



Representations and derivations of quasi $\ast$ -algebras induced by *local modifications* of states

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ABSTRACT

The relationship between the GNS representations associated to states on a quasi $\ast$ -algebra, which are *local modifications* of each other (in a sense which we will discuss) is examined. The role of local modifications on the spatiality of the corresponding induced derivations describing the dynamics of a given quantum system with infinite degrees of freedom is discussed.

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1. Introduction and preliminaries

In two recent papers [1,2], we have investigated the role of derivations of quasi $\ast$ -algebras and the possibility of finding a certain symmetric operator which *implements the derivation*, in the sense that in a suitable representation the derivation can be written as a commutator with an operator which in the physical literature is usually called the *effective hamiltonian*. This is useful for physical applications and produces an algebraic framework in which the time evolution of some physical model can be analyzed [3].

Here we continue our analysis, taking inspiration again from physical motivations: it is known [4] that in a physical context *local modifications do not affect much the main physical results*. Our interest here is to understand this statement more in detail, mainly in the framework of quasi $\ast$ -algebras [5,6], which, as we have discussed in several other places, see [6,3,7], in our opinion play an important role in the mathematical description of quantum mechanical systems with infinite degrees of freedom.

Just as an introductory example, let us consider a $C^\ast$ -algebra $\mathcal{A}$ with unit e , and let ω and ω' be two (different) positive linear functionals on $\mathcal{A}$. Let further $(\pi_\omega, \xi_\omega, \mathcal{H}_\omega)$ and $(\pi_{\omega'}, \xi_{\omega'}, \mathcal{H}_{\omega'})$ be their associated GNS-representations. An interesting problem is the following: under which conditions on ω and ω' are the representations π_ω and $\pi_{\omega'}$ unitarily equivalent?

It is somehow more convenient to consider first the following preliminary problem: *how must ω and ω' be related for $\pi_{\omega'}$ to be unitarily equivalent to a sub- $\ast$ -representation of π_ω ?* An easy proof shows that

$\pi_{\omega'}$ is unitarily equivalent to a sub- $\ast$ -representation of π_ω if, and only if, there exists a sequence $\{b_n\}$ of elements of $\mathcal{A}$ such that $\omega'(a) = \lim_{n \rightarrow \infty} \omega(b_n^\ast a b_n)$, $\forall a \in \mathcal{A}$, and the sequence $\{\pi_\omega(b_n)\xi_\omega\}$ converges in $\mathcal{H}_\omega$.

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We refer to [4] for the physical implications of this result. Here we observe that, in particular, if ω is a positive linear functional on $\mathcal{A}$, and $b \in \mathcal{A}$ a fixed element such that $\omega(b^*b) \neq 0$, then the GNS-representation associated to $\omega_b(\cdot) = \omega(b^* \cdot b)$ is unitarily equivalent to a sub- $*$ -representation of π_ω . This means that there exists a subspace $\mathcal{H}_\omega^b$ of $\mathcal{H}_\omega$ and a unitary operator $U : \mathcal{H}_{\omega_b} \rightarrow \mathcal{H}_\omega^b$ such that $\pi_{\omega_b}(a) = U^* \pi_\omega^b(a) U$ for all $a \in \mathcal{A}$, where $\pi_\omega^b(a) := \pi_\omega(a) \upharpoonright_{\mathcal{H}_\omega^b}$.

Going back to our original question, i.e. to the unitary equivalence of π_ω and $\pi_{\omega'}$, we will postpone this analysis to the next section, where the more relevant case of quasi $*$ -algebras is discussed.

Let now δ be a $*$ -derivation on $\mathcal{A}$ and let us define $\delta_{\pi_\omega^b}(a) = \pi_\omega^b(\delta(a))$ and $\delta_{\pi_{\omega_b}}(a) = \pi_{\omega_b}(\delta(a))$, $a \in \mathcal{A}$. The first obvious remark is that, under our assumptions,

$$\delta_{\pi_{\omega_b}}(a) = \pi_{\omega_b}(\delta(a)) = U^* \pi_\omega^b(\delta(a)) U = U^* \delta_{\pi_\omega^b}(a) U.$$

Secondly, if $\delta_{\pi_\omega^b}(a)$ is spatial, i.e. there exists an element $H_{\pi_\omega^b} \in B(\mathcal{H}_\omega^b)$ such that $\delta_{\pi_\omega^b}(a) = i[H_{\pi_\omega^b}, \pi_\omega^b(a)]$, $a \in \mathcal{A}$, then $\delta_{\pi_{\omega_b}}$ is also spatial and the implementing operator is $H_{\pi_{\omega_b}} = U^* H_{\pi_\omega^b} U$, which belongs to $B(\mathcal{H}_{\omega_b})$.

From a physical point of view we can interpret this result as follows: it is well known that no hamiltonian operator exists in general which implements the dynamics of an infinitely extended system [4]. For this reason one has to consider a finite-volume approximation of the system, for which a self-adjoint energy operator H_V can be defined. Associated to H_V we can introduce a finite-volume derivation $\delta_V(X) = i[H_V, X]$, for each observable X localized in V , and a time evolution $\alpha_V^t(X) = e^{iH_V t} X e^{-iH_V t}$. However, usually, neither $\delta_V(X)$ nor $\alpha_V^t(X)$ converge in the uniform, strong or weak topology. One usually has to consider some representation of the abstract algebra and, as in [1], the corresponding family of *effective derivations*, i.e. derivations in the given representation. This net of derivations may now be converging and, under suitable conditions, it still defines a derivation whose implementing operator is the effective hamiltonian. Therefore the choice of the representations in this procedure is crucial. Our results show that, in fact, there is no essential difference between the effective hamiltonians that we obtain starting from two different representations, at least if they are GNS generated by a fixed positive linear functional ω and by a different positive linear functional $\omega' = \omega_b$, for each possible choice of $b \in \mathcal{A}$. In particular this implies that, if b is a local observable (meaning by this that it belongs to some of the $\mathcal{A}_V$'s which produce the quasi-local C^* -algebra [4,3]), then the two related derivations are unitarily equivalent and, consequently, the two effective hamiltonians are unitarily equivalent as well. Hence their physical content is essentially the same, as claimed before.

2. The case of quasi $*$ -algebras

We begin this section with recalling briefly the definitions of quasi $*$ -algebras and their $*$ -representations and sub- $*$ -representations. More details can be found in [5,6].

Let $\mathcal{A}$ be a complex vector space and $\mathcal{A}_0$ a $*$ -algebra contained in $\mathcal{A}$. We say that $\mathcal{A}$ is a quasi $*$ -algebra with distinguished $*$ -algebra $\mathcal{A}_0$ (or, simply, over $\mathcal{A}_0$) if

- (i) the left multiplication ax and the right multiplication xa of an element a of $\mathcal{A}$ and an element x of $\mathcal{A}_0$ which extend the multiplication of $\mathcal{A}_0$ are always defined and bilinear;
- (ii) $x_1(x_2a) = (x_1x_2)a$ and $x_1(ax_2) = (x_1a)x_2$, for each $x_1, x_2 \in \mathcal{A}_0$ and $a \in \mathcal{A}$;
- (iii) an involution $*$ which extends the involution of $\mathcal{A}_0$ is defined in $\mathcal{A}$ with the property $(ax)^* = x^*a^*$ and $(xa)^* = a^*x^*$ for each $x \in \mathcal{A}_0$ and $a \in \mathcal{A}$.

Let now $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ the set of all (closable) linear operators X such that $\mathcal{D}(X) = \mathcal{D}$, $\mathcal{D}(X^*) \supseteq \mathcal{D}$.

The set $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is a partial $*$ -algebra with respect to the following operations: the usual sum $X_1 + X_2$, the scalar multiplication λX , the involution $X \mapsto X^\dagger = X^* \upharpoonright_{\mathcal{D}}$ and the (weak) partial multiplication $X_1 \square X_2 = X_1^{\dagger*} X_2$, defined whenever X_2 is a weak right multiplier of X_1 (we shall write $X_2 \in R^w(X_1)$ or $X_1 \in L^w(X_2)$), that is, iff $X_2 \mathcal{D} \subset \mathcal{D}(X_1^{\dagger*})$ and $X_1^* \mathcal{D} \subset \mathcal{D}(X_2^*)$.

Let $\mathcal{L}^\dagger(\mathcal{D})$ be the subspace of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ consisting of all its elements which leave, together with their adjoints, the domain $\mathcal{D}$ invariant. Then $\mathcal{L}^\dagger(\mathcal{D})$ is a $*$ -algebra with respect to the usual operations.

Let $(\mathcal{A}, \mathcal{A}_0)$ be a quasi $*$ -algebra with identity e and $\mathcal{D}_\pi$ a dense domain in a certain Hilbert space $\mathcal{H}_\pi$. A linear map π from $\mathcal{A}$ into $\mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)$ such that:

- (i) $\pi(a^*) = \pi(a)^\dagger$, $\forall a \in \mathcal{A}$,
- (ii) if $a \in \mathcal{A}$, $x \in \mathcal{A}_0$, then $\pi(a) \square \pi(x)$ is well defined and $\pi(ax) = \pi(a) \square \pi(x)$, is called a $*$ -representation of $\mathcal{A}$. Moreover,
- (iii) if $\pi(\mathcal{A}_0) \subset \mathcal{L}^\dagger(\mathcal{D}_\pi)$, then π is said to be a $*$ -representation of the quasi $*$ -algebra $(\mathcal{A}, \mathcal{A}_0)$.

If π is a $*$ -representation of $(\mathcal{A}, \mathcal{A}_0)$, then the *closure* $\tilde{\pi}$ of π is defined, for each $x \in \mathcal{A}$, as the restriction of $\overline{\pi(x)}$ to the domain $\widehat{\mathcal{D}_\pi}$, which is the completion of $\mathcal{D}_\pi$ under the *graph topology* t_π [5] defined by the seminorms $\xi \in \mathcal{D}_\pi \rightarrow \|\pi(a)\xi\|$, $a \in \mathcal{A}$. If $\pi = \tilde{\pi}$ the representation is said to be *closed*.

The adjoint of a $*$ -representation π of a quasi $*$ -algebra $(\mathcal{A}, \mathcal{A}_0)$ is defined as follows:

$$\mathcal{D}_{\pi^*} \equiv \bigcap_{x \in \mathcal{A}} \mathcal{D}(\pi(x)^*) \quad \text{and} \quad \pi^*(x) = \pi(x^*)^* \upharpoonright_{\mathcal{D}_{\pi^*}}, \quad x \in \mathcal{A}.$$

The representation π is said to be *self-adjoint* if $\pi = \pi^*$.

The representation π is said to be *ultra-cyclic* if there exists $\xi_0 \in \mathcal{D}_\pi$ such that $\mathcal{D}_\pi = \pi(\mathcal{A}_0)\xi_0$, while is said to be *cyclic* if there exists $\xi_0 \in \mathcal{D}_\pi$ such that $\pi(\mathcal{A}_0)\xi_0$ is dense in $\mathcal{D}_\pi$ w.r.t. t_π .

Definition 1. Let π be a $*$ -representation of $\mathcal{A}$. A subspace $\mathcal{M} \subset \mathcal{D}_\pi$ is said to be *quasi-invariant* for π if $\pi(\mathcal{A}_0)\mathcal{M} \subset \mathcal{M}$ and $\pi(\mathcal{A})\mathcal{M} \subset \overline{\mathcal{M}}$, the closure of $\mathcal{M}$ in the Hilbert norm of $\mathcal{H}_\pi$. Moreover the quasi-invariant subspace $\mathcal{M}$ is called *ultra-cyclic* if there exists $\xi_0 \in \mathcal{M}$ such that $\mathcal{M} = \pi(\mathcal{A}_0)\xi_0$. $\mathcal{M}$ is called *cyclic* if there exists $\xi_0 \in \mathcal{M}$ such that $\pi(\mathcal{A}_0)\xi_0$ is dense in $\mathcal{M}$ w.r.t. t_π .

Proposition 2. Let π be a $*$ -representation of $\mathcal{A}$ and $\mathcal{M}$ a quasi-invariant subspace of $\mathcal{D}_\pi$ for π . We put

$$\begin{cases} \mathcal{D}_{\pi \upharpoonright \mathcal{M}} := \mathcal{M}, \\ (\pi \upharpoonright \mathcal{M})(x) := \pi(x) \upharpoonright \mathcal{M}, \quad x \in \mathcal{A}. \end{cases}$$

Then $\pi \upharpoonright \mathcal{M}$ is a $*$ -representation of $\mathcal{A}$ with domain $\mathcal{M}$ in $\overline{\mathcal{M}}$. Let $\pi_{\mathcal{M}}$ denote the closure of $\pi \upharpoonright \mathcal{M}$. Then

- (i) if $\mathcal{M}$ is ultra-cyclic then $\pi \upharpoonright \mathcal{M}$ is ultra-cyclic and $\pi_{\mathcal{M}}$ is cyclic;
- (ii) if $\mathcal{M}$ is cyclic then $\pi \upharpoonright \mathcal{M}$ and $\pi_{\mathcal{M}}$ are cyclic.

In the sequel we will also need the following definitions:

Definition 3. Let ρ and π be $*$ -representations of $\mathcal{A}$ respectively on $\mathcal{D}_\rho \subset \mathcal{H}_\rho$ and $\mathcal{D}_\pi \subset \mathcal{H}_\pi$. Then ρ and π are unitarily equivalent if there exists a unitary operator $U: \mathcal{H}_\rho \rightarrow \mathcal{H}_\pi$ such that $U\mathcal{D}_\rho = \mathcal{D}_\pi$ and $\rho(x) = U^*\pi(x)U$, for all $x \in \mathcal{A}$.

Definition 4. Let π be a $*$ -representation of $\mathcal{A}$. Then π' is a sub- $*$ -representation of π if and only if $\pi' = \pi \upharpoonright \mathcal{M}$, for a certain quasi-invariant subspace $\mathcal{M}$ of $\mathcal{D}_\pi$. Furthermore π' is a closed sub- $*$ -representation of π if and only if $\pi' = \pi_{\mathcal{M}}$, for a certain quasi-invariant subspace $\mathcal{M}$ of $\mathcal{D}_\pi$.

The following proposition, proved by one of us in [8], extends the GNS construction to quasi $*$ -algebras.

Proposition 5. Let ω be a linear functional on $\mathcal{A}$ satisfying the following requirements:

- (L1) $\omega(a^*a) \geq 0$ for all $a \in \mathcal{A}_0$;
- (L2) $\omega(b^*x^*a) = \overline{\omega(a^*xb)}$, $\forall a, b \in \mathcal{A}_0, x \in \mathcal{A}$;
- (L3) $\forall x \in \mathcal{A}$ there exists $\gamma_x > 0$ such that $|\omega(x^*a)| \leq \gamma_x \omega(a^*a)^{1/2}$.

Then there exists a triple $(\pi_\omega, \lambda_\omega, \mathcal{H}_\omega)$ such that

- π_ω is an ultra-cyclic $*$ -representation of $\mathcal{A}$ with ultra-cyclic vector ξ_ω ;
- λ_ω is a linear map of $\mathcal{A}$ into $\mathcal{H}_\omega$ with $\lambda_\omega(\mathcal{A}_0) = \mathcal{D}_{\pi_\omega}$, $\xi_\omega = \lambda_\omega(e)$ and $\pi_\omega(x)\lambda_\omega(a) = \lambda_\omega(xa)$, for every $x \in \mathcal{A}, a \in \mathcal{A}_0$;
- $\omega(x) = \langle \pi_\omega(x)\xi_\omega | \xi_\omega \rangle$, for every $x \in \mathcal{A}$.

The representation π_ω satisfies the properties: (1) $\pi_{\omega_0} = \pi_\omega \upharpoonright_{\mathcal{A}_0}$; (2) $\pi_\omega(x)\lambda_\omega(a) = \lambda_\omega(xa)$, $x \in \mathcal{A}, a \in \mathcal{A}_0$, and (3) $\pi_\omega^*(a)\lambda_\omega(x) = \lambda_\omega(ax)$, $x \in \mathcal{A}, a \in \mathcal{A}_0$. Here π_ω^* denotes the adjoint representation of π , see [5,6].

For shortness, a linear functional ω on $\mathcal{A}$ satisfying (L1)–(L3) will be called a *representable* functional on $\mathcal{A}$. If ω is representable, $(\pi_\omega, \lambda_\omega, \mathcal{H}_\omega)$ will be called, as usual, the GNS construction for ω .

It is possible to check that conditions (L1)–(L3) are stable under the map $\omega \rightarrow \omega_b$, with $b \in \mathcal{A}_0$. This means that, if ω is representable, then ω_b is representable, for every $b \in \mathcal{A}_0$. We only prove (L3) since (L1) and (L2) are trivial. We have

$$|\omega_b(x^*a)| = |\omega((xb)^*a)| \leq \gamma_{xb}\omega((ab)^*ab)^{1/2} = \gamma_{xb}\omega_b(a^*a)^{1/2}.$$

Hence ω_b produces a GNS representation as well, so that it is worth comparing the two representations arising from ω and ω_b , in view of extending to quasi $*$ -algebras what we discussed in the first section for C^* -algebras.

We start with considering the following question: *when a representable linear functional ω' can be written as $\omega' = \omega_b$, for some $b \in \mathcal{A}_0$?* To answer this question we give the following

Proposition 6. Let ω' and ω be representable linear functionals on $\mathcal{A}$. Then $\omega' = \omega_b$ for some $b \in \mathcal{A}_0$ if and only if $\pi_{\omega'}$ is unitarily equivalent to a sub- $*$ -representation of π_ω .

Proof. Suppose first that $\omega' = \omega_b$ for some $b \in \mathcal{A}_0$. For every $x \in \mathcal{A}$ and $a, c \in \mathcal{A}_0$, we have

$$\omega_b(c^*xa) = \langle \pi_{\omega_b}(x)\lambda_{\omega_b}(a) | \lambda_{\omega_b}(c) \rangle. \quad (2.1)$$

On the other hand,

$$\omega_b(c^*xa) = \omega(b^*c^*xab) = \langle \pi_\omega(x)\pi_\omega(a)\lambda_\omega(b) | \pi_\omega(c)\lambda_\omega(b) \rangle. \quad (2.2)$$

Now put $\mathcal{H}_\omega^b := \overline{\pi_\omega(\mathcal{A}_0)\lambda_\omega(b)}$. Then, from equality (2.1), it follows that there exists a unitary operator $U : \mathcal{H}_\omega^b \rightarrow \mathcal{H}_{\omega_b}$ such that

$$U\pi_\omega(a)\lambda_\omega(b) = \lambda_{\omega_b}(a), \quad \forall a \in \mathcal{A}_0.$$

From (2.2) we deduce that, for every $a \in \mathcal{A}$ and $a, c \in \mathcal{A}_0$,

$$\begin{aligned} \langle \pi_\omega(x)\pi_\omega(a)\lambda_\omega(b) | \pi_\omega(c)\lambda_\omega(b) \rangle &= \langle \pi_{\omega_b}(x)\lambda_{\omega_b}(a) | \lambda_{\omega_b}(c) \rangle \\ &= \langle \pi_{\omega_b}(x)U\pi_\omega(a)\lambda_\omega(b) | U\pi_\omega(c)\lambda_\omega(b) \rangle \\ &= \langle U^*\pi_{\omega_b}(x)U\pi_\omega(a)\lambda_\omega(b) | \pi_\omega(c)\lambda_\omega(b) \rangle. \end{aligned}$$

This implies that

$$\pi_\omega^b(x) := \pi_\omega(x)|_{\pi_\omega(\mathcal{A}_0)\lambda_\omega(b)} = U^*\pi_{\omega_b}(x)U|_{\pi_\omega(\mathcal{A}_0)\lambda_\omega(b)}.$$

Hence, $\pi_\omega(\mathcal{A}_0)\lambda_\omega(b)$ is a quasi-invariant subspace for π_ω , that is, $\pi_\omega(\mathcal{A})\pi_\omega(\mathcal{A}_0)\lambda_\omega(b) \subseteq \overline{\pi_\omega(\mathcal{A}_0)\lambda_\omega(b)}$ and so π_ω^b is a sub- $*$ -representation of π_ω with ultra-cyclic vector $\lambda_\omega(b)$, and it is unitarily equivalent to π_{ω_b} .

Conversely, suppose that $\pi_{\omega'}$ is unitarily equivalent to a sub- $*$ -representation of π_ω . Then there exists a quasi-invariant subspace $\mathcal{M}$ of $\mathcal{D}_{\pi_\omega}$, and a unitary operator $U : \mathcal{H}_{\omega'} \rightarrow \overline{\mathcal{M}} \subset \mathcal{H}_\omega$ such that $U\lambda_{\omega'}(\mathcal{A}_0) = \mathcal{M} \subset \lambda_\omega(\mathcal{A}_0) = \mathcal{D}_{\pi_\omega}$ and $\pi_{\omega'}(x) = U^*(\pi|_{\mathcal{M}})(x)U$, $\forall x \in \mathcal{A}$. Since $U\lambda_{\omega'}(e) \in \mathcal{M} \subset \lambda_\omega(\mathcal{A}_0)$, then there exists $b \in \mathcal{A}_0$ such that $U\lambda_{\omega'}(e) = \lambda_\omega(b)$. Thus, for every $x \in \mathcal{A}$,

$$\begin{aligned} \omega'(x) &= \langle \pi_{\omega'}(x)\lambda_{\omega'}(e) | \lambda_{\omega'}(e) \rangle = \langle \pi_\omega(x)U\lambda_{\omega'}(e) | U\lambda_{\omega'}(e) \rangle \\ &= \langle \pi_\omega(x)\lambda_\omega(b) | \lambda_\omega(b) \rangle = \omega_b(x). \quad \square \end{aligned}$$

We now consider a slightly generalized problem, looking for conditions under which a representable linear functional ω' on $\mathcal{A}$ can be written as $\omega' = \lim_\alpha \omega_{b_\alpha}$ for some net $\{b_\alpha\}$ in $\mathcal{A}_0$.

Proposition 7. Let ω' and ω be representable linear functionals on $\mathcal{A}$. Then $\omega' = \lim_\alpha \omega_{b_\alpha}$ for some net $\{b_\alpha\}$ in $\mathcal{A}_0$ such that $\{\pi_\omega(b_\alpha)\xi_\omega\}$ converges w.r.t. t_{π_ω} if, and only if, $\pi_{\omega'}$ is unitarily equivalent to a sub- $*$ -representation of $\tilde{\pi}_\omega$.

Proof. Suppose that $\omega' = \lim_\alpha \omega_{b_\alpha}$, for some net $\{b_\alpha\}$ in $\mathcal{A}_0$ such that $\{\pi_\omega(b_\alpha)\xi_\omega\}$ converges w.r.t. t_{π_ω} . Then, it is easily shown that $\mathcal{M} := \tilde{\pi}_\omega(\mathcal{A}_0)\xi_0$ is a quasi-invariant subspace of $\mathcal{D}_{\tilde{\pi}_\omega}$, where $\xi_0 := t_{\pi_\omega} - \lim_\alpha \pi_\omega(b_\alpha)\xi_\omega$. For every $x \in \mathcal{A}$ and every $a, c \in \mathcal{A}_0$, we have

$$\begin{aligned} \langle \pi_{\omega'}(x)\lambda_{\omega'}(a) | \lambda_{\omega'}(c) \rangle &= \omega'(c^*xa) \\ &= \lim_\alpha \omega(b_\alpha^*c^*xab_\alpha) \\ &= \lim_\alpha \langle \pi_\omega(xa)\lambda_\omega(b_\alpha) | \pi_\omega(c)\lambda_\omega(b_\alpha) \rangle \\ &= \langle \tilde{\pi}_\omega(xa)\xi_0 | \tilde{\pi}_\omega(c)\xi_0 \rangle \\ &= \langle (\tilde{\pi}_\omega|_{\mathcal{M}})(x)\tilde{\pi}_\omega(a)\xi_0 | \tilde{\pi}_\omega(c)\xi_0 \rangle. \end{aligned} \quad (2.3)$$

Here we put

$$U\tilde{\pi}_\omega(a)\xi_0 = \lambda_{\omega'}(a), \quad a \in \mathcal{A}_0.$$

Then U extends to a unitary operator of $\overline{\mathcal{M}}$ onto $\mathcal{H}_{\omega'}$, which we denote with the same symbol, such that $U\mathcal{M} = \lambda_{\omega'}(\mathcal{A}_0) = \mathcal{D}_{\pi_{\omega'}}$. Furthermore, by (2.3), we have

$$\begin{aligned} \langle \pi_{\omega'}(x)\lambda_{\omega'}(a) | \lambda_{\omega'}(c) \rangle &= \langle (\tilde{\pi}_\omega|_{\mathcal{M}})(x)\tilde{\pi}_\omega(a)\xi_0 | \tilde{\pi}_\omega(c)\xi_0 \rangle \\ &= \langle (\tilde{\pi}_\omega|_{\mathcal{M}})(x)U^*\lambda_{\omega'}(a) | U^*\lambda_{\omega'}(c) \rangle \\ &= \langle U(\tilde{\pi}_\omega|_{\mathcal{M}})(x)U^*\lambda_{\omega'}(a) | \lambda_{\omega'}(c) \rangle, \end{aligned}$$

for each $a, c \in \mathcal{A}_0$ and $x \in \mathcal{A}$, which implies that

$$\pi_{\omega'}(x) = U(\tilde{\pi}_\omega|_{\mathcal{M}})(x)U^*, \quad \forall x \in \mathcal{A}.$$

Thus $\pi_{\omega'}$ is unitarily equivalent to a sub- $*$ -representation $\tilde{\pi}_{\omega} \upharpoonright \mathcal{M}$ of $\tilde{\pi}_{\omega}$. Conversely, suppose $\pi_{\omega'}$ is unitarily equivalent to a sub- $*$ -representation of $\tilde{\pi}_{\omega}$. Then, there exists a quasi-invariant subspace of $\mathcal{D}_{\tilde{\pi}_{\omega}}$, $\mathcal{M}$, and a unitary operator $U : \mathcal{H}_{\omega'} \rightarrow \overline{\mathcal{M}}$ such that $U\lambda_{\omega'}(\mathcal{A}_0) = \mathcal{M} \subset \mathcal{D}_{\tilde{\pi}_{\omega}}$ and $\pi_{\omega'}(x) = U^*(\pi_{\omega} \upharpoonright \mathcal{M})(x)U$, $\forall x \in \mathcal{A}$. Since $U\lambda_{\omega'}(e) \in \mathcal{M} \subset \mathcal{D}_{\tilde{\pi}_{\omega}}$, there exists $\{b_{\alpha}\} \subset \mathcal{A}_0$ such that $\lambda_{\omega}(b_{\alpha}) = \pi_{\omega}(b_{\alpha})\xi_{\omega} \rightarrow U\lambda_{\omega'}(e)$, in the topology $t_{\pi_{\omega}}$. Hence,

$$\begin{aligned}\omega'(x) &= \langle \pi_{\omega'}(x)\lambda_{\omega'}(e) | \lambda_{\omega'}(e) \rangle \\ &= \langle \pi_{\omega}(x)U\lambda_{\omega'}(e) | U\lambda_{\omega'}(e) \rangle \\ &= \lim_{\alpha} \langle \pi_{\omega}(x)\lambda_{\omega}(b_{\alpha}) | \lambda_{\omega}(b_{\alpha}) \rangle \\ &= \lim_{\alpha} \omega_{b_{\alpha}}(x),\end{aligned}$$

for every $x \in \mathcal{A}$. $\square$

The previous propositions, and in particular Proposition 6, show that, for every $b \in \mathcal{A}_0$ such that $\omega(b^*b) \neq 0$, ω and ω_b produce close GNS representations and the same physical considerations given in Section 1 can also be repeated here, with no major change. In particular we consider now some consequences of our results on the theory of spatial derivations in the quasi $*$ -algebraic setting discussed in [1,2]. To keep the paper self-contained, let us first recall few definitions. Let $(\mathcal{A}, \mathcal{A}_0)$ be a quasi $*$ -algebra. A $*$ -derivation of $\mathcal{A}_0$ is a map $\delta : \mathcal{A}_0 \rightarrow \mathcal{A}$ with the following properties:

- (i) $\delta(a^*) = \delta(a)^*$, $\forall a \in \mathcal{A}_0$;
- (ii) $\delta(\alpha a + \beta b) = \alpha \delta(a) + \beta \delta(b)$, $\forall a, b \in \mathcal{A}_0$, $\forall \alpha, \beta \in \mathbb{C}$;
- (iii) $\delta(ab) = a\delta(b) + \delta(a)b$, $\forall a, b \in \mathcal{A}_0$.

Further, let π be a $*$ -representation of $(\mathcal{A}, \mathcal{A}_0)$. As in [1] we will always assume that whenever $a \in \mathcal{A}_0$ is such that $\pi(a) = 0$, then $\pi(\delta(a)) = 0$ as well. Under this assumption, the linear map $\delta_{\pi}(\pi(a)) = \pi(\delta(a))$, $a \in \mathcal{A}_0$, is well defined on $\pi(\mathcal{A}_0)$ with values in $\pi(\mathcal{A})$ and it is a $*$ -derivation of $\pi(\mathcal{A}_0)$. We call δ_{π} the $*$ -derivation induced by π . Given such a representation π and its dense domain $\mathcal{D}_{\pi}$, we consider the usual graph topology $t_{\dagger}$ generated by the seminorms $\xi \in \mathcal{D}_{\pi} \rightarrow \|\mathcal{A}\xi\|$, $A \in \mathcal{L}^{\dagger}(\mathcal{D}_{\pi})$.

If $\mathcal{D}'_{\pi}$ denotes the conjugate dual of $\mathcal{D}_{\pi}$, we get the usual rigged Hilbert space $\mathcal{D}_{\pi}[t_{\dagger}] \subset \mathcal{H}_{\pi} \subset \mathcal{D}'_{\pi}[t'_{\dagger}]$, where $t'_{\dagger}$ is the strong dual topology of $\mathcal{D}'_{\pi}$. As usual, we denote by $\mathcal{L}(\mathcal{D}_{\pi}, \mathcal{D}'_{\pi})$ the space of all continuous linear maps from $\mathcal{D}_{\pi}[t_{\dagger}]$ into $\mathcal{D}'_{\pi}[t'_{\dagger}]$. In this case, $\mathcal{L}^{\dagger}(\mathcal{D}_{\pi}) \subset \mathcal{L}(\mathcal{D}_{\pi}, \mathcal{D}'_{\pi})$. Each operator $A \in \mathcal{L}^{\dagger}(\mathcal{D}_{\pi})$ can be extended to the whole $\mathcal{D}'_{\pi}$ by putting

$$\langle \hat{A}\xi', \eta \rangle = \langle \xi', A^{\dagger}\eta \rangle, \quad \forall \xi' \in \mathcal{D}'_{\pi}, \quad \eta \in \mathcal{D}_{\pi},$$

where $\langle \cdot, \cdot \rangle$ denotes the form which puts $\mathcal{D}_{\pi}$ and $\mathcal{D}'_{\pi}$ in conjugate duality. Hence the multiplication of $X \in \mathcal{L}(\mathcal{D}_{\pi}, \mathcal{D}'_{\pi})$ and $A \in \mathcal{L}^{\dagger}(\mathcal{D}_{\pi})$ can always be defined. Indeed [1], $(X \circ A)\xi = X(A\xi)$, and $(A \circ X)\xi = \hat{A}(X\xi)$, $\forall \xi \in \mathcal{D}_{\pi}$. With these definitions, however, $(\mathcal{L}(\mathcal{D}_{\pi}, \mathcal{D}'_{\pi}), \mathcal{L}^{\dagger}(\mathcal{D}_{\pi}))$ may fail to be a quasi $*$ -algebra, since the operator $X \circ A$ need not be continuous from $\mathcal{D}_{\pi}[t_{\dagger}]$ into $\mathcal{D}'_{\pi}[t'_{\dagger}]$, unless some additional condition, like the reflexivity of $\mathcal{D}_{\pi}[t_{\dagger}]$, is fulfilled. From now on, we will assume that $\mathcal{D}_{\pi}[t_{\dagger}]$ is a reflexive space. This assumption (which was missing in [1]) even though restrictive, is fulfilled in most of the physical models considered so far [3].

Given a derivation δ of $(\mathcal{A}, \mathcal{A}_0)$ and a $*$ -representation π of $(\mathcal{A}, \mathcal{A}_0)$, that we suppose to be cyclic with cyclic vector ξ_0 , the induced derivation δ_{π} is spatial if there exists $H_{\pi} = H_{\pi}^{\dagger} \in \mathcal{L}(\mathcal{D}_{\pi}, \mathcal{D}'_{\pi})$ such that $H_{\pi}\xi_0 \in \mathcal{H}_{\pi}$ and

$$\delta_{\pi}(\pi(x)) = i\{H_{\pi} \circ \pi(x) - \pi(x) \circ H_{\pi}\}, \quad \forall x \in \mathcal{A}_0.$$

Let $(\mathcal{A}, \mathcal{A}_0)$ be a locally convex quasi $*$ -algebra with locally convex topology τ . In [1] we have found necessary and sufficient conditions for an induced derivation to be spatial. One of these conditions is the following:

there exists a positive linear functional f on $\mathcal{A}_0$ such that:

$$f(a^*a) \leq p(a)^2, \quad \forall a \in \mathcal{A}_0, \quad (2.4)$$

for some continuous seminorm p of τ and, denoting with $\tilde{f}$ the continuous extension of f to $\mathcal{A}$, the following inequality holds:

$$|\tilde{f}(\delta(a))| \leq C(\sqrt{f(a^*a)} + \sqrt{f(aa^*)}), \quad \forall a \in \mathcal{A}_0, \quad (2.5)$$

for some positive constant C .

Suppose now that ω_0 is a positive linear representable functional on $\mathcal{A}_0$ satisfying condition (2.4). Let $\omega := \tilde{\omega}_0$ be the continuous extension of ω_0 to $\mathcal{A}$, that is

$$\omega(x) = \lim_{\alpha} \omega_0(a_{\alpha}), \quad x \in \mathcal{A},$$

where a_α is a net in $\mathcal{A}_0$ which converges to x w.r.t. τ . Then ω automatically satisfies conditions (L1), (L2) and (L3). Indeed, (L1) is clear since ω_0 is positive by assumption. As for (L2), let $x \in \mathcal{A}$ and $\{x_\alpha\} \subset \mathcal{A}_0$ be a net τ -converging to x . Since ω_0 is hermitian we have $\omega_0(b^*x_\alpha^*a) = \overline{\omega_0(a^*x_\alpha b)}$, for all $a, b \in \mathcal{A}_0$. Because of (2.4), taking the limit on α of this equality we get (L2). To prove (L3) we first use the Schwarz inequality on $\mathcal{A}_0$: $|\omega_0(x_\alpha a)| \leq \omega_0(x_\alpha^*x_\alpha)^{1/2}\omega_0(a^*a)^{1/2}$. But $\omega_0(x_\alpha^*x_\alpha)^{1/2} \leq p(x_\alpha)^2 \rightarrow p(x)^2$ so that

$$|\omega(xa)| = \lim_\alpha |\omega_0(x_\alpha a)| \leq p(x)\omega(a^*a)^{1/2}$$

which is (L3).

Suppose that ω_0 is a positive linear representable functional on $\mathcal{A}_0$ satisfying both conditions (2.4) and (2.5). Then we consider the question as to whether $(\omega_0)_b$ satisfies these same conditions. This is important for the following reason. If both ω_0 and $(\omega_0)_b$ satisfy (2.4) and (2.5), then they have continuous extensions ω and $(\omega_0)_b$ respectively to $\mathcal{A}$ and it turns out that $(\omega_0)_b = \omega_b$. Thus ω_b satisfies conditions (L1), (L2) and (L3) and both δ_{π_ω} and $\delta_{\pi_{\omega_b}}$ are spatial. Hence a relation between the effective hamiltonians can be found.

First we notice that, because of the continuity of the multiplication, we have

$$(\omega_0)_b(a^*a) = \omega_0((ab)^*ab) \leq p(ab)^2 \leq q(a)^2, \quad a \in \mathcal{A}_0$$

for some continuous seminorm q of τ .

Thus we have the following

Proposition 8. *Let $(\mathcal{A}, \mathcal{A}_0)$ be a locally convex quasi $*$ -algebra with locally convex topology τ , δ a $*$ -derivation of $(\mathcal{A}, \mathcal{A}_0)$ and ω_0 a positive linear functional on $\mathcal{A}_0$.*

(1) *Suppose that ω_0 satisfies the condition*

$$\omega_0(a^*a) \leq p(a)^2, \quad \forall a \in \mathcal{A}_0$$

for some continuous seminorm p of τ . Then the continuous extension $\omega := \tilde{\omega}_0$ of ω_0 to $\mathcal{A}$ and every ω_b , $b \in \mathcal{A}_0$, produce the ultra-cyclic GNS-representations π_ω and π_{ω_b} .

(2) *Furthermore, suppose that*

$$|\omega(\delta(a))| \leq C(\sqrt{\omega(a^*a)} + \sqrt{\omega(aa^*)}), \quad \forall a \in \mathcal{A}_0$$

for some positive constant C . Then the $*$ -derivation δ_{π_ω} induced by π_ω is spatial. If $\pi_\omega \upharpoonright \mathcal{A}_0$ is bounded, in particular in the case where $\mathcal{A}_0$ is a C^* -algebra, then the $*$ -derivation $\delta_{\pi_{\omega_b}}$ induced by π_{ω_b} is also spatial for every $b \in \mathcal{A}_0$.

Proof. We need only to prove the last statement in (2). For this we notice that if $b \in \mathcal{A}_0$ is such that $\pi_\omega(b)$ is bounded, then ω_b satisfies (2.5). Indeed, taking into account that, for every $a \in \mathcal{A}_0$, the equality $b^*\delta(a)b = \delta(b^*ab) - \delta(b^*)ab - b^*a\delta(b)$ holds, we have

$$|\omega_b(\delta(a))| = |\omega(b^*\delta(a)b)| \leq |\omega(\delta(b^*ab))| + |\omega(\delta(b^*)ab)| + |\omega(b^*a\delta(b))|.$$

Using (2.5) for the first and introducing π_ω for the second and the third contributions above, we find that, for every $a \in \mathcal{A}_0$,

$$\begin{aligned} |\omega_b(\delta(a))| &\leq C(\omega(b^*a^*bb^*ab)^{1/2} + \omega(b^*abb^*a^*b)^{1/2}) + |\langle \lambda_\omega(ab) | \lambda_\omega(\delta(b)) \rangle| + |\langle \lambda_\omega(\delta(b)) | \lambda_\omega(a^*b) \rangle| \\ &= C(\|\pi_\omega(b)^*\lambda_\omega(ab)\| + \|\pi_\omega(b)^*\lambda_\omega(a^*b)\|) + |\langle \lambda_\omega(ab) | \lambda_\omega(\delta(b)) \rangle| + |\langle \lambda_\omega(\delta(b)) | \lambda_\omega(a^*b) \rangle| \\ &\leq (C\|\overline{\pi_\omega(b)}\| + \|\lambda_\omega(\delta(b))\|)(\omega_b(a^*a)^{1/2} + \omega_b(aa^*)^{1/2}). \quad \square \end{aligned}$$

The conclusion is therefore that, under mild conditions on π_ω , and therefore on ω , both δ_{π_ω} and $\delta_{\pi_{\omega_b}}$ turn out to be spatial so that two different effective hamiltonians H_ω and H_{ω_b} do exist, and they are related as in Section 1. Once again, the physical contents of the two representations is essentially the same.

We end this section with some further results on the GNS representations of a quasi $*$ -algebra $(\mathcal{A}, \mathcal{A}_0)$.

Let $(\mathcal{A}, \mathcal{A}_0)$ be a locally convex quasi $*$ -algebra, ω_0 a positive linear functional on $\mathcal{A}_0$ satisfying (2.4) and $\omega = \tilde{\omega}_0$ its continuous extension on $\mathcal{A}$. As we have shown, both ω and ω_b , $b \in \mathcal{A}_0$, satisfy conditions (L1), (L2) and (L3), and so the GNS-constructions $(\pi_\omega, \lambda_\omega, \mathcal{H}_\omega)$ and $(\pi_{\omega_b}, \lambda_{\omega_b}, \mathcal{H}_{\omega_b})$ are defined. Let $\tilde{\pi}_\omega$ and $\tilde{\pi}_{\omega_b}$ be the closures of π_ω and π_{ω_b} , respectively. In this section we find conditions which imply that $\tilde{\pi}_\omega$ is unitarily equivalent to the direct sum of a family of $\tilde{\pi}_{\omega_b}$, $b \in \mathcal{A}_0$.

Lemma 9. *Let $x \in \mathcal{A}$ and $\{x_\alpha\} \subset \mathcal{A}_0$ such that $\tau - \lim_\alpha x_\alpha = x$, then $\lambda_\omega(x_\alpha) = \lambda_{\omega_0}(x_\alpha) \rightarrow \lambda_\omega(x)$.*

Proof. We begin with proving that $\{\lambda_\omega(x_\alpha)\}$ is a Cauchy net in the Hilbert space $\mathcal{H}_\omega$:

$$\|\lambda_\omega(x_\alpha) - \lambda_\omega(x_\beta)\|^2 = \omega((x_\alpha - x_\beta)^*(x_\alpha - x_\beta)) \leq p(x_\alpha - x_\beta)^2 \rightarrow 0.$$

Therefore there exists a vector $\xi \in \mathcal{H}_\omega$ such that $\lambda_\omega(x_\alpha) \rightarrow \xi$. We now prove that $\xi = \lambda_\omega(x)$. Indeed we have, for every $c \in \mathcal{A}_0$, $\langle \lambda_\omega(x_\alpha) | \lambda_\omega(c) \rangle \rightarrow \langle \xi | \lambda_\omega(c) \rangle$ and, on the other hand, $\langle \lambda_\omega(x_\alpha) | \lambda_\omega(c) \rangle = \omega(c^* x_\alpha) \rightarrow \tilde{\omega}(c^* x) = \langle \lambda_\omega(x) | \lambda_\omega(c) \rangle$, due to the definition of $\tilde{\omega}$. Therefore $\xi = \lambda_\omega(x)$. $\square$

We recall that the weak commutant $\mathfrak{M}'_\omega$ of a $*$ -invariant subset $\mathfrak{M}$ of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is defined as

$$\mathfrak{M}'_\omega = \{C \in \mathcal{B}(\mathcal{H}): \langle X\xi | C^* \eta \rangle = \langle C\xi | X^\dagger \eta \rangle, \forall X \in \mathfrak{M}, \xi, \eta \in \mathcal{D}\}.$$

Then we can prove the following

Proposition 10. $\pi_\omega(\mathcal{A})'_\omega = \pi_\omega(\mathcal{A}_0)'_\omega$.

Proof. The inclusion $\pi_\omega(\mathcal{A})'_\omega \subset \pi_\omega(\mathcal{A}_0)'_\omega$ is clear. To prove the converse inclusion we take $C \in \pi_\omega(\mathcal{A}_0)'_\omega$ and $x \in \mathcal{A}$, $c_1, c_2 \in \mathcal{A}_0$. Then we have, using the previous lemma,

$$\begin{aligned} \langle C\pi_\omega(x)\lambda_\omega(c_1) | \lambda_\omega(c_2) \rangle &= \lim_\alpha \langle C\pi_\omega(x_\alpha)\lambda_\omega(c_1) | \lambda_\omega(c_2) \rangle \\ &= \lim_\alpha \langle C\lambda_\omega(c_1) | \pi_\omega(x_\alpha^*)\lambda_\omega(c_2) \rangle = \langle C\lambda_\omega(c_1) | \pi_\omega(x^*)\lambda_\omega(c_2) \rangle. \quad \square \end{aligned}$$

Let $b \in \mathcal{A}_0$. We denote by P_ω^b the projection of $\mathcal{H}_\omega$ onto $\mathcal{H}_\omega^b = \overline{\pi_\omega(\mathcal{A}_0)\lambda_\omega(b)}$. By Lemma 9 we deduce the following

Lemma 11. Suppose that $\pi_\omega(a)$ is bounded for every $a \in \mathcal{A}_0$. Then $\pi_\omega(\mathcal{A})'_\omega$ is a von Neumann algebra and $P_\omega^b \in \pi_\omega(\mathcal{A})'_\omega$.

Even if $\pi_\omega|_{\mathcal{A}_0}$ is bounded, $P_\omega^b D(\tilde{\pi}_\omega) \neq D(\tilde{\pi}_\omega)$ in general. Hence we introduce the following notion:

Definition 12. Let $b \in \mathcal{A}_0$. We say that b is a self-adjoint element for π_ω if $\tilde{\pi}_{\omega_b}$ is a self-adjoint $*$ -representation of $\mathcal{A}$.

By [6, Theorem 7.4.4] we have the following

Lemma 13. Let b be a self-adjoint element for π_ω . Then

- (1) $P_\omega^b \in \pi_\omega(\mathcal{A}_0)'_\omega = \pi_\omega(\mathcal{A})'_\omega$;
- (2) $P_\omega^b \mathcal{D}(\tilde{\pi}_\omega) = \mathcal{D}(\tilde{\pi}_\omega^b)$;
- (3) $\tilde{\pi}_\omega^b = (\tilde{\pi}_\omega)_{P_\omega^b} := P_\omega^b \tilde{\pi}_\omega(\cdot) P_\omega^b$.

By Proposition 6, $\tilde{\pi}_\omega^b$ is unitarily equivalent to $\tilde{\pi}_{\omega_b}$, and by the above lemma we have the following result, which answer our original question.

Proposition 14. Suppose that $\tilde{\pi}_\omega$ is self-adjoint. If there exist a family $\{b_\gamma\}_{\gamma \in \Gamma}$ of self-adjoint elements for π_ω , such that $\{P_\omega^{b_\gamma}\}$ consists of mutually orthogonal projections and $\sum_{\gamma \in \Gamma} P_\omega^{b_\gamma} = I$, then $\tilde{\pi}_\omega$ is unitarily equivalent to $\bigoplus_{\gamma \in \Gamma} \tilde{\pi}_{\omega_{b_\gamma}}$.

3. Local modifications of states

We consider now the particular case in which the C^* -algebra $\mathcal{A}$ is endowed with a *local structure*. Following [9] we construct the local C^* -algebra as follows.

Let $\mathcal{F}$ be a set of indexes directed upward and with an orthonormality relation $\perp$ such that (i) $\forall \alpha \in \mathcal{F}$ there exists $\beta \in \mathcal{F}$ such that $\alpha \perp \beta$; (ii) if $\alpha \leq \beta$ and $\beta \perp \gamma$, $\alpha, \beta, \gamma \in \mathcal{F}$, then $\alpha \perp \gamma$; (iii) if, for $\alpha, \beta, \gamma \in \mathcal{F}$, $\alpha \perp \beta$ and $\alpha \perp \gamma$, there exists $\delta \in \mathcal{F}$ such that $\alpha \perp \delta$ and $\delta \geq \beta, \gamma$.

Let now $\{\mathcal{A}_\alpha(\|\cdot\|_\alpha), \alpha \in \mathcal{F}\}$ be a family of C^* -algebras with C^* -norm $\|\cdot\|_\alpha$, indexed by $\mathcal{F}$, such that (a) if $\alpha \geq \beta$ then $\mathcal{A}_\alpha \supset \mathcal{A}_\beta$; (b) there exists a unique identity e for all $\mathcal{A}_\alpha$'s; (c) if $\alpha \perp \beta$ then $xy = yx$ for all $x \in \mathcal{A}_\alpha$, $y \in \mathcal{A}_\beta$. Let further $\mathcal{A}_0 := \bigcup_\alpha \mathcal{A}_\alpha$. The uniform completion of $\mathcal{A}_0$ is, as it is well known, the quasi-local C^* -algebra¹ with the norm $\|\cdot\|$ inherited by the $\|\cdot\|_\alpha$'s. If we take instead the completion of $\mathcal{A}_0$ w.r.t. a locally convex topology τ which makes the involution and the multiplications continuous we get, in general, a locally convex quasi $*$ -algebra $\mathcal{A}$ which we call a *quasi-local quasi $*$ -algebra*.

¹ This terminology is due to the fact that, in concrete applications, α is quite often a given bounded open region in a d -dimensional space.

Given $x \in \mathcal{A}_0$, there will be some $\beta \in \mathcal{F}$ such that $x \in \mathcal{A}_\beta$. But of course, x also belongs to many other $\mathcal{A}_{\beta'}$, for instance to all those algebras which contains $\mathcal{A}_\beta$ as a sub-algebra. For this reason we introduce a set J_x , related to $x \in \mathcal{A}_0$, which is defined as follows: $J_x = \{\alpha \in \mathcal{F} \text{ such that } x \in \mathcal{A}_\alpha\}$. If we now define $\mathcal{A}_\infty = \bigcap_{\alpha \in \mathcal{F}} \mathcal{A}_\alpha$, then we will work here under the assumption, which is verified for very general discrete and continuous models [4], that $\forall x \in \mathcal{A}_0$, $x \notin \mathcal{A}_\infty$, there exists $\alpha_x \in \mathcal{F}$ such that $\bigcap_{\beta \in J_x} \mathcal{A}_\beta = \mathcal{A}_{\alpha_x}$. We call α_x the *support* of x .

The following definition selects states on $\mathcal{A}$ with a *reasonable asymptotic behavior*. These states, indeed, factorize on regions far enough from the support of a given element.

Definition 15. A state ω over $\mathcal{A}$ is said to be almost clustering (AC) if, $\forall b \in \mathcal{A}_0$ and $\forall \epsilon > 0$, there exists $\alpha_b \in \mathcal{F}$, $\alpha \geq \alpha_b$, such that, $\forall \gamma \perp \alpha$ we have $|\omega(ab) - \omega(a)\omega(b)| \leq \epsilon \|a\|$, $\forall a \in \mathcal{A}_\gamma$.

Similar definitions are given in many textbooks, like [4,9,10], where the physical motivations are discussed in detail. Related to the notion of factorization is also that of *local modification* of a given state. Of course, several definitions of local modifications can be introduced. The most natural one is perhaps the following: ω' is a local modification of ω if there exists $\alpha \in \mathcal{F}$ such that $\forall \gamma \in \mathcal{F}$, $\gamma \perp \alpha$, $\omega'(a) = \omega(a)$ for all $a \in \mathcal{A}_\gamma$. This simply implies that, outside a fixed *region* α , the two states coincide. However this condition is rather strong and has no counterpart in the existing literature on this subject and for this reason will not be considered here. To stay in touch with the existing literature, we rather consider the following definitions.

Definition 16. Given two states ω and ω' over $\mathcal{A}$, ω' is said to be a local modification of type 1 (1LM) of ω if, calling $\pi_{\omega'}$ and π_ω their associated GNS-representations, $\pi_{\omega'}$ is unitarily equivalent to a sub- $*$ -representation of π_ω .

Also, ω' is said to be a local modification of type 2 (2LM) of ω if $\forall \epsilon > 0$, there exists $\alpha_\epsilon \in \mathcal{F}$ such that, $\forall \gamma \in \mathcal{F}$, $\gamma \perp \alpha_\epsilon$, $|\omega'(x) - \omega(x)| \leq \epsilon \|x\|$, $\forall x \in \mathcal{A}_\gamma$.

These definitions are physically motivated essentially from what is discussed in [4]. Just to clarify the situation if, for instance, ω' is a 2LM of ω then they coincide, but for an error of order ϵ , outside a region whose size is, in general, proportional to $1/\epsilon$.

There is an apparent difference between the conditions 1LM and 2LM: if ω' is a 2LM of ω , then ω is a 2LM of ω' . This symmetry is not shared by 1LM. We argue that 2LM could be used for the mathematical description of reversible local operations on a given state while 1LM seems to be more appropriate for describing the action of irreversible operations (like a quantum mechanical measurement).

One immediate consequence of the results of Section 2 and of these definitions is that if $b \in \mathcal{A}_\alpha$ for some $\alpha \in \mathcal{F}$ then the state $\omega_b(\cdot)$ is a 1LM of ω . Less trivial is the proof of the following statement: *let the state ω be AC and $b \in \mathcal{A}_0$ with $\omega(b^\dagger b) = 1$. Then ω_b is a 2LM of ω .* This is not the end of the story. Indeed, let us suppose that ω is AC and that ω' is a 1LM of ω . Therefore there exists a sequence $\{b_n\}$ of elements of $\mathcal{A}_0$ such that $\omega'(a) = \lim_{n \rightarrow \infty} \omega(b_n^* a b_n)$, $\forall a \in \mathcal{A}$, and the sequence $\{\pi_\omega(b_n) \xi_\omega\}$ converges in $\mathcal{H}_\omega$. We suppose now that there exists $n_0 \in \mathbb{N}$ and $\lambda \in \mathcal{F}$ such that, for all $n \geq n_0$, $b_n \in \mathcal{A}_\lambda$. Then ω' is also a 2LM of ω . The proof of these statements are easy and will be omitted here.

We end this section, and the paper, with the following example of what a concrete local modification of a state could be.

Discrete System. Let V be a finite region of a d -dimensional lattice Λ and $|V|$ the number of points in V . The local C^* -algebra $\mathcal{A}_V$ is generated by the Pauli operators $\vec{\sigma}_p = (\sigma_p^1, \sigma_p^2, \sigma_p^3)$ and by the unit 2×2 matrix e_p at every point $p \in V$. The $\vec{\sigma}_p$'s are copies of the Pauli matrices localized in p .

If $V \subset V'$ and $A_V \in \mathcal{A}_V$, then $A_V \rightarrow A_{V'} = A_V \otimes (\bigotimes_{p \in V' \setminus V} e_p)$ defines the natural imbedding of $\mathcal{A}_V$ into $\mathcal{A}_{V'}$.

Let $\vec{n} = (n_1, n_2, n_3)$ be a unit vector in $\mathbb{R}^3$, and put $(\vec{\sigma} \cdot \vec{n}) = n_1 \sigma^1 + n_2 \sigma^2 + n_3 \sigma^3$. Then, denoting as $Sp(\vec{\sigma} \cdot \vec{n})$ the spectrum of $\vec{\sigma} \cdot \vec{n}$, we have $Sp(\vec{\sigma} \cdot \vec{n}) = \{1, -1\}$. Let $|\vec{n}\rangle \in \mathbb{C}^2$ be a unit eigenvector associated with 1.

Let now denote by $\mathbf{n} := \{\vec{n}_p\}_{p \in \Lambda}$ an infinite sequence of unit vectors in $\mathbb{R}^3$ and $|\mathbf{n}\rangle = \bigotimes_p |\vec{n}_p\rangle$ the corresponding unit vector in the infinite tensor product $\mathcal{H}_\infty = \bigotimes_p \mathbb{C}_p^2$. We put $\mathcal{A}_0 = \bigcup_V \mathcal{A}_V$ and $\mathcal{D}_\mathbf{n}^0 = \mathcal{A}_0 |\mathbf{n}\rangle$ and we denote the closure of $\mathcal{D}_\mathbf{n}^0$ in $\mathcal{H}_\infty$ by $\mathcal{H}_\mathbf{n}$. As we saw above, to any sequence $\mathbf{n}$ of three-vectors there corresponds a state $|\mathbf{n}\rangle$ of the system. Such a state defines a realization $\pi_\mathbf{n}$ of $\mathcal{A}_0$ in the Hilbert space $\mathcal{H}_\mathbf{n}$. This representation is faithful, since the norm completion $\mathcal{A}_S$ of $\mathcal{A}_0$ is a simple C^* -algebra. A special basis for $\mathcal{H}_\mathbf{n}$ is obtained from the *ground state* $|\mathbf{n}\rangle$ by *flipping* a finite number of spins using the following strategy:

Let $\vec{n}$ be a unit vector in $\mathbb{R}^3$, as above, and $|\vec{n}\rangle$ the corresponding vector of $\mathbb{C}^2$. Let us choose two other unit vectors $\vec{n}^1, \vec{n}^2$ so that $(\vec{n}, \vec{n}^1, \vec{n}^2)$ form an orthonormal basis of $\mathbb{R}^3$. We put $\vec{n}_\pm = \frac{1}{2}(\vec{n}^1 \pm i\vec{n}^2)$ and define $|m, \vec{n}\rangle := (\vec{\sigma} \cdot \vec{n}_-)^m |\vec{n}\rangle$ ($m = 0, 1$). Then we have

$$(\vec{\sigma} \cdot \vec{n})|m, \vec{n}\rangle = (-1)^m |m, \vec{n}\rangle \quad (m = 0, 1).$$

Thus the set $\{|\mathbf{m}, \mathbf{n}\rangle = \bigotimes_p |m_p, \vec{n}_p\rangle; m_p = 0, 1, \sum_p m_p < \infty\}$ forms an orthonormal basis in $\mathcal{H}_\mathbf{n}$ [11].

The representation $\pi_\mathbf{n}$ is defined on the basis vectors $\{|\mathbf{m}, \mathbf{n}\rangle\}$ by

$$\pi_\mathbf{n}(\sigma_p^i) |\mathbf{m}, \mathbf{n}\rangle = \sigma_p^i |m_p, \vec{n}_p\rangle \otimes \left(\prod_{p' \neq p} \otimes |m_{p'}, \vec{n}_{p'}\rangle \right) \quad (i = 1, 2, 3).$$

This definition is then extended in obvious way to the whole space $\mathcal{H}_{\mathbf{n}}$. It turns out that $\pi_{\mathbf{n}}$ is a *bounded* representation of $\mathcal{A}_0$ into $\mathcal{H}_{\mathbf{n}}$. More details on this construction, particularly in connection with quasi $*$ -algebras, can be found in [7,12].

Let now $\varphi = \bigotimes_{j \in \Lambda} \varphi_j$ be a fixed normalized vector in $\mathcal{H}_{\mathbf{n}}$ and ω the related vector state: if $a \in \mathcal{A}_0$ then $\omega(a) = \langle \varphi, \pi_{\mathbf{n}}(a)\varphi \rangle$. Let now $x = \prod_{p \in \lambda} \otimes x_p$, for some bounded subset λ in Λ . Here x_p acts on $\mathbb{C}_p^2$ and λ is the support of x . Let furthermore γ be another bounded subset of Λ , orthogonal to λ : this means that the sets λ and γ have empty intersection. Then, we fix $b = \prod_{p \in \gamma} \otimes b_p$, where as before b_p acts on $\mathbb{C}_p^2$. We further assume that $\langle \pi_{\mathbf{n}}(b)\varphi, \pi_{\mathbf{n}}(b)\varphi \rangle = 1$. Then we can check that $\omega(a)$ coincides with $\omega_b(a) = \langle \pi_{\mathbf{n}}(b)\varphi, \pi_{\mathbf{n}}(a)\pi_{\mathbf{n}}(b)\varphi \rangle$, and this is true for all possible choices of a and b which are supported in separated regions. So ω_b is a local modification of ω in the strongest sense and, in particular, is a 2LM of ω .

This example shows that the definitions of local modification given here are really physically motivated. States sharing the same properties in the case of continuous physical systems [4] could also be constructed with no major difficulty. To [4] we also refer for a more physically-minded discussion on 1LM of states.

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