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Homoclinic solutions for a class of nonperiodic and noneven second-order Hamiltonian systems ☆

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ABSTRACT

The existence of homoclinic solutions is obtained for second-order Hamiltonian systems $-\ddot{u}(t)+L(t)u(t)=\nabla W(t,u(t))-f(t)$, as the limit of the solutions of a sequence of nil-boundary-value problems which are obtained by the Mountain Pass theorem, when $L(t)$ and $W(t,x)$ are neither periodic nor even with respect to t.

1. Introduction and main results

Consider the second-order Hamiltonian systems

$$-\ddot{u}(t)+L(t)u(t)=\nabla W\big(t,u(t)\big)-f(t), \tag{1}$$

where $L:R\to R^N$ is a matrix valued function, $W:R\times R^N\to R$ is a C^1-map and $f:R\to R^N$. As usual, we say that a solution $u(t)$ of problem (1) is nontrivial homoclinic (to 0) if $u\neq 0$, $u(t)\to 0$ and $\dot{u}(t)\to 0$ as $t\to\pm\infty$. Subsequently, $\nabla W(t,x)$ denotes the gradient with respect to the x variable, $(\cdot,\cdot):R^N\times R^N\to R$ denotes the standard inner product in R^N and $|\cdot|$ is the induced norm.

Homoclinic orbits of dynamical systems are important in applications. Recently the existence and multiplicity of homoclinic orbits for problem (1) have been studied in many papers via critical point theory. In particular, the second-order systems were considered in [1,2,4–14] and that of the first-order in [3]. In 1990, Rabinowitz in [11] obtained the existence of homoclinic orbits of problem (1) for the superquadratic case as the limit of the $2kT$-periodic solutions of problem (1) when $f=0$, which is the following theorem.

Theorem A. *(See [11].) Assume that $f=0$ and the following conditions hold*

(A_1) *L is a continuous T-periodic matrix valued function and $W\in C^1(R\times R^N,R)$ is T-periodic with respect to t, $T>0$,*
(A_2) *$L(t)$ is positive definite symmetric for all $t\in[0,T]$,*

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(A_3) *there is a constant* $\theta > 2$ *such that*

$$0 < \theta W(t,x) \leqslant \big(x, \nabla W(t,x)\big) \tag{2}$$

for every $t \in [0,T]$ *and* $x \in R^N \setminus \{0\}$,

(A_4) $\nabla W(t,x) = o(|x|)$ *as* $|x| \to 0$ *uniformly with respect to* t.

Then system (1) *possesses a nontrivial homoclinic solution.*

When $L(t)$ and $W(t,x)$ are not periodic in t, the problem becomes much complicated, because of the lack of compactness of the Sobolev's embedding. Rabinowitz and Tanaka in [12] proved that problem (1) possesses a homoclinic orbit under the condition that the smallest eigenvalue of $L(t)$ tends to $+\infty$ as $|t| \to \infty$ without the periodicity assumptions, using a variant of the Mountain Pass theorem without the Palais–Smale condition. In 1994, Korman and Lazer in [7] proved that problem (1) possesses a nontrivial homoclinic solution on an even case still when $f = 0$ without periodicity conditions in $L(t)$ and $W(t,x)$. They proved the following theorem.

Theorem B. *(See [7].) Suppose* (A_3) *holds uniformly in* $t \in R$ *and* L *satisfies the following conditions*

(B_1) $L \in C^1(R, R^{N^2})$ *is a positive definite matrix and* $L(-t) = L(t)$ *for all* t,

(B_2) $(L'(t)x,x) - W_t(t,x) \geqslant 0$ *for all* $x \in R^N$ *and* $t \geqslant 0$.

Then problem (1) *possesses an even nontrivial homoclinic orbits.*

In 2007, for $L = 0$, $f = 0$ and $W(t,x)$ even in t, Lv and Tang in [8] showed the existence of the even homoclinic orbits for (1) as the limit of the solutions of nil-boundary-value problems, which are obtained by using the Mountain Pass theorem and appropriate estimates for passing to a nontrivial limit, which is the following theorem.

Theorem C. *(See [8].) Assume that* $L = 0$, $f = 0$ *and* $W \in C^1(R \times R^N, R)$ *satisfies*

(C_1) $W(t,0) \equiv 0$ *and* $W(-t,x) = W(t,x)$ *for all* $t \in R$ *and* $x \in R^N$,

(C_2) $W_t(t,x) \leqslant 0$ *for all* $t \geqslant 0$ *and* $x \in R^N$,

(C_3) *there exist constants* $a_1 > 0$, $q \geqslant 2$ *such that*

$$W(t,x) \leqslant a_1|x|^q$$

for all $(t,x) \in R \times R^N$,

(C_4) $\nabla W(t,x) \to 0$ *as* $|x| \to 0$ *uniformly in* $t \in R$,

(C_5) *there exist constants* $a_2 > 0$, $\tau > q-2$ *and a function* $d \in L^1(R, R^+)$ *such that*

$$\big(x, \nabla W(t,x)\big) - 2W(t,x) \geqslant a_2|x|^\tau - d(t), \quad \forall (t,x) \in R \times R^N,$$

(C_6) $\limsup_{|x|\to 0} \frac{W(t,x)}{|x|^2} < 0$ *uniformly in* $t \in R$,

(C_7) *there exists* $T_0 > 0$ *such that*

$$\liminf_{|x|\to\infty} \frac{W(t,x)}{|x|^2} > \frac{2\pi^2}{T_0^2}$$

uniformly in $t \in [-T_0, T_0]$.

Then system (1) *possesses one even homoclinic solution.*

Motivated by these papers, in this paper, we will obtain the existence of homoclinic solution for problem (1) when $L(t)$ and $W(t,x)$ are not periodic and $W(t,x)$ is not even in t. Set $A = \inf\{W(t,x)\colon t \in R,\ |x| = 1\}$, $B = \sup\{W(t,x)\colon t \in R,\ |x| = 1\}$, then we have the following theorems.

Theorem 1.1. *Suppose* $0 < A \leqslant B < +\infty$, (C_4) *and the following conditions hold*

(L) $L(t)$ *is a positive definite symmetric matrix for all* $t \in R$ *and satisfies*

$$\sup_{t\in R} \big|L_{ij}(t)\big| < \infty, \tag{3}$$

(W_1) *there is a constant* $\lambda > 2$ *such that*

$$0 < \lambda W(t,x) \leqslant \big(x, \nabla W(t,x)\big) \tag{4}$$

for every $t \in R$ *and* $x \in R^N \setminus \{0\}$,

(W_2) *for every* $M > 0$ *the following inequality holds*

$$\sup_{t\in R,\, |x|\leqslant M} \big|\nabla W(t,x)\big| < \infty.$$

Then there is a constant $\delta > 0$ *such that, for any* $f \neq 0$ *satisfying*

$$\left(\int_R \big|f(t)\big|^2\,dt\right)^{1/2} < \delta, \tag{5}$$

system (1) *possesses at least one nontrivial homoclinic solution* $u \in W^{1,2}(R, R^N)$.

Theorem 1.2. *Suppose* $L = 0$, W *satisfies* (C_4), (C_6), (W_2) *and the following conditions*

(W_1') $W(t,0) \equiv 0$ *and there exist constants* $d_1 > 0$, $\beta \geqslant 2$ *such that*

$$W(t,x) \leqslant d_1|x|^\beta$$

for all $(t,x) \in R \times R^N$,

(W_2') *there is a constant* $T > 0$ *such that*

$$\liminf_{|x|\to\infty} \frac{W(t,x)}{|x|^2} > \frac{\pi^2}{2T^2}$$

uniformly in $t \in [-T, T]$,

(W_3) *there are constants* $d_2 > 0$, $\mu > \max\{\beta - 2, 1\}$ *and a function* $g \in L^1(R, R^+)$ *such that*

$$\big(x, \nabla W(t,x)\big) - 2W(t,x) \geqslant d_2|x|^\mu - g(t)$$

for all $(t,x) \in R \times R^N$,

(f) $f \neq 0$ *is a bounded function and* $\int_R |f(t)|^{\mu/(\mu-1)}\,dt < \infty$.

Then there is a constant $\delta > 0$ *such that, for any* f *satisfying* (5), *system* (1) *possesses at least one nontrivial homoclinic solution* $u \in W^{1,2}(R, R^N)$.

Corollary 1.1. *Assume that* W *satisfies* (C_4), (C_6), (W_2), (W_3), (W_1'), (f) *and the following conditions*

(W_2'') $\liminf_{|x|\to\infty} \frac{W(t,x)}{|x|^2} > 0$ *uniformly in* $t \in R$.

Then there is a constant $\delta > 0$ *such that, for any* f *satisfying* (5), *system* (1) *possesses at least one nontrivial homoclinic solution* $u \in W^{1,2}(R, R^N)$.

Remark 1. Compared with Theorem A, $L(t)$ and $W(t,x)$ are not periodic in t in Theorem 1.1. Moreover, conditions (L), (W_2) can be easily obtained from (A_1) and (A_2). And condition (C_4) is weaker than (A_4). Since $W(t,x)$ is not periodic in t, we assume that condition (A_3) holds for all $t \in R$ and $x \in R^N \setminus \{0\}$, which is condition (W_2).

Remark 2. In Theorem 1.2, $W(t,x)$ is not supposed to be even in t, and we obtain the same result without condition (C_2) when $f \neq 0$.

Similar to the method used by Lv and Tang in [8], we obtain the existence of the homoclinic solutions as the limit of the solutions of nil-boundary-value problems. We consider a sequence of systems of differential equations

$$\begin{cases} -\ddot{u}(t) + L(t)u(t) = \nabla W\big(t, u(t)\big) - f(t) & \text{for } t \in (-kT, kT), \\ u(-kT) = u(kT) = 0 \end{cases} \tag{6}$$

for all $k \in N$, where T comes from (W_2') for convenience. We will prove the existence of at least one homoclinic solution of (1) as the limit of the solutions of (6) as $k \to \infty$.

2. Proof of theorems

For each $k \in N$, $p \geqslant 1$, let

$$L^p_{2kT}(R, R^N) = \{u : [-kT, kT] \to R^N \mid \|u\|_{L^p_{2kT}(R,R^N)} < \infty\},$$

where

$$\|u\|_{L^p_{2kT}(R,R^N)} := \left(\int_{-kT}^{kT} |u(t)|^p \, dt\right)^{1/p}$$

and $L^\infty_{2kT}(R, R^N)$ be a space of essentially bounded and measurable functions from R into R^N under the norm

$$\|u\|_{L^\infty_{2kT}(R,R^N)} := \operatorname{ess\,sup}\{|u(t)| : t \in [-kT, kT]\}.$$

For each $k \in N$, set

$$E_k = \{u : [-kT, kT] \to R^N \mid u \text{ is absolutely continuous, } u(-kT) = u(kT) = 0\},$$

with the norm

$$\|u\|_{E_k} := \left(\int_{-kT}^{kT} |\dot{u}(t)|^2 \, dt + \int_{-kT}^{kT} |u(t)|^2 \, dt\right)^{1/2}.$$

Let $\eta_k : E_k \to [0, +\infty)$ be given by

$$\eta_k(u) = \left(\int_{-kT}^{kT} \left(|\dot{u}(t)|^2 + (L(t)u, u)\right) dt\right)^{1/2}. \tag{7}$$

By (L), there are constants $b_1, b_2 > 0$ such that

$$b_1\|u\|^2_{E_k} \leqslant \eta^2_k(u) \leqslant b_2\|u\|^2_{E_k}. \tag{8}$$

Moreover, let $I_k : E_k \to R$ be the corresponding functional of (6) defined by

$$I_k(u) = \int_{-kT}^{kT} \left(\frac{1}{2}|\dot{u}(t)|^2 + \frac{1}{2}\big(L(t)u(t), u(t)\big) - W\big(t, u(t)\big) + \big(f(t), u(t)\big)\right) dt. \tag{9}$$

Then one can easily check that $I_k \in C'(E_k, R)$ and

$$\langle I'_k(u), v\rangle = \int_{-kT}^{kT} \Big(\big(\dot{u}(t), \dot{v}(t)\big) + \big(L(t)u(t), v(t)\big) - \big(\nabla W\big(t, u(t)\big), v(t)\big) + \big(f(t), v(t)\big)\Big) dt.$$

It follows from (7) that

$$I_k(u) = \frac{1}{2}\eta^2_k(u) - \int_{-kT}^{kT} W\big(t, u(t)\big) + \int_{-kT}^{kT} \big(f(t), u(t)\big) dt. \tag{10}$$

Furthermore, the critical points of I_k are classical solutions of (6). From (4) we can obtain the following properties of the function $W(t, x)$.

Lemma 2.1. *For every $t \in R$ the following inequalities hold*

$$W(t, x) \leqslant W\left(t, \frac{x}{|x|}\right)|x|^\lambda \quad \textit{for } 0 < |x| \leqslant 1 \tag{11}$$

and

$$W(t, x) \geqslant W\left(t, \frac{x}{|x|}\right)|x|^\lambda \quad \textit{for } |x| \geqslant 1.$$

From the condition (W_1) we can obtain that $s \to W(t, s^{-1}u)s^{\lambda}$ is a nonincreasing function, which suffices to prove this lemma.

Lemma 2.2. *(See [5].) For every $r \in R \setminus \{0\}$ and $u \in E_k \setminus \{0\}$ the following inequality holds*

$$\int_{-kT}^{kT} W\big(t, ru(t)\big)\,dt \geqslant A|r|^{\lambda} \int_{-kT}^{kT} \big|u(t)\big|^{\lambda}\,dt - 2kTA. \tag{12}$$

By the Sobolev's embedding theorem, $H^1(R)$ is continuous embedded into $L^{\infty}(R)$. Then there is a constant $C_0 > 0$ such that

$$\|u\|_{L^{\infty}} \leqslant C_0 \|u\|_{H^1}$$

for all $u \in H^1(R)$. Since $E_k \subset E_{k+1} \subset H^1$ for all $k \in N$, when $u \in E_k$, we can extend it by zero in $R \setminus [-kT, kT]$. Then we have the following lemma.

Lemma 2.3. *There is a positive constant $C > 0$ which is independent of k such that the following inequality holds*

$$\|u\|_{L^{\infty}_{2kT}(R,R^N)} \leqslant C\|u\|_{E_k} \tag{13}$$

for each $k \in N$ and $u \in E_k$.

The proof of Theorem 1.1 is divided into a sequence of lemmas. Firstly, we prove that I_k possesses at least one point via the Mountain Pass theorem which is the classical solution of (6).

Lemma 2.4. *Suppose the conditions of Theorem* 1.1 *hold, then there is a constant $\delta > 0$ such that, for any f satisfying* (5), *system* (6) *possesses one solution $u_k \in E_k$ for every $k \in N$.*

Proof. It is known that a deformation lemma can be proved when the usual (PS) condition is replaced by condition (C) which means the Mountain Pass theorem holds under condition (C). Then we apply the Mountain Pass theorem to obtain the critical point of I_k under condition (C).

Our proof involves three steps.

Step 1: I_k satisfies condition (C). From (W_1), we can see $I_k(0) = 0$. Then we show that I_k satisfies (C) condition. Suppose that $\{u_j\}_{j\in N} \subset E_k$ is a sequence such that $\{I_k(u_j)\}_{j\in N}$ is bounded and $\|I_k'(u_j)\|(1 + \|u_j\|_{E_k}) \to 0$ as $j \to \infty$. Then there exists a constant $C_k > 0$ such that

$$I_k(u_j) \leqslant C_k, \qquad \big\|I_k'(u_j)\big\|\big(1 + \|u_j\|_{E_k}\big) \leqslant C_k. \tag{14}$$

It follows from (14), (10), (8) and (W_1) that

$$\begin{aligned}
(\lambda+1)C_k &\geqslant \lambda I_k(u_j) + \big\|I_k'(u_j)\big\|\big(1 + \|u_j\|_{E_k}\big) \\
&\geqslant \lambda I_k(u_j) - \big\langle I_k'(u_j), u_j \big\rangle \\
&\geqslant \left(\frac{\lambda}{2} - 1\right)\eta_k^2(u_j) + (\lambda - 1)\int_{-kT}^{kT} \big(f(t), u_j(t)\big)\,dt \\
&\geqslant b_1\left(\frac{\lambda}{2} - 1\right)\|u_j\|_{E_k}^2 - (\lambda - 1)\|f\|_{L^2}\|u_j\|_{E_k}.
\end{aligned}$$

Since $\lambda > 2$, then $\{u_j\}_{j\in N}$ is bounded in E_k. By a standard argument, we see that $\{u_j\}_{j\in N}$ has a convergent subsequence in E_k. Hence I_k satisfies (C) condition.

Step 2: Now we show that there exist constants $\varrho_1, \alpha_1 > 0$ independent of k such that $I_k|_{\partial B_{\varrho_1}(0)} \geqslant \alpha_1$. We choose $\delta < \min\{(\frac{b_1}{4C})^2, (\frac{b_1}{4B})^{\frac{2}{\lambda-2}}, 1\}$, then set

$$\varrho_1 = C^{-1}\delta^{\frac{1}{2}}, \qquad \alpha_1 = \frac{b_1}{4C} - \delta^{\frac{1}{2}} > 0.$$

It follows from (13) that $0 < \|u\|_{L^{\infty}_{2kT}} < \delta^{\frac{1}{2}} < 1$ if $\|u\|_{E_k} = \varrho_1$. By (10), (11), (5), (8) and (13) we obtain

$$\begin{aligned}
I_k(u) &= \frac{1}{2}\eta_k^2(u) - \int_{-kT}^{kT} W\big(t, u(t)\big)\,dt + \int_{-kT}^{kT} \big(f(t), u(t)\big)\,dt \\
&\geqslant \frac{b_1}{2}\|u\|_{E_k}^2 - \int_{-kT}^{kT} W\left(t, \frac{u(t)}{|u(t)|}\right)\big|u(t)\big|^{\lambda}\,dt + \int_{-kT}^{kT} \big(f(t), u(t)\big)\,dt \\
&\geqslant \frac{b_1}{2}\|u\|_{E_k}^2 - B\int_{-kT}^{kT} \big|u(t)\big|^{\lambda}\,dt + \int_{-kT}^{kT} \big(f(t), u(t)\big)\,dt \\
&\geqslant \frac{b_1}{2}\|u\|_{E_k}^2 - B\|u\|_{L_{2kT}^{\infty}}^{\lambda-2}\int_{-kT}^{kT} \big|u(t)\big|^{2}\,dt - \delta\|u\|_{E_k} \\
&\geqslant \frac{b_1}{2}\|u\|_{E_k}^2 - BC^{\lambda-2}\|u\|_{E_k}^{\lambda} - \delta\|u\|_{E_k} \\
&= \left(\frac{b_1}{4}\|u\|_{E_k}^2 - BC^{\lambda-2}\|u\|_{E_k}^{\lambda}\right) + \left(\frac{b_1}{4}\|u\|_{E_k}^2 - \delta\|u\|_{E_k}\right).
\end{aligned} \tag{15}$$

By the definition of ϱ_1 and α_1, (15) implies $I_k|_{\partial B_{\varrho_1}(0)} \geqslant \alpha_1$.

Step 3: Now we only need to prove that for each $k \in N$ there is $e_k \in E_k$ such that $\|e_k\|_{E_k} > \varrho_1$ and $I_k(e_k) \leqslant 0$. By (12), (10), (8) for every $r \in R \setminus \{0\}$, the following inequality holds

$$\begin{aligned}
I_k(ru) &= \frac{1}{2}\eta_k^2(ru) - \int_{-kT}^{kT} W\big(t, ru(t)\big) + \int_{-kT}^{kT} \big(f(t), ru(t)\big)\,dt \\
&\leqslant \frac{b_2|r|^2}{2}\|u\|_{E_k}^2 - A|r|^{\lambda}\int_{-kT}^{kT} \big|u(t)\big|^{\lambda}\,dt + |r|\delta\|u\|_{E_k} + 2kTA.
\end{aligned} \tag{16}$$

Fix $Q_1 \in C_0^{\infty}(-T, T) \setminus \{0\} \subset E_1$. Since $A > 0$ and $\lambda > 2$, then (16) implies that there exists $r_1 \in R \setminus \{0\}$ such that $\|r_1 Q_1\|_{E_1} > \varrho_1$ and $I_1(r_1 Q_1) < 0$. Set $e_1(t) = r_1 Q_1(t)$ and $e_k(t) = e_1(t)$. Then $e_k \in E_k$, $\|e_k\|_{E_k} = \|e_1\|_{E_1} > \varrho_1$ and $I_k(e_k) = I_1(e_1) < 0$ for each $k \in N$. By the Mountain Pass theorem, I_k possesses a critical value $c_k \geqslant \alpha_1$ given by

$$c_k = \inf_{g \in \Gamma_k} \max_{s \in [0,1]} I_k\big(g(s)\big), \tag{17}$$

where

$$\Gamma_k = \big\{g \in C\big([0,1], E_k\big) \bigm| g(0) = 0,\ g(1) = e_k\big\}.$$

Hence, for each $k \in N$, there exists $u_k \in E_k$ such that

$$I_k(u_k) = c_k, \qquad I_k'(u_k) = 0. \tag{18}$$

Then the function u_k is a desired classical solution of system (6). □

Lemma 2.5. *Let $u_k \in E_k$ be the solution of system* (6) *which satisfies* (18) *for all $k \in N$. Then there is a constant $M_1 > 0$ independent of k such that*

$$\|u_k\|_{E_k} \leqslant M_1 \tag{19}$$

for all $k \in N$.

Proof. For each $k \in N$, let $g_k : [0,1] \to E_k$ be a curve given by $g_k(s) = se_k$ where e_k is defined in Lemma 2.4. Then $g_k \in \Gamma_k$ and $I_k(g_k(s)) = I_1(g_1(s))$ for all $k \in N$ and $s \in [0,1]$. Therefore, by (17) we have

$$c_k \leqslant \max_{s \in [0,1]} I_1\big(g_1(s)\big) \equiv M_0,$$

where M_0 is independent of $k \in N$, then from (18) we obtain

$$I_k(u_k) \leqslant M_0, \qquad \|I_k'(u_k)\|\big(1+\|u_k\|_{E_k}\big)=0.$$

The following proof is the same to Step 1 in Lemma 2.4; then we see that there exists $M_1 > 0$ independent of k such that

$$\|u_k\|_{E_k} \leqslant M_1$$

for all $k \in N$, which completes the proof. □

Lemma 2.6. *Let $u_k \in E_k$ be the solution of system* (6) *which satisfies* (18) *for $k \in N$. Then there exists a subsequence $\{u_{k_j}\}$ of $\{u_k\}_{k\in N}$ convergent to u_0 in $C^1_{loc}(R, R^N)$.*

Proof. By Lemma 2.5, we know that $\{u_k\}_{k\in N}$ is uniformly bounded in E_k. Now we show that $\{\dot u_k\}_{k\in N}$ and $\{\ddot u\}_{k\in N}$ are also uniformly bounded sequences. By Lemma 2.3 and (19), we have

$$\|u_k\|_{L^\infty_{2kT}} \leqslant C\|u_k\|_{E_k} \leqslant CM_1. \tag{20}$$

Since u_k is a solution of system (6), it follows that

$$-\ddot u_k(t) + L(t)u_k(t) = \nabla W\big(t, u_k(t)\big) - f(t)$$

for every $t \in (-kT, kT)$. By (20) and the boundedness of f, we obtain

$$\begin{aligned}\big|\ddot u_k(t)\big| &\leqslant \big|\nabla W\big(t,u_k(t)\big)\big| + \big|L(t)u_k(t)\big| + \big|f(t)\big| \\ &\leqslant \sup_{(t,x)\in[-kT,kT]\times[-CM_1,CM_1]} \big|\nabla W(t,x)\big| + CM_1 \sup_{t\in[-kT,kT]} \big|L_{ij}(t)\big| + \sup_{t\in R}\big|f(t)\big|\end{aligned}$$

for $k \in N$. It follows from (W_2) and (3) that there is $M_2 > 0$ independent of k such that

$$\|\ddot u_k\|_{L^\infty_{2kT}} \leqslant M_2. \tag{21}$$

We can suppose that $u_k(t) = (u_{k_1}(t), u_{k_2}(t), \dots, u_{k_N}(t))$ for each $t \in R$. By the Mean Value theorem, there exits $t_{k_i} \in [t-1, t]$, for all $t \in R$, such that

$$\dot u_{k_i}(t_{k_i}) = \int_{t-1}^{t} \dot u_{k_i}(s)\,ds = u_{k_i}(t) - u_{k_i}(t-1)$$

for any $i \in \{1, 2, \dots, N\}$. Then by (21) we have

$$\begin{aligned}\big|\dot u_{k_i}(t)\big| &= \left|\int_{t_{k_i}}^{t} \ddot u_{k_i}(s)\,ds + \dot u_{k_i}(t_{k_i})\right| \\ &\leqslant \int_{t-1}^{t} \big|\ddot u_{k_i}(s)\big|\,ds + \big|\dot u_{k_i}(t_{k_i})\big| \\ &\leqslant \int_{t-1}^{t} \big|\ddot u_k(s)\big|\,ds + \big|u_{k_i}(t) - u_{k_i}(t-1)\big| \\ &\leqslant M_2 + 2CM_1 \equiv M_3.\end{aligned}$$

Consequently, there exists a constant $M_4 > 0$ such that

$$\|\dot u_k\|_{L^\infty_{2kT}} \leqslant M_4.$$

In order to finish the proof via the Arzelà–Ascoli theorem, we need to prove that $\{u_k\}_{k\in N}$ and $\{\dot u_k\}_{k\in N}$ are equicontinuous. Actually, by (21) we have

$$\big|\dot u_k(t_1) - \dot u_k(t_2)\big| \leqslant \left|\int_{t_2}^{t_1} \ddot u_k(s)\,ds\right| \leqslant \int_{t_2}^{t_1} \big|\ddot u_k(s)\big|\,ds \leqslant M_2|t_1 - t_2|$$

for each $k \in N$ and $t_1, t_2 \in R$, which shows that $\{\dot u_k\}_{k\in N}$ is equicontinuous, and $\{u_k\}_{k\in N}$ remains in the same way. Then there is a subsequence $\{u_{k_j}\}_{j\in N}$ convergent to u_0 in $C^1_{loc}(R, R^N)$ by the Arzelà–Ascoli theorem. □

To prove that u_0 is a desired solution of problem (1), we need to state an estimation made by Tang and Xiao in [13].

Lemma 2.7. *(See [13].) Let $u : R \to R^N$ be a continuous mapping such that $\dot{u} \in L^2_{loc}(R, R^N)$. For every $t \in R$ the following inequality holds*

$$\left|u(t)\right| \leqslant \sqrt{2}\left(\int_{t-1/2}^{t+1/2} \left(\left|u(s)\right|^2 + \left|\dot{u}(s)\right|^2\right) ds\right)^{1/2}. \tag{22}$$

Lemma 2.8. *Let $u_0 : R \to R^N$ be a function determined by Lemma* 2.6. *Then u_0 is a nontrivial homoclinic solution of problem* (1).

Proof. *Step 1*: We will show that $u_0(t)$ satisfies (1) and $u_0(t) \to 0$ as $t \to \pm\infty$. By Lemmas 2.4 and 2.6, we have $u_{k_j} \to u_0$ in $C^1_{loc}(R, R^N)$ as $j \to \infty$, and

$$-\ddot{u}_{k_j}(t) + L(t)u_{k_j}(t) = \nabla W\left(t, u_{k_j}(t)\right) - f(t)$$

for each $j \in N$ and $t \in (-k_jT, k_jT)$. Take $a, b \in R$ such that $a < b$. There exists $j_0 \in N$ such that

$$-\ddot{u}_{k_j}(t) + L(t)u_{k_j}(t) = \nabla W\left(t, u_{k_j}(t)\right) - f(t)$$

for all $j > j_0$ and $t \in [a, b]$. Since, for $j > j_0$, $\ddot{u}_{k_j}(t)$ is continuous in $[a, b]$ and $\ddot{u}_{k_j}(t) \to -\nabla W(t, u_0(t)) + L(t)u_0(t) + f(t)$ uniformly on $[a, b]$. So it follows that $\ddot{u}_{k_j}$ is a classical derivative of $\dot{u}_{k_j}$ in (a, b) for each $j > j_0$. Moreover, since $\dot{u}_{k_j} \to \dot{u}_0$ uniformly on $[a, b]$, we get

$$-\ddot{u}_0(t) + L(t)u_0(t) = \nabla W\left(t, u_0(t)\right) - f(t)$$

for all $t \in [a, b]$. Since a and b are arbitrary, we conclude that u_0 satisfies (1).

Subsequently, we will show that $u_0(t) \to 0$ as $t \to \pm\infty$. Then for every $l \in N$ there exists $j_0 \in N$ such that

$$\int_{-lT}^{lT} \left(\left|u_{k_j}(t)\right|^2 + \left|\dot{u}_{k_j}(t)\right|^2\right) dt \leqslant \|u_{k_j}\|^2_{E_{k_j}} \leqslant M_1^2$$

for any $j \geqslant j_0$. From this and Lemma 2.6 it follows that

$$\int_{-lT}^{lT} \left(\left|u_0(t)\right|^2 + \left|\dot{u}_0(t)\right|^2\right) dt \leqslant M_1^2$$

for each $l \in N$. Letting $l \to +\infty$, we have

$$\int_{-\infty}^{+\infty} \left(\left|u_0(t)\right|^2 + \left|\dot{u}_0(t)\right|^2\right) dt \leqslant M_1^2,$$

then

$$\int_{|t| \geqslant r} \left(\left|u_0(t)\right|^2 + \left|\dot{u}_0(t)\right|^2\right) dt \to 0 \tag{23}$$

as $r \to +\infty$. Then by (22), we obtain $u_0(t) \to 0$ as $t \to \pm\infty$.

Step 2: We will prove that $\dot{u}_0(t) \to 0$ as $t \to \pm\infty$. Observe that

$$\left|\dot{u}_0(t)\right|^2 \leqslant 2\int_{t-1}^{t} \left(\left|\dot{u}_0(s)\right|^2 + \left|\ddot{u}_0(s)\right|^2\right) ds$$

for each $t \in R$. From (23), one has

$$\int_{t-1}^{t} \left|\dot{u}_0(s)\right|^2 ds \to 0$$

as $t \to \pm\infty$. And since u_0 is a solution of problem (1), by Hölder inequality we have

$$\int_{t-1}^{t} \left|\ddot{u}_0(s)\right|^2 ds = \int_{t-1}^{t} \left|-\nabla W\big(t, u_0(t)\big) + L(t)u_0(t) + f(s)\right|^2 ds$$
$$\leqslant \int_{t-1}^{t} \left(\left|-\nabla W\big(t, u_0(t)\big) + L(t)u_0(t)\right| + \left|f(s)\right|\right)^2 ds$$
$$\leqslant 2\int_{t-1}^{t} \left(\left|-\nabla W\big(t, u_0(t)\big) + L(t)u_0(t)\right|^2 + \left|f(s)\right|^2\right) ds.$$

It follows from (C_4) and (L) that for every $\varepsilon > 0$, there is a constant $\rho > 0$ such that

$$\left|-\nabla W(s,x) + L(s)x\right| < \varepsilon$$

for all $|x| < \rho$ and uniformly in $s \in R$. Since $u_0(s) \to 0$ as $s \to \pm\infty$, there exists a constant $p > 0$ such that $|u_0(s)| < \rho$ for $|s| \geqslant p$. Hence, when $|t| \geqslant p + 1$,

$$\int_{t-1}^{t} \left|-\nabla W\big(s, u_0(s)\big) + L(s)u_0(s)\right|^2 ds < \varepsilon^2.$$

It follows from $\int_{t-1}^{t} |f(s)|^2 ds \to 0$ as $t \to \pm\infty$ that

$$\int_{t-1}^{t} \left|\ddot{u}_0(s)\right|^2 ds \to 0,$$

then we obtain our conclusion.

Since $f \neq 0$ and by (4), it can easily check that $\nabla W(t,0) = 0$ uniformly in $t \in R$, we can conclude that $u = 0$ is not a solution for system (1), hence $u_0 \neq 0$. The proof of Theorem 1.1 is completed. □

Proof of Theorem 1.2. It is standard to prove that I_k satisfies the geometric conditions of the Mountain Pass theorem. Some details are different from Theorem 1.1.

It follows from (C_6) that, there exist $\varepsilon_0 \in (0, \frac{1}{2}]$ and $\sigma > 0$ such that

$$W(t,x) \leqslant -\varepsilon_0 |x|^2 \tag{24}$$

for all $|x| \leqslant \sigma$ and $t \in R$.

Step 1: I_k satisfies condition (C). From (W_1'), we can see $I_k(0) = 0$. Then we show that I_k satisfies (C) condition. Suppose that $\{u_j\}_{j\in N} \subset E_k$ is a sequence such that $\{I_k(u_j)\}_{j\in N}$ is bounded and $\|I_k'(u_j)\|(1 + \|u_j\|_{E_k}) \to 0$ as $j \to \infty$. Then there exists a constant $C_k > 0$ such that

$$I_k(u_j) \leqslant C_k, \qquad \left\|I_k'(u_j)\right\|\big(1 + \|u_j\|_{E_k}\big) \leqslant C_k. \tag{25}$$

By (25), (9), (W_3) and (f) we have

$$\begin{aligned}
3C_k &\geqslant 2I_k(u_j) + \left\|I_k'(u_j)\right\|\big(1 + \|u_j\|_{E_k}\big)\\
&\geqslant 2I_k(u_j) - \left\langle I_k'(u_j), u_j\right\rangle\\
&\geqslant \int_{-kT}^{kT} \big(\big(\nabla W(t,u_j), u_j\big) - 2W(t,u_j)\big)\, dt + \int_{-kT}^{kT} \big(f(t), u_j(t)\big)\, dt\\
&\geqslant d_2 \int_{-kT}^{kT} \left|u_j(t)\right|^\mu dt - \int_{-kT}^{kT} g(t)\, dt - \left(\int_{-kT}^{kT} \left|f(t)\right|^{\mu/(\mu-1)} dt\right)^{(\mu-1)/\mu} \left(\int_{-kT}^{kT} \left|u_j(t)\right|^\mu dt\right)^{1/\mu},
\end{aligned}$$

which implies that

$$3C_k \geqslant d_2 \int_{-kT}^{kT} \left|u_j(t)\right|^\mu dt - \|f\|_{L^{\mu/(\mu-1)}} \left(\int_{-kT}^{kT} \left|u_j(t)\right|^\mu dt\right)^{1/\mu} - \|g\|_{L^1}.$$

Since $\mu > 1$, there exists $D_k > 0$ such that

$$\int_{-kT}^{kT} \left|u_j(t)\right|^{\mu} dt \leqslant D_k. \tag{26}$$

Moreover, from (W_1') and (W_3) we can conclude $\beta \geqslant \mu$, then from (9), (25), (W_1'), (24), (26) and Lemma 2.3 we obtain

$$\begin{aligned}\frac{1}{2}\|\dot{u}_j\|^2_{L^2_{2kT}} &= I_k(u_j) + \int_{-kT}^{kT} W\big(t, u_j(t)\big)\,dt - \int_{-kT}^{kT} \big(f(t), u_j(t)\big)\,dt \\ &\leqslant C_k + d_1 \int_{\{t\in[-kT,kT]\,|\,|u_j|>\sigma\}} \left|u_j(t)\right|^{\beta} dt \\ &\quad - \varepsilon_0 \int_{\{t\in[-kT,kT]\,|\,|u_j|\leqslant\sigma\}} \left|u_j(t)\right|^2 dt + \|f\|_{L^{\mu/(\mu-1)}} \left(\int_{-kT}^{kT} \left|u_j(t)\right|^{\mu} dt\right)^{1/\mu},\end{aligned}$$

which implies that

$$\begin{aligned}\varepsilon_0\|u_j\|^2_{E_k} &\leqslant d_1 \int_{\{t\in[-kT,kT]\,|\,|u_j|>\sigma\}} \left|u_j(t)\right|^{\beta} dt + \varepsilon \int_{\{t\in[-kT,kT]\,|\,|u_j|>\sigma\}} \left|u_j(t)\right|^2 dt + \|f\|_{L^{\mu/(\mu-1)}} \left(\int_{-kT}^{kT} \left|u_j(t)\right|^{\mu} dt\right)^{1/\mu} + C_k \\ &\leqslant \big(d_1 + \varepsilon\sigma^{2-\beta}\big) \int_{\{t\in[-kT,kT]\,|\,|u_j|>\sigma\}} \left|u_j(t)\right|^{\beta} dt + \|f\|_{L^{\mu/(\mu-1)}} \left(\int_{-kT}^{kT} \left|u_j(t)\right|^{\mu} dt\right)^{1/\mu} + C_k \\ &\leqslant \big(d_1 + \varepsilon\sigma^{2-\beta}\big) \int_{-kT}^{kT} \left|u_j(t)\right|^{\beta} dt + \|f\|_{L^{\mu/(\mu-1)}} D_k^{1/\mu} + C_k \\ &\leqslant \big(d_1 + \varepsilon\sigma^{2-\beta}\big) D_k C^{\beta-\mu} \|u_j\|^{\beta-\mu}_{E_k} + \|f\|_{L^{\mu/(\mu-1)}} D_k^{1/\mu} + C_k.\end{aligned}$$

Since $\beta - \mu < 2$, then $\{u_j\}_{j\in N}$ is bounded in E_k. By a standard argument, we see that $\{u_j\}_{j\in N}$ has a convergent subsequence in E_k. Hence I_k satisfies (C) condition.

Step 2: Now we choose $\delta > 0$ such that

$$\delta < \varepsilon_0\left(\frac{\sigma}{C}\right). \tag{27}$$

With this, we prove that there exist constants $\varrho_2, \alpha_2 > 0$ independent of k such that $I_k|_{\partial B_{\varrho_2}(0)} \geqslant \alpha_2$. By (27), we can set

$$\varrho_2 = \frac{\sigma}{C}, \qquad \alpha_2 = \varepsilon_0\left(\frac{\sigma}{C}\right)^2 - \delta\left(\frac{\sigma}{C}\right) > 0,$$

which implies $0 < \|u\|_{L^\infty_{2kT}} \leqslant \sigma$ if $\|u\|_{E_k} = \varrho_2$. Then it follows from (9), (24) and (5) that

$$\begin{aligned}I_k(u) &= \frac{1}{2}\int_{-kT}^{kT} \left|\dot{u}(t)\right|^2 dt - \int_{-kT}^{kT} W\big(t, u(t)\big)\,dt + \int_{-kT}^{kT} \big(f(t), u(t)\big)\,dt \\ &\geqslant \frac{1}{2}\int_{-kT}^{kT} \left|\dot{u}(t)\right|^2 dt + \varepsilon_0 \int_{-kT}^{kT} \left|u_j(t)\right|^2 dt - \delta\|u\|_{E_k} \\ &\geqslant \varepsilon_0\|u\|^2_{E_k} - \delta\|u\|_{E_k}.\end{aligned} \tag{28}$$

By the definition of ϱ_2 and α_2, (28) implies $I_k|_{\partial B_{\varrho_2}(0)} \geqslant \alpha_2$.

Step 3: Now we only need to prove that for each $k \in N$ there is $\gamma_k \in E_k$ such that $\|\gamma_k\|_{E_k} > \varrho_2$ and $I_k(\gamma_k) \leqslant 0$. It follows from (W_2') that there exist $\xi > 0$ and $\varepsilon_1 > 0$ such that

$$W(t,x) \geqslant \left(\frac{\pi^2}{2T^2}+\varepsilon_1\right)|x|^2$$

for all $t \in [-T,T]$ and $|x| > \xi$. Set $\zeta = \max\{|W(t,x)| \mid t \in [-T,T],\ |x| \leqslant \xi\}$, hence we have

$$W(t,x) \geqslant \left(\frac{\pi^2}{2T^2}+\varepsilon_1\right)\left(|x|^2-\xi^2\right)-\zeta.$$

Set

$$Q_2(t) = \begin{cases} \sin(\omega t)e, & t \in [-T,T], \\ 0 & t \in [-kT,kT] \setminus [-T,T] \end{cases}$$

where $\omega = \frac{\pi}{T}$, $e = (1,0,\dots,0)$. It can easily check that $(\frac{\pi^2}{2T^2}+\varepsilon_1)m > M$, where

$$M = \frac{1}{2}\int_{-T}^{T} \left|\dot{Q}_2(t)\right|^2 dt, \qquad m = \int_{-T}^{T} \left|Q_2(t)\right|^2 dt.$$

By (9), for every $r \in R \setminus \{0\}$, the following inequality holds

$$\begin{aligned} I_1(rQ_2) &= \frac{1}{2}\int_{-T}^{T} \left|r\dot{Q}_2(t)\right|^2 dt - \int_{-T}^{T} W\left(t, rQ_2(t)\right) dt + \int_{-T}^{T} \left(f(t), rQ_2(t)\right) dt \\ &\leqslant \frac{|r|^2}{2}\int_{-T}^{T} \left|\dot{Q}_2(t)\right|^2 dt - \left(\frac{\pi^2}{2T^2}+\varepsilon_1\right)|r|^2 \int_{-T}^{T} \left|Q_2(t)\right|^2 dt + |r|\delta m^{1/2} + 2T\left(\left(\frac{\pi^2}{2T^2}+\varepsilon_1\right)\xi^2+\zeta\right) \\ &= -\left(\left(\frac{\pi^2}{2T^2}+\varepsilon_1\right)m - M\right)|r|^2 + |r|\delta m^{1/2} + 2T\left(\left(\frac{\pi^2}{2T^2}+\varepsilon_1\right)\xi^2+\zeta\right), \end{aligned}$$

which implies that there exists $r_2 \in R \setminus \{0\}$ such that $\|r_2 Q_2\|_{E_1} > \varrho$ and $I_1(r_2 Q_2) < 0$. Set $e_2(t) = r_2 Q_2(t)$ and $\gamma_k(t) = e_2(t)$. Then $\gamma_k \in E_k$, $\|\gamma_k\|_{E_k} = \|e_2\|_{E_1} > \varrho_2$ and $I_k(\gamma_k) = I_1(e_2) < 0$ for each $k \in N$. By the Mountain Pass theorem, I_k possesses a critical value $d_k \geqslant \alpha_2$ given by

$$d_k = \inf_{g \in \Gamma_k} \max_{s \in [0,1]} I_k\left(g(s)\right), \tag{29}$$

where

$$\Gamma_k = \left\{g \in C\left([0,1], E_k\right) \;\middle|\; g(0)=0,\ g(1)=\gamma_k\right\}.$$

Hence, for each $k \in N$, there exists $u_k \in E_k$ such that

$$I_k(u_k) = d_k, \qquad I_k'(u_k) = 0. \tag{30}$$

Then the function u_k is a desired classical solution of system (6).

Lemma 2.9. *Let $u_k \in E_k$ be the solution of system* (6) *which satisfies* (30) *for all $k \in N$. Then there is a constant $M_6 > 0$ independent of k such that*

$$\|u_k\|_{E_k} \leqslant M_6 \tag{31}$$

for all $k \in N$.

Proof. Similar to Lemma 2.5, it follows from (29) that

$$d_k \leqslant \max_{s \in [0,1]} I_1\left(g_1(s)\right) \equiv M_5,$$

where M_0 is independent of $k \in N$, then from (18) we obtain

$$I_k(u_k) \leqslant M_5, \qquad \left\|I_k'(u_k)\right\|\left(1+\|u_k\|_{E_k}\right) = 0. \tag{32}$$

It follows from (30), (9), (W_3) and (f) that

$$
\begin{aligned}
2M_5 &\geqslant 2I_k(u_j) + \big\|I_k'(u_j)\big\|\big(1+\|u_j\|_{E_k}\big)\\
&\geqslant 2I_k(u_j) - \big\langle I_k'(u_j), u_j\big\rangle\\
&\geqslant \int_{-kT}^{kT} \big(\big(\nabla W(t,u_j), u_j\big) - 2W(t,u_j)\big)\,dt + \int_{-kT}^{kT} \big(f(t), u_j(t)\big)\,dt\\
&\geqslant d_2 \int_{-kT}^{kT} \big|u_j(t)\big|^{\mu}\,dt - \int_{-kT}^{kT} g(t)\,dt\\
&\quad - \left(\int_{-kT}^{kT} \big|f(t)\big|^{\mu/(\mu-1)}\,dt\right)^{(\mu-1)/\mu} \left(\int_{-kT}^{kT} \big|u_j(t)\big|^{\mu}\,dt\right)^{1/\mu},
\end{aligned}
$$

which implies that

$$
2M_5 \geqslant d_2 \int_{-kT}^{kT} \big|u_j(t)\big|^{\mu}\,dt - \|f\|_{L^{\mu/(\mu-1)}} \left(\int_{-kT}^{kT} \big|u_j(t)\big|^{\mu}\,dt\right)^{1/\mu} - \|g\|_{L^1}.
$$

Since $\mu > 1$, there exists $D > 0$ independent of k such that

$$
\int_{-kT}^{kT} \big|u_j(t)\big|^{\mu}\,dt \leqslant D.
$$

Then from (9), (32), (W_1'), (24) and Lemma 2.3 we obtain

$$
\begin{aligned}
\frac{1}{2}\|\dot u_j\|^2_{L^2_{2kT}} &= I_k(u_j) + \int_{-kT}^{kT} W\big(t,u_j(t)\big)\,dt - \int_{-kT}^{kT} \big(f(t), u_j(t)\big)\,dt\\
&\leqslant M_5 + d_1 \int_{\{t\in[-kT,kT]\,|\,|u_j|>\sigma\}} \big|u_j(t)\big|^{\beta}\,dt\\
&\quad - \varepsilon_0 \int_{\{t\in[-kT,kT]\,|\,|u_j|\leqslant\sigma\}} \big|u_j(t)\big|^{2}\,dt + \|f\|_{L^{\mu/(\mu-1)}} \left(\int_{-kT}^{kT} \big|u_j(t)\big|^{\mu}\,dt\right)^{1/\mu},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\varepsilon_0\|u_j\|^2_{E_k} &\leqslant d_1 \int_{\{t\in[-kT,kT]\,|\,|u_j|>\sigma\}} \big|u_j(t)\big|^{\beta}\,dt + \varepsilon \int_{\{t\in[-kT,kT]\,|\,|u_j|>\sigma\}} \big|u_j(t)\big|^{2}\,dt\\
&\quad + \|f\|_{L^{\mu/(\mu-1)}} \left(\int_{-kT}^{kT} \big|u_j(t)\big|^{\mu}\,dt\right)^{1/\mu} + M_5\\
&\leqslant \big(d_1 + \varepsilon\sigma^{2-\beta}\big) \int_{\{t\in[-kT,kT]\,|\,|u_j|>\sigma\}} \big|u_j(t)\big|^{\beta}\,dt + \|f\|_{L^{\mu/(\mu-1)}} \left(\int_{-kT}^{kT} \big|u_j(t)\big|^{\mu}\,dt\right)^{1/\mu} + M_5\\
&\leqslant \big(d_1 + \varepsilon\sigma^{2-\beta}\big) \int_{-kT}^{kT} \big|u_j(t)\big|^{\beta}\,dt + \|f\|_{L^{\mu/(\mu-1)}} D^{1/\mu} + M_5\\
&\leqslant \big(d_1 + \varepsilon\sigma^{2-\beta}\big) D C^{\beta-\mu} \|u_j\|^{\beta-\mu}_{E_k} + \|f\|_{L^{\mu/(\mu-1)}} D^{1/\mu} + M_5.
\end{aligned}
$$

Since $\beta - \mu < 2$, u_k is uniformly bounded in E_k. We obtain our conclusion.

The following proof is the same to Lemmas 2.6 and 2.8. We complete the proof. □

Proof of Corollary 1.1. It follows from (W_2'') that there exists a $T > 0$ such that

$$\liminf_{|x|\to\infty} \frac{W(t,x)}{|x|^2} > \frac{\pi^2}{2T^2}$$

uniformly in $t \in [-T, T]$, which implies the condition (W_2') in Theorem 1.2. Similar to the proof of Theorem 1.2, Corollary 1.1 holds. The proof is completed. □

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