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Moduleability, algebraic structures, and nonlinear properties

F.J. García-Pacheco [a], C. Pérez-Eslava [b], J.B. Seoane-Sepúlveda [c,*,1]

[a] Department of Mathematics, Texas A&M University, College Station, TX 77843, USA
[b] Departamento de Matemáticas, Universidad de Cádiz, Puerto Real, 11510, Spain
[c] Departamento de Análisis Matemático, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, Plaza de Ciencias, 3, 28040 Madrid, Spain

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Dedicated to the memory of our friend Antonio Aizpuru (1954–2008)

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ABSTRACT

We show that some pathological phenomena occur *more often* than one could expect, existing *large* algebraic structures (infinite dimensional vector spaces, algebras, positive cones or infinitely generated modules) enjoying certain *special* properties. In particular we construct infinite dimensional vector spaces of non-integrable, measurable functions, completing some recent results shown in García-Pacheco et al. (2009) [13], García-Pacheco and Seoane-Sepúlveda (2006) [15], Muñoz-Fernández et al. (2008) [20]. We prove, as well, the existence of dense and not barrelled spaces of sequences every non-zero element of which has a finite number of zero coordinates (giving partial answers to a problem originally posed by R.M. Aron and V.I. Gurariy in 2003).

1. Introduction and preliminary results

This paper is a contribution to the search for *special* properties among function and sequence spaces. Given such a special property, we say that subset M of functions satisfying it is *lineable* (respectively *spaceable*) if $M \cup \{0\}$ contains an infinite dimensional (respectively closed) vector space. We will say that the set M is μ-lineable if it contains a vector space of dimension μ. The terminology of lineable/spaceable was first introduced in [4,5,14]. One of the earliest results in this direction was proved by Gurariy [18,19], who showed that the set of nowhere differentiable functions on $[0,1]$ is lineable. Let us denote by $\aleph_0$ and c the cardinalities of $\mathbb{N}$ and the power set of $\mathbb{N}$, $\mathcal{P}(\mathbb{N})$, respectively. In [4], it was shown that the set of everywhere surjective functions is 2^c-lineable. Moreover, there exist infinitely generated algebras every non-zero element of which is an everywhere surjective function from $\mathbb{C}$ to $\mathbb{C}$ [2,6], and in this case the set is said to be *algebrable* [6, Definition 1.1]. Some of these pathological behaviours occur in really interesting ways. In [15] vector spaces of non-measurable functions of dimension β, for any cardinal β, were constructed. Also, Aron, Pérez-García, and the third author [3] constructed, given any set $E \subset \mathbb{T}$ of measure zero, an infinite dimensional, infinitely generated dense subalgebra of $\mathcal{C}(\mathbb{T})$ every non-zero element of which has a Fourier series expansion divergent in E. We refer the interested reader to [2,7–12,16,21] for more recent results on this topic.

Besides vector spaces or algebras, one could also study the existence of positive or negative cones, introducing the following concept (see [1, Definition 1.1]):

* Corresponding author.
E-mail addresses: fgarcia@math.tamu.edu (F.J. García-Pacheco), sonriencanto@hotmail.com (C. Pérez-Eslava), jseoane@mat.ucm.es (J.B. Seoane-Sepúlveda).
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Definition 1.1 *(Coneability).* A set of functions in $\mathcal{F}(\mathbb{K},\mathbb{K})$ is said to be coneable if it possesses a positive (or negative) cone containing an infinite linearly independent set.

In this paper we will also introduce the new notion of *moduleability*, as follows:

Definition 1.2 *(Moduleability).* Let $\mathcal{A}$ be an infinitely generated algebra over the commutative field $\mathbb{K}$. Let L be a subset of $\mathcal{A}$. We say that L is moduleable if there exists an infinitely generated subalgebra M of $\mathcal{A}$ and a non-zero additive subgroup G of $\mathcal{A}$ such that G is a $(\mathbb{K},M)$-sub-bimodule of ${}_{\mathbb{K}}\mathcal{A}_M$ and $L\cup\{0\}\supseteq G$.

This notion is closely related to lineability and algebrability, as we see in the following remarks.

Remark 1.3. Let $\mathcal{A}$ be an infinitely generated algebra over the commutative field $\mathbb{K}$. If M is an infinitely generated subalgebra of $\mathcal{A}$, then M is trivially a $(\mathbb{K},M)$-sub-bimodule of ${}_{\mathbb{K}}\mathcal{A}_M$. Therefore, algebrability implies moduleability.

The converse to the previous remark does not remain true.

Remark 1.4. Let $\mathcal{A}$ be an infinitely generated algebra over the commutative field $\mathbb{K}$. Assume that $\mathcal{A}$ is commutative and has no divisors of 0. Let $g\in\mathcal{A}$ and M an infinitely generated subalgebra of $\mathcal{A}$ such that $gM\neq\{0\}$ and $gM\cap M=\{0\}$. Then, $G:=gM$ is moduleable but not algebrable. Indeed, G is trivially a $(\mathbb{K},M)$-sub-bimodule of ${}_{\mathbb{K}}\mathcal{A}_M$. Now, G cannot contain any subalgebra of $\mathcal{A}$ because of the following: if $m_1,m_2,m_3\in M$ are such that $(gm_1)(gm_2)=gm_3$, then $g(m_1m_2)=m_3$, which means that $m_1=m_2=m_3=0$. A concrete example of this situation can be $\mathbb{K}:=\mathbb{R}$, $\mathcal{A}:=\mathbb{R}[X]$ the ring of polynomials over an infinite set X of variables, $g:=x\in X$, and $M:=\mathbb{R}[X\setminus\{x\}]$.

The next proposition shows that the interesting case of moduleability happens when G contains at least one non-divisor of 0 of $\mathcal{A}$.

Proposition 1.5. *Let $\mathcal{A}$ be an infinitely generated algebra over the commutative field $\mathbb{K}$. Let M be an infinitely generated subalgebra of $\mathcal{A}$. Let G be a non-zero additive subgroup of $\mathcal{A}$ such that G is a $(\mathbb{K},M)$-sub-bimodule of ${}_{\mathbb{K}}\mathcal{A}_M$. If G contains an element $g\in G$ that is not a divisor of 0, then G is $\mathbb{K}$-infinite dimensional. Therefore, moduleability implies lineability.*

Proof. In the first place, since M is an infinitely generated subalgebra, we have that M is $\mathbb{K}$-infinite dimensional (if $(m_n)_{n\in\mathbb{N}}\subset M$ is an algebraically independent sequence, then it is linearly independent). Now, the mapping

$$M\to G,$$
$$m\mapsto gm,$$

is a monomorphism of $\mathbb{K}$-vector spaces because g is not a divisor of 0, which concludes the proof. $\square$

At this point an example of a lineable set that is not moduleable is due.

Remark 1.6. Let X be an infinite set. Let $\mathcal{A}:=\mathbb{R}[X]$, the ring of polynomials over the set X of variables. Let $x\in X$. Then, $\mathbb{R}[x]$ is lineable but not moduleable.

This paper is arranged into four main sections, in which we focus on finding algebraic structures inside certain subsets of functions or sequences. For instance, we study:

- Spaceability, algebrability and moduleability of the set of non-integrable measurable functions (Theorems 2.6 and 2.9).
- Lineability and dense (and not barrelled) subspaces of Banach spaces with a Schauder basis every non-zero element of which has a finite number of zero coordinates (Theorems 3.1–3.4).
- Lineability or coneability of the following sets of functions: Differentiable functions with discontinuous derivative, differentiable functions having an extreme value at a point where the derivative does not make a simple change in sign, and differentiable functions whose derivative is positive at a point but not monotonic in any neighbourhood of the point (Theorems 4.2–4.4).

Set theoretical considerations, abstract algebra, measure theory, cardinal theory, usual real analysis techniques, and classical Banach spaces techniques are used.

2. Non-integrable, measurable functions

In this section we will complete and close some results from [13,15,20], for which we will need to recall some theorems appearing in the latter references. In [15, Theorem 2.5] the authors proved the following result:

Theorem 2.1. *There exists an infinite dimensional vector space every non-zero element of which is a non-measurable function. Moreover, this vector space can be chosen to be closed and to have dimension β for any cardinality β.*

Later on, in [20, Section 3] the following result was proved.

Theorem 2.2. *Let $1 \leqslant q < p$. Then*:

(1) $\mathsf{L}^q[0,1] \setminus \mathsf{L}^p[0,1]$ *is c-lineable.*
(2) *If I is any unbounded interval, then $\mathsf{L}^p(I) \setminus \mathsf{L}^q(I)$ is c-lineable.*
(3) $\ell_p \setminus \ell_q$ *is c-lineable.*

Finally, in [13] the authors showed the following theorem.

Theorem 2.3. *Given an interval I, we have the following*:

(1) *If I is unbounded, then the set of all almost everywhere continuous bounded functions on I which are not Riemann-integrable contains an infinitely generated closed subalgebra. In particular, this set is spaceable and also algebrable.*
(2) *If I is unbounded, then the set of Riemann-integrable functions on I that are not Lebesgue-integrable is lineable (but not algebrable).*
(3) *The set of Lebesgue-integrable functions on I that are not Riemann-integrable is spaceable.*

In this section we complete all these results by means of proving, particularly, that the set of non-integrable, measurable functions can be made spaceable, algebrable, or even moduleable (see Definition 1.2).

Lemma 2.4. *Let (Ω, Σ, μ) be a measure space such that there exists a family $(A_n)_{n\in\mathbb{N}} \subseteq \Sigma$ of pairwise disjoint measurable sets with $\mu(A_n) > 0$ for all $n \in \mathbb{N}$. Then, the mapping*

$$\begin{aligned} &\ell_\infty \to \mathsf{L}^\infty(\Omega, \Sigma, \mu), \\ &(a_n)_{n\in\mathbb{N}} \mapsto \sum_{n=1}^{\infty} a_n \chi_{A_n} \end{aligned} \tag{1}$$

is a $\|\cdot\|_\infty$-$\|\cdot\|_\infty$ isometry and an algebra-isomorphism over its image.

Proof. It is sufficient to observe that

$$\left\|(a_n)_{n\in\mathbb{N}}\right\|_\infty = \sup\left\{|a_n|\colon n \in \mathbb{N}\right\} = \left\|\sum_{n=1}^{\infty} a_n \chi_{A_n}\right\|_\infty. \qquad \square$$

Now we want to make a useful remark related to closed algebras of bounded sequences that not converging to 0 (see [13]).

Remark 2.5. According to [13, Proposition 2.1] $(\ell_\infty \setminus c_0) \cup \{0\}$ contains a closed and infinitely generated subalgebra M. In particular, let $P = \{p_1, p_2, p_3, \ldots\}$ denote the set of all prime numbers (assuming that $p_1 < p_2 < p_3 < \cdots$), and for every $p \in P$ we consider the bounded sequence $x^p \in \ell_\infty$ given by

$$x_j^p = \begin{cases} 1, & \text{if } j = p^k \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the subspace of ℓ_∞

$$M := \left\{\sum_{i=1}^{\infty} \lambda_i x^{p_i}\colon (\lambda_i)_{i\in\mathbb{N}} \in \ell_\infty\right\}$$

is a $\|\cdot\|_\infty$-closed subalgebra of ℓ_∞ minimally generated by $\{x^p\colon p \in P\}$.

We are now ready to prove the main result of this section.

Theorem 2.6. *Let (Ω, Σ, μ) be a measure space such that there exists $\varepsilon > 0$ and an infinite family $(A_n)_{n\in\mathbb{N}} \subseteq \Sigma$ of pairwise disjoint measurable sets with $\mu(A_n) \geqslant \varepsilon$ for all $n \in \mathbb{N}$. Then,*

$$\bigcap_{p=1}^{\infty} \mathsf{L}^{\infty}(\Omega, \Sigma, \mu) \setminus \mathsf{L}^{p}(\Omega, \Sigma, \mu)$$

is $\|\cdot\|_\infty$-spaceable and algebrable.

Proof. Let M be as in Remark 2.5. Let us consider the mapping ϕ given in (1). Since ϕ is a $\|\cdot\|_\infty$-$\|\cdot\|_\infty$ isometry and M is $\|\cdot\|_\infty$-closed, we have that $\phi(M)$ is $\|\cdot\|_\infty$-closed. Thus, it remains to check that

$$\phi(M) \subseteq \mathsf{L}^{\infty}(\Omega, \Sigma, \mu) \setminus \mathsf{L}^{p}(\Omega, \Sigma, \mu) \cup \{0\}$$

for all $1 \leqslant p < \infty$. Indeed, let us fix $1 \leqslant p < \infty$ and let $(a_n)_{n\in\mathbb{N}}$ be in $M \setminus \{0\}$. Then,

$$\left|\sum_{n=1}^{\infty} a_n \chi_{A_n}\right|^p = \sum_{n=1}^{\infty} |a_n|^p \chi_{A_n},$$

thus

$$\int_{\Omega} \left|\sum_{n=1}^{\infty} a_n \chi_{A_n}\right|^p d\mu = \int_{\Omega} \sum_{n=1}^{\infty} |a_n|^p \chi_{A_n}\, d\mu = \sum_{n=1}^{\infty} |a_n|^p \mu(A_n) = \varepsilon \sum_{n=1}^{\infty} |a_n|^p = \infty$$

since $(a_n)_{n\in\mathbb{N}}$ does not converge to 0. $\square$

To summarise, we have found spaceability and algebrability of essentially bounded measurable functions that are not p-integrable for $1 \leqslant p < \infty$. The purpose now is to find a general result that includes finite measure spaces. Observe that then we will need to consider measurable functions that are not essentially bounded, as indicated in the following well-known fact:

Let (Ω, Σ, μ) be a finite measure space. Then $\mathsf{L}^{\infty}(\Omega, \Sigma, \mu) \subseteq \mathsf{L}^{1}(\Omega, \Sigma, \mu)$.

Notice that having to consider non-essentially bounded measurable functions will make us lose topological strength and will take us to moduleability. However, our next step is to modify Lemma 2.4 so that we can consider measurable functions that are not essentially bounded.

Lemma 2.7. *Let (Ω, Σ, μ) be a measure space such that there exists an infinite family $(A_n)_{n\in\mathbb{N}} \subseteq \Sigma$ of pairwise disjoint measurable sets with $\mu(A_n) > 0$ for all $n \in \mathbb{N}$. Then, the mapping*

$$\begin{aligned} \mathbb{K}^{\mathbb{N}} &\to \mathsf{L}^{0}(\Omega, \Sigma, \mu), \\ (a_n)_{n\in\mathbb{N}} &\mapsto \sum_{n=1}^{\infty} a_n \chi_{A_n} \end{aligned} \tag{2}$$

is τ-μ sequentially continuous and an algebra-isomorphism over its image, where τ denotes the topology of pointwise convergence in $\mathbb{K}^{\mathbb{N}}$ and μ denotes the topology of local convergence in measure in $\mathsf{L}^{0}(\Omega, \Sigma, \mu)$. If (Ω, Σ, μ) is σ-finite, then:

(1) *The inverse of the previous mapping is μ-τ sequentially continuous.*
(2) *The range of the previous mapping is μ-sequentially closed.*

Proof. Let $((a_n^k)_{n\in\mathbb{N}})_{k\in\mathbb{N}}$ be a sequence in $\mathbb{K}^{\mathbb{N}}$ that is τ-convergent to $(a_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$. Then, $(a_n^k)_{k\in\mathbb{N}}$ converges to a_n for all $n \in \mathbb{N}$, therefore the sequence $(\sum_{n=1}^{\infty} a_n^k \chi_{A_n})_{k\in\mathbb{N}}$ converges almost everywhere to $\sum_{n=1}^{\infty} a_n \chi_{A_n}$. Since almost everywhere convergence implies local convergence in measure, we deduce that $(\sum_{n=1}^{\infty} a_n^k \chi_{A_n})_{k\in\mathbb{N}}$ is μ-convergent to $\sum_{n=1}^{\infty} a_n \chi_{A_n}$. Assume now that (Ω, Σ, μ) is σ-finite. Then:

(1) Let $(\sum_{n=1}^{\infty} a_n^k \chi_{A_n})_{k\in\mathbb{N}}$ be μ-convergent to $\sum_{n=1}^{\infty} a_n \chi_{A_n}$. We will show that $(a_n^k)_{k\in\mathbb{N}}$ converges to a_n for all $n \in \mathbb{N}$. Otherwise, there exists $n \in \mathbb{N}$, $\varepsilon > 0$, and a subsequence $(a_n^{k_i})_{i\in\mathbb{N}}$ such that $|a_n^{k_i} - a_n| \geqslant \varepsilon$ for all $i \in \mathbb{N}$. Since (Ω, Σ, μ) is σ-finite, there exists another subsequence $(\sum_{n=1}^{\infty} a_n^{k_{i_j}} \chi_{A_n})_{j\in\mathbb{N}}$ that is almost everywhere convergent to $\sum_{n=1}^{\infty} a_n \chi_{A_n}$. Since $\mu(A_n) > 0$, we deduce that $(a_n^{k_{i_j}})_{j\in\mathbb{N}}$ converges to a_n, which is impossible since $|a_n^{k_{i_j}} - a_n| \geqslant \varepsilon$ for all $j \in \mathbb{N}$.

(2) Let $(\sum_{n=1}^{\infty} a_n^k \chi_{A_n})_{k\in\mathbb{N}}$ be μ-convergent to $f \in \mathsf{L}^0(\Omega, \Sigma, \mu)$. Since (Ω, Σ, μ) is σ-finite, there exists a subsequence $(\sum_{n=1}^{\infty} a_n^{k_i} \chi_{A_n})_{i\in\mathbb{N}}$ that is almost everywhere convergent to f. This clearly implies that

$$f = \sum_{n=1}^{\infty} \Bigl(\lim_{i\to\infty} a_n^{k_i}\Bigr) \chi_{A_n}. \qquad \square$$

Before stating and proving the main result in this section we want to make a useful remark related to Remark 2.5.

Remark 2.8. Let $\mathcal{A}$ be any subalgebra of $\mathbb{K}^{\mathbb{N}}$. Let

$$N_{\mathcal{A}} := \Biggl\{ \sum_{i=1}^{\infty} \lambda_i x^{p_i}\colon\ (\lambda_i)_{i\in\mathbb{N}} \in \mathcal{A} \Biggr\}.$$

We have that $N_{\mathcal{A}}$ is a subalgebra of $\mathbb{K}^{\mathbb{N}}$ whose intersection with c_0 is $\{0\}$. Let us show that $N_{\mathcal{A}}$ is τ-sequentially closed in $\mathbb{K}^{\mathbb{N}}$ if $\mathcal{A}$ is τ-sequentially closed in $\mathbb{K}^{\mathbb{N}}$. Indeed, let $(\sum_{i=1}^{\infty} \lambda_i^n x^{p_i})_{n\in\mathbb{N}}$ be a sequence in $N_{\mathcal{A}}$ that is τ-convergent to an element $(\alpha_j)_{j\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$. Observe that if j is not of the form p^k for some $p \in P$ and some $k \in \mathbb{N}$ then $\alpha_j = 0$. Now, if we fix $i \in \mathbb{N}$, then

$$\alpha_{p_i^k} = \lim_{n\to\infty} \lambda_i^n$$

for all $k \in \mathbb{N}$. This proves that

$$(\alpha_j)_{j\in\mathbb{N}} = \sum_{i=1}^{\infty} \Bigl(\lim_{n\to\infty} \lambda_i^n\Bigr) x^{p_i} \in N_{\mathcal{A}}.$$

Observe that in general it can be showed that

$$\mathrm{cl}_\tau^s(N_{\mathcal{A}}) = N_{\mathrm{cl}_\tau^s(\mathcal{A})},$$

where $\mathrm{cl}_\tau^s(C)$ denotes the τ-sequential closure of any subset C of $\mathbb{K}^{\mathbb{N}}$. Finally, notice also that if M is as in Remark 2.5 then $M = N_{\ell_\infty}$ and M is τ-sequentially closed in ℓ_∞.

Now we are in the right position to state and prove the main theorem in this section.

Theorem 2.9. *Let (Ω, Σ, μ) be a measure space such that there exists an infinite family $(A_n)_{n\in\mathbb{N}} \subseteq \Sigma$ of pairwise disjoint measurable sets with $\mu(A_n) > 0$ for all $n \in \mathbb{N}$. Then, $\mathsf{L}^0(\Omega, \Sigma, \mu) \setminus \mathsf{L}^1(\Omega, \Sigma, \mu)$ is moduleable. If (Ω, Σ, μ) is σ-finite, then the bimodule can be chosen to be μ-sequentially closed.*

Proof. Let $N := N_{\mathbb{K}^{\mathbb{N}}}$ be as in Remark 2.8. Let

$$G := \bigl\{ (a_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}\colon\ \bigl(a_n \mu(A_n)\bigr)_{n\in\mathbb{N}} \in N \bigr\}.$$

Note that G is a $(\mathbb{K}, N)$-sub-bimodule of ${}_{\mathbb{K}}\mathbb{K}^{\mathbb{N}}_N$. Now, let ϕ be as in (2) and show that $\phi(G) \subseteq \mathsf{L}^0(\Omega, \Sigma, \mu) \setminus \mathsf{L}^1(\Omega, \Sigma, \mu) \cup \{0\}$. Indeed, let $(a_n)_{n\in\mathbb{N}}$ be in $G \setminus \{0\}$. Then,

$$\Biggl| \sum_{n=1}^{\infty} a_n \chi_{A_n} \Biggr| = \sum_{n=1}^{\infty} |a_n| \chi_{A_n},$$

therefore

$$\int_\Omega \Biggl| \sum_{n=1}^{\infty} a_n \chi_{A_n} \Biggr| d\mu = \int_\Omega \sum_{n=1}^{\infty} |a_n| \chi_{A_n}\, d\mu = \sum_{n=1}^{\infty} |a_n| \mu(A_n) = \infty$$

since $(a_n \mu(A_n))_{n\in\mathbb{N}}$ does not converge to 0 in virtue of Remark 2.8. Assume now that (Ω, Σ, μ) is σ-finite. Since N is τ-sequentially closed in $\mathbb{K}^{\mathbb{N}}$ in virtue of Remark 2.8 again, we deduce that G is also τ-closed in $\mathbb{K}^{\mathbb{N}}$. Finally, since ϕ is a τ-μ sequentially homeomorphism over its image and its image is μ-sequentially closed in $\mathsf{L}^0(\Omega, \Sigma, \mu)$, we deduce that $\phi(G)$ is μ-sequentially closed in $\mathsf{L}^0(\Omega, \Sigma, \mu)$. $\square$

In summary, we have found moduleability and (sequential) spaceability of non-integrable measurable functions.

3. Subspaces of sequences with finitely many zero coordinates

R.M. Aron and V.I. Gurariy posed, in 2003, the question of whether there exists an infinite dimensional and closed subspace of ℓ_∞ every non-zero element of which has a finite number of zero coordinates. Later on, in [20, Remark 1], the authors also studied this problem partially. One could as well-consider the previous question but replacing the word closed for dense. Also, it would be interesting, from our viewpoint, to study the existence of (either dense or closed) vector subspaces of a Banach space with Schauder basis every non-zero vector of which has a finite number of zero coordinates.

We will first consider the particular case of the spaces c, c_0, and ℓ_p $(1 \leqslant p < \infty)$. It is well known that, in the above spaces, the subspace c_{00} of sequences of finite range and convergent to 0 is dense and not barrelled. The purpose of our work in this section is to show that, in the above spaces, there exists a dense and non-barrelled subspace non-zero vectors of which have only a finite number of zero coordinates. From now on, and as usual, $\mathbb{R}[x]$ will denote the ring of polynomials with coefficients in $\mathbb{R}$.

Theorem 3.1. *There exists a vector subspace H of ℓ_p $(1 \leqslant p < \infty)$, verifying the following properties*:

(1) *If* $0 \neq a \in H$, *then* $\operatorname{card}\{i \in \mathbb{N}\colon a_i = 0\} < \infty$.
(2) *If* $a, b \in H$ *then* $ab \in H$.
(3) *H is dense and not barrelled.*

Proof. Similarly as before, let us consider the vector space

$$H = \left\{ \left(q\left(\frac{1}{n^2}\right)\right)_{n\in\mathbb{N}} :\ q \in \mathbb{R}[x],\ q(0) = 0 \right\}.$$

Let $q \in \mathbb{R}[x]$ with $q(0) = 0$. Take $M = \sup\{|q(x)|\colon\ x \in [0,1]\}$. By the Mean Value Theorem

$$\left| q\left(\frac{1}{n^2}\right)\right| \leqslant \frac{M}{n^2}$$

for every $n \in \mathbb{N}$. Then, it follows that $H \subseteq \ell_p$. On the other hand, H verifies the first two properties. Therefore, it remains to show that H is dense and not barrelled.

In first place, let us prove that H is not barrelled. Let

$$H_n = \left\{ \left(q\left(\frac{1}{n^2}\right)\right)_{n\in\mathbb{N}} :\ q \in \mathbb{R}_n[x],\ q(0) = 0 \right\},$$

then we have that H_n is vector subspace of H of dimension n, furthermore $H = \bigcup_{n\in\mathbb{N}} H_n$ thus H is not barrelled.

Finally, we will show that H is dense in ℓ_p. Suppose not. We can find then a sequence $(a_n)_{n\in\mathbb{N}}$ in ℓ_p^* such that

$$\sum_{n\in\mathbb{N}} a_n \cdot q\left(\frac{1}{n^2}\right) = 0,$$

for every $q \in \mathbb{R}[x]$ with $q(0) = 0$. Particularising in polynomials of the form $q_m(x) = x^m$, we obtain, for every $m \in \mathbb{N}$, that

$$a_1 + a_2 \cdot \left(\frac{1}{2^2}\right)^m + a_3 \cdot \left(\frac{1}{3^2}\right)^m + \cdots + a_n \cdot \left(\frac{1}{n^2}\right)^m + \cdots = 0.$$

There also exists $M > 0$ such that $|a_n| \leqslant M$ for $n \in \mathbb{N}$, therefore

$$|a_1| \leqslant M \cdot \left[\left(\frac{1}{2^2}\right)^m + \left(\frac{1}{3^2}\right)^m + \cdots + \left(\frac{1}{n^2}\right)^m + \cdots \right]$$

for every $m \in \mathbb{N}$ and necessarily $a_1 = 0$, obtaining that

$$a_2 \cdot \left(\frac{1}{2^2}\right)^m + a_3 \cdot \left(\frac{1}{3^2}\right)^m + \cdots + a_n \cdot \left(\frac{1}{n^2}\right)^m + \cdots = 0,$$

that is

$$a_2 + a_3 \cdot \left(\frac{2^2}{3^2}\right)^m + \cdots + a_n \cdot \left(\frac{2^2}{n^2}\right)^m + \cdots = 0.$$

Similarly, it follows that $a_2 = 0$ as well. Inductively, we have that $a_n = 0$ for every $n \in \mathbb{N}$, reaching a contradiction. $\square$

Let us see now the general case of Banach spaces with a Schauder basis. It is well known that if X is a Banach space with a Schauder basis, then the vectors whose coordinates with respect to this basis are all zero but a finite number of them form a dense and not barrelled vector subspace of X. This arises the question of finding a dense and not barrelled vector subspace every non-zero element of which has a finite number of zero coordinates. If X denotes an infinite dimensional Banach space and $(e_n)_{n\in\mathbb{N}}$ is a Schauder basis for X contained in S_X, the unit sphere of X, then

$$\ell_1 \subseteq \left\{(t_n)_{n\in\mathbb{N}}\colon \sum_{n=1}^{\infty} t_n e_n \text{ converges}\right\} \subseteq c_0.$$

Moreover, the operator

$$\begin{aligned} \ell_1 &\to X, \\ (t_n)_{n\in\mathbb{N}} &\mapsto \sum_{n=1}^{\infty} t_n e_n \end{aligned} \tag{3}$$

verifies that

(1) it is linear,
(2) it is one-to-one,
(3) it is continuous with norm 1, and
(4) the image of ℓ_1 is dense in X (because it contains the image of c_{00}).

Theorem 3.2. *Let X be an infinite dimensional Banach space with a Schauder basis $(e_n)_{n\in\mathbb{N}}$ contained in S_X. There exists a vector subspace V of X such that*:

(1) *If $0 \neq a \in V$, then* $\operatorname{card}\{n \in \mathbb{N}\colon a[n] = 0\} < \infty$.
(2) *If $a, b \in V$ then $\sum_{n=1}^{\infty} a[n]b[n]e_n \in V$.*
(3) *V is dense and not barrelled.*

Proof. Let us denote by ϕ the operator given in (3). According to Theorem 3.1 for $p = 1$, there exists a vector subspace H of ℓ_1 verifying that

(1) if $0 \neq a \in H$, then $\operatorname{card}\{i \in \mathbb{N}\colon a_i = 0\} < \infty$;
(2) if $a, b \in H$ then $ab \in H$;
(3) H is dense and not barrelled.

Now, we consider $V = \phi(H)$. Taking into account the properties enjoyed by ϕ, we deduce that V verifies (1) and (2) of the theorem. Besides, since ϕ is continuous and $\phi(\ell_1)$ is dense in X, we deduce that V is dense in X. Finally, according to the proof of Theorem 3.1, we can choose H in such a way it can be written as $H = \bigcup_{n\in\mathbb{N}} H_n$ where each H_n is a vector subspace of dimension n. Now, since ϕ is one-to-one, we deduce that $V = \bigcup_{n\in\mathbb{N}} \phi(H_n)$ and each $\phi(H_n)$ is a vector space of dimension n. As a consequence, V is not barrelled. $\square$

From the techniques appearing in proofs of the previous theorems we can obtain the following results as well:

Theorem 3.3. *There exists a vector subspace E of c verifying the following properties*:

(1) *If $0 \neq a \in E$, then* $\operatorname{card}\{i \in \mathbb{N}\colon a_i = 0\} < \infty$.
(2) *If $a, b \in E$ then $ab \in E$.*
(3) *E is dense and non-barrelled.*

Theorem 3.4. *There exists a vector subspace F of c_0 such that*:

(1) *If $0 \neq a \in F$, then* $\operatorname{card}\{i \in \mathbb{N}\colon a_i = 0\} < \infty$.
(2) *If $a, b \in F$ then $ab \in F$.*
(3) *F is dense and not barrelled.*

Next, given any Banach space X with Schauder basis $(e_n)_{n\in\mathbb{N}}$ contained in S_X, we provide an explicit construction of a non-barrelled dense vector subspace V of X every non-zero element of which has a finite number of zero coordinates. Consider a dense sequence $(y_n)_{n\in\mathbb{N}}$ in X such that for every $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ with $y_n[j] = 0$ for every $j \geqslant k$, where $y_n[j]$ denotes the j-th coordinate of y_n with respect to the Schauder basis $(e_n)_{n\in\mathbb{N}}$. Now, given the sequence $(x_n)_{n\in\mathbb{N}}$ in X by

$$x_n = y_n + \frac{1}{n}\left(\frac{1}{2^n}, \frac{1}{3^n}, \frac{1}{4^n}, \dots\right),$$

one can show that the linear space $V = \operatorname{span}\{x_n\colon n > 1\}$ enjoys the required properties (we spare the details of this proof to the interested reader, since the technique is similar to that of Theorems 3.1 and 3.2).

To conclude this section, we will consider the case of ℓ_∞. From the previous results it follows that ℓ_∞ contains a subspace H such that

(1) if $0 \neq a \in H$, then $\operatorname{card}\{i \in \mathbb{N}\colon a_i = 0\} < \infty$, and
(2) if $a, b \in H$ then $ab \in H$.

In particular, if $(\alpha_n)_{n\in\mathbb{N}}$ is any bounded sequence of real numbers, then

$$\left\{\left(q(\alpha_n)\right)_{n\in\mathbb{N}}\colon q \in \mathbb{R}[x]\right\} \tag{4}$$

verifies (1) and (2). However, let us notice that if H is of the form given in (4), being $(\alpha_n)_{n\in\mathbb{N}}$ a convergent sequence, then H can be neither closed nor dense. This follows from the fact that we can adapt the proof of Theorem 3.3 to prove that H is dense in c, therefore H cannot be either closed or dense in ℓ_∞.

We finish the paper with a section devoted to some pathologies amongst the set of differentiable functions.

4. Some pathologies in subsets of differentiable functions

In [17, p. 36, ex. 2] is given an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable and whose derivative is discontinuous at the origin. Namely, the function given is the following:

$$f(x) = \begin{cases} x^2 \cdot \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

This function is differentiable and

$$f'(x) = \begin{cases} 2x \cdot \sin(1/x) - \cos(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Also, $\lim_{x\to 0} f'(x)$ does not exist.

One could wonder whether this kind of behaviour amongst functions occurs often, namely, whether there exists an infinite dimensional vector space of such kind of functions. We will show that this is the case, but before that, let us recall the following result, of simple proof, which is left as an exercise to the interested reader.

Lemma 4.1. *Let P denote the set of odd prime numbers. The sets of functions $\{\sin(1/px)\colon p \in P\}$ and $\{\cos(1/px)\colon p \in P\}$ are linearly independent families.*

Theorem 4.2. *The set of differentiable functions on $\mathbb{R}$ with discontinuous derivative at the origin, D, is lineable, i.e. there exists an infinite dimensional vector space E, with $E \setminus \{0\} \subset D$.*

Proof. For every $p \in P$ (the set of odd prime numbers) define the following sequence of functions:

$$f_p(x) = \begin{cases} x^2 \cdot \sin(1/px), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Let $E = \operatorname{span}\{f_p\colon p \in P\}$ and let $g \in E \setminus \{0\}$, i.e. $g(x) = \sum_{i=1}^k \alpha_i \cdot x^2 \cdot \sin(1/p_i x)$, with $p_i \in P$ and $\alpha_i \neq 0$ for every $i \in \{1, \dots, k\}$. It is clear that g is also differentiable and that g' can be written in the form $g'(x) = Z(x) - C(x)$, where $Z(x) \to 0$ as $x \to 0$ and $C(x) = \sum_{i=1}^k \frac{\alpha_i}{p_i} \cdot \cos(1/p_i x)$. Next, we will see that $\lim_{x\to 0} C(x)$ does not exist, and we will be done. To do this, let us consider the sequence $(x_n)_n$ given, for $n \in \mathbb{N}$, by $x_n = (p_2 \cdot p_3 \cdot \dots \cdot p_k)^{-1} \cdot (\frac{\pi}{2} + 2n\pi)^{-1}$. We have that $x_n \to 0$ as $n \to \infty$, and

$$\begin{aligned} C(x_n) &= \sum_{i=1}^k \frac{\alpha_i}{p_i} \cdot \cos\left[\frac{p_2 \cdot p_3 \cdot \dots \cdot p_k}{p_i} \cdot \left(\frac{\pi}{2} + 2n\pi\right)\right] \\ &= (\alpha_1/p_1) \cdot \cos\left[\frac{p_2 \cdot p_3 \cdot \dots \cdot p_k}{p_1} \cdot \left(\frac{\pi}{2} + 2n\pi\right)\right] \\ &= (\alpha_1/p_1) \cdot \cos\left(q \cdot \left[\frac{\pi}{2} + 2n\pi\right]\right), \end{aligned}$$

where $q \in \mathbb{Q} \setminus \mathbb{Z}$. Thus, $\lim_{n\to\infty} g'(x_n) = -\frac{\alpha_1}{p_1} \cdot \lim_{n\to\infty} \cos[q \cdot (\frac{\pi}{2} + 2n\pi)]$, which does not exist, therefore $\lim_{x\to 0} g'(x)$ does not exist, and $g \in D$. Thus we have proved that D is lineable (E is infinite dimensional by Lemma 4.1). $\square$

As in the proof of the previous results (and with help of the basic examples appearing in [17]) we also obtain the following results on lineability/coneability.

Theorem 4.3. *The set of differentiable functions having an extreme value at a point where the derivative does not make a simple change in sign is lineable.*

Theorem 4.4. *The set of differentiable functions whose derivative is positive at a point but which is not monotonic in any neighbourhood of the point, N, is coneable and not lineable. Moreover, if $V \subset N \cup \{0\}$ is a vector space then $V = \{0\}$.*

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