) \subset X$ be a basic sequence. Then (x_i) is an unconditional basic sequence if and only if $[x_i]$ can be renormed so that it is an order continuous Banach lattice with order induced by the cone*

$$C_+^{(x_i)} := \left\{ \sum_{i=1}^\infty \alpha_i x_i \in [x_i] : \alpha_i \geq 0 \text{ for each } i \in \mathbb{N} \right\}.$$

Proof. We have that (x_i) is a K -m.d.s. in $[x_i]$ relative the associated natural projections. Since (e_i) is a basis of $A^{(x_i)}$ with $(e_i) \approx (x_i)$, it follows that $([x_i], C_+^{(x_i)})$ is Riesz isomorphic to $(A^{(x_i)}, A_+^{(x_i)})$ when (x_i) is unconditional. The result now follows by inducing the (equivalent) norm $\|\cdot\|_{A^{(x_i)}}$ on $[x_i]$. \square

5. Martingale difference sequences with positive equivalence

For an unconditional K -m.d.s. (d_i) in a Banach lattice E , we are now faced with the problem of relating the sequential ordering on $A^{(d_i)}$ to the ordering on E . We introduce the following property for a K -m.d.s. in a Banach lattice.

Definition 5.1. Let E be a Banach lattice. A K -m.d.s. (d_i) in E is said to have *positive equivalence* if $(d_i) \sim (|d_i|)$.

Note that if (d_i) is a K -m.d.s. in a Banach lattice with $(d_i) \sim (|d_i|)$, it does not follow that $(|d_i|)$ is a K -m.d.s. Indeed, let (χ_i) denote the Haar system, which is a classical m.d.s. It is obvious that $(|\chi_i|)$ is not an m.d.s. because $|\chi_1| = |\chi_2| = \mathbf{1}$ which implies that $(|\chi_i|)$ is not a basic sequence.

Let (r_i) denote the Rademacher functions. In [13, pp. 27–28] it is shown, as a consequence of the Khintchine inequality, that for any sequence $x_1, \dots, x_n \in L^p(\mu)$, where $1 \leq p < \infty$, there exist constants K_1 and K_p (K_p dependent on p) for which

$$\begin{aligned} K_1 \left\| \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} \right\|_{L^p(\mu)} &\leq \int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|_{L^p(\mu)} dt \\ &\leq K_p \left\| \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} \right\|_{L^p(\mu)} \end{aligned} \quad (5.1)$$

holds. In the case where (x_i) is an unconditional basic sequence with unconditional constant M , the above inequality yields

$$\begin{aligned} K_1 M^{-1} \left\| \left(\sum_{i=1}^n |\alpha_i x_i|^2 \right)^{1/2} \right\|_{L^p(\mu)} &\leq \left\| \sum_{i=1}^n \alpha_i x_i \right\|_{L^p(\mu)} \\ &\leq M K_p \left\| \left(\sum_{i=1}^n |\alpha_i x_i|^2 \right)^{1/2} \right\|_{L^p(\mu)} \end{aligned} \quad (5.2)$$

for all scalars $\alpha_1, \dots, \alpha_n$. Combining inequalities (5.1) and (5.2) we obtain

$$K_1 (M K_p)^{-1} \left\| \sum_{i=1}^n \alpha_i x_i \right\|_{L^p(\mu)} \leq \int_0^1 \left\| \sum_{i=1}^n r_i(t) \alpha_i |x_i| \right\|_{L^p(\mu)} dt \quad (5.3)$$

$$\leq K_1^{-1} (M K_p) \left\| \sum_{i=1}^n \alpha_i x_i \right\|_{L^p(\mu)} \quad (5.4)$$

for all scalars $\alpha_1, \dots, \alpha_n$. Using the unconditionality of (x_i) , it follows from inequalities (5.3) and (5.4) that $(x_i) \sim (|x_i|)$.

Burkholder (and Gundy) showed in [3, Theorem 9] that every classical m.d.s. in $L^p(\mu)$, for $1 < p < \infty$, is unconditional. Thus, every classical m.d.s. in $L^p(\mu)$ has positive equivalence provided $1 < p < \infty$.

More generally, using the functional calculus of Krivine (cf. [14] or [17, Proposition 1.d.1]), inequalities (5.3) and (5.4) hold in any p -concave Banach lattice (cf. [17, Definition 1.d.3]) for unconditional basic sequences. Thus, every unconditional K -m.d.s. in a p -concave Banach lattice has positive equivalence. These inequalities were noted by Maurey (cf. [18] or [17, Proposition 1.d.6]).

Theorem 5.2. *Let E be a Banach lattice and let $(d_i) \subset E$ be an unconditional K -m.d.s. with positive equivalence. Then the co-ordinate map from the Banach lattice $A^{(d_i)}$ into E is regular and order continuous.*

Proof. Since $(d_i) \sim (|d_i|)$, we have that $A^{(d_i)} = A^{(|d_i|)}$ (as sets). Moreover, since (d_i) is a basic sequence, it follows from [23, Chapter I, Theorem 8.1] that $(d_i) \succ (|d_i|)$. Thus there exists a bounded linear map $u : [d_i] \rightarrow [|d_i|]$ such that $u(d_i) = |d_i|$. Let $(\alpha_i) \in A^{(|d_i|)}$, then

$$\begin{aligned} \|(\alpha_i)\|_{A^{(|d_i|)}} &= \sup_n \left\| \sum_{i=1}^n \alpha_i |d_i| \right\| = \sup_n \left\| \sum_{i=1}^n \alpha_i u(d_i) \right\| \\ &\leq \|u\| \sup_n \left\| \sum_{i=1}^n \alpha_i d_i \right\| = \|u\| \|(\alpha_i)\|_{A^{(d_i)}}. \end{aligned}$$

Since $A^{(d_i)}$ and $A^{(|d_i|)}$ are both Banach spaces under their respective norms, it follows from the open mapping theorem that the norms $\|\cdot\|_{A^{(d_i)}}$ and $\|\cdot\|_{A^{(|d_i|)}}$ are equivalent on $A^{(d_i)}$.

Define the respective maps R_1 and R_2 from $A^{(d_i)}$ into E by

$$R_1((\alpha_i)) = \frac{1}{2} \left(\sum_{i=1}^{\infty} \alpha_i |d_i| + \sum_{i=1}^{\infty} \alpha_i d_i \right)$$

and

$$R_2((\alpha_i)) = \frac{1}{2} \left(\sum_{i=1}^{\infty} \alpha_i |d_i| - \sum_{i=1}^{\infty} \alpha_i d_i \right)$$

for each $(\alpha_i) \in A^{(d_i)}$. As a consequence of the equivalence of the norms $\|\cdot\|_{A^{(d_i)}}$ and $\|\cdot\|_{A^{(|d_i|)}}$, we have that R_1 and R_2 are well defined, linear and bounded.

Let R denote the co-ordinate map from $A^{(d_i)}$ into E . It follows from $R_1((\alpha_i)) = \sum_{i=1}^{\infty} \alpha_i (d_i)^+$, $R_2((\alpha_i)) = \sum_{i=1}^{\infty} \alpha_i (d_i)^-$ and

$$(R_1 - R_2)((\alpha_i)) = \sum_{i=1}^{\infty} \alpha_i (d_i)^+ - \sum_{i=1}^{\infty} \alpha_i (d_i)^- = \sum_{i=1}^{\infty} \alpha_i d_i = R((\alpha_i))$$

for each $(\alpha_i) \in A^{(d_i)}$, that $R = R_1 - R_2$, where R_1 and R_2 are positive; i.e., R is regular.

Since Theorem 4.5 implies that $A^{(d_i)}$ has order continuous norm, the regularity of R implies that R is order continuous. \square

6. The l -tensor product of martingale difference sequences

If X and Y are Banach spaces and α is a norm on $X \otimes Y$, we denote the normed space $(X \otimes Y, \alpha)$ by $X \otimes_{\alpha} Y$, its norm completion by $X \widetilde{\otimes}_{\alpha} Y$ and its continuous dual by $(X \otimes_{\alpha} Y)'$. The norm of an element $u \in X \widetilde{\otimes}_{\alpha} Y$ will be denoted $\alpha_{X,Y}(u)$ when there is a need to distinguish the Banach spaces involved or simply $\alpha(u)$ if there is no risk of ambiguity. A norm α on $X \otimes Y$ is called a *reasonable cross norm* (cf. [6–8,12]) if α satisfies the conditions:

- (a) For $x \in X$ and $y \in Y$, $\alpha(x \otimes y) \leq \|x\| \|y\|$.
- (b) For $x' \in X'$ and $y' \in Y'$, $x' \otimes y' \in (X \otimes_{\alpha} Y)'$ and $\|x' \otimes y'\| \leq \|x'\| \|y'\|$.

It is well known that the inequalities in (a) and (b) may be replaced by equality.

Let X , X_0 , Y and Y_0 be Banach spaces. If $S : X_0 \rightarrow X$ and $T : Y_0 \rightarrow Y$ are bounded linear maps, then a reasonable cross norm α is called a *uniform cross norm* if

$$\|S \otimes T\| \leq \|S\| \|T\|.$$

Since the inequality $\|S \otimes T\| \geq \|S\| \|T\|$ holds for all reasonable cross norms α , equality holds in the definition of uniform cross norms. In the case where X_0 is a closed subspace of X , Y_0 is a closed subspace of Y and α is a uniform cross norm, we have that $\alpha_{X,Y}(u) \leq \alpha_{X_0,Y_0}(u)$. This inequality can be strict and thus $E_0 \widetilde{\otimes}_\alpha Y_0$ need not be a subspace of $E \widetilde{\otimes}_\alpha Y$. A uniform cross norm for which $\alpha_{X_0,Y_0}(u) = \alpha_{X,Y}(u)$ holds for each closed subspace X_0 of X and Y_0 of Y is called *injective*.

Pisier noted that the Bochner norm Δ_p is not an injective uniform cross norm for $1 < p < \infty$ (cf. [6, p. 147]). However, for $1 \leq p < \infty$, it is known the Bochner norm Δ_p on $L^p(\mu, X)$ has the property that if $0 \leq S : L^p(\mu) \rightarrow L^p(\mu)$ (which implies that S is bounded) and $T : X \rightarrow X$ is a bounded map, then $S \otimes T : L^p(\mu, X) \rightarrow L^p(\mu, X)$ has the property that

$$\|S \otimes T\| = \|S\| \|T\| \quad (6.1)$$

(cf. [8,15]).

Chaney and Schaefer extended the Bochner norm to the tensor product of a Banach lattice and a Banach space (cf. [4,22]). If E is a Banach lattice and Y is a Banach space, then the l -norm of $u = \sum_{i=1}^n x_i \otimes y_i \in E \otimes Y$ is given by

$$\|u\|_l = \inf \left\{ \left\| \sum_{i=1}^n \|y_i\| |x_i| \right\| : u = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

Furthermore, if $E = L^p(\mu)$ where (Ω, Σ, μ) is a σ -finite measure space, then we have that $E \widetilde{\otimes}_l Y$ is isometric to $L^p(\mu, Y)$.

Property (6.1) extends to the l -tensor product as stated below; proofs of which may be found in [15]:

Let E_1 and E_2 be Banach lattices and Y_1 and Y_2 Banach spaces. If $S : E_1 \rightarrow E_2$ is a positive linear operator and $T : Y_1 \rightarrow Y_2$ a bounded linear operator, then

$$\|(S \otimes T)u\|_l \leq \|S\| \|T\| \|u\|_l \quad \text{for all } u \in E_1 \otimes Y_1.$$

In [15] it is shown that if E and E_0 are Banach lattices, Y and Y_0 are Banach spaces, $S : E_0 \rightarrow E$ is a Riesz isometry and $T : Y_0 \rightarrow Y$ is an isometry, then both $S \otimes_l \text{id}_Y : E_0 \widetilde{\otimes}_l Y \rightarrow E \widetilde{\otimes}_l Y$ and $\text{id}_E \otimes_l T : E \widetilde{\otimes}_l Y_0 \rightarrow E \widetilde{\otimes}_l Y$ are isometries. It now follows from the fact that $S \otimes_l \text{id}_{Y_0} : E_0 \widetilde{\otimes}_l Y_0 \rightarrow E \widetilde{\otimes}_l Y_0$ is an isometry that the composition

$$S \otimes_l T = (\text{id}_E \otimes_l T)(S \otimes_l \text{id}_{Y_0}) : E_0 \widetilde{\otimes}_l Y_0 \rightarrow E \widetilde{\otimes}_l Y$$

is also an isometry. Thus the l -norm exhibits a weaker form of injectivity: if E_0 is a closed Riesz subspace of E and Y_0 is a closed subspace of Y , then $E_0 \widetilde{\otimes}_l Y_0$ is a closed subspace of $E \widetilde{\otimes}_l Y$.

Definition 6.1. If E and E_0 are Banach lattices, Y and Y_0 are Banach spaces, $0 \leq S : E_0 \rightarrow E$ and $T : Y_0 \rightarrow Y$ are bounded linear maps, then a reasonable cross norm α is called

- (a) *left order uniform* (or in short, *left uniform*) if $\|S \otimes T\| \leq \|S\| \|T\|$;
- (b) *left order injective* (or in short, *left injective*) if $S \otimes T : E_0 \widetilde{\otimes}_\alpha Y_0 \rightarrow E \widetilde{\otimes}_\alpha Y$ is an isometry, provided that S is a Riesz isometry and T is an isometry.

For further reading on the tensor product of filtrations with respect to left order, left injective cross norms, see [5].

Using the idea of Gelbaum and Gil de Lamadrid in [11] for constructing the tensor product basis with respect to a uniform cross norm, we construct the l -tensor product of two martingale difference sequences.

Definition 6.2. Let (ξ_i) and (η_j) be sequences in the Banach spaces X and Y , respectively. We define the *square ordering* on the sequence of tensors $(\xi_i \otimes \eta_j)$ to be the ordering of the indices (i, j) along the squares, i.e., $(i_1, j_1) \leq (i_2, j_2)$ when one of the following conditions hold:

- (a) $\max\{i_1, j_1\} < \max\{i_2, j_2\}$,
- (b) $\max\{i_1, j_1\} = \max\{i_2, j_2\}$ and $i_1 < i_2$, or
- (c) $\max\{i_1, j_1\} = \max\{i_2, j_2\} = i_1 = i_2$ and $j_1 \geq j_2$.

Thus, $(\xi_i \otimes \eta_j)$ with the square ordering is the sequence $\xi_1 \otimes \eta_1, \xi_1 \otimes \eta_2, \xi_2 \otimes \eta_2, \xi_2 \otimes \eta_1, \xi_1 \otimes \eta_3, \xi_2 \otimes \eta_3, \dots$. We shall use the notation S_k for the set consisting of the first k ordered pairs of indices (i, j) in the square ordering.

Let E be a Banach lattice and Y a Banach space. Suppose that $(\xi_i) \subset E$ and $(\eta_j) \subset Y$ are basic sequences with $[\xi_i]$ a Riesz subspace of E . Since the l -norm is a reasonable cross norm, it follows that $[\xi_i] \tilde{\otimes}_l [\eta_j] = [\xi_i \otimes \eta_j]$. Moreover, the left order injectivity of the l -norm implies that $[\xi_i] \tilde{\otimes}_l [\eta_j]$ is a closed subspace of $E \tilde{\otimes}_l Y$.

Definition 6.3. A K -filtration (T_i) on a Banach lattice E is said to be *positive* if T_i is positive for each $i \in \mathbb{N}$.

Note that every classical m.d.s. in $L^p(\mu)$ is relative to a filtration that is a sequence of conditional expectation operators on $L^p(\mu)$. Thus, every classical m.d.s. is an m.d.s. relative to a positive filtration.

Proposition 6.4. Let (S_i) be a positive K_1 -filtration on the Banach lattice E , (T_j) be a K_2 -filtration on the Banach space Y and define the sequence (P_k) by

$$P_k = \begin{cases} S_i \otimes_l T_i, & k = i^2, \\ S_i \otimes_l T_i + S_{k-i^2} \otimes_l (T_{i+1} - T_i), & i^2 < k \leq i^2 + i + 1, \\ S_{i+1} \otimes_l T_{i+1} - (S_{i+1} - S_i) \otimes_l T_{(i+1)^2-k}, & i^2 + i + 1 < k < (i+1)^2, \end{cases}$$

for each $k \in \mathbb{N}$. Then (P_k) is a K -filtration on $E \tilde{\otimes}_l Y$ where $K \leq 3K_1K_2$. Moreover, if (ξ_i) and (η_j) are martingale difference sequences relative to (S_i) and (T_j) , respectively, then the sequence $(\xi_i \otimes \eta_j)$ with the square ordering is a K -m.d.s. in $E \tilde{\otimes}_l Y$ relative to (P_k) .

Proof. Since (S_i) is a positive filtration and (T_j) is a filtration we have, for each $i \in \mathbb{N}$, that

$$\|S_i \otimes_l T_i\| = \|S_i\| \|T_i\| \leq K_1 K_2,$$

$$\|S_{k-i^2} \otimes_l (T_{i+1} - T_i)\| = \|S_{k-i^2}\| \|(T_{i+1} - T_i)\| \leq 2K_1 K_2$$

and

$$\begin{aligned}
\|(S_{i+1} - S_i) \otimes_l T_{(i+1)^2-k}\| &\leq \|S_{i+1} \otimes_l T_{(i+1)^2-k}\| + \|S_i \otimes_l T_{(i+1)^2-k}\| \\
&= \|T_{(i+1)^2-k}\| (\|S_{i+1}\| + \|S_i\|) \\
&\leq 2K_1 K_2,
\end{aligned}$$

from which we deduce $\sup_{k \in \mathbb{N}} \|P_k\| \leq 3K_1 K_2$. Hence (P_k) is uniformly bounded on $E \widetilde{\otimes}_l Y$.

Using the fact that (S_i) and (T_j) are filtrations, we first show that P_k is a projection for each $k \in \mathbb{N}$. The case where k is a perfect square is trivial. For the case $i^2 < k \leq i^2 + i + 1$, for some $i \in \mathbb{N}$, we have

$$\begin{aligned}
P_k^2 &= (S_i \otimes_l T_i + S_{k-i^2} \otimes_l (T_{i+1} - T_i))^2 \\
&= (S_i \otimes_l T_i)^2 + (S_i \otimes_l T_i)(S_{k-i^2} \otimes_l (T_{i+1} - T_i)) \\
&\quad + (S_{k-i^2} \otimes_l (T_{i+1} - T_i))(S_i \otimes_l T_i) + (S_{k-i^2} \otimes_l (T_{i+1} - T_i))^2 \\
&= S_i \otimes_l T_i + (S_i S_{k-i^2}) \otimes_l (T_i T_{i+1} - T_i^2) \\
&\quad + (S_{k-i^2} S_i) \otimes_l (T_{i+1} T_i - T_i^2) + S_{k-i^2} \otimes_l (T_{i+1} - T_i) \\
&= S_i \otimes_l T_i + 0 + 0 + S_{k-i^2} \otimes_l (T_{i+1} - T_i) \\
&= P_k.
\end{aligned}$$

For the case $i^2 + i + 1 < k < (i+1)^2$, for some $i \in \mathbb{N}$, we have

$$\begin{aligned}
P_k^2 &= (S_{i+1} \otimes_l T_{i+1} - (S_{i+1} - S_i) \otimes_l T_{(i+1)^2-k})^2 \\
&= (S_{i+1} \otimes_l T_{i+1})^2 - (S_{i+1} \otimes_l T_{i+1})((S_{i+1} - S_i) \otimes_l T_{(i+1)^2-k}) \\
&\quad - ((S_{i+1} - S_i) \otimes_l T_{(i+1)^2-k})(S_{i+1} \otimes_l T_{i+1}) + ((S_{i+1} - S_i) \otimes_l T_{(i+1)^2-k})^2 \\
&= S_{i+1} \otimes_l T_{i+1} - (S_{i+1}^2 - S_{i+1} S_i) \otimes_l T_{i+1} T_{(i+1)^2-k} \\
&\quad - (S_{i+1}^2 - S_i S_{i+1}) \otimes_l T_{(i+1)^2-k} T_{i+1} + (S_{i+1} - S_i) \otimes_l T_{(i+1)^2-k} \\
&= S_{i+1} \otimes_l T_{i+1} - 2(S_{i+1} - S_i) \otimes_l T_{(i+1)^2-k} + (S_{i+1} - S_i) \otimes_l T_{(i+1)^2-k} \\
&= S_{i+1} \otimes_l T_{i+1} - (S_{i+1} - S_i) \otimes_l T_{(i+1)^2-k} \\
&= P_k.
\end{aligned}$$

To prove that (P_k) is a K -filtration, we need only to show that $P_k = P_k P_{k+1} = P_{k+1} P_k$ for each $k \in \mathbb{N}$. This presents us with five cases for each $i \in \mathbb{N}$: $k = i^2$, $i^2 < k < i^2 + i + 1$, $k = i^2 + i + 1$, $i^2 + i + 1 < k < (i+1)^2 - 1$ and $k = (i+1)^2 - 1$. The verification of these cases is a tedious but trivial exercise and will be omitted.

For the last part of the proof, it follows from the definition of the square ordering that

$$P_n \left(\sum_{(i,j) \in S_m} \xi_i \otimes \eta_j \right) = \sum_{(i,j) \in S_n} \xi_i \otimes \eta_j$$

for $n \leq m$. This gives $\xi_i \otimes \eta_j \in \mathcal{R}(P_k - P_{k-1})$ for each $k \in \mathbb{N}$, where $\{(i, j)\} = S_k \setminus S_{k-1}$. Here, P_0 is defined to be zero and S_0 to be the empty set. \square

In the case where (S_i) and (T_j) are the natural projections associated with the bases (ξ_i) and (η_j) , respectively, it is evident that (P_k) are the natural projections associated with the basic sequence $(\xi_i \otimes \eta_j)$ with the square ordering and so we obtain the following corollary.

Corollary 6.5. *Let E be a Banach lattice with a basis (ξ_i) possessing the property that the natural projections associated with (ξ_i) are positive. If Y is a Banach space with basis (basic sequence) (η_j) , then the sequence $(\xi_i \otimes \eta_j)$ with the square ordering is a basis (basic sequence) of $E \tilde{\otimes}_l Y$.*

Proof. The case where (η_j) is a basis for Y follows from Proposition 6.4 and the fact that $[\xi_i \otimes \eta_j] = E \tilde{\otimes}_l Y$. For the case where (η_j) is a basic sequence in Y , it follows from the first part of the proof that $(\xi_i \otimes \eta_j)$ is a basis of $E \tilde{\otimes}_l [\eta_j]$. The left order injectivity of the l -norm now implies that $E \tilde{\otimes}_l [\eta_j]$ is a closed subspace of $E \tilde{\otimes}_l Y$. Thus $(\xi_i \otimes \eta_j)$ is a basic sequence in $E \tilde{\otimes}_l Y$. \square

In particular, when $1 \leq p < \infty$ and $E = L^p(\mu)$, we obtain the following result:

Corollary 6.6. *Let $1 \leq p < \infty$ and (y_j) be a basic sequence in a Banach space Y . If (d_i) is a classical m.d.s. in $L^p(\mu)$, then the sequence $(d_i \otimes y_j)$ with the square ordering is a basic sequence in $L^p(\mu, Y)$. If, in addition, we have $[d_i] = L^p(\mu)$ then the sequence $(d_i \otimes y_j)$ with the square ordering is a basis of $L^p(\mu, Y)$ provided (y_j) is a basis of Y .*

Proof. Since (d_i) is an m.d.s. relative to a positive filtration, the result follows from Proposition 6.4 and Corollary 6.5 and the fact that $L^p(\mu, Y)$ is isometric to $L^p(\mu) \tilde{\otimes}_l Y$. \square

Note that the Haar system is an example of a classical m.d.s. for which the linear span is dense in $L^p(\mu)$, thus $L^p(\mu, Y)$ has a basis if Y has a basis and $1 \leq p < \infty$. Aldous showed in [1, Proposition 3] that if a classical m.d.s. formed a basis of $L^p(\mu, Y)$ for a Banach space Y , then Y is necessarily one-dimensional. It is important to note that the definition of a K -m.d.s. determines a larger class of sequences than the definition of a classical m.d.s. In the latter case, it is therefore possible to have a K -m.d.s. as a basis of $L^p(\mu, Y)$ for which Y could be infinite dimensional.

7. The l -tensor product of unconditional martingale difference sequences

Let E and F denote Banach lattices. We denote the *projective cone* of $E \otimes F$ by

$$E_+ \otimes F_+ := \left\{ \sum_{i=1}^n x_i \otimes y_i : (x_i, y_i) \in E_+ \times F_+, n \in \mathbb{N} \right\}.$$

Chaney and Schaefer showed, in [4, Theorem 1.7] and [22, Chapter IV, §7, Theorem 7.2], respectively, that $E \tilde{\otimes}_l F$ is a Banach lattice and that the positive cone of $E \tilde{\otimes}_l F$ is the l -closure of the projective cone $E_+ \otimes F_+$. Moreover, Popa showed in [21] that $E \tilde{\otimes}_l F$ is an order continuous Banach lattice if E and F are order continuous Banach lattices. We use these results in the following proposition.

Proposition 7.1. *Let X and Y be Banach spaces. If $(\xi_i) \subset X$ and $(\eta_j) \subset Y$ are both unconditional K -m.d.s.s, then $(e_i \otimes e_j)$ is an unconditional basis of $A^{(\xi_i)} \tilde{\otimes}_l A^{(\eta_j)}$ with unconditional constant one.*

Proof. By Lemma 4.4, we have that (e_i) and (e_j) are unconditional bases of the respective Banach lattices $A^{(\xi_i)}$ and $A^{(\eta_j)}$. It follows from the above remarks that $A^{(\xi_i)} \tilde{\otimes}_l A^{(\eta_j)}$ is a Banach

lattice with positive cone the l -closure of the projective cone $A^{(\xi_i)}_+ \otimes A^{(\eta_j)}_+$. Corollary 6.5 implies that the sequence $(e_i \otimes e_j)$ with the square ordering is a basis of $A^{(\xi_i)} \widetilde{\otimes}_l A^{(\eta_j)}$. We claim that

$$(A^{(\xi_i)} \widetilde{\otimes}_l A^{(\eta_j)})_+ = \left\{ \sum_{i,j \in \mathbb{N}} \alpha_{ij} (e_i \otimes e_j) : \alpha_{ij} \geq 0 \text{ for each } i, j \in \mathbb{N} \right\}. \quad (7.1)$$

Indeed, it is clear that $\alpha_{ij} \geq 0$ for each $i, j \in \mathbb{N}$ implies that $\sum_{i,j \in \mathbb{N}} \alpha_{ij} (e_i \otimes e_j) \geq 0$. Conversely, suppose $\sum_{i,j \in \mathbb{N}} \alpha_{ij} (e_i \otimes e_j) \geq 0$, we wish to show that $\alpha_{ij} \geq 0$ for each $i, j \in \mathbb{N}$. Let $i, j, r, s \in \mathbb{N}$ and assume $j \neq s$. Using the fact that \otimes is a Riesz bimorphism, i.e., $|x \otimes y| = |x| \otimes |y|$ for all $(x, y) \in E \times F$ (cf. [10,15]), we deduce $(e_i \otimes e_j) \wedge (e_r \otimes e_s) = 0$ from the mutual disjointness of $(e_j) \subset A^{(\eta_j)}$. Similarly, if $i \neq r$, then $(e_i \otimes e_j) \wedge (e_r \otimes e_s) = 0$ follows from the mutual disjointness of $(e_i) \subset A^{(\xi_i)}$. Thus, $(e_i \otimes e_j)$ is a mutually disjoint set so that

$$\sum_{i,j \in \mathbb{N}} \alpha_{ij} (e_i \otimes e_j) = \left| \sum_{i,j \in \mathbb{N}} \alpha_{ij} (e_i \otimes e_j) \right| = \sum_{i,j \in \mathbb{N}} |\alpha_{ij}| (e_i \otimes e_j),$$

giving $\alpha_{ij} \geq 0$ for each $i, j \in \mathbb{N}$. This proves (7.1).

By Theorem 4.5 we have that $A^{(\xi_i)}$ and $A^{(\eta_j)}$ are order continuous Banach lattices. Thus, by the theorem of Popa, $A^{(\xi_i)} \widetilde{\otimes}_l A^{(\eta_j)}$ is also an order continuous Banach lattice. It now follows from (7.1) and Corollary 4.6 that $(e_i \otimes e_j)$ is an unconditional basis of $A^{(\xi_i)} \widetilde{\otimes}_l A^{(\eta_j)}$.

To see that the unconditional constant of $(e_i \otimes e_j)$ is one, let $\theta = (\theta_{ij})$ be any choice of signs and $\sum_{i,j \in \mathbb{N}} \alpha_{ij} (e_i \otimes e_j) \in A^{(\xi_i)} \widetilde{\otimes}_l A^{(\eta_j)}$. Then, using the fact that $\|\cdot\|_l$ is a Riesz norm on $A^{(\xi_i)} \widetilde{\otimes}_l A^{(\eta_j)}$, we obtain

$$\begin{aligned} & \left\| M_\theta \left(\sum_{i,j \in \mathbb{N}} \alpha_{ij} (e_i \otimes e_j) \right) \right\|_l \\ &= \left\| \sum_{i,j \in \mathbb{N}} \theta_{ij} \alpha_{ij} (e_i \otimes e_j) \right\|_l = \left\| \sum_{i,j \in \mathbb{N}} |\alpha_{ij}| (e_i \otimes e_j) \right\|_l = \left\| \sum_{i,j \in \mathbb{N}} \alpha_{ij} (e_i \otimes e_j) \right\|_l. \end{aligned}$$

Hence $\{M_\theta\}$ is uniformly bounded by one. This completes the proof. \square

If (ξ_i) is an unconditional K_1 -m.d.s. in a Banach lattice E relative to a positive filtration and (η_j) is an unconditional K_2 -m.d.s. in a Banach space Y , then it does not follow that $(\xi_i \otimes \eta_j)$ is an unconditional K -m.d.s. in $E \widetilde{\otimes}_l Y$. Indeed, let $E = L^p(\mu)$, (χ_i) denote the Haar system in E , $Y = \ell_1$ and (e_j) denote the unit vector basis in Y . Note that (χ_i) is an unconditional m.d.s. in E relative to a positive filtration and (e_j) is an unconditional m.d.s. in Y . Aldous showed that Y is super reflexive if $L^p(\mu, Y)$ possesses an unconditional basis (cf. [1, Theorem 1]). This fact, together with Corollary 6.6, imply that if $(\chi_i \otimes e_j)$ were an unconditional K -m.d.s., then ℓ_1 would be reflexive, which is certainly not the case. In sight of this, we pursue a weaker result.

Theorem 7.2. *Let E be a Banach lattice and Y be a Banach space. Assume $(\xi_i) \subset E$ is an unconditional K_1 -m.d.s., having positive equivalence, relative to a positive K_1 -filtration. Then the following statements hold:*

- If $(\eta_j) \subset Y$ is a sequence, then the sequence $(\xi_i \otimes \eta_j) \subset E \widetilde{\otimes}_l Y$ is strictly dominated by $(e_i \otimes e_j)$ which is a basis of $A^{(\xi_i)} \widetilde{\otimes}_l A^{(\eta_j)}$.*
- If $(\eta_j) \subset Y$ is a K_2 -m.d.s., then the sequence $(\xi_i \otimes \eta_j) \subset E \widetilde{\otimes}_l Y$ with the square ordering is a K -m.d.s. that is strictly dominated by $(e_i \otimes e_j)$ which is a basis of $A^{(\xi_i)} \widetilde{\otimes}_l A^{(\eta_j)}$.*

(c) If $(\eta_j) \subset Y$ is an unconditional K_2 -m.d.s., then the sequence $(\xi_i \otimes \eta_j) \subset E \widetilde{\otimes}_l Y$ with the square ordering is a K -m.d.s. that is strictly dominated by $(e_i \otimes e_j)$ which is an unconditional basis of $A^{(\xi_i)} \widetilde{\otimes}_l A^{(\eta_j)}$.

Proof. Let $(\eta_j) \subset Y$ be a sequence, then Corollary 6.5 implies $(e_i \otimes e_j)$ is a basis of $A^{(\xi_i)} \widetilde{\otimes}_l A^{(\eta_j)}$. Let S denote the co-ordinate map from $A^{(\xi_i)}$ into E and T denote the co-ordinate map from $A^{(\eta_j)}$ into Y . Note that S and T both have norm one. Since (ξ_i) has positive equivalence, Theorem 5.2 implies S is regular. Thus $S = S_1 - S_2$ where S_1 and S_2 are positive maps. Since the l -norm is a left order uniform cross norm, the map

$$S \otimes T : A^{(\xi_i)} \otimes_l A^{(\eta_j)} \rightarrow E \otimes_l Y$$

is bounded because

$$\|S \otimes T\| = \|S_1 \otimes T - S_2 \otimes T\| \leq \|S_1\| \|T\| + \|S_2\| \|T\| = \|S_1\| + \|S_2\|.$$

Thus, the unique continuous extension

$$S \otimes_l T : A^{(\xi_i)} \widetilde{\otimes}_l A^{(\eta_j)} \rightarrow E \widetilde{\otimes}_l Y \quad (7.2)$$

has the properties $(S \otimes_l T)(e_i \otimes e_j) = \xi_i \otimes \eta_j$ for each $i, j \in \mathbb{N}$ and

$$\|S \otimes_l T\| \leq \|S_1\| + \|S_2\| := K_S < \infty. \quad (7.3)$$

This shows that $(e_i \otimes e_j) \succ (\xi_i \otimes \eta_j)$ which proves part (a). For part (b) observe that Proposition 6.4 implies $(\xi_i \otimes \eta_j)$ with the square ordering is a K -m.d.s. in $E \widetilde{\otimes}_l Y$. Part (c) now follows from Proposition 7.1. \square

Corollary 7.3. Let $1 < p < \infty$ and Y be a Banach space. Assume $(d_i) \subset L^p(\mu)$ is a classical m.d.s. Then the following statements hold:

- (a) If $(y_j) \subset Y$ is a sequence, then the sequence $(d_i \otimes y_j) \subset L^p(\mu, Y)$ is strictly dominated by $(e_i \otimes e_j)$ which is a basis of $A^{(d_i)} \widetilde{\otimes}_l A^{(y_j)}$.
- (b) If $(y_j) \subset Y$ is a basic sequence, then the sequence $(d_i \otimes y_j) \subset L^p(\mu, Y)$ with the square ordering is a basic sequence that is strictly dominated by $(e_i \otimes e_j)$ which is a basis of $A^{(d_i)} \widetilde{\otimes}_l A^{(y_j)}$.
- (c) If $(y_j) \subset Y$ is an unconditional basic sequence, then the sequence $(d_i \otimes y_j) \subset L^p(\mu, Y)$ with the square ordering is a basic sequence that is strictly dominated by $(e_i \otimes e_j)$ which is an unconditional basis of $A^{(d_i)} \widetilde{\otimes}_l A^{(y_j)}$.

Proof. Since (d_i) is a classical m.d.s. in $L^p(\mu)$, it follows from the remarks in Section 5 that (d_i) is an unconditional m.d.s. relative to a positive filtration and that (d_i) has positive equivalence. Thus, the result follows directly from the above theorem. \square

Corollary 7.4. Let $1 < p < \infty$ and (d_i) be a classical m.d.s. in $L^p(\mu)$. Then there exists a constant $K > 0$ for which

$$\left\| \sum_{i,j \in \mathbb{N}} (\alpha_{ij} y_j) d_i \right\|_{L^p(\mu, Y)} \leq K \left\| \sum_{i,j \in \mathbb{N}} \|\alpha_{ij} y_j\| d_i \right\|_{L^p(\mu)}$$

holds for any sequence $(y_j) \subset Y$ and any choice of finitely supported scalars (α_{ij}) .

Proof. Let S denote the co-ordinate map from $A^{(d_i)}$ into E and, for an arbitrary sequence $(y_j) \subset Y$, let T denote the co-ordinate map from $A^{(y_j)}$ into Y . As in the previous corollary, the requirements for Theorem 7.2 are satisfied and the map $S \otimes_l T$ given by (7.2) is bounded. Note that $\|S \otimes_l T\|$ is less than some constant $K_S > 0$ given by (7.3) which depends only on the co-ordinate map S . Thus

$$\left\| \sum_{(i,j) \in \mathbb{N}} \alpha_{ij} (d_i \otimes y_j) \right\|_l \leq K_S \left\| \sum_{(i,j) \in \mathbb{N}} \alpha_{ij} (e_i \otimes e_j) \right\|_l \quad (7.4)$$

for every sequence $(y_j) \subset Y$ and every choice of finitely supported scalars (α_{ij}) . The result now follows from calculating the l -norm in the above inequality. Indeed, let $n = \min\{k \in \mathbb{N} : (i, j) \in S_k \forall \alpha_{ij} \neq 0\}$, then we may write $\sum_{(i,j) \in S_n} \alpha_{ij} (d_i \otimes y_j) = \sum_{i=1}^m \sum_{j=1}^m \alpha_{ij} (d_i \otimes y_j)$ where $m = \max(S_n \setminus S_{n-1})$ and $\alpha_{ij} = 0$ for each $(i, j) \in S_{m^2} \setminus S_n$. On the left-hand side of inequality (7.4) we have

$$\begin{aligned} \left\| \sum_{(i,j) \in S_n} \alpha_{ij} (d_i \otimes y_j) \right\|_l &= \left\| \sum_{i=1}^m d_i \otimes \left(\sum_{j=1}^m \alpha_{ij} y_j \right) \right\|_l = \left\| \sum_{i=1}^m \left(\sum_{j=1}^m \alpha_{ij} y_j \right) d_i \right\|_{L^p(\mu, Y)} \\ &= \left\| \sum_{(i,j) \in S_n} (\alpha_{ij} y_j) d_i \right\|_{L^p(\mu, Y)}. \end{aligned}$$

Using the positive mutual disjointness of (e_i) in $A^{(d_i)}$ we obtain from the right-hand side of inequality (7.4)

$$\begin{aligned} &K_S \left\| \sum_{(i,j) \in S_n} \alpha_{ij} (e_i \otimes e_j) \right\|_l \\ &= K_S \left\| \sum_{j=1}^m \left(\sum_{i=1}^m \alpha_{ij} e_i \right) \otimes e_j \right\|_l \leq K_S \left\| \sum_{j=1}^m \left| \sum_{i=1}^m \alpha_{ij} e_i \right| \|e_j\|_{A^{(y_j)}} \right\|_{A^{(d_i)}} \\ &= K_S \left\| \sum_{j=1}^m \left(\sum_{i=1}^m |\alpha_{ij}| e_i \right) \|y_j\| \right\|_{A^{(d_i)}} = K_S \left\| \sum_{i=1}^m \left(\sum_{j=1}^m \|\alpha_{ij} y_j\| \right) e_i \right\|_{A^{(d_i)}} \\ &\leq K_S \|S^{-1}\| \left\| \sum_{i=1}^m \left(\sum_{j=1}^m \|\alpha_{ij} y_j\| \right) d_i \right\|_{L^p(\mu)} = K_S \|S^{-1}\| \left\| \sum_{(i,j) \in S_n} \|\alpha_{ij} y_j\| d_i \right\|_{L^p(\mu)}. \end{aligned}$$

Setting $K = K_S \|S^{-1}\|$ completes the proof. \square

In particular, suppose $1 < p < \infty$ and $(d_i) \subset L^p(\mu)$ is a classical m.d.s. Then the above corollary, together with the unconditionality of (d_i) , imply that there exists a constant $K > 0$ such that

$$\left(\int_{\Omega} \left\| \sum_{i \in \mathbb{N}} \alpha_i y_i d_i(\omega) \right\|^p d\mu(\omega) \right)^{1/p} \leq K \left(\int_{\Omega} \left| \sum_{i \in \mathbb{N}} \alpha_i \|y_i\| d_i(\omega) \right|^p d\mu(\omega) \right)^{1/p}$$

holds for every sequence $(y_i) \subset Y$ and every choice of finitely supported scalars (α_i) .

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