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Characterizations of linear Volterra integral equations with nonnegative kernels

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Abstract

We first introduce the notion of positive linear Volterra integral equations. Then, we offer a criterion for positive equations in terms of the resolvent. In particular, equations with nonnegative kernels are positive. Next, we obtain a variant of the Paley–Wiener theorem for equations of this class and its extension to perturbed equations. Furthermore, we get a Perron–Frobenius type theorem for linear Volterra integral equations with nonnegative kernels. Finally, we give a criterion for positivity of the initial function semigroup of linear Volterra integral equations and provide a necessary and sufficient condition for exponential stability of the semigroups.

Keywords: Linear Volterra integral equation; Positive system; Paley–Wiener theorem; Perron–Frobenius theorem; C_0-semigroup

1. Introduction

Generally speaking, a dynamical system is called *positive* if an input of the system is nonnegative, then the corresponding output of the system is also nonnegative. In particular, a dynamical

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system with state space $\mathbb{R}^n$ is positive if any trajectory of the system starting at an initial state in the positive orthant $\mathbb{R}^n_+$ remains forever in $\mathbb{R}^n_+$. Positive dynamical systems play an important role in the modelling of dynamical phenomena whose variables are restricted to be nonnegative. They are often encountered in applications, for example, networks of reservoirs, industrial processes involving chemical reactors, heat exchangers, distillation columns, storage systems, hierarchical systems, compartmental systems used for modelling transport and accumulation phenomena of substances, see e.g. [3,4,15]. Concrete examples of positive systems are such as an electrical circuit consisting of resistors, capacitors and voltage sources or an electrically heated oven.

The mathematical theory of positive systems is based on the theory of nonnegative matrices founded by Perron and Frobenius. As references we mention [3,4]. Positive systems are objects for many interesting problems in Mathematics, Physics, Economics, Biology, ... Moreover, obtained results of problems for a class of positive systems are often very interesting, see e.g. [3,4,6–11,14,15,18,21–29,33,34]. In recent time, problems of positive systems have attracted a lot of attention from many researchers, see e.g. [2,6–11,21–29,33,34].

In the literature, there are some criteria for familiar positive linear systems such as positive linear invariant-time differential (difference) system, positive linear time delay system of retarded type. For example, it is well known that a linear time-delay system of the form $\dot{x}(t) = A_0x(t) + A_1x(t-h)$, $t \geqslant 0$, is positive if and only if A_0 is a Metzler matrix and A_1 is a nonnegative matrix and a linear discrete system of the form $x(k+1) = A_0x(k) + A_1x(k-h)$, $k \in \mathbb{N}$, $k \geqslant h$, is positive if and only if A_0, A_1 are nonnegative matrices, see e.g. [21,22,33]. In the recent paper [19], we showed that a linear Volterra integro-differential equation of the convolution type

$$\dot{x}(t) = Ax(t) + \int_0^t B(t-s)x(s)\,ds, \quad x(t) \in \mathbb{R}^n,\ t \geqslant 0, \tag{1}$$

is positive if and only if A is a Metzler matrix and $B(t)$ is a nonnegative matrix for every $t \geqslant 0$. Furthermore, stability and robust stability of the system (1) have been explored in the paper.

In the present paper, we first introduce the notion of positive linear Volterra integral equations. Then, we offer a criterion for positive equations in terms of the resolvent. As a direct consequence of this result, equations with nonnegative kernels are positive. Next, we obtain a variant of the Paley–Wiener theorem for equations of this class and its extension to perturbed equations. Moreover, we get a Perron–Frobenius type theorem for linear Volterra integral equations with nonnegative kernels. Finally, we give a criterion for positivity of the initial function semigroup of linear Volterra integral equations and offer a simple necessary and sufficient condition for exponential stability of the semigroups.

2. Preliminaries

In this section we shall define some notations and recall some well-known results which will be used in the subsequent sections. Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ where $\mathbb{C}$ and $\mathbb{R}$ denote the sets of all complex and all real numbers, respectively. For integers $l, q \geqslant 1$, $\mathbb{K}^l$ denotes the l-dimensional vector space over $\mathbb{K}$, $(\mathbb{K}^l)^*$ is its dual and $\mathbb{K}^{l\times q}$ stands for the set of all $(l \times q)$-matrices with entries in $\mathbb{K}$. Inequalities between real matrices or vectors will be understood componentwise, i.e. for two real matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{R}^{l\times q}$, we write $A \geqslant B$ if and only if $a_{ij} \geqslant b_{ij}$ for $i = 1, \ldots, l$, $j = 1, \ldots, q$. In particular, if $a_{ij} > b_{ij}$ for $i = 1, \ldots, l$, $j = 1, \ldots, q$, then we write $A \gg B$ instead of $A \geqslant B$. We denote by $\mathbb{R}^{l\times q}_+$ the set of all nonnegative matrices $A \geqslant 0$. Similar notations are adopted for vectors. For $x \in \mathbb{K}^n$ and $P \in \mathbb{K}^{l\times q}$ we define

$|x| = (|x_i|)$ and $|P| = (|p_{ij}|)$. For any matrix $A \in \mathbb{K}^{n\times n}$ the *spectral radius* and *spectral abscissa* of A are denoted by $\rho(A) = \max\{|\lambda|;\ \lambda \in \sigma(A)\}$ and $\mu(A) = \max\{\operatorname{Re}\lambda;\ \lambda \in \sigma(A)\}$, where $\sigma(A) := \{s \in \mathbb{C}; \det(sI_n - A) = 0\}$ is the spectrum of A.

A norm $\|\cdot\|$ on $\mathbb{K}^n$ is said to be *monotonic* if $\|x\| \leqslant \|y\|$ whenever $|x| \leqslant |y|, x, y \in \mathbb{K}^n$. Every p-norm on $\mathbb{K}^n$, $1 \leqslant p \leqslant \infty$, is monotonic. Throughout the paper, if otherwise not stated, the norm of a matrix $P \in \mathbb{K}^{l\times q}$ is understood as its operator norm associated with a given pair of monotonic vector norms on $\mathbb{K}^l$ and $\mathbb{K}^q$, that is $\|P\| = \max\{\|Py\|;\ \|y\| = 1\}$. We note that the operator norm is in general not monotonic norm on $\mathbb{K}^{l\times q}$ even if $\mathbb{K}^l, \mathbb{K}^q$ are provided with monotonic norms. However, such monotonicity holds for nonnegative matrices. Moreover, we have (see, e.g. [12,32])

$$P \in \mathbb{K}^{l\times q},\ Q \in \mathbb{R}_+^{l\times q},\ |P| \leqslant Q \quad \Rightarrow \quad \|P\| \leqslant \big\||P|\big\| \leqslant \|Q\|. \tag{2}$$

The following theorem summarizes some existing results on properties of nonnegative matrices which will be used in the sequel (see, e.g. [32]).

Theorem 2.1. *Suppose that $A \in \mathbb{R}^{n\times n}$ is a nonnegative matrix. Then*

(i) (*Perron–Frobenius*) $\rho(A)$ *is an eigenvalue of A and there exists a nonnegative eigenvector $x \geqslant 0$, $x \neq 0$ such that $Ax = \rho(A)x$.*
(ii) *Given $\alpha \in \mathbb{R}_+$, there exists a nonzero vector $x \geqslant 0$ such that $Ax \geqslant \alpha x$ if and only if $\rho(A) \geqslant \alpha$.*
(iii) $(tI_n - A)^{-1}$ *exists and is nonnegative if and only if $t > \rho(A)$.*
(iv) *Given $B \in \mathbb{R}_+^{n\times n}$, $C \in \mathbb{C}^{n\times n}$. Then*

$$|C| \leqslant B \quad \Rightarrow \quad \rho(A + C) \leqslant \rho(A + B). \tag{3}$$

To make a presentation self-contained, we present here some basic facts on vector functions of bounded variation and relative knowledge. A matrix function $\eta(\cdot) : [\alpha, \beta] \to \mathbb{R}^{l\times q}$ is called an increasing matrix function, if

$$\eta(\theta_2) \geqslant \eta(\theta_1) \quad \text{for } \alpha \leqslant \theta_1 \leqslant \theta_2 \leqslant \beta.$$

A matrix function $\eta(\cdot) : [\alpha, \beta] \to \mathbb{K}^{m\times n}$ is said to be of bounded variation if

$$\operatorname{Var}(\eta; \alpha, \beta) := \sup_{P[\alpha,\beta]} \sum_k \big\|\eta(\theta_k) - \eta(\theta_{k-1})\big\| < +\infty, \tag{4}$$

where the supremum is taken over the set of all finite partitions of the interval $[\alpha, \beta]$. The set $BV([\alpha, \beta], \mathbb{K}^{m\times n})$ of all matrix functions $\eta(\cdot)$ of bounded variation on $[\alpha, \beta]$ satisfying $\eta(\alpha) = 0$ is a Banach space endowed with the norm $\|\eta\| = \operatorname{Var}(\eta; \alpha, \beta)$.

Given $\eta(\cdot) \in BV([\alpha, \beta], \mathbb{K}^{m\times n})$ then for any continuous functions $\gamma \in C([\alpha, \beta], \mathbb{K})$ and $\phi \in C([\alpha, \beta], \mathbb{K}^n)$, the integrals

$$\int_\alpha^\beta \gamma(\theta)\, d\big[\eta(\theta)\big] \quad \text{and} \quad \int_\alpha^\beta d\big[\eta(\theta)\big]\, \phi(\theta)$$

exist and are defined respectively as the limits of $S_1(P) := \sum_{k=1}^p \gamma(\zeta_k)(\eta(\theta_k) - \eta(\theta_{k-1}))$ and $S_2(P) := \sum_{k=1}^p (\eta(\theta_k) - \eta(\theta_{k-1}))\phi(\zeta_k)$ as $d(P) := \max_k |\theta_k - \theta_{k-1}| \to 0$, where $P = \{\theta_1 = \alpha \leqslant \theta_2 \leqslant \cdots \leqslant \theta_p = \beta\}$ is any finite partition of the interval $[\alpha, \beta]$ and $\zeta_k \in [\theta_{k-1}, \theta_k]$. Let $\mathbb{K}^n$

be endowed with a vector norm $\|\cdot\|$ and $C([\alpha,\beta],\mathbb{K}^n)$ be a Banach space of all continuous functions on $[\alpha,\beta]$ with values in $\mathbb{K}^n$ normed by the maximum norm $\|\phi\| = \max_{\theta\in[\alpha,\beta]}\|\phi(\theta)\|$.

Let $L : C([\alpha,\beta],\mathbb{K}^n) \to \mathbb{K}^n$ be a linear bounded operator. Then, by the Riesz representation theorem, there exists a unique matrix function $\eta = (\eta_{ij}(\cdot)) \in BV([\alpha,\beta],\mathbb{K}^{n\times n})$ which is *continuous from the left* (or briefly c.f.l.) on (α,β) such that

$$L\phi = \int_\alpha^\beta d\big[\eta(\theta)\big]\phi(\theta), \quad \forall\phi \in C\big([\alpha,\beta],\mathbb{K}^n\big). \tag{5}$$

Let J be an interval of $\mathbb{R}$. For a function $\phi : J \to \mathbb{R}^{l\times q}$, we say that ϕ is nonnegative and write $\phi \geqslant 0$ if $\phi(t) \in \mathbb{R}_+^{l\times q}$ almost everywhere on J. Let X be a subspace of $C([\alpha,\beta];\mathbb{R}^n)$. Then the operator L is called positive on X if $L\phi \geqslant 0$ for every $\phi \in X$, $\phi \geqslant 0$.

In the subsequent sections the following subspace of $BV([\alpha,\beta],\mathbb{K}^{m\times n})$ will be used:

$$NBV\big([\alpha,\beta],\mathbb{K}^{m\times n}\big) := \big\{\eta \in BV\big([\alpha,\beta],\mathbb{K}^{m\times n}\big);\ \eta(\alpha) = 0,\ \eta \text{ is c.f.l. on } [\alpha,\beta]\big\}. \tag{6}$$

It is clear that $NBV([\alpha,\beta],\mathbb{K}^{m\times n})$ is closed in $BV([\alpha,\beta],\mathbb{K}^{m\times n})$ and thus it is a Banach space with the norm $\|\delta\| = \mathrm{Var}(\delta;\alpha,\beta)$.

3. Positive linear Volterra integral equations

Consider a linear Volterra integral equation of the convolution type

$$x(t) = f(t) + \int_0^t K(t-s)x(s)\,ds, \quad t \geqslant 0, \tag{7}$$

where $f : \mathbb{R}_+ \to \mathbb{R}^n$ and $K : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ are given (vector) matrix functions. We say that K is the kernel and that f is the forcing function and we assume that these functions are (at least) locally integrable. That is, $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^n)$ and $K \in L^1_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^{n\times n})$.

It is important to note that a comprehensive theory of linear Volterra integral equations can be found in [5]. In particular, to prove that Eq. (7) has a solution in $L^1_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^n)$ whenever $K \in L^1_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^{n\times n})$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^n)$, one first shows that it has a fundamental solution R.

Let a, b be matrix functions defined on an interval $[0,T)$, $0 < T \leqslant +\infty$. The convolution of a and b on the interval $[0,T)$ is, by definition, the function

$$(a * b)(t) = \int_0^t a(t-s)b(s)\,ds, \tag{8}$$

which is defined for all those $t \in [0,T)$ for which the integral exists. Here the integrals are interpreted in the sense of Lebesgue.

Theorem 3.1. *(See [5].) Let $K \in L^1_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^{n\times n})$ be given. Then there is a solution $R \in L^1_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^{n\times n})$ of the two equations*

$$R = K + R * K = K + K * R. \tag{9}$$

This solution R is unique and depends continuously on K in the topology of $L^1_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^{n\times n})$.

The matrix function R given in Theorem 3.1 is called the resolvent of K. Then, existence and uniqueness of the solution of Eq. (7) is given in the following.

Theorem 3.2. *(See [5].) Let $K \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{n\times n})$. Then, for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, there exists a unique solution $x(\cdot, f) \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ of Eq.* (7)*. This solution is given by the variation of constants formula*

$$x(t, f) = f(t) + (R * f)(t), \quad t \geqslant 0, \tag{10}$$

where R is the resolvent of K.

Definition 3.3. Equation (7) is called positive if for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ being nonnegative, the corresponding solution $x(\cdot, f)$ is also nonnegative.

The following theorem provides a criterion for positive equations of the form (7).

Theorem 3.4. *Equation* (7) *is positive if and only if R is nonnegative.*

To prove the above theorem, we need the following technical lemma.

Lemma 3.5. *Let $T > 0$ and $C_0([0, T], \mathbb{R}^n) := \{\phi \in C([0, T], \mathbb{R}^n)\colon\ \phi(T) = 0\}$. Suppose that the linear operator L is defined by*

$$L : C_0\big([0, T], \mathbb{R}^n\big) \to \mathbb{R}^n; \quad \phi \mapsto L\phi = \int_0^T d\big[\eta(\theta)\big]\phi(\theta),$$

where $\eta \in NBV([0, T], \mathbb{R}^{n\times n})$ is given. Then L is a positive operator if and only if η is an increasing matrix function.

Proof. Let η be an increasing matrix function then by the definition of Riemann–Stieltjes integrals, we have

$$L\phi = \lim_{d(P)\to 0} \sum_{k=1}^{p} \big(\eta(\theta_k) - \eta(\theta_{k-1})\big)\phi(\zeta_k) \geqslant 0,$$

for every $\phi \in C_0([0, T], \mathbb{R}^n)$, $\phi \geqslant 0$. It means that L is positive.

Conversely, assume that L is positive on $C_0([0, T], \mathbb{R}^n)$. Let $\eta(\cdot) = (\eta_{ij}(\cdot))$. We show that $\eta_{ij}(\cdot) \in NBV([0, T], \mathbb{R})$ is an increasing scalar function for every $i, j \in \{1, 2, \ldots, n\}$. Since L is positive, it is easy to see that the operator

$$L_{ij} : C_0\big([0, T], \mathbb{R}\big) \to \mathbb{R}; \quad \phi \mapsto L_{ij}\phi := \int_0^T \phi(\theta)\, d\big[\eta_{ij}(\theta)\big],$$

is also positive for every $i, j \in \{1, 2, \dots, n\}$. Fix $\theta_1, \theta_2 \in (0, T)$, $\theta_1 < \theta_2$ and $k \in \mathbb{N}$, $k > \max\{\frac{1}{\theta_1}, \frac{1}{\theta_2}, \frac{1}{\theta_2 - \theta_1}\}$, consider the continuous function ϕ_k defined by

$$\phi_k(\theta) := \begin{cases} 0 & \text{if } \theta \in [0, \theta_1 - \frac{1}{k}], \\ k\theta + 1 - k\theta_1 & \text{if } \theta \in (\theta_1 - \frac{1}{k}, \theta_1], \\ 1 & \text{if } \theta \in (\theta_1, \theta_2 - \frac{1}{k}], \\ -k\theta + k\theta_2 & \text{if } \theta \in (\theta_2 - \frac{1}{k}, \theta_2], \\ 0 & \text{if } \theta \in (\theta_2, T]. \end{cases} \tag{11}$$

Since ϕ_k is a continuous on $[0, T]$, it follows from a standard property of Riemann–Stieltjes integrals that

$$\int_0^T \phi_k(\theta)\, d\big[\eta_{ij}(\theta)\big] = \left(\int_0^{\theta_1 - \frac{1}{k}} + \int_{\theta_1 - \frac{1}{k}}^{\theta_1} + \int_{\theta_1}^{\theta_2 - \frac{1}{k}} + \int_{\theta_2 - \frac{1}{k}}^{\theta_2} + \int_{\theta_2}^{T} \right) \phi_k(\theta)\, d\big[\eta_{ij}(\theta)\big],$$

see e.g. [30]. This gives

$$\int_{\theta_1 - \frac{1}{k}}^{\theta_1} \phi_k(\theta)\, d\big[\eta_{ij}(\theta)\big] + \eta_{ij}\left(\theta_2 - \frac{1}{k}\right) - \eta_{ij}(\theta_1) + \int_{\theta_2 - \frac{1}{k}}^{\theta_2} \phi_k(\theta)\, d\big[\eta_{ij}(\theta)\big] \geqslant 0,$$

for every $k \in \mathbb{N}$ large enough. Taking into account that η_{ij} is continuous from the left at θ_1, θ_2 and letting $k \to +\infty$, we have $\eta_{ij}(\theta_2) \geqslant \eta_{ij}(\theta_1)$ for every $\theta_1, \theta_2 \in (0, T)$, $\theta_2 \geqslant \theta_1$. In case of $\theta_1 = 0 < \theta_2 < T$, by a similar way, we also get $\eta_{ij}(\theta_2) \geqslant \eta_{ij}(\theta_1)$. Finally, since η_{ij} is continuous from the left at T, we have $\eta_{ij}(T) \geqslant \eta_{ij}(\theta)$ for $\theta \in [0, T]$. This completes our proof. □

Remark 3.6. Actually, from the proof of Lemma 3.5, it is easy to see that the conclusion of the lemma holds true for $C_0([\alpha, \beta], \mathbb{R}^n)$, with arbitrary interval $[\alpha, \beta]$. Moreover, the following statements are equivalent:

(i) L is a positive operator on $C_0([\alpha, \beta], \mathbb{R}^n)$;
(ii) L is a positive operator on $C([\alpha, \beta], \mathbb{R}^n)$;
(iii) η is an increasing matrix function on $[\alpha, \beta]$.

Proof of Theorem 3.4. The part "if" of the theorem follows from the variation of constants formula (10). We now prove that if Eq. (7) is positive then R is nonnegative. To do so, we fix $k \in \mathbb{N}$ and consider an arbitrary function $\phi \in C_0([0, k], \mathbb{R}^n)$. Let us define

$$f(t) = \begin{cases} \phi(t) & \text{if } t \in [0, k], \\ 0 & \text{if } t > k. \end{cases}$$

Let R be the resolvent of K. From $f \in C([0, k+1], \mathbb{R}^n)$ and $R \in L^1([0, k+1], \mathbb{R}^n)$, it follows that $R * f$ is continuous on $[0, k+1]$. By the variation of constants formula (10), so is the solution $x(\cdot, f)$. Assume that $\phi \geqslant 0$ and we thus get $f \geqslant 0$. Since Eq. (7) is positive, it follows that

$$x(k, f) = f(k) + \int_0^k R(k-s) f(s)\, ds = \int_0^k R(k-s)\phi(s)\, ds \geqslant 0,$$

by the variation of constants formula. Thus, the linear operator defined by

$$L : C_0\big([0,k],\mathbb{R}^n\big) \to \mathbb{R}^n; \quad \phi \mapsto L\phi := \int_0^k R(k-s)\phi(s)\,ds,$$

is a positive operator. Applying Lemma 3.5 to the positive operator L, we conclude that the function

$$\eta(t) = \int_0^t R(k-s)\,ds, \quad t \in [0,k],$$

is an increasing matrix function. From $R \in L^1([0,k],\mathbb{R}^n)$, it follows that $R(t) \geqslant 0$ almost everywhere on $[0,k]$. Since $k \in \mathbb{N}$ is arbitrary, it follows that $R(t) \geqslant 0$ almost everywhere on R_+. This completes our proof. □

Corollary 3.7. *If K is nonnegative, then Eq.* (7) *is positive.*

Proof. It is sufficient to show that the resolvent R of K is nonnegative, provided K is nonnegative. The argument below is based on that of the proof of Theorem 3.1 of [5, p. 43].

Suppose for a moment that $\int_0^T \|K(s)\|\,ds < 1$ for some $T > 0$. Consider the sequence defined by $R_k := \sum_{i=1}^k K^{*i}$, $k \in \mathbb{N}$, where K^{*i} is the $(i-1)$-fold convolution of K by itself. Since $\|K^{*i}\|_{L^1([0,T],\mathbb{R}^n)} \leqslant \|K\|^i_{L^1([0,T],\mathbb{R}^n)}$, it follows that $(R_k)_k$ is a Cauchy sequence in $L^1([0,T],\mathbb{R}^n)$. Thus, $(R_k)_k$ converges to R_T in $L^1([0,T],\mathbb{R}^n)$. Moreover, each $R_k, k \geqslant 2$, satisfies

$$R_k = K + K * R_{k-1} = K + R_{k-1} * K.$$

Letting $k \to +\infty$, we get $R_T = K + K * R_T = K + R_T * K$. By the uniqueness, $R_T = R$ on $[0,T]$. Furthermore, there exists a subsequence of $(R_k)_k$ such that it converges pointwise to R almost everywhere on $[0,T]$. Since R_k is nonnegative almost everywhere on $[0,T]$ for every $k \in \mathbb{N}$, so is R. Now let $T > 0$ be arbitrary. Choosing $\sigma > 0$ large enough, we can assume that $K_1(t) = e^{-\sigma t}K(t), t \in [0,T]$ satisfies that $\int_0^T \|K_1(s)\|\,ds < 1$. Let R_1 be the solution of the equations $R_1 = K_1 + R_1 * K_1 = K_1 + K_1 * R_1$ in $L^1([0,T],\mathbb{R}^n)$. Then $R(t) = e^{\sigma t}R_1(t)$, $t \in [0,T]$, is the solution of the equations $R = K + R * K = K + K * R$. From the result of the above, we conclude that R_1 is nonnegative almost everywhere on $[0,T]$. So is R. The conclusion of the corollary easily follows from this fact. □

Remark 3.8. It is worth noticing that the converse of the above corollary is not true. To see this, let us consider the function $K(t) := e^{-t}\sin t$, $t \in \mathbb{R}_+$. Then, it is easy to check that the resolvent of K is given by $R(t) = te^{-t}$, $t \in \mathbb{R}_+$. Since R is nonnegative, Eq. (7) is positive. However, K is not nonnegative.

4. Paley–Wiener theorem revised for linear Volterra integral equations with nonnegative kernels

We now deal with asymptotic behavior of the solution of Eq. (7). It is well known that under the hypothesis, $K \in L^1(\mathbb{R}_+,\mathbb{R}^{n\times n})$, the asymptotic behavior of the solution of Eq. (7) is determined by the kernel K through a famous theorem of the Paley–Wiener.

Let $h:\mathbb{R}_+ \to \mathbb{R}$. Then the Laplace transform of h is formally defined to be

$$\hat{h}(s) := \int_0^{+\infty} e^{-st} h(t)\,dt.$$

If $\beta \in \mathbb{R}$ and $\int_0^{+\infty} e^{-\beta t}|h(t)|\,dt < +\infty$, then $\hat{h}(s)$ exists for $s \in \mathbb{C}$, $\Re s \geqslant \beta$. Furthermore, $\hat{h}(s)$ is an analytic function in the domain $\{s \in \mathbb{C}\colon\ \Re s > \beta\}$. If $D(t) = (d_{ij}(t))$ is a matrix function then we define

$$\widehat{D} := (\hat{d}_{ij}).$$

Theorem 4.1. *(See [5].) Let $K \in L^1(\mathbb{R}+, \mathbb{R}^{n\times n})$ and let V be one of the following spaces: $L^p(\mathbb{R}_+, \mathbb{R}^n)$, $p \in [1, +\infty]$; $BC(\mathbb{R}_+, \mathbb{R}^n)$; $BUC(\mathbb{R}_+, \mathbb{R}^n)$; $BV(\mathbb{R}_+, \mathbb{R}^n)$. Then, the following statements are equivalent:*

(i) *the resolvent R of K belongs to $L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$;*
(ii) $\det(I_n - \widehat{K}(z)) \neq 0$, *for all $z \in \mathbb{C}$, $\Re z \geqslant 0$;*
(iii) *for every $f \in V$, the solution $x(\cdot, f)$ of* (7) *belongs to V and x depends continuously on f in the norm of V.*

Remark 4.2. Actually, the space V stated in the above theorem can be taken in a large class of normed spaces. For further information, see Theorem 4.5 of [5, p. 47].

Combining Theorems 2.1 and 4.1, we get a variant of the Paley–Wiener theorem for linear Volterra integral equations with nonnegative kernels.

Theorem 4.3. *Let $K \in L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$ and let V be one of the spaces stated in Theorem* 4.1. *If $K \geqslant 0$, then the following statements are equivalent:*

(i) *the resolvent R of K belongs to $L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$;*
(ii) $\rho(\widehat{K}(0)) < 1$;
(iii) *for every $f \in V$, the solution $x(\cdot, f)$ of* (7) *belongs to V and x depends continuously on f in the norm of V;*
(iv) *for any vector $b \in \mathbb{R}^n$, $b \gg 0$, the solution of the equation $x = b + K * x$ satisfies*

$$\lim_{t\to+\infty} x(t) =: x^* \gg 0;$$

(v) *for some vector $b \in \mathbb{R}^n$, $b \gg 0$, the solution of the equation $x = b + K * x$ satisfies*

$$\lim_{t\to+\infty} x(t) =: x^* \gg 0.$$

Proof. To prove (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii), it is sufficient to show that

$$\det\bigl(I_n - \widehat{K}(z)\bigr) \neq 0, \quad \text{for all } z \in \mathbb{C},\ \Re z \geqslant 0,$$

if and only if $\rho(\widehat{K}(0)) < 1$. Assume that $\det(I_n - \widehat{K}(z)) = 0$, for some $z \in \mathbb{C}$, $\Re z \geqslant 0$. This implies that $1 \leqslant \rho(\widehat{K}(z))$. From $K \geqslant 0$, it follows that $|e^{-zs}K(s)| \leqslant K(s)$ almost everywhere on $[0, +\infty)$. Using Theorem 2.1(iv), we get $1 \leqslant \rho(\widehat{K}(z)) \leqslant \rho(\widehat{K}(0))$. Conversely, suppose that

$\det(I_n - \widehat{K}(z)) \neq 0$, for all $z \in \mathbb{C}$, $\Re z \geqslant 0$. We show that $\rho(\widehat{K}(0)) < 1$. Assume contrary that $\rho(\widehat{K}(0)) \geqslant 1$. Consider the following real function:

$$g(t) := 1 - \rho\big(\widehat{K}(t)\big) = 1 - \rho\left(\int_0^{+\infty} e^{-ts} K(s)\,ds\right), \quad t \in [0, +\infty). \tag{12}$$

By $K \in L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$, the function g is well defined in $[0, +\infty)$. Moreover, since the spectral radius is continuous in the matrix space and $\widehat{K}$ is continuous in t on $[0, +\infty)$, it follows that g is continuous in t on $[0, +\infty)$. Moreover, by the assumption, we have $g(0) \leqslant 0$ and $\lim_{t\to+\infty} g(t) = 1$. It implies that $g(t_0) = 0$, for some $t_0 \geqslant 0$. This gives $1 = \rho(\widehat{K}(t_0))$. Since $\widehat{K}(t_0)$ is a nonnegative matrix, we get $\det(I_n - \widehat{K}(t_0)) = 0$, by Theorem 2.1(i). However, this conflicts with our assumption.

We now prove that (i) $\Rightarrow$ (iv). By the variation of constants formula, we have $x(t) = b + \int_0^t R(t-s)b\,ds$. Thus, $\lim_{t\to+\infty} x(t) = b + (\int_0^{+\infty} R(t)\,dt)b =: x^*$. Since $K \geqslant 0$, it implies that $R(t) \geqslant 0$, by Corollary 3.7. This results that $x^* \gg 0$.

To complete the proof, we show that (v) $\Rightarrow$ (i). Let $R(t) = (R_{ij}(t)) \in \mathbb{R}^{n\times n}$, $t \geqslant 0$, and $b^T = (b_1, b_2, \dots, b_n)$. Since $\lim_{t\to+\infty} x(t) = x^*$, it follows that $\lim_{t\to+\infty} \int_0^t R(s)b\,ds = x^* - b$. Taking into account that $R_{ij} \geqslant 0$, and $b_j > 0$, $\forall i, j \in \{1, 2, \dots, n\}$, it implies that

$$\lim_{t\to+\infty} \int_0^t R_{ij}(s) b_j\,ds < +\infty,$$

for every $i, j \in \{1, 2, \dots, n\}$. We thus get

$$\int_0^{+\infty} R_{ij}(s)\,ds = \lim_{t\to+\infty} \int_0^t R_{ij}(s)\,ds < +\infty,$$

for every $i, j \in \{1, 2, \dots, n\}$. This means that $R \in L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$. $\square$

Furthermore, we give here an extension of Theorem 4.3 to perturbed equations.

Theorem 4.4. *Let $K \in L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$. Suppose that $K \geqslant 0$ and $\rho(\widehat{K}(0)) < 1$. Then, for every $\Delta \in L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$ satisfying,*

$$\|\Delta\| < \frac{1}{\|(I_n - \widehat{K}(0))^{-1}\|},$$

the resolvent of $K + \Delta$ still belongs to $L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$.

Proof. Assume that the resolvent of $K + \Delta$ does not belong to $L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$ for some $\Delta \in L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$. By Theorem 4.1, there exist a complex number z_0, $\Re z_0 \geqslant 0$, and a vector $x_0 \in \mathbb{C}^n$, $x_0 \neq 0$, such that

$$\big(I_n - \widehat{K}(z_0) - \widehat{\Delta}(z_0)\big)x_0 = 0. \tag{13}$$

Since $K \geqslant 0$ and $\rho(\widehat{K}(0)) < 1$, Theorems 4.1 and 4.3 show that $I_n - \widehat{K}(z)$ is invertible for every $z \in \mathbb{C}$, $\Re z \geqslant 0$. Hence, it follows from (13) that

$$\big(I_n - \widehat{K}(z_0)\big)^{-1} \widehat{\Delta}(z_0) x_0 = x_0.$$

This implies $\|(I_n - \widehat{K}(z_0))^{-1}\| \|\widehat{\Delta}(z_0)\| \geqslant 1$. From $\Re z_0 \geqslant 0$, it is clear that

$$\|\widehat{\Delta}(z_0)\| = \left\| \int_0^{+\infty} e^{-z_0 s} \Delta(s)\, ds \right\| \leqslant \int_0^{+\infty} \|\Delta(s)\|\, ds = \|\Delta\|.$$

We thus get $\|(I_n - \widehat{K}(z_0))^{-1}\| \|\Delta\| \geqslant 1$, or equivalently,

$$\|\Delta\| \geqslant \frac{1}{\|(I_n - \widehat{K}(z_0))^{-1}\|}.$$

To end the proof, it is sufficient to show that

$$\left\|\left(I_n - \widehat{K}(z_0)\right)^{-1}\right\| \leqslant \left\|\left(I_n - \widehat{K}(0)\right)^{-1}\right\|. \tag{14}$$

In fact, from $\Re z_0 \geqslant 0$ and $K \geqslant 0$, it follows that $|\widehat{K}(z_0)| \leqslant \widehat{K}(0)$. By Theorem 2.1(iv), we have $\rho(\widehat{K}(z_0)) \leqslant \rho(\widehat{K}(0)) < 1$. So, we can rewrite $(I_n - K(z_0))^{-1}$ as

$$\left(I_n - \widehat{K}(z_0)\right)^{-1} = \sum_{k=0}^{+\infty} \widehat{K}(z_0)^k.$$

Therefore,

$$\left|\left(I_n - \widehat{K}(z_0)\right)^{-1}\right| \leqslant \sum_{k=0}^{+\infty} \widehat{K}(0)^k = \left(I_n - \widehat{K}(0)\right)^{-1}.$$

By the property of operator norm (2), this results (14) which ends our proof. □

Remark 4.5. By using an argument used in many our papers on stability radius of positive systems (see e.g., [11,24,25]), we can show that $\gamma_0 := \frac{1}{\|(I_n - \widehat{K}(0))^{-1}\|}$ is the largest number of positive numbers γ such that the resolvent of $K + \Delta$ belongs to $L^1(\mathbb{R}+, \mathbb{R}^{n\times n})$ for all $\Delta \in L^1(\mathbb{R}+, \mathbb{R}^{n\times n})$ satisfying $\|\Delta\| < \gamma$.

5. A Perron–Frobenius type theorem for linear Volterra integral equations with nonnegative kernels

It is well known that the classical Perron–Frobenius theorem and its extensions are principal tools for analysis of stability and robust stability of positive linear time-invariant systems, see e.g. [3,4,6–11,23–29,33,34]. To our knowledge, there is a large number of extensions of the classical Perron–Frobenius theorem, see e.g. [1,13,23,26,27,31] and references therein. Recently, we gave some extensions of the classical Perron–Frobenius theorem to positive linear time-delay systems (see [23,27]), to positive linear functional equations (see [26,28]) and to positive linear Volterra equations (see [19,20]). Furthermore, the obtained results are used to analyze stability and robust stability of the corresponding systems.

In this section, we give a Perron–Frobenius type theorem for linear Volterra integral equations with nonnegative kernels. The obtained results are used to study the asymptotic behavior of solutions of linear integral Volterra equations in Sections 3 and 6.

Let us define

$$Q(z) := \left(I_n - \widehat{K}(z)\right),$$

for appropriate $z \in \mathbb{C}$. Set

$$\mu(K) := \sup\left\{\Re z\colon\ z \in \mathbb{C},\ \int_0^{+\infty} e^{-\Re z t}\left\|K(t)\right\|dt < +\infty,\ \det Q(z) = 0\right\}, \tag{15}$$

where we define $\mu(K) = -\infty$ if $\det Q(z) \neq 0$, for every $z \in \mathbb{C}$ with $\int_0^{+\infty} e^{-\Re z t}\|K(t)\|\,dt < +\infty$. Then, $\mu(K)$ is called spectral abscissa of Eq. (7).

Theorem 5.1. *Let $K \geqslant 0$ and suppose that*

$$\beta := \inf\left\{\gamma \in \mathbb{R}\colon\ \int_0^{+\infty} e^{-\gamma t}\left\|K(t)\right\|dt < +\infty\right\} < +\infty$$

and $\alpha \in [\beta, +\infty)$. If $\mu(K) > -\infty$, then

(i) (*Perron–Frobenius theorem for linear Volterra integral equations with nonnegative kernels*) *$\mu(K)$ is a root of the characteristic equation, that is* $\det Q(\mu(K)) = 0$. *Moreover, there exists a nonzero vector $x \in \mathbb{R}^n$, $x \geqslant 0$, such that*

$$\widehat{K}\big(\mu(K)\big)x = x.$$

(ii) *There exists a nonzero vector $x \in \mathbb{R}^n$, $x \geqslant 0$, such that $\widehat{K}(\alpha)x \geqslant x$ if and only if $\mu(K) \geqslant \alpha$. Here, we assume that $\widehat{K}(\alpha)$ exists if $\alpha = \beta$.*

(iii) *$Q(\alpha)^{-1} = (I_n - \widehat{K}(\alpha))^{-1}$ exists and is nonnegative if and only if $\alpha > \mu(K)$.*

Proof. First, we show that $\mu(K) < +\infty$. In fact, if $\det Q(z) = 0$, for some $z \in \mathbb{C}$, then $1 \leqslant \rho(\int_0^{+\infty} e^{-zt}K(t)\,dt)$. Since $K \geqslant 0$, using Theorem 2.1(iv), we get $\rho(\int_0^{+\infty} e^{-zt}K(t)\,dt) \leqslant \rho(\int_0^{+\infty} e^{-\Re z t}K(t)\,dt) = \rho(\widehat{K}(\Re z))$, for $z \in \mathbb{C}$, $\Re z > \max\{0, \beta\}$. On the other hand, it follows from $e^{-(\beta+1)s}K(s) \in L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$ that $\widehat{K}(\Re z) \to 0$ as $\Re z \to +\infty$. By the continuity of the spectral radius in matrix space, we have $\rho(\widehat{K}(\Re z)) < 1$, for $z \in \mathbb{C}$, $\Re z > 0$ large enough. Therefore, $\det Q(z) \neq 0$, for $z \in \mathbb{C}$, $\Re z > 0$ large enough.

Next, by the assumption, there exists $z_0 \in \mathbb{C}$ such that $\det Q(z_0) = 0$. This implies that $\beta \leqslant \Re z_0 \leqslant \mu(K)$. If $\Re z_0 = \mu(K)$ then taking into account $K \geqslant 0$, Theorem 2.1(iv) gives

$$1 \leqslant \rho\left(\int_0^{+\infty} e^{-z_0 s}K(s)\,ds\right) \leqslant \rho\left(\int_0^{+\infty} e^{-\Re z_0 s}K(s)\,ds\right) = \rho\left(\int_0^{+\infty} e^{-\mu(K)s}K(s)\,ds\right).$$

If $\beta \leqslant \Re z_0 < \mu(K)$, then there exists a sequence of complex numbers $(z_k)_k$ such that $\det Q(z_k) = 0$, $\beta < \Re z_k \leqslant \mu(K)$, $k \in \mathbb{N}$ and $\Re z_k \to \mu(K)$ as $k \to +\infty$. By Theorem 2.1(iv)

$$1 \leqslant \rho\left(\int_0^{+\infty} e^{-z_k s}K(s)\,ds\right) \leqslant \rho\left(\int_0^{+\infty} e^{-\Re z_k s}K(s)\,ds\right). \tag{16}$$

Letting $k \to +\infty$ in (16), we get $1 \leqslant \rho(\int_0^{+\infty} e^{-\mu(K)s}K(s)\,ds)$. Therefore, we always have $1 \leqslant \rho(\int_0^{+\infty} e^{-\mu(K)s}K(s)\,ds)$. Consider again the real function g defined by (12), where $t \in [\beta, +\infty)$ if $\int_0^{+\infty} e^{-\beta s}\|K(s)\|\,ds < +\infty$, otherwise $t \in (\beta, +\infty)$. Then, from the above result, we have $g(\mu(K)) \leqslant 0$. Assume that $g(\mu(K)) < 0$. Since, clearly, $\lim_{t\to+\infty} g(t) = 1$, it implies that $g(t_0) = 0$, for some $t_0 > \mu(K)$. This gives $1 = \rho(\int_0^{+\infty} e^{-t_0 s} K(s)\,ds)$. By Theorem 2.1(i), $\det(I_n - \int_0^{+\infty} e^{-t_0 s} K(s)\,ds) = 0$. However, this conflicts with the definition of $\mu(K)$. Thus $g(\mu(K)) = 0$, or equivalently, $1 = \rho(\int_0^{+\infty} e^{-\mu(K)s} K(s)\,ds)$. Then, (i) now follows from Theorem 2.1(i).

Furthermore, by Theorem 2.1(iv), $\rho(\int_0^{+\infty} e^{-t_2 s} K(s)\,ds) \leqslant \rho(\int_0^{+\infty} e^{-t_1 s} K(s)\,ds)$, $\beta < t_1 \leqslant t_2$ ($\beta \leqslant t_1 \leqslant t_2$, if $\int_0^{+\infty} e^{-\beta s} \|K(s)\|\,ds < +\infty$). Therefore, g is increasing. Moreover, from the above arguments, it must be $g(t) > 0$, for every $t > \mu(K)$. Now, it is easy to see that (ii), (iii) follow from (i) and Theorem 2.1(ii), (i) and Theorem 2.1(iii), respectively. □

Remark 5.2. (i) Let $K \in L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$ and $K \geqslant 0$. It follows from Theorems 4.3 and 5.1 that the resolvent R of K belongs to $L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$ if and only if $\mu(K) < 0$. Since $K \in L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$, it follows that $\beta \leqslant 0$. Then, we consider separately two cases as follows:

(a) $\beta = 0$.

If $\mu(K) = -\infty$ then $\rho(\widehat{K}(0)) < 1$. If $\mu(K) > -\infty$ then $\mu(K) \geqslant 0 = \beta$. From the proof of Theorem 5.1, we showed that $\rho(\widehat{K}(\mu(K))) = 1$. Using Theorem 2.1(iv) again, we get $1 = \rho(\widehat{K}(\mu(K))) \leqslant \rho(\widehat{K}(0))$. Hence, we have

$$\mu(K) = -\infty \quad \text{if and only if} \quad \rho\big(\widehat{K}(0)\big) < 1.$$

(b) $\beta < 0$.

By a similar argument, we get

$$\mu(K) < 0 \quad \text{if and only if} \quad \rho\big(\widehat{K}(0)\big) < 1.$$

Hence, we get back Theorem 4.3. Furthermore, it is worth noticing that if $\beta < 0$, then there exists a positive number α such that

$$\int_0^{+\infty} e^{\alpha t} \|R(t)\|\,dt < +\infty. \tag{17}$$

Conversely, (17) implies $\beta < 0$, provided $0 \leqslant K \in L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$ with $\rho(\widehat{K}(0)) < 1$. In fact, $\beta < 0$ implies that $\int_0^{+\infty} e^{\gamma t} \|K(t)\|\,dt < +\infty$, for some $\gamma > 0$. Moreover, since $\rho(\widehat{K}(0)) < 1$ it follows that $\rho(\widehat{K}_1(0)) < 1$, where $K_1(t) := K(t)e^{\alpha t}$, for some $0 < \alpha < \gamma$. By Theorem 4.3, the resolvent R_1 of K_1 belongs to $L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$. Since R is the resolvent of K, it is easy to see that $R_1 = e^{\alpha t} R$. We thus get (17). Conversely, if $0 \leqslant K \in L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$ with $\rho(\widehat{K}(0)) < 1$ and (17) is valid then by an argument as in the proof of Theorem 2 of [16], we can show that $\int_0^{+\infty} e^{\gamma t} \|K(t)\|\,dt < +\infty$, for some $\gamma > 0$. That means $\beta < 0$. We omit the details here.

(ii) Let $K(t) = A \in \mathbb{R}_+^{n\times n}$, $t \geqslant 0$, and assume that $\rho(A) > 0$. Then by a simple computation, it is easy to see that Theorem 5.1 comes back the classical Perron–Frobenius Theorem 2.1.

6. A criterion for positivity of the initial function semigroup of linear Volterra integral equations

Consider the initial value problem of a homogeneous linear Volterra integral equation of the form

$$x(t) = \int_0^\sigma K(s)x(t-s)\,ds, \quad t \geqslant 0, \tag{18}$$

$$x(t) = \psi(t) \in L^2\big((-\sigma,0),\mathbb{R}^n\big), \quad -\sigma < t < 0. \tag{19}$$

Here $0 < \sigma < +\infty$ and $K \in L^1((0,\sigma),\mathbb{R}^{n\times n})$ are given. It is important to note that if we define $K(t) = 0$, $\forall t \geqslant \sigma$, then (18)–(19) can be rewritten in the form of (7) where f is defined by

$$f(t) = \begin{cases} \int_t^\sigma K(s)\psi(t-s)\,ds & \text{if } t \in [0,\sigma), \\ 0 & \text{if } t \geqslant \sigma. \end{cases} \tag{20}$$

Let x be the solution of (18)–(19). Fix $h \geqslant 0$ and define $x_h(t) = x(t+h)$, $t \geqslant -\sigma$. The following theorem shows that the mappings $\psi \mapsto x_h$ produce a strongly continuous semigroup.

Theorem 6.1. *(See [5].) Let $K \in L^1((0,\sigma),\mathbb{R}^{n\times n})$. For each $\psi(t) \in L^2((-\sigma,0),\mathbb{R}^n)$, let x be the solution of* (18)–(19) *and define*

$$\big(T(h)\psi\big)(t) := x_h(t), \quad t \in (-\sigma,0), \tag{21}$$

for $h \geqslant 0$. Then T is a strongly continuous semigroup on $L^2((-\sigma,0),\mathbb{R}^n)$. Moreover, the infinitesimal generator $\mathcal{A}$ of the semigroup is given by

$$\mathcal{A}\psi = \frac{d\psi}{ds}, \quad \psi \in \mathcal{D}(\mathcal{A}), \tag{22}$$

where

$$\mathcal{D}(\mathcal{A}) = \left\{\psi \in W^{1,2}\big([-\sigma,0],\mathbb{R}^n\big)\colon\ \psi(0) - \int_0^\sigma K(s)\psi(-s) = 0\right\}. \tag{23}$$

Definition 6.2. (See [5].) The semigroup $(T(h))_{h\geqslant 0}$ described in Theorem 6.1 is called the initial function semigroup determined by K on $(0,\sigma)$.

Recall that a semigroup $(T(h))_{h\geqslant 0}$ is positive, by definition, if $T(h)$ is a positive operator for every $h \geqslant 0$. We are now interested in the problem of finding conditions under which the semigroup $(T(h))_{h\geqslant 0}$ is positive. The following theorem gives us the solution of this problem.

Theorem 6.3. *Let $K \in L^1((0,\sigma),\mathbb{R}^{n\times n})$. The semigroup $(T(h))_{h\geqslant 0}$ is positive if and only if $K \geqslant 0$.*

To prove the above theorem, we need the following.

Lemma 6.4. *For $z \in \mathbb{C}$ satisfying* $\det(I_n - \widehat{K}(z)) \neq 0$*, we have*

$$(zI-\mathcal{A})^{-1}\phi(t)=e^{zt}\big(I_n-\widehat{K}(z)\big)^{-1}\left(\int_{-\sigma}^{0}K(-s)\left(\int_{s}^{0}e^{z(s-\tau)}\phi(\tau)\,d\tau\right)ds\right)$$
$$+\int_{t}^{0}e^{z(t-s)}\phi(s)\,ds,\quad t\in[-\sigma,0], \tag{24}$$

for every $\phi\in L^2((-\sigma,0),\mathbb{R}^n)$.

Proof. Assume that $(zI-\mathcal{A})^{-1}\phi(t)=\psi$ where $\phi\in L^2((-\sigma,0),\mathbb{R}^n)$ and $\psi\in\mathcal{D}(\mathcal{A})$. By (22), we have

$$\frac{d\psi}{dt}(t)-z\psi(t)=-\phi(t),\quad \text{a.e. on } (-\sigma,0).$$

It follows that

$$\psi(t)=e^{zt}\psi(0)+\int_{t}^{0}e^{z(t-s)}\phi(s)\,ds,\quad t\in[-\sigma,0].$$

By $\psi\in\mathcal{D}(\mathcal{A})$, $\psi(0)=\int_0^\sigma K(s)\psi(-s)\,ds=\int_{-\sigma}^0 K(-s)\psi(s)\,ds$. Then we get

$$\psi(0)=\int_{-\sigma}^{0}e^{zs}K(-s)\,ds\,\psi(0)+\int_{-\sigma}^{0}K(-s)\left(\int_{s}^{0}e^{z(s-\tau)}\phi(\tau)\,d\tau\right)ds.$$

This is equivalent to

$$\big(I_n-\widehat{K}(z)\big)\psi(0)=\left(I_n-\int_{-\sigma}^{0}e^{zs}K(-s)\,ds\right)\psi(0)=\int_{-\sigma}^{0}K(-s)\left(\int_{s}^{0}e^{z(s-\tau)}\phi(\tau)\,d\tau\right)ds.$$

From the above argument, it is easy to see that the operator $(zI-\mathcal{A})^{-1}$ exists for each $z\in\mathbb{C}$, satisfying $\det(I_n-\widehat{K}(z))\neq 0$. Moreover, if it is the case then

$$\psi(0)=\big(I_n-\widehat{K}(z)\big)^{-1}\int_{-\sigma}^{0}K(-s)\left(\int_{s}^{0}e^{z(s-\tau)}\phi(\tau)\,d\tau\right)ds.$$

This gives

$$\psi(t)=e^{zt}\big(I_n-\widehat{K}(z)\big)^{-1}\left(\int_{-\sigma}^{0}K(-s)\left(\int_{s}^{0}e^{z(s-\tau)}\phi(\tau)\,d\tau\right)ds\right)+\int_{t}^{0}e^{z(t-s)}\phi(s)\,ds,$$
$$t\in[-\sigma,0],$$

which ends our proof. □

Proof of Theorem 6.3. ($\Leftarrow$) By a standard property of a C_0-semigroup,

$$T(h)\phi=\lim_{k\to+\infty}\left[\frac{k}{h}R\left(\frac{k}{h},\mathcal{A}\right)\right]^k\phi,\quad h>0,$$

for every $\phi \in L^2((-\sigma, 0), \mathbb{R}^n)$, see e.g. [17]. Thus, we only have to show that $R(s, \mathcal{A}) \geqslant 0$, for $s > 0$ large enough. In view of (24), it is sufficient to show that $(I_n - \widehat{K}(s))^{-1} \geqslant 0$ for $s > 0$ large enough. Since $K \geqslant 0$ it follows that $\widehat{K}(s) \geqslant 0$, for every $s \in \mathbb{R}$. Taking into account that $\lim_{s\to+\infty} \widehat{K}(s) = 0$, by the continuity of the spectral radius in the matrix space, we have

$$\rho\big(\widehat{K}(s)\big) < 1, \quad \forall s \geqslant s_1,$$

for some $s_1 > 0$. Finally, it follows from Theorem 2.1(iii) that $(I_n - \widehat{K}(s))^{-1} \geqslant 0$, for every $s > s_1$.

($\Rightarrow$) Assume that the semigroup $(T(h))_{h\geqslant 0}$ is positive, we show that $K \geqslant 0$. Let $\psi \in C([-\sigma, 0], \mathbb{R}^n)$. It follows that the function f defined by (20) is bounded continuous on $\mathbb{R}$. Since the resolvent R of K is locally integrable, it follows from the variation of constants formula that the solution x of Eq. (7) where f is given by (20) (therefore, it is also the solution of (18)–(19)) is continuous on $[0, \sigma)$. In particular, x is continuous from the right at zero. Suppose $\psi \geqslant 0$. Since the semigroup $(T(h))_{h\geqslant 0}$ is positive, it follows that $x(0) = \int_0^\sigma K(s)x(-s)\,ds = \int_0^\sigma K(s)\psi(-s)\,ds \geqslant 0$. Using Lemma 3.5, it is easy to see that $K \geqslant 0$. □

Finally, we give a criterion for exponential stability of the semigroup $(T(h))_{h\geqslant 0}$.

Theorem 6.5. *Let $K \in L^1((0, \sigma), \mathbb{R}^{n\times n})$ and $K \geqslant 0$. Then the semigroup $(T(h))_{h\geqslant 0}$ is exponentially stable if and only if $\rho(\int_0^\sigma K(s)\,ds) < 1$.*

Proof. Since $(T(h))_{h\geqslant 0}$ is a positive semigroup on $L^2((-\sigma, 0), \mathbb{R}^n)$, the spectral mapping theorem holds true, see e.g. [17, Theorem 1.1, p. 334]. That is, $s(\mathcal{A}) = \omega(\mathcal{A})$, where $s(\mathcal{A}), \omega(\mathcal{A})$, is the spectral abscissa and growth bound of the semigroup, respectively. Therefore, the semigroup is exponentially stable if and only if $s(\mathcal{A}) < 0$. On the other hand, the spectrum of the generator $\mathcal{A}$ is the pure point spectrum and it is defined by $\{z \in \mathbb{C}\colon \det(I_n - \widehat{K}(z)) = 0\}$, see [5, Theorem 2.7, p. 213] (we also see that from the proof of Lemma 6.4). We thus get, $s(\mathcal{A}) = \mu(K)$ where $\mu(K)$ is given by (15). Furthermore, taking Remark 5.2 into account, it follows from Theorem 5.1 that $s(\mathcal{A}) = \mu(K) < 0$ if and only if $\rho(\widehat{K}(0)) = \rho(\int_0^\sigma K(s)\,ds) < 1$. This completes our proof. □

Remark 6.6. It is important to know that if $\sigma = +\infty$ then the semigroup $(T(h))_{h\geqslant 0}$ is still existing. However, in this case, the definition of the generator of the semigroup is quite complicated. Moreover, the spectrum of $\mathcal{A}$ contains the half plane $\{z \in \mathbb{C}\colon \Re z \geqslant 0\}$, see [5, p. 213]. Thus, the semigroup $(T(h))_{h\geqslant 0}$ is not exponentially stable.

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