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Positive solutions for a second-order differential system

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ABSTRACT

This paper investigates the existence of positive solutions for a second-order differential system by using the fixed point theorem of cone expansion and compression.

1. Introduction

In this paper we are concerned with the existence of positive solutions for the second-order boundary value problem

$$\begin{cases} -u'' + \lambda u = \varphi u + f(t,u), & 0 < t < 1, \\ -\varphi'' = \mu u, & 0 < t < 1, \\ u(0) = u(1) = 0, \\ \varphi(0) = \varphi(1) = 0, \end{cases} \tag{1}$$

where $\lambda > -\pi^2$, μ is a positive parameter, $f \in C[(0,1) \times R^+, R^+]$, that is, f is probably singular at $t = 0$ and $t = 1$.

Problem (1) is related to the stationary version of the reaction–diffusion system

$$\begin{cases} u_{1t} - \Delta u_1 = u_1 u_2 - b u_1, & x \in \Omega,\ t > 0, \\ u_{2t} - \Delta u_2 = a u_1, & x \in \Omega,\ t > 0, \\ u_1 = u_2 = 0, & x \in \partial\Omega,\ t > 0, \\ u_1(x,0) = u_{10}(x) \geqslant 0, \quad u_2(x,0) = u_{20}(x) \geqslant 0, & x \in \overline{\Omega}, \end{cases} \tag{2}$$

where $\Omega \in R^N$ is a smooth bounded domain, a, $b > 0$ are constants, u_{10}, u_{20} are continuous nonnegative functions on $\overline{\Omega}$. As a model to describe the neutron flux and temperature of the nuclear reactors, the system (2) is studied by the authors in [3]. It is proved that there is at least one positive stationary solution if $2 \leqslant n < 6$. In addition, it is also proved that every positive stationary solution is a threshold when Ω is a ball.

Recently, boundary value problem of fourth-order ordinary differential equations has been extensively studied (see [6,8] and references therein). Naturally, we can find that problem (1) is similar to the following fourth-order boundary prob-

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$$\begin{cases} \varphi^{(4)} + A(\varphi)\varphi'' = f(t, -\varphi''), & 0 < t < 1, \\ \varphi(0) = \varphi(1) = \varphi''(0) = \varphi''(1) = 0. \end{cases} \tag{3}$$

So, it is useful for us to study problem (1). In addition, there are many authors have studied the differential system, such as [1,4,7,9–12], and they obtain fruitful results. Inspired by the above references, we will study problem (1) by the following fixed point theorem of cone expansion and compression in [2]:

Lemma 1.1. *Let E be a Banach space, and $K \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(i) $\|Tu\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_2$; *or*
(ii) $\|Tu\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

The organization of this paper is as follows: In Section 2, we introduce some preliminaries. In Section 3, we state and prove our main results. In Section 4, we give two examples to illustrate our main results.

2. Preliminaries

Let $G(t,s)$ be the Green function of linear boundary value problem

$$-u'' + \lambda u = 0, \qquad u(0) = u(1) = 0,$$

where the constant $\lambda > -\pi^2$. Thus, from [5], we have the following lemmas.

Lemma 2.1. *Let $\omega = \sqrt{|\lambda|}$, then $G(t,s)$ can be expressed by*

(i) $$G(t,s) = \begin{cases} \frac{\sinh \omega t \sinh \omega(1-s)}{\omega \sinh \omega}, & 0 \leqslant t \leqslant s \leqslant 1, \\ \frac{\sinh \omega s \sinh \omega(1-t)}{\omega \sinh \omega}, & 0 \leqslant s \leqslant t \leqslant 1, \end{cases} \quad \text{if } \lambda > 0.$$

(ii) $$G(t,s) = \begin{cases} t(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \\ s(1-t), & 0 \leqslant s \leqslant t \leqslant 1, \end{cases} \quad \text{if } \lambda = 0.$$

(iii) $$G(t,s) = \begin{cases} \frac{\sin \omega t \sin \omega(1-s)}{\omega \sin \omega}, & 0 \leqslant t \leqslant s \leqslant 1, \\ \frac{\sin \omega s \sin \omega(1-t)}{\omega \sin \omega}, & 0 \leqslant s \leqslant t \leqslant 1, \end{cases} \quad \text{if } -\pi^2 < \lambda < 0.$$

Lemma 2.2. *The function $G(t,s)$ has the following properties*:

(i) $G(t,s) > 0$, $\forall t, s \in (0,1)$,
(ii) $G(t,s) \leqslant CG(s,s)$, $\forall t, s \in [0,1]$,
(iii) $G(t,s) \geqslant \delta G(t,t)G(s,s)$, $\forall t, s \in [0,1]$,

where $C = 1$, $\delta = \omega/\sinh\omega$, if $\lambda > 0$; $C = 1$, $\delta = 1$, if $\lambda = 0$; $C = 1/\sin\omega$, $\delta = \omega\sin\omega$, if $-\pi^2 < \lambda < 0$.

Problem (1) can be easily transformed into a nonlinear second-order ordinary differential equation with a nonlocal term. Briefly, the boundary value problem

$$-\varphi'' = \mu u, \qquad \varphi(0) = \varphi(1) = 0,$$

is solved by Lemma 2.1, namely,

$$\varphi(t) = \mu \int_0^1 K(t,s)u(s)\,ds \tag{4}$$

where $K(t,s)$ denotes the Green function $G(t,s)$ when $\lambda = 0$. Then, inserting (4) into the first equation of (1), we have

$$\begin{cases} -u'' + \lambda u = \mu u \int_0^1 K(t,s)u(s)\,ds + f(t,u), & 0 < t < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{5}$$

Now we consider the existence of positive solutions of (5). the function $u \in C^2(0,1) \cap C[0,1]$ is a positive solution of (5), if u satisfies (5), and $u \geqslant 0$, $t \in [0,1]$, $u \neq 0$. Then by Lemma 2.1, the solution of (5) can be expressed as following

$$u = \mu \int_0^1 \int_0^1 G(t,s)u(s)K(s,\tau)u(\tau)\,d\tau\,ds + \int_0^1 G(t,s) f(s,u)\,ds.$$

We now define a mapping $T : C[0,1] \to C[0,1]$ by

$$Tu(t) = \mu \int_0^1 \int_0^1 G(t,s)u(s)K(s,\tau)u(\tau)\,d\tau\,ds + \int_0^1 G(t,s) f(s,u)\,ds.$$

It is clear that $T : C[0,1] \to C[0,1]$ is completely continuous.

Set

$$P = \left\{ u \in C[0,1]\colon\ u(t) \geqslant \sigma \|u\|,\ t \in \left[\frac{1}{4}, \frac{3}{4}\right] \right\},$$

where $\sigma = \frac{\delta m}{C} \in (0,1)$ and

$$m = \min_{t \in [1/4,3/4]} G(t,t) = \begin{cases} \frac{\sinh \frac{\omega}{4} \sinh \frac{3}{4}\omega}{\omega \sinh \omega}, & \text{if } \lambda > 0, \\ \frac{3}{16}, & \text{if } \lambda = 0, \\ \frac{\sin \frac{\omega}{4} \sin \frac{3}{4}\omega}{\omega \sin \omega}, & \text{if } -\pi^2 < \lambda < 0. \end{cases}$$

It is well known that P is a cone in $C[0,1]$.

Lemma 2.3. *(See [5].) $T(P) \subset P$ and $T : P \to P$ is completely continuous.*

3. Main result

Theorem 3.1. *Assume that the following conditions hold*:

(H1) $\lambda > -\pi^2$;

(H2) $f \in C[(0,1) \times R^+, R^+]$, *and*

$$f(t,u) \leqslant p(t)q(u), \quad t \in (0,1),\ u \in R^+,$$

where $p(t) \in C[(0,1), R^+]$, $q(u) \in C[R^+, R^+]$, *and*

$$\int_0^1 G(s,s)p(s)\,ds < +\infty;$$

(H3) $$\overline{\lim_{u \to 0^+}} \frac{q(u)}{u} = 0, \qquad \underline{\lim_{u \to +\infty}} \min_{[1/4,3/4]} \frac{f(t,u)}{u} = +\infty.$$

If $\mu \in (0, \frac{1}{2C \int_0^1 \int_0^1 G(s,s)K(s,\tau)\,d\tau\,ds}]$, *then problem* (1) *has at least one positive solution.*

Proof. By condition (H3), there exist $c_1 > 0$, $0 < r < 1$ such that

$$q(u) \leqslant c_1 u, \quad u \in [0,r],$$

and satisfying

$$C \cdot c_1 \cdot \int_0^1 G(s,s)p(s)\,ds \leqslant \frac{1}{2}.$$

Thus, by Lemma 2.2, (H2) and (H3), we have

$$\begin{aligned}(Tu)(t) &= \mu\int_0^1\int_0^1 G(t,s)u(s)K(s,\tau)u(\tau)\,d\tau\,ds + \int_0^1 G(t,s)f(s,u)\,ds\\ &\leqslant \mu C\int_0^1\int_0^1 G(s,s)K(s,\tau)\,d\tau\,ds\,\|u\|^2 + C\int_0^1 G(s,s)p(s)q(u)\,ds\\ &\leqslant \mu C\int_0^1\int_0^1 G(s,s)K(s,\tau)\,d\tau\,ds\,\|u\|^2 + C\cdot c_1\cdot\int_0^1 G(s,s)p(s)\,ds\,\|u\|\\ &\leqslant \frac{1}{2}\|u\|^2 + \frac{1}{2}\|u\|\\ &\leqslant \|u\|, \quad \forall u\in\partial B_r\cap P,\ t\in[0,1].\end{aligned}$$

Consequently,

$$\|Tu\|\leqslant\|u\|, \quad \forall u\in\partial B_r\cap P.$$

On the other hand, by condition (H3), there exist $c_2>0$, $R_1>0$ such that

$$f(t,u)\geqslant c_2u, \quad \forall u\geqslant R_1,\ t\in[1/4,3/4],$$

and satisfying

$$c_2\cdot\sigma\cdot\int_{1/4}^{3/4} G\left(\frac{1}{2},s\right)ds\geqslant 1.$$

Set $R>\frac{R_1}{\sigma}$, it is easy to know that

$$\min_{t\in[\frac14,\frac34]} u\geqslant\sigma\|u\|=\sigma R>R_1, \quad \forall u\in B_R\cap P.$$

Then, from conditions (H3) and Lemma 2.2, we have

$$\begin{aligned}(Tu)\left(\frac{1}{2}\right) &= \mu\int_0^1\int_0^1 G\left(\frac{1}{2},s\right)u(s)K(s,\tau)u(\tau)\,d\tau\,ds + \int_0^1 G\left(\frac{1}{2},s\right)f(s,u)\,ds\\ &\geqslant \mu\int_{\frac14}^{\frac34}\int_{\frac14}^{\frac34} G\left(\frac{1}{2},s\right)u(s)K(s,\tau)u(\tau)\,d\tau\,ds + \int_{\frac14}^{\frac34} G\left(\frac{1}{2},s\right)f(s,u)\,ds\\ &\geqslant \mu\sigma^2\int_{\frac14}^{\frac34}\int_{\frac14}^{\frac34} G\left(\frac{1}{2},s\right)K(s,\tau)\,d\tau\,ds\,\|u\|^2 + \sigma c_2\int_{\frac14}^{\frac34} G\left(\frac{1}{2},s\right)ds\,\|u\|\\ &\geqslant \sigma c_2\int_{\frac14}^{\frac34} G\left(\frac{1}{2},s\right)ds\,\|u\|\\ &\geqslant \|u\|\end{aligned}$$

for all $u\in\partial B_R\cap P$.

Consequently,

$$\|Tu\|\geqslant\|u\|, \quad \forall u\in\partial B_R\cap P.$$

Then by Lemma 1.1, problem (1) has at least one positive solution. □

Theorem 3.2. *Assume that* (H1) *and* (H2) *hold. In addition, also assume that the following conditions hold*:

(H4) $$\varliminf_{u\to 0^+} \min_{[1/4,3/4]} \frac{f(t,u)}{u} = +\infty, \qquad \varliminf_{u\to +\infty} \min_{[1/4,3/4]} \frac{f(t,u)}{u} = +\infty;$$

(H5) *There exists* $0 < \rho \leqslant 1$ *such that*

$$\sup_{u\in[0,1]} q(u) \leqslant \frac{\rho}{2C\int_0^1 G(s,s)p(s)\,ds}.$$

If $\mu \in (0, \frac{1}{2C\int_0^1\int_0^1 G(s,s)K(s,\tau)\,d\tau\,ds}]$, *then problem* (1) *has at least two positive solutions.*

Proof. By conditions (H1) and (H5), for $\forall u \in \partial B_\rho \cap P$ and $t \in [0,1]$, we have

$$\begin{aligned}
(Tu)(t) &= \mu \int_0^1\int_0^1 G(t,s)u(s)K(s,\tau)u(\tau)\,d\tau\,ds + \int_0^1 G(t,s)f(s,u)\,ds \\
&\leqslant \mu C \int_0^1\int_0^1 G(s,s)K(s,\tau)\,d\tau\,ds\,\|u\|^2 + C\int_0^1 G(s,s)p(s)q(u)\,ds \\
&\leqslant \mu C \int_0^1\int_0^1 G(s,s)K(s,\tau)\,d\tau\,ds\,\|u\|^2 + C\cdot \frac{\rho}{2C\int_0^1 G(s,s)p(s)\,ds}\cdot \int_0^1 G(s,s)p(s)\,ds \\
&\leqslant \frac{1}{2}\rho^2 + \frac{1}{2}\rho \leqslant \rho.
\end{aligned}$$

Consequently, we get

$$\|Tu\| \leqslant \|u\|, \quad \forall u \in \partial B_\rho \cap P.$$

On the other hand, by (H4), there exist $c_3 > 0$, $0 < r < \rho$ such that

$$f(t,u) \geqslant c_3 u, \quad \forall u \in [0,r],\ t \in [1/4,3/4],$$

and satisfying

$$c_3 \cdot \sigma \cdot \int_{1/4}^{3/4} G\left(\frac{1}{2},s\right) ds \geqslant 1.$$

Then, from conditions (H4) and Lemma 2.2, we have

$$\begin{aligned}
(Tu)\left(\frac{1}{2}\right) &= \mu \int_0^1\int_0^1 G\left(\frac{1}{2},s\right)u(s)K(s,\tau)u(\tau)\,d\tau\,ds + \int_0^1 G\left(\frac{1}{2},s\right)f(s,u)\,ds \\
&\geqslant \mu \int_{\frac14}^{\frac34}\int_{\frac14}^{\frac34} G\left(\frac{1}{2},s\right)u(s)K(s,\tau)u(\tau)\,d\tau\,ds + \int_{\frac14}^{\frac34} G\left(\frac{1}{2},s\right)f(s,u)\,ds \\
&\geqslant \mu\sigma^2 \int_{\frac14}^{\frac34}\int_{\frac14}^{\frac34} G\left(\frac{1}{2},s\right)K(s,\tau)\,d\tau\,ds\,\|u\|^2 + \sigma c_3 \int_{\frac14}^{\frac34} G\left(\frac{1}{2},s\right) ds\,\|u\| \\
&\geqslant \sigma c_3 \int_{\frac14}^{\frac34} G\left(\frac{1}{2},s\right) ds\,\|u\| \\
&\geqslant \|u\|
\end{aligned}$$

for all $u \in \partial B_r \cap P$.

Consequently,

$$\|Tu\| \geqslant \|u\|, \quad \forall u \in \partial B_r \cap P.$$

From the proof of Theorem 3.1, for sufficiently large $R > 1$, we also have

$$\|Tu\| \geqslant \|u\|, \quad \forall u \in \partial B_R \cap P,$$

by (H4) and Lemma 2.2.

Then by Lemma 1.1, we know that T has at least two fixed points in $(\overline{B}_R \backslash B_\rho) \cap P$ and $(\overline{B}_\rho \backslash B_r) \cap P$, namely, problem (1) has at least two positive solutions. □

4. Examples

In this section, we will give two examples to illustrate Theorem 3.1 and Theorem 3.2 when $\lambda = 0$.

Example 1. In problem (1), let $f(t,u) = \frac{u^2}{t(1-t)}$, we can choose $p(t) = \frac{1}{t(1-t)}$ and $q(u) = u^2$. It is easy to see that (H1) is satisfied. In addition, we can verify that

$$\int_0^1 G(s,s)p(s)\,ds = \int_0^1 s(1-s)\frac{1}{s(1-s)}\,ds = 1 < +\infty.$$

Then (H2) and (H3) are satisfied. Therefore, by Theorem 3.1, problem (1) has at least one positive solution when $\mu \in (0, \frac{1}{60}]$.

Example 2. In problem (1), let $f(t,u) = \frac{u^2+u^{\frac{1}{2}}}{6t(1-t)}$, we choose $p(t) = \frac{1}{6t(1-t)}$, $q(u) = u^2 + u^{\frac{1}{2}}$. Then (H1), (H2) and (H4) are obviously hold. In addition, let $\rho = 1$, then

$$2C\int_0^1 G(s,s)p(s)\,ds \sup_{u\in[0,1]} q(u) = 2 \cdot 1 \cdot \int_0^1 s(1-s)\frac{1}{6s(1-s)} \sup_{u\in[0,1]}\bigl(u^2 + u^{\frac{1}{2}}\bigr) = \frac{2}{3} < 1 = \rho,$$

namely, (H5) is satisfied. Therefore, by Theorem 3.2, problem (1) has at least two positive solutions when $\mu \in (0, \frac{1}{60}]$.

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