

The uniform convergence and precise asymptotics of generalized stochastic order statistics[☆]

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Abstract

Let $X(i, n, m, k)$, $i = 1, \dots, n$, be generalized order statistics based on F . For fixed $r \in \mathcal{N}$, and a suitable counting process $N(t)$, $t > 0$, we mainly discuss the precise asymptotic of the generalized stochastic order statistics $X(N(n) - r + 1, N(n), m, k)$. It not only makes the results of Yan, Wang and Cheng [J.G. Yan, Y.B. Wang, F.Y. Cheng, Precise asymptotics for order statistics of a non-random sample and a random sample, J. Systems Sci. Math. Sci. 26 (2) (2006) 237–244] as the special case of our result, and presents many groups of weighted functions and boundary functions, but also permits a unified approach to several models of ordered random variables.

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1. Introduction and main results

Let X_1, X_2, \dots be non-degenerate i.i.d. random variables, $S_n = \sum_{i=1}^n X_i$, $n \geq 1$, and $f(x)$, $\varphi(x)$ be the positive functions defined on $[n_0, \infty)$, which satisfy $\sum_{n=n_0}^{\infty} \varphi(n) = \infty$, and might as well let n_0 be a positive integer. Since Hsu and Robbins [12] introduced the concept of complete convergence, there have been many results. One of the starting points is to discuss the convergence conditions of the series

$$\sum_{n=1}^{\infty} \varphi(n) P(|S_n| \geq \varepsilon f(n)), \quad \forall \varepsilon > 0, \tag{1.1}$$

for example, Hsu and Robbins [12], Erdős [5,6] got the results of $\varphi(x) \equiv 1$, $f(x) = x$, $x \geq 1$; Baum and Katz [2] achieved the results of $\varphi(x) = x^{\frac{r}{p}-2}$, $f(x) = x^{\frac{1}{p}}$, $x \geq 1$, $0 < p < 2$ and $r \geq p$, etc.

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On the other hand, let $\varepsilon \downarrow 0$, the series in (1.1) tends to ∞ . We are interested in the convergence rates, limit positions and the corresponding conditions of (1.1). At first, Heyde [11] proved that when $EX^2 < \infty$ and $EX = 0$,

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \sum_{n=1}^{\infty} P(|S_n| \geq \varepsilon n) = EX^2. \quad (1.2)$$

For analogous results in the more general case, see Chen [3], Spătaru [17], Gut and Spătaru [9,10], Gut [8], etc. The research in these fields is called precise asymptotic.

Till now, the research in precise asymptotic is focused on the part sum S_n and its corresponding objects, but not touch on the samples' maximum $M_n = \max_{i \leq n} X_i$, $n \geq 1$. M_n is one of the main objects in studying of extreme value theory and its applications. On the other hand, M_n is the special case of order statistics. Moreover, order statistics and record values play a crucial role in statistics and its applications; both models describe random variables arranged in order of magnitude. In this paper, we will study the precise asymptotic for generalized order statistics introduced by Kamps [13], which permits a unified approach to several models of ordered random variables, e.g. (ordinary) order statistics, record values, sequential order statistics, progressive censoring, etc. We first introduce some concepts, notations and basic properties.

Definition 1.1. Say that X (or F) belongs to the maximum domain of attraction of the extreme value distribution $G(x)$, if there exist normalizing constants $c_n > 0$, centering constants $d_n \in \mathcal{R}$, $n \geq 1$, and d.f. $G(x)$, satisfying

$$c_n^{-1}(M_n - d_n) \xrightarrow{d} Z \sim G,$$

denoted by $X \in MDA(G)$ (or $F \in MDA(G)$).

By the famous Fisher–Tippett theorem (see Embrechts et al. [4, Theorem 3.2.7]), extreme value distributions have only three types: Fréchet distributions, Gumbel distributions and Weibull distributions. In this paper we are only interested in Fréchet distributions. It is well known that the standard Fréchet distribution has the form

$$G_{1,\alpha}(x) = e^{-x^{-\alpha}} I(x > 0), \quad \text{for some } \alpha > 0,$$

and a general Fréchet distribution is denoted by $H = G_{1,\alpha}^\theta$, for some $\theta > 0$. By Lemma 2.1 in Section 2, we only restrict ourselves to the standard Fréchet distribution. And by Embrechts et al. [4, Theorem 3.3.7], $F \in MDA(G_{1,\alpha})$ holds if and only if $\bar{F}(x) = P(X > x) = x^{-\alpha} L(x)$, $x > 0$, where $L(x)$ is a positive slowly varying function, denoted by $L \in \mathcal{R}_0$, and $\bar{F}(x)$ is a regular varying function with the index $-\alpha$, denoted by $F \in \mathcal{R}_{-\alpha}$, for some $\alpha \geq 0$. If $F \in MDA(G_{1,\alpha})$, we can choose $d_n = 0$, $n \geq 1$, and $\bar{F}(c_n) \sim n^{-1}$, $n \rightarrow \infty$.

Definition 1.2. Let $n \in \mathcal{N}$, $k > 0$, $m_1, \dots, m_{n-1} \in \mathcal{R}$, $M_r = \sum_{j=r}^{n-1} m_j$, $1 \leq r \leq n-1$, be parameters such that $\gamma_r = k + n - r + M_r \geq 1$ for all $r \in \{1, \dots, n-1\}$, and let $\tilde{m} = (m_1, \dots, m_{n-1})$, if $n \geq 2$, $\tilde{m} \in \mathcal{R}$ arbitrary, if $n = 1$. If the random variables $U(r, n, \tilde{m}, k)$, $r = 1, \dots, n$, possess a joint density function of the form

$$f^{U(1,n,\tilde{m},k), \dots, U(n,n,\tilde{m},k)}(u_1, \dots, u_n) = k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{i=1}^{n-1} (1 - u_i)^{m_i} \right) (1 - u_n)^{k-1},$$

on the cone $0 \leq u_1 \leq \dots \leq u_n < 1$, then they are called uniform generalized order statistics.

Let F be an arbitrary distribution function. The random variables $X(r, n, \tilde{m}, k) = F^{-1}(U(r, n, \tilde{m}, k))$, $r = 1, \dots, n$, are called generalized order statistics based on F . In the particular case $m_1 = \dots = m_{n-1} = m$ the random variables are denoted by $U(r, n, m, k)$ and $X(r, n, m, k)$, $r = 1, \dots, n$, respectively.

The parameters k, m_j , $j = 1, \dots, n-1$, determine the model of ordered random variables. For example, in the case $k = 1$, $m_j = 0$ one gets ordinary order statistics, and in the case $k = 1$, $m_j = -1$ one gets record values. Further examples can be found in Kamps [13].

The paper of Nasri-Roudsari [15] started to develop the extreme value theory of generalized order statistics. It was shown that well-known results of extreme value theory for ordinary order statistics concerning the (weak) domain of attraction and normalizing constants carry over to generalized order statistics. In particular, the possible limit

distributions of extreme generalized order statistics (called generalized extreme value distributions) were established (Nasri-Roudsari [15, Theorem 3.3]) and it was shown that an underlying distribution function F of an i.i.d. sequence X_1, X_2, \dots , belongs to the (weak) domain of attraction of an extreme value distribution if and only if F belongs to the (weak) domain of attraction of generalized extreme value distribution (Nasri-Roudsari [15, Corollary 3.7]). Marohn [14] deduced that the Von Mises conditions also imply that the underlying distribution function F belongs to the strong domain of attraction of a generalized extreme value distribution.

As in Nasri-Roudsari [15] and Nasri-Roudsari and Cramer [16] we assume in the following that the underlying parameters m_1, \dots, m_{n-1} of the generalized order statistics are equal, i.e. $m_i = m > -1$. Such a condition seems to be restrictive, nevertheless, various interesting models are still included. For a discussion we refer to Nasri-Roudsari and Cramer [16].

Throughout the following, we assume that $g(x)$, $h(x)$, $x \geq n_0$, be positive and derivable functions, which both strictly increase to ∞ , $g(h(x))$ is defined on $[n_0, \infty)$, and $g^{-1}(x)$, $h^{-1}(x)$ are the inverse functions of $f(x)$ and $h(x)$ respectively. Let $\varphi(x) = (g(h(x)))'$, $x \geq n_0$. $G_{1,\alpha}$, $\alpha > 0$ denotes the Fréchet extreme value distribution, i.e. $G_{1,\alpha}(x) = \exp(-x^{-\alpha})I(x > 0)$.

Let $N(t)$, $t > 0$, be a nonnegative integer counting process, which is independent of $X(r, n, m, k)$, $r = 1, \dots, n$. $X(r, N(n), m, k)$ is called the r th order statistic of the random samples, $r = 1, \dots, N(n)$.

Now we can present our results.

Theorem 1.1. Assume that the underlying distribution function $F \in MDA(G_{1,\alpha})$, $\alpha > 0$, and the counting process $N(t)$, $t > 0$, satisfies

$$t^{-1}N(t) \xrightarrow{P} \lambda > 0, \quad t \rightarrow \infty, \quad \sup_{t>0} \frac{E(N(t))^\tau}{t^\tau} < \infty, \quad \text{for fixed } r \geq 1, \quad (1.3)$$

where $\tau = \frac{k}{m+1} + r - 1$. And assume that $g(x)$, $x \geq n_0$, satisfy the following conditions: $\forall r \geq 1$, $\forall \varepsilon > 0$,

$$\tilde{G}_r(\varepsilon) := \frac{1}{\Gamma(\tau)} \int_0^{\lambda(\varepsilon h(n_0))^{-\alpha(m+1)}} g(\varepsilon^{-1}(\lambda t^{-1})^{\frac{1}{\alpha(m+1)}}) t^{\tau-1} e^{-t} dt < \infty, \quad (1.4)$$

$$\lim_{\varepsilon \downarrow 0} \tilde{G}_r(\varepsilon) = \infty. \quad (1.5)$$

And for all $G_r(\varepsilon) \sim (\tilde{G}_r(\varepsilon))^{-1}$, $\varepsilon \downarrow 0$,

$$\lim_{M \rightarrow \infty} \lim_{\varepsilon \downarrow 0} G_r(\varepsilon) \int_0^{\lambda(\varepsilon^{-1} g^{-1}(M \tilde{G}_r(\varepsilon)))^{-\alpha(m+1)}} g(\varepsilon^{-1}(\lambda t^{-1})^{\frac{1}{\alpha(m+1)}}) t^{\tau-1} e^{-t} dt = 0, \quad (1.6)$$

for some $\theta > 0$,

$$\lim_{M \rightarrow \infty} \lim_{\varepsilon \downarrow 0} G_r(\varepsilon) \varepsilon^{-(\alpha-\theta)(k+(r-1)(m+1))} \int_{g^{-1}(\tilde{G}_r(\varepsilon)M)}^{\infty} y^{-(\alpha-\theta)(k+(r-1)(m+1))} dg(y) = 0. \quad (1.7)$$

Suppose that $\varphi(x)$, $x \geq n_0$, be monotone, and when $\varphi(x)$ is nondecreasing

$$\lim_{n \rightarrow \infty} \frac{\varphi(n+1)}{\varphi(n)} = 1. \quad (1.8)$$

Then

$$\lim_{\varepsilon \downarrow 0} G_r(\varepsilon) \sum_{n=n_0}^{\infty} \varphi(n) P(X(N(n) - r + 1, N(n), m, k) \geq \varepsilon \alpha_n h(n)) = 1, \quad (1.9)$$

where α_n as in the following Lemma 2.2.

Corollary 1.1. Suppose that $F \in MDA(G_{1,\alpha})$, then for $\forall 0 < s \leq p < t \leq \alpha$, we have

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{s(\frac{r}{p}-1)} \sum_{n \geq 1} n^{\frac{t}{p}-2} P(X_{n-r+1,n} > \varepsilon n^{\frac{1}{s}} c_n) = \frac{p}{(r-1)!(t-p)} \Gamma\left(\frac{s}{\alpha} \left(1 - \frac{t}{p}\right) + r\right). \quad (1.10)$$

Corollary 1.2. Suppose that $F \in MDA(G_{1,\alpha})$, then for $\forall 0 < p < \alpha$, we have

$$\lim_{\varepsilon \downarrow 0} \frac{1}{-\log \varepsilon} \sum_{n \geq 1} n^{-1} P(X_{n-r+1,n} > \varepsilon n^{\frac{1}{s}} c_n) = p. \quad (1.11)$$

Remark 1.1. Corollaries 1.1 and 1.2 are Theorems 1.1 and 1.2 of Yan, Wang and Cheng [19] respectively. Moreover, Corollary 1.2 is the special case of Corollary 1.1 when $p = t$, but the forms of their convergence rates are different.

Remark 1.2. By Kamps [13], one can give a lot of corollaries as before, such as results for order statistics with non-integral sample size, etc. we omit them.

2. Some lemmas

Lemma 2.1. Assume that the counting process $N(t)$, $t > 0$, satisfies condition (1.3), $F \in MDA(G)$. Then there exist $c_n > 0$, $d_n \in \mathcal{R}$, $n \geq 1$, such that

$$\lim_{n \rightarrow \infty} P(X_{k,N(n)} \leq c_n x + d_n) = G^\lambda(x) \sum_{i=0}^{k-1} \frac{(-\ln G^\lambda(x))^i}{i!}, \quad k \geq 1, x \in \mathcal{R}. \quad (2.1)$$

The proof can refer to Embrechts et al. [4, Theorem 4.3.2], etc.

Lemma 2.2. Suppose that the counting process $N(t)$, $t > 0$, satisfies condition (1.3), $F \in MDA(G)$. Then there exist $\alpha_n > 0$, $\beta_n \in \mathcal{R}$, $n \geq 1$ such that when $n \rightarrow \infty$,

$$\Delta_{n,r} = \sup_x |P(\alpha_n^{-1}(X(N(n) - r + 1, N(n), m, k) - \beta_n) > x) - \bar{H}_{r,m,k,\lambda}(x)| \rightarrow 0. \quad (2.2)$$

Here $H_{r,m,k,\lambda}(x) = \frac{1}{\Gamma(\tau)} \Gamma(\tau, \lambda(-\log G(x))^{m+1})$ and $\Gamma(\alpha, x)$ denotes the incomplete gamma function which is defined by $\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t} dt$, $\alpha > 0$, $x \geq 0$.

Proof. Similarly to the proof of Theorem 3.3 in Nasri-Roudsari [15], we have that

$$P(\alpha_n^{-1}(X(N(n) - r + 1, N(n), m, k) - \beta_n) \leq x) \rightarrow H_{r,m,k,\lambda}(x), \quad n \rightarrow \infty.$$

By Lemma 2.10.1 in Galambos [7] and the continuity of H , we end the proof. \square

To be simple, in the following we still put

$$H_{r,m,k,\lambda}(x) = \frac{1}{\Gamma(\tau)} \Gamma(\tau, \lambda(-\log G_{1,\alpha}(x))^{m+1}) = \frac{1}{\Gamma(\tau)} \Gamma(\tau, \lambda x^{-\alpha(m+1)}), \quad x > 0.$$

Lemma 2.3. Let $n \in \mathcal{N}$, $l > 0$. Then

$$\left| \frac{\int_x^1 t^{n-1} (1-t)^{l-1} dt}{B(n, l)} \right| \leq C(1-x)^l n^l,$$

where $C > 0$ is a suitable constant.

Proof. By Lemma 2.4 in Nasri-Roudsari [15], we have

$$\frac{\int_x^1 t^{n-1}(1-t)^{l-1} dt}{B(n, l)} = (1-x)^l \sum_{i=0}^{n-1} \binom{-l}{i} (-x)^i.$$

Then

$$\left| \frac{\int_x^1 t^{n-1}(1-t)^{l-1} dt}{B(n, l)} \right| \leq (1-x)^l \sum_{i=0}^{n-1} \binom{l+i-1}{i} = (1-x)^l \binom{l+n-1}{l} \leq C(1-x)^l n^l,$$

where $C > 0$ is a suitable constant. \square

Lemma 2.4. Suppose that $F \in MDA(G_{1,\alpha})$, $\alpha > 0$. Then for $\forall \varepsilon > 0$, $\forall \theta > 0$, $\forall r \geq 1$, there exists $0 < C < \infty$ such that

$$P(X(N(n) - r + 1, N(n), m, k) > \varepsilon h(n)\alpha_n) \leq C(\varepsilon h(n))^{-(\alpha-\theta)(k+(r-1)(m+1))}.$$

Proof. By (6) in Marohn [14] and Lemma 2.2, for every $i \geq r$,

$$\begin{aligned} P(X(i - r + 1, i, m, k) > \varepsilon h(n)\alpha_n) &= \frac{1}{B(i - r + 1, \tau)} \int_{1-(1-F(\varepsilon h(n)\alpha_n))^{m+1}}^1 t^{i-r}(1-t)^{\tau-1} dt \\ &\leq C(\bar{F}(\varepsilon h(n)\alpha_n))^{(m+1)(\tau)} (i - r + 1)^\tau \\ &\leq C(\bar{F}(\varepsilon h(n)\alpha_n))^{k+(r-1)(m+1)} i^\tau. \end{aligned} \quad (2.3)$$

By (2.2) and the basic renewal theorem, we have

$$\begin{aligned} P(X(N(n) - r + 1, N(n), m, k) > \varepsilon h(n)\alpha_n) &= \sum_{i=r}^{\infty} P(X(i - r + 1, i, m, k) > \varepsilon h(n)\alpha_n) P(N(n) = i) \\ &\leq C(\bar{F}(\varepsilon h(n)\alpha_n))^{k+(m+1)(r-1)} \sum_{i=r}^{\infty} i^\tau P(N(n) = i) \\ &\leq C(\bar{F}(\varepsilon h(n)\alpha_n))^{k+(m+1)(r-1)} E(N(n))^\tau \\ &\leq C(\bar{F}(\varepsilon h(n)\alpha_n))^{k+(r-1)(m+1)} n^\tau. \end{aligned} \quad (2.4)$$

Since $F \in MDA(G_{1,\alpha})$, there exists a function $L \in \mathcal{R}_0$, such that $\bar{F}(x) = x^{-\alpha} L(x)$, $x > 0$, and $\bar{F}(a_n) \sim n^{-1}$, $n \rightarrow \infty$, i.e. $a_n^{-\alpha} L(a_n) \sim n^{-1}$, $n \rightarrow \infty$. Therefore

$$\bar{F}(\alpha_n) \sim n^{-\frac{1}{m+1}} \quad \text{i.e.} \quad \alpha_n^{-\alpha} L(\alpha_n) \sim n^{-\frac{1}{m+1}}.$$

Hence, for $\forall n \geq 1$, there exists $0 < C < \infty$,

$$\alpha_n^{-\alpha} \leq C n^{-\frac{1}{m+1}} (L(\alpha_n))^{-1}. \quad (2.5)$$

By (2.3), (2.4) and Potter's theorem (cf. Bingham et al. [1, Theorem 1.5.6]), for $\forall \theta > 0$, we have

$$\begin{aligned} P(X(N(n) - r + 1, N(n), m, k) > \varepsilon h(n)\alpha_n) &\leq C((\varepsilon h(n)\alpha_n)^{-\alpha} L(\varepsilon h(n)\alpha_n))^{k+(r-1)(m+1)} (\alpha_n^\alpha (L(\alpha_n))^{-1})^{k+(r-1)(m+1)} \\ &= C\varepsilon^{-\alpha(k+(r-1)(m+1))} (h(n))^{-\alpha(k+(r-1)(m+1))} \left(\frac{L(\varepsilon h(n)\alpha_n)}{L(\alpha_n)} \right)^{k+(r-1)(m+1)} \\ &\leq C\varepsilon^{-\alpha(k+(r-1)(m+1))} (h(n))^{-\alpha(k+(r-1)(m+1))} (\varepsilon h(n))^{\theta(k+(r-1)(m+1))} \\ &= C(\varepsilon h(n))^{-(\alpha-\theta)(k+(r-1)(m+1))}. \quad \square \end{aligned}$$

3. Proof of results

Proof of Theorem 1.1. When $\varphi(x)$ is nonincreasing, we have

$$\begin{aligned} \sum_{n=n_0+1}^{\infty} \varphi(n) \bar{H}_{r,m,k,\lambda}(\varepsilon h(n)) &\leq \sum_{n=n_0}^{\infty} \int_n^{n+1} \varphi(x) \bar{H}_{r,m,k,\lambda}(\varepsilon h(x)) dx = \int_{n_0}^{\infty} \varphi(x) \bar{H}_{r,m,k,\lambda}(\varepsilon h(x)) dx \\ &\leq \sum_{n=n_0}^{\infty} \varphi(n) \bar{H}_{r,m,k,\lambda}(\varepsilon h(n)). \end{aligned} \quad (3.1)$$

Note that $G_r(\varepsilon) \sim (\tilde{G}_r(\varepsilon))^{-1}$, $\varepsilon \downarrow 0$, by (3.1) and integration by parts, we have

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} G_r(\varepsilon) \sum_{n=n_0}^{\infty} \varphi(n) \bar{H}_{r,m,k,\lambda}(\varepsilon h(n)) &= \lim_{\varepsilon \downarrow 0} G_r(\varepsilon) \int_{n=n_0}^{\infty} \varphi(x) \bar{H}_{r,m,k,\lambda}(\varepsilon h(x)) dx \\ &= \lim_{\varepsilon \downarrow 0} G_r(\varepsilon) \int_{n=n_0}^{\infty} \bar{H}_{r,m,k,\lambda}(\varepsilon h(x)) dg(h(x)) \\ &= \lim_{\varepsilon \downarrow 0} G_r(\varepsilon) \int_{\varepsilon h(n_0)}^{\infty} \bar{H}_{r,m,k,\lambda}(t) dg(\varepsilon^{-1}t) \\ &= \lim_{\varepsilon \downarrow 0} G_r(\varepsilon) \left[\bar{H}_{r,m,k,\lambda}(t) g(\varepsilon^{-1}t) \Big|_{\varepsilon h(n_0)}^{\infty} + \int_{\varepsilon h(n_0)}^{\infty} g(\varepsilon^{-1}t) dH_{r,m,k,\lambda}(t) \right] \\ &= \lim_{\varepsilon \downarrow 0} G_r(\varepsilon) \left[\lim_{t \rightarrow \infty} \bar{H}_{r,m,k,\lambda}(t) g(\varepsilon^{-1}t) + \frac{\alpha(m+1)\lambda^\tau}{\Gamma(\tau)} \int_{\varepsilon h(n_0)}^{\infty} g(\varepsilon^{-1}t) t^{-\alpha k - \alpha(m+1)(r-1) - 1} e^{-\lambda t^{-\alpha(m+1)}} dt \right] \\ &= \lim_{\varepsilon \downarrow 0} G_r(\varepsilon) \left[\lim_{t \rightarrow \infty} \bar{H}_{r,m,k,\lambda}(t) g(\varepsilon^{-1}t) + \frac{1}{\tau} \int_0^{\lambda(\varepsilon h(n_0))^{-\alpha(m+1)}} g(\varepsilon^{-1}(\lambda y^{-1})^{\frac{1}{\alpha(m+1)}}) y^{\tau-1} e^{-y} dy \right] \\ &= \lim_{\varepsilon \downarrow 0} G_r(\varepsilon) \lim_{t \rightarrow \infty} \bar{H}_{r,m,k,\lambda}(t) g(\varepsilon^{-1}t) + 1. \end{aligned}$$

Note that for $\forall \beta > 0$,

$$\beta \int_0^x t^{\beta-1} e^{-t} dt \sim x^\beta e^{-x}, \quad x \rightarrow 0.$$

In fact, by L'Hospital Principle

$$\lim_{x \rightarrow 0} \frac{\beta \int_0^x t^{\beta-1} e^{-t} dt}{x^\beta e^{-x}} = \lim_{x \rightarrow 0} \frac{\beta}{\beta - x} = 1.$$

Hence

$$\bar{H}_{r,m,k,\lambda}(t) = \frac{1}{\tau} \int_0^{\lambda t^{-\alpha(m+1)}} x^{\tau-1} e^{-x} dx \sim \frac{(\tau)^{-1}}{\Gamma(\tau)} (\lambda t^{-\alpha(m+1)})^\tau e^{-\lambda t^{-\alpha(m+1)}}, \quad t \rightarrow \infty.$$

Thus, by (1.4), we have

$$\begin{aligned} & \frac{1}{\alpha(k+(r-1)(m+1))} g(\varepsilon^{-1}t) t^{-\alpha(k+(r-1)(m+1))} e^{-\lambda t^{-\alpha(m+1)}} \\ &= g(\varepsilon^{-1}t) e^{-\lambda t^{-\alpha(m+1)}} \int_t^\infty y^{-\alpha(k+(r-1)(m+1)) - 1} dy \\ &\leq \int_t^\infty g(\varepsilon^{-1}y) y^{-\alpha(k+(r-1)(m+1)) - 1} e^{-\lambda y^{-\alpha(m+1)}} dy \rightarrow 0, \quad t \rightarrow \infty, \end{aligned}$$

hence, we get

$$\lim_{\varepsilon \downarrow 0} G_r(\varepsilon) \sum_{n=n_0}^\infty \varphi(n) \bar{H}_{r,m,k,\lambda}(\varepsilon h(n)) = 1. \quad (3.2)$$

When $\varphi(x)$ is nondecreasing, by (1.8), we know for $\forall \delta > 0$, there exists $n_1 \in \mathcal{N}$, such that for $\forall n \geq n_1$,

$$\varphi(n) \leq \varphi(n+1) \leq (1+\delta)\varphi(n). \quad (3.3)$$

Then we have

$$\begin{aligned} (1+\delta)^{-1} \sum_{n=n_1+1}^\infty \varphi(n) \bar{H}_{r,m,k,\lambda}(\varepsilon h(n)) &= (1+\delta)^{-1} \sum_{n=n_1}^\infty \varphi(n+1) \bar{H}_{r,m,k,\lambda}(\varepsilon h(n+1)) \\ &\leq \sum_{n=n_1}^\infty \varphi(n) \bar{H}_{r,m,k,\lambda}(\varepsilon h(n+1)) \leq \int_{n_1}^\infty \varphi(x) \bar{H}_{r,m,k,\lambda}(\varepsilon h(x)) dx \\ &= \sum_{n=n_1}^\infty \int_n^{n+1} \varphi(x) \bar{H}_{r,m,k,\lambda}(\varepsilon h(x)) dx \leq \sum_{n=n_1}^\infty \varphi(n+1) \bar{H}_{r,m,k,\lambda}(\varepsilon h(n)) \\ &\leq (1+\delta) \sum_{n=n_1}^\infty \varphi(n) \bar{H}_{r,m,k,\lambda}(\varepsilon h(n)). \end{aligned} \quad (3.4)$$

Similarly, we can get (3.2). For $\forall M > 1$, let

$$b(\varepsilon) = h^{-1}(g^{-1}(\tilde{G}_r(\varepsilon)M)), \quad \text{for } \forall \varepsilon > 0. \quad (3.5)$$

By Lemma 2.3 in Wang and Yang [18], for ε small enough,

$$2\tilde{G}_r(\varepsilon)M = 2g(h(b(\varepsilon))) \geq g(h(b(\varepsilon)+1)) \geq \int_{n_0}^{b(\varepsilon)+1} dg(h(x)) = \int_{n_0}^{b(\varepsilon)+1} \varphi(x) dx \geq \sum_{n=n_0}^{[b(\varepsilon)]} \int_n^{n+1} \varphi(x) dx. \quad (3.6)$$

If $\varphi(x)$ is nonincreasing, then by (3.6)

$$2\tilde{G}_r(\varepsilon)M \geq \sum_{n=n_0}^{[b(\varepsilon)]} \varphi(n+1) = \sum_{n=n_0}^{[b(\varepsilon)]+1} \varphi(n) - \varphi(n_0). \quad (3.7)$$

If $\varphi(x)$ is nondecreasing, then by (3.3) we have

$$\begin{aligned}
 2\tilde{G}_r(\varepsilon)M &\geq \sum_{n=n_0}^{[b(\varepsilon)]} \varphi(n) = \sum_{n=n_0}^{n_1-1} \varphi(n) + \sum_{n=n_1}^{[b(\varepsilon)]} \varphi(n) \\
 &\geq \sum_{n=n_0}^{n_1-1} \varphi(n) + (1+\delta)^{-1} \sum_{n=n_1}^{[b(\varepsilon)]} \varphi(n+1) = \sum_{n=n_0}^{n_1-1} \varphi(n) + (1+\delta)^{-1} \sum_{n=n_1+1}^{[b(\varepsilon)]+1} \varphi(n) \\
 &\geq (1+\delta)^{-1} \sum_{n=n_0}^{[b(\varepsilon)]+1} \varphi(n) - \varphi(n_1).
 \end{aligned} \tag{3.8}$$

By (3.7), (3.8), Lemma 2.1 and Toeplitz's lemma, we have

$$\lim_{\varepsilon \downarrow 0} G_r(\varepsilon) \sum_{n=n_0}^{[b(\varepsilon)]+1} \varphi(n) \Delta_{n,r} = 0. \tag{3.9}$$

Taking into account (3.2) and (3.9), to prove (1.9), it suffices to show that

$$\lim_{\varepsilon \downarrow 0} G_r(\varepsilon) \sum_{n=[b(\varepsilon)]+2}^{\infty} \varphi(n) \bar{H}_{r,m,k,\lambda}(\varepsilon h(n)) = 0, \tag{3.10}$$

$$\lim_{\varepsilon \downarrow 0} G_r(\varepsilon) \sum_{n=[b(\varepsilon)]+2}^{\infty} \varphi(n) P(X(N(n) - r + 1, N(n), m, k) > \varepsilon \alpha_n h(n)) = 0. \tag{3.11}$$

First we prove (3.10). Obviously

$$\int_{b(\varepsilon)}^{\infty} \varphi(x) \bar{H}_{r,m,k,\lambda}(\varepsilon h(x)) dx \geq \sum_{n=[b(\varepsilon)]+1}^{\infty} \int_n^{n+1} \varphi(x) \bar{H}_{r,m,k,\lambda}(\varepsilon h(x)) dx. \tag{3.12}$$

If $\varphi(x)$ is nonincreasing, then from (3.12),

$$\begin{aligned}
 \int_{b(\varepsilon)}^{\infty} \varphi(x) \bar{H}_{r,m,k,\lambda}(\varepsilon h(x)) dx &\geq \sum_{n=[b(\varepsilon)]+1}^{\infty} \varphi(n+1) \bar{H}_{r,m,k,\lambda}(\varepsilon h(n+1)) \\
 &= \sum_{n=[b(\varepsilon)]+2}^{\infty} \varphi(n) \bar{H}_{r,m,k,\lambda}(\varepsilon h(n)).
 \end{aligned} \tag{3.13}$$

If $\varphi(x)$ is nondecreasing, then for every ε small enough, by (3.3) and (3.12),

$$\begin{aligned}
 \int_{b(\varepsilon)}^{\infty} \varphi(x) \bar{H}_{r,m,k,\lambda}(\varepsilon h(x)) dx &\geq \sum_{n=[b(\varepsilon)]+1}^{\infty} \varphi(n) \bar{H}_{r,m,k,\lambda}(\varepsilon h(n+1)) \\
 &\geq (1+\delta)^{-1} \sum_{n=[b(\varepsilon)]+1}^{\infty} \varphi(n+1) \bar{H}_{r,m,k,\lambda}(\varepsilon h(n+1)) \\
 &= (1+\delta)^{-1} \sum_{n=[b(\varepsilon)]+2}^{\infty} \varphi(n) \bar{H}_{r,m,k,\lambda}(\varepsilon h(n)).
 \end{aligned} \tag{3.14}$$

From (3.13), (3.14), by using integration by parts, we get

$$\begin{aligned}
 G_r(\varepsilon) \int_{b(\varepsilon)}^{\infty} \varphi(x) \bar{H}_{r,m,k,\lambda}(\varepsilon h(x)) dx &= G_r(\varepsilon) \int_{b(\varepsilon)}^{\infty} \bar{H}_{r,m,k,\lambda}(\varepsilon h(x)) dg(h(x)) \\
 &= G_r(\varepsilon) \int_{h(b(\varepsilon))}^{\infty} \bar{H}_{r,m,k,\lambda}(\varepsilon x) dg(x) = G_r(\varepsilon) \int_{\varepsilon^{-1}h(b(\varepsilon))}^{\infty} \bar{H}_{r,m,k,\lambda}(t) dg(\varepsilon^{-1}t) \\
 &\leq CG_r(\varepsilon) \int_{\varepsilon^{-1}h(b(\varepsilon))}^{\infty} g(\varepsilon^{-1}t) dH_{r,m,k,\lambda}(t) \\
 &= CG_r(\varepsilon) \int_{\varepsilon^{-1}h(b(\varepsilon))}^{\infty} g(\varepsilon^{-1}t) t^{-\alpha k - \alpha(m+1)(r-1) - 1} e^{-\lambda t^{-\alpha(m+1)}} dt \\
 &= CG_r(\varepsilon) \int_0^{\lambda(\varepsilon^{-1}g^{-1}(M\tilde{G}_r(\varepsilon)))^{-\alpha(m+1)}} g(\varepsilon^{-1}(\lambda y^{-1})^{\frac{1}{\alpha(m+1)}}) y^{\tau-1} e^{-y} dy. \quad (3.15)
 \end{aligned}$$

Hence by (1.6), we get (3.10) immediately.

Next we prove (3.11). By Lemma 2.3

$$\begin{aligned}
 G_r(\varepsilon) \sum_{n=[b(\varepsilon)]+2}^{\infty} \varphi(n) P(X(N(n) - r + 1, N(n), m, k) > \varepsilon \alpha_n h(n)) \\
 \leq CG_r(\varepsilon) \sum_{n=[b(\varepsilon)]+2}^{\infty} \varphi(n) \varepsilon^{-(\alpha-\theta)(k+(r-1)(m+1))} (h(n))^{-(\alpha-\theta)(k+(r-1)(m+1))} \\
 \leq CG_r(\varepsilon) \varepsilon^{-(\alpha-\theta)(k+(r-1)(m+1))} \int_{b(\varepsilon)}^{\infty} \varphi(x) (h(x))^{-(\alpha-\theta)(k+(r-1)(m+1))} dx \\
 = CG_r(\varepsilon) \varepsilon^{-(\alpha-\theta)(k+(r-1)(m+1))} \int_{b(\varepsilon)}^{\infty} (h(x))^{-(\alpha-\theta)(k+(r-1)(m+1))} dg(h(x)) \\
 = CG_r(\varepsilon) \varepsilon^{-(\alpha-\theta)(k+(r-1)(m+1))} \int_{g^{-1}(\tilde{G}_r(\varepsilon)M)}^{\infty} (y)^{-(\alpha-\theta)(k+(r-1)(m+1))} dg(y).
 \end{aligned}$$

Together with (1.7), we get (3.11) at once. \square

By Nasri-Roudsari [15], when $F \in MDA(G_{1,\alpha})$, we have $\alpha_n = c_n$. Thus, the proofs of Corollaries 1.1 and 1.2 are just to verify the conditions of Theorem 1.1 straightly, we omit them.

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