



## Concerning the Strauss conjecture on asymptotically Euclidean manifolds

Chengbo Wang<sup>1</sup>, Xin Yu<sup>\*</sup>

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, United States

### ARTICLE INFO

#### Article history:

Received 25 September 2010

Available online 27 January 2011

Submitted by T. Witelski

#### Keywords:

Strauss conjecture

KSS estimates (Keel–Smith–Sogge estimates)

Weighted Strichartz estimates

### ABSTRACT

In this paper we verify the Strauss conjecture for semilinear wave equations on asymptotically Euclidean manifolds when  $n = 3, 4$ . We also give an almost sharp lifespan for the subcritical case  $2 \leq p < p_c$  when  $n = 3$ . The main ingredients include a Keel–Smith–Sogge type estimate with  $0 < \mu < 1/2$  and weighted Strichartz estimates of order two.

© 2011 Elsevier Inc. All rights reserved.

### 1. Introduction and main results

This paper is devoted to the study of the semilinear wave equation on asymptotically Euclidean non-trapping Riemannian manifolds with small initial data. In particular, we verify the Strauss conjecture in this setting when  $n = 3, 4$  and  $p > p_c$ . Moreover, we obtain an almost sharp lifespan for the solution when  $2 \leq p < p_c$  and  $n = 3$ .

In the Minkowski space-time, this problem has been thoroughly studied. The work on global existence part (i.e.  $p > p_c$ ) is initiated by John [10] for  $n = 3$  and ended by Georgiev, Lindblad and Sogge [5] and Tataru [19]. It is known that  $p > p_c$  is necessary for global existence, even with small data, see [16,20,23] and reference therein. Moreover, when  $n = 3$  and  $p \leq p_c$ , the sharp lifespan is known in Zhou [22] (see also [14] for lower bound of the lifespan  $p \leq p_c$  and  $n \geq 3$ , and [24] for upper bound of the lifespan when  $p < p_c$  and  $n \geq 3$ ).

When dealing with semilinear wave equations, we know that the Keel–Smith–Sogge (KSS) estimate plays an important role, which is originated by Keel, Smith and Sogge [11] and states that

$$(\log(2+T))^{-1/2} \|\langle x \rangle^{-1/2} u'\|_{L^2([0,T] \times \mathbb{R}^3)} \lesssim \|u'(0, \cdot)\|_{L^2(\mathbb{R}^3)} + \int_0^T \|F(s, \cdot)\|_{L^2(\mathbb{R}^3)} ds, \quad (1.1)$$

where  $u$  solves the equation  $\square u = F$  and  $u' = (\partial_t u, \partial_x u)$ . This estimate has been generalized for general weight of form  $\langle x \rangle^{-a}$  with  $a \geq 0$  (see [9] and references therein).

Recently, Bony and Häfner [2] obtained a weaker version of the KSS estimates for asymptotically Euclidean space when the metric is non-trapping. With this estimate, they were able to show the global and long time existence for quadratic semilinear wave equations with dimension  $n \geq 4$  and  $n = 3$ . Then Sogge and Wang [17] proved the almost global existence for 3-D quadratic semilinear equations by obtaining the sharp KSS estimates for  $a = 1/2$ . Together with the KSS estimates, they also proved the Strauss conjecture for  $n = 3$  and  $p > p_c$  with spherically symmetric metric. The proof is based on

<sup>\*</sup> Corresponding author.

E-mail addresses: [wangcbo@jhu.edu](mailto:wangcbo@jhu.edu) (C. Wang), [yuxin@jhu.edu](mailto:yuxin@jhu.edu) (X. Yu).

<sup>1</sup> The first author was supported in part by NSFC 10871175 and 1091120383.

weighted Strichartz estimates, and there are the weighted Strichartz estimates of higher order where the additional symmetric assumption is posed to avoid the technical difficulties when commuting the Laplacian with the vector fields.

In this work, we are able to overcome the difficulties of commuting vector fields and verify the weighted Strichartz estimates and energy estimates with derivatives up to second order, for a general metric. This enables us to prove the Strauss conjecture with  $p > p_c$  for  $n = 3, 4$ . Moreover, we are able to get the KSS estimates for  $0 < a < 1/2$ , by applying the corresponding estimates for wave equations with variable coefficients (see [15,8]). With these estimates in hand, we can also prove the local existence for  $2 \leq p < p_c$  when  $n = 3$  with almost sharp lifespan.

Let us now state our results precisely. First, we introduce the necessary notations. We consider asymptotically Euclidean manifolds  $(\mathbb{R}^n, g)$  with  $n \geq 3$  and

$$g = \sum_{i,j=1}^n g_{ij}(x) dx^i dx^j.$$

We suppose  $g_{ij}(x) \in C^\infty(\mathbb{R}^n)$  and, for some  $\rho > 0$ ,

$$\forall \alpha \in \mathbb{N}^n \quad \partial_x^\alpha (g_{ij} - \delta_{ij}) = \mathcal{O}(\langle x \rangle^{-|\alpha|-\rho}), \quad (\text{H1})$$

with  $\delta_{ij} = \delta^{ij}$  being the Kronecker delta function. We also assume that

$$g \text{ is non-trapping.} \quad (\text{H2})$$

Let  $g(x) = (\det(g))^{1/4}$ . The Laplace–Beltrami operator associated with  $g$  is given by

$$\Delta_g = \sum_{ij} \frac{1}{g^2} \partial_i g^{ij} g^2 \partial_j,$$

where  $g^{ij}(x)$  denotes the inverse matrix of  $g_{ij}(x)$ . It is easy to see that  $-\Delta_g$  is a self-adjoint non-negative operator on  $L^2(\mathbb{R}^n, g^2 dx)$ , while  $P = -g \Delta_g g^{-1}$  is a self-adjoint non-negative operator on  $L^2(\mathbb{R}^n, dx)$ .

Let  $p > 1$ ,

$$s_c = \frac{n}{2} - \frac{2}{p-1}, \quad s_d = \frac{1}{2} - \frac{1}{p}$$

and  $p_c$  be the positive root for

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

Note that  $p_c = 1 + \sqrt{2}$  for  $n = 3$  and  $p_c = 2$  for  $n = 4$ . The semilinear wave equations we will consider are

$$\begin{cases} (\partial_t^2 - \Delta_g)u(t, x) = F_p(u(t, x)), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \quad (1.2)$$

We will assume that the nonlinear term behaves like  $|u|^p$ , and so we assume that

$$\sum_{0 \leq j \leq 2} |u|^j |\partial_u^j F_p(u)| \lesssim |u|^p, \quad \text{for } |u| \text{ small.} \quad (1.3)$$

Finally we introduce the notation for vector fields  $Z = \{\partial_{x_i}, \Omega_{ij}: 1 \leq i \leq j \leq 3\}$ ,  $\Gamma = \{\partial_t\} \cup Z$ , where  $\Omega_{ij} = x_i \partial_j - x_j \partial_i$  is the rotational vector field, and define  $\tilde{\partial}_i = \partial_i g^{-1}$ ,  $\tilde{\Omega}_{ij} = \Omega_{ij} g^{-1}$ .

Now we can state our main results.

**Theorem 1.1.** Suppose (H1) and (H2) hold with  $\rho > 2$ ,  $n = 3, 4$ , and  $p_c < p < 1 + 4/(n-1)$ . Then for any  $\epsilon > 0$  such that (recall that  $s_c > s_d$  since  $p > p_c$ )

$$s = s_c - \epsilon \in \left(s_d, \frac{1}{2}\right) \quad (1.4)$$

there is a  $\delta > 0$  depending on  $p$  so that (1.2) has a global solution satisfying  $(Z^\alpha u(t, \cdot), \partial_t Z^\alpha u(t, \cdot)) \in \dot{H}^s \times \dot{H}^{s-1}$ ,  $|\alpha| \leq 2$ ,  $t \in \mathbb{R}_+$ , whenever the initial data satisfy

$$\sum_{|\alpha| \leq 2} (\|Z^\alpha u_0\|_{\dot{H}^s} + \|Z^\alpha u_1\|_{\dot{H}^{s-1}}) < \delta. \quad (1.5)$$

Moreover, in the case  $n = 3$ , we can relax the assumption for  $\rho$  to  $\rho > 1$ . More precisely, if  $F_p(u)$  satisfies

$$\sum_{0 \leq j \leq 1} |u|^j |\partial_u^j F_p(u)| \lesssim |u|^p \quad (1.6)$$

instead of (1.3), for any  $p_c < p < 3$  and any  $\epsilon > 0$  such that (1.4) is true, the problem (1.2) has a global solution satisfying  $(Z^\alpha u(t, \cdot), \partial_t Z^\alpha u(t, \cdot)) \in \dot{H}^s \times \dot{H}^{s-1}$ ,  $|\alpha| \leq 1$ ,  $t \in \mathbb{R}_+$ , whenever the initial data satisfy

$$\sum_{|\alpha| \leq 1} (\|Z^\alpha u_0\|_{\dot{H}^s} + \|Z^\alpha u_1\|_{\dot{H}^{s-1}}) < \delta. \quad (1.7)$$

We also have the following existence result for  $2 \leq p < p_c$  when  $n = 3$ , where the lifespan is almost sharp (see [22] for the blow up results).

**Theorem 1.2.** Suppose (H1) and (H2) hold with  $\rho > 2$ ,  $n = 3$ , and  $2 \leq p < p_c = 1 + \sqrt{2}$ . Then there exist  $c > 0$  and  $\delta_0 > 0$  depending on  $p$  so that (1.2) has a solution in  $[0, T_\delta] \times \mathbb{R}^3$  satisfying  $(Z^\alpha u(t, \cdot), \partial_t Z^\alpha u(t, \cdot)) \in \dot{H}^s \times \dot{H}^{s-1}$ ,  $|\alpha| \leq 2$ ,  $t \in [0, T_\delta]$ , with

$$s = s_d, \quad T_\delta = c\delta^{\frac{p(p-1)}{p^2-2p-1} + \epsilon}, \quad (1.8)$$

whenever  $\epsilon > 0$  and the initial data satisfy (1.7) with  $\delta < \delta_0$ . Moreover, we can relax the assumption for  $\rho$  to  $\rho > 1$ , when  $F$  satisfies (1.6) and  $s = s_d + \epsilon'$  for some small  $\epsilon' > 0$ .

**Remark 1.1.** The above result for  $p < p_c$  is a natural extension of Theorem 4.1 in Chapter 4 of Sogge [18] and Theorem 4.2 of Hidano [6]. See also Theorem 4.1 of Yu [21] and Theorem 6.1 of [9] for closely related  $\dot{H}^{s_d}$ -results.

For convenience we define the norm  $Y_{s,\epsilon}$  as

$$\|f(x)\|_{Y_{s,\epsilon}} = \|\langle x \rangle^{-(1/2)-s-\epsilon} f(x)\|_{L_x^2}.$$

The main estimate we will need to prove Theorem 1.1 is as follows.

**Theorem 1.3.** Let  $u$  be a solution of the linear equation

$$\begin{cases} (\partial_t^2 + P)u(t, x) = F(t, x), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0, x) = u_0(x), & \partial_t u(0, x) = u_1(x), \quad x \in \mathbb{R}^n \end{cases} \quad (1.9)$$

with  $F = 0$ . Assume that (H1) and (H2) hold with  $\rho > 2$ ,  $n \geq 3$ ,  $2 < p \leq \infty$  and  $s \in (s_d, 1)$ . For all  $\epsilon > 0$  and  $\eta > 0$  small enough, we have

$$\sum_{|\alpha| \leq 2} \|Z^\alpha u\|_{L_t^2 Y_{s,\epsilon}} + \||x|^{n/2-(n+1)/p-s-\epsilon} Z^\alpha u\|_{L_t^p L_{|x|}^p L_\omega^{2+\eta}(\{|x|>1\})} \lesssim \sum_{|\alpha| \leq 2} (\|Z^\alpha u_0\|_{\dot{H}^s} + \|Z^\alpha u_1\|_{\dot{H}^{s-1}}), \quad (1.10)$$

and for  $s \in [0, 1]$ ,

$$\sum_{|\alpha| \leq 2} (\|Z^\alpha u\|_{L_t^\infty \dot{H}^s} + \|\partial_t Z^\alpha u\|_{L_t^\infty \dot{H}^{s-1}} + \|Z^\alpha u\|_{L_t^{q_s} L_{|x|}^{q_s}(\{|x| \leq 1\})}) \lesssim \sum_{|\alpha| \leq 2} (\|Z^\alpha u_0\|_{\dot{H}^s} + \|Z^\alpha u_1\|_{\dot{H}^{s-1}}), \quad (1.11)$$

where  $q_s = 2n/(n-2s)$ . On the other hand, if we assume  $\rho > 1$  instead of  $\rho > 2$ , we have the same estimates of first order ( $|\alpha| \leq 1$ ).

Here, the angular mixed-norm space  $L_{|x|}^p L_\omega^r$  is defined as follows

$$\|f\|_{L_{|x|}^p L_\omega^r(\mathbb{R}^n)} = \left( \int_0^\infty \left( \int_{\mathbb{S}^{n-1}} |f(\lambda\omega)|^r d\omega \right)^{p/r} \lambda^{n-1} d\lambda \right)^{1/p},$$

which is consistent with the usual Lebesgue space  $L_x^p$  when  $p = r$ .

Recall that Theorem 1.3, with order 0 ( $|\alpha| = 0$ ) and  $\rho > 0$ , has been proved in Theorem 1.6 of [17] for any  $s \in (s_d, 1]$  in general. However, the estimates with higher order derivatives are much more complicated. As we will see, one of the main difficulties in the proof is that we need to establish the relation between  $P$  and the vector fields  $Z$ , where only the powers of  $P$  can be commuted with the equation  $\partial_t^2 + P$ . The most difficult part of the commutators comes from the commutator of  $P$  and the rotational vector fields  $\Omega_{ij}$ . Another difficulty arises from the estimates with second order derivatives, and the techniques we use here will require the assumption  $\rho > 2$  instead of  $\rho > 1$ .

To obtain Theorem 1.2 we will need the following local in time weighted Strichartz estimates.

**Theorem 1.4.** Let  $u$  be a solution of (1.9) with  $F = 0$ . Assume that (H1) and (H2) hold with  $\rho > 2$ ,  $n \geq 3$ ,  $0 < a < 1/p$ ,  $2 \leq p < \infty$  and  $s = s_d$ . Then we have

$$\sum_{|\alpha| \leq 2} \|\langle x \rangle^{-a} |x|^{(n-1)s} Z^\alpha u\|_{L_t^p L_{|x|}^p L_\omega^2([0, T] \times \mathbb{R}^n)} \lesssim (1+T)^{(1/p)-a+\epsilon} \sum_{|\alpha| \leq 2} (\|Z^\alpha u_0\|_{\dot{H}^s} + \|Z^\alpha u_1\|_{\dot{H}^{s-1}}). \quad (1.12)$$

On the other hand, if we assume  $\rho > 1$  instead of  $\rho > 2$ , we have the same estimates of first order ( $|\alpha| \leq 1$ ), with  $s = s_d + \epsilon'$  for small enough  $\epsilon' > 0$ .

**Remark 1.2.** Note that the estimates in the above two theorems are given for solutions of  $(\partial_t^2 + P)u = F$ , which has the benefit that the solution can be represented by the following formula

$$u(t) = \cos(tP^{1/2})u_0 + P^{-1/2} \sin(tP^{1/2})u_1 + \int_0^t P^{-1/2} \sin((t-s)P^{1/2})F(s) ds.$$

All of the operators occurring in this formula commute with the wave operator  $\partial_t^2 + P$ . In general, an estimate for  $-\Delta_g$  will correspond to another estimate for  $P$ . For example, if we have the estimate (1.10) for  $P$ , consider the equation

$$\begin{cases} (\partial_t^2 - \Delta_g)v(t, x) = G(t, x), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0, x) = v_0(x), & \partial_t v(0, x) = v_1(x), \quad x \in \mathbb{R}^n. \end{cases} \quad (1.13)$$

Notice that if we let  $u = gv$  and  $F = gG$ , then

$$(\partial_t^2 - \Delta_g)v = G \Leftrightarrow (\partial_t^2 + P)u = F. \quad (1.14)$$

Thus we have also the estimate (1.10) for  $-\Delta_g$ .

The paper is arranged as follows. In Section 2 we prove the weighted Strichartz estimates and energy estimates (i.e. Theorem 1.3). In Section 3 we prove higher order KSS estimates and local in time weighted Strichartz estimates (i.e. Theorem 1.4). Finally in Section 4 we will see how Theorem 1.3 and Theorem 1.4 imply the Strauss conjecture when  $n = 3, 4$ .

## 2. Weighted Strichartz and energy estimates

In this section, we will give the proof of our main estimates (1.10) and (1.11).

In what follows, “remainder terms”,  $r_j$ ,  $j \in \mathbb{N}$ , will denote any smooth functions such that

$$\partial_x^\alpha r_j(x) = O(\langle x \rangle^{-\rho-j-|\alpha|}), \quad \forall \alpha, \quad (2.1)$$

thus  $P = -g\Delta_g g^{-1} = -\Delta + r_0\partial^2 + r_1\partial + r_2$ .

### 2.1. Preparation

Before we go through the proof of the main theorems, we will present several useful lemmas. The first one is the KSS estimates (Keel–Smith–Sogge estimates) on asymptotically Euclidean manifolds obtained in [2] and [17], and the second one gives the relation between the operators  $P^{1/2}$  and  $\partial_x$ .

**Lemma 2.1** (KSS estimates). Assume that (H1) and (H2) hold with  $\rho > 1$ . Let  $N \geq 0$ ,  $\mu \geq 1/2$  and

$$A_\mu(T) = \begin{cases} (\log(2+T))^{-1/2}, & \mu = 1/2, \\ 1, & \mu > 1/2. \end{cases}$$

Then the solution of (1.9) satisfies

$$\begin{aligned} & \sup_{0 \leq t \leq T} \sum_{1 \leq k+j \leq N+1} \|\partial_t^k P^{j/2} g u(t, \cdot)\|_{L_x^2} + \sum_{|\alpha| \leq N} A_\mu(T) \left\| \langle x \rangle^{-\mu} \left( |(F^\alpha u)'| + \frac{|F^\alpha u|}{\langle x \rangle} \right) \right\|_{L_T^2 L_x^2} \\ & \lesssim \sum_{|\alpha| \leq N} \|(Z^\alpha u)'(0, \cdot)\|_{L_x^2} + \sum_{|\alpha| \leq N} \|F^\alpha F(s, \cdot)\|_{L_T^1 L_x^2}, \end{aligned} \quad (2.2)$$

where  $L_T^q L_x^r = L^q([0, T]; L^r(\mathbb{R}^n))$ .

**Proof.** This is Theorem 1.3 in [17].  $\square$

**Remark 2.1.** Here, we notice that the estimate (2.2) still holds if we replace  $F$  and  $Z$  with  $\partial_x$  in (2.2) (see (3.6) in [17]). Moreover, we will see later in Proposition 3.2 that the corresponding estimates for  $0 < \mu < 1/2$  also hold.

The next lemma gives the relation between the operators  $\partial_x$  and  $P^{1/2}$ .

**Lemma 2.2.** If  $s \in [-1, 1]$ , then

$$\|u\|_{\dot{H}^s} \simeq \|P^{s/2}u\|_{L_x^2}. \quad (2.3)$$

If  $s \in [0, 1]$ ,

$$\|\tilde{\partial}_j u\|_{\dot{H}^{-s}} \lesssim \|P^{1/2}u\|_{\dot{H}^{-s}}, \quad (2.4)$$

$$\|P^{1/2}u\|_{\dot{H}^s} \lesssim \sum_j \|\tilde{\partial}_j u\|_{\dot{H}^s}. \quad (2.5)$$

Moreover, we have for  $s \in (0, 2]$  and  $1 < q < n/s$ ,

$$\|P^{s/2}u\|_{L_x^q} \lesssim \|u\|_{\dot{H}^{s,q}}. \quad (2.6)$$

**Proof.** This is just Lemma 2.4 in [17].  $\square$

The three following lemmas are proved to deal with the commutator terms we will encounter in the proof of our higher order estimates (1.10) and (1.11).

**Lemma 2.3.** Let  $u$  solve the wave equation (1.9). Then for any  $s \in [0, 1]$  and  $\epsilon > 0$ , we have:

$$\|u\|_{L_t^2 Y_{s,\epsilon}} \lesssim \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}} + \|\langle x \rangle^{(1/2)+\epsilon} F\|_{L_t^2 \dot{H}^{s-1}}. \quad (2.7)$$

**Proof.** We give first the proof in the case  $u_0 = u_1 = 0$ . First, from Remark 2.1 in [17] we know

$$\|\langle x \rangle^{-(3/2)-\epsilon} u\|_{L^2(\mathbb{R} \times \mathbb{R}^n)} \lesssim \|\langle x \rangle^{(1/2)+\epsilon} F\|_{L^2(\mathbb{R} \times \mathbb{R}^n)}. \quad (2.8)$$

Next, using the KSS estimates on asymptotically Euclidean manifolds (Lemma 2.1 in [17]) together with (2.8), we have

$$\begin{aligned} \|\langle x \rangle^{-(1/2)-\epsilon} u\|_{L_t^2 \dot{H}^1(\mathbb{R} \times \mathbb{R}^n)} &\lesssim \|\langle x \rangle^{-(3/2)-\epsilon} u\|_{L^2(\mathbb{R} \times \mathbb{R}^n)} + \|\langle x \rangle^{-(1/2)-\epsilon} u'\|_{L^2(\mathbb{R} \times \mathbb{R}^n)} \\ &\lesssim \|\langle x \rangle^{(1/2)+\epsilon} F\|_{L^2(\mathbb{R} \times \mathbb{R}^n)}. \end{aligned} \quad (2.9)$$

Since  $P$  is self-adjoint, for any fixed  $T > 0$ , if we let  $\square_P v = (\partial_t^2 + P)v = G$  with vanishing initial data at  $T$ , then

$$\begin{aligned} \|\langle x \rangle^{-(1/2)-\epsilon} u\|_{L^2([0,T] \times \mathbb{R}^n)} &= \sup_{\|\langle x \rangle^{1/2+\epsilon} G\|_{L^2([0,T] \times \mathbb{R}^n)} \leq 1} \langle u, G \rangle \\ &= \sup_{\|\langle x \rangle^{1/2+\epsilon} G\|_{L^2([0,T] \times \mathbb{R}^n)} \leq 1} \langle \square_P u, v \rangle \\ &\lesssim \|\langle x \rangle^{(1/2)+\epsilon} \square_P u\|_{L_t^2 \dot{H}^{-1}([0,T] \times \mathbb{R}^n)} \|\langle x \rangle^{-(1/2)-\epsilon} v\|_{L_t^2 \dot{H}^1([0,T] \times \mathbb{R}^n)} \\ &\lesssim \|\langle x \rangle^{(1/2)+\epsilon} F\|_{L_t^2 \dot{H}^{-1}([0,T] \times \mathbb{R}^n)} \|\langle x \rangle^{(1/2)+\epsilon} G\|_{L^2([0,T] \times \mathbb{R}^n)} \\ &\lesssim \|\langle x \rangle^{(1/2)+\epsilon} F\|_{L_t^2 \dot{H}^{-1}(\mathbb{R} \times \mathbb{R}^n)}. \end{aligned}$$

Since the constants in the inequality are independent of  $T$ , we get

$$\|\langle x \rangle^{-(1/2)-\epsilon} u\|_{L^2(\mathbb{R} \times \mathbb{R}^n)} \lesssim \|\langle x \rangle^{(1/2)+\epsilon} F\|_{L_t^2 \dot{H}^{-1}(\mathbb{R} \times \mathbb{R}^n)}. \quad (2.10)$$

Now we can get the desired estimate (2.7) for  $u_0 = u_1 = 0$  by an interpolation between (2.8) and (2.10). The estimate with  $F = 0$  follows just from the estimate (1.10) of order 0, which is proved in [17].  $\square$

**Lemma 2.4.** Let  $w$  solve the wave equation (1.9) with  $u_0 = u_1 = 0$ . Then for  $s \in [0, 1]$  and  $\epsilon > 0$ ,

$$\|w\|_{L_t^\infty \dot{H}_x^s} \lesssim \|\langle x \rangle^{1/2+\epsilon} F\|_{L_t^2 \dot{H}_x^{s-1}}. \quad (2.11)$$

**Proof.** We will show this estimate by interpolation. For  $s = 1$ , notice that KSS estimates in Lemma 2.1 give us

$$\|\langle x \rangle^{-1/2-\epsilon} e^{itP^{1/2}} f\|_{L_{t,x}^2} \lesssim \|f\|_{L_x^2}.$$

After the standard  $TT^*$  argument, we get

$$\left\| \int_{\mathbb{R}} e^{-isP^{1/2}} G(s, \cdot) ds \right\|_{L_x^2} \lesssim \|\langle x \rangle^{1/2+\epsilon} G(t, x)\|_{L_{t,x}^2},$$

and so

$$\begin{aligned} \left\| \int_{\mathbb{R}} e^{i(t-s)P^{1/2}} F(s) ds \right\|_{L_t^\infty L_x^2} &\lesssim \left\| \int_{\mathbb{R}} e^{-isP^{1/2}} F(s) ds \right\|_{L_x^2} \\ &\lesssim \|\langle x \rangle^{(1/2)+\epsilon} F\|_{L_{t,x}^2}. \end{aligned}$$

Thus by the Christ–Kiselev lemma (cf. [3]) we have

$$\left\| \int_0^t e^{i(t-s)P^{1/2}} F(s) ds \right\|_{L_t^\infty L_x^2} \lesssim \|\langle x \rangle^{(1/2)+\epsilon} F\|_{L_{t,x}^2}.$$

Recall that  $w = P^{-1/2} \int_0^t \sin((t-s)P^{1/2}) F(s) ds$ , then we get the proof of (2.11) for the case  $s = 1$  as follows,

$$\|w\|_{L_t^\infty \dot{H}^1} \approx \|P^{1/2} w\|_{L_t^\infty L_x^2} \lesssim \|\langle x \rangle^{(1/2)+\epsilon} F\|_{L_{t,x}^2}.$$

For  $s = 0$ , by (2.2),

$$\|\langle x \rangle^{-(1/2)-\epsilon} w\|_{L_t^2 \dot{H}^1} \lesssim \|\langle x \rangle^{-(3/2)-\epsilon} w\|_{L_{t,x}^2} + \|\langle x \rangle^{-(1/2)-\epsilon} \partial_x w\|_{L_{t,x}^2} \lesssim \|F\|_{L_t^1 L_x^2}.$$

The above inequality, combined with a similar duality argument for (2.10), gives

$$\|w\|_{L_t^\infty L_x^2} \lesssim \|\langle x \rangle^{(1/2)+\epsilon} F\|_{L_t^2 \dot{H}^{-1}},$$

which is just the estimate for  $s = 0$ . This completes the proof if we interpolate between the estimates for  $s = 0$  and  $s = 1$ .  $\square$

On the basis of the above two lemmas, we can control the commutator terms by a kind of weighted  $L_t^2 \dot{H}_x^{s-1}$  norm. Then with the following lemma we will be able to bound this norm by the good terms, thus we can use the argument as in [17] to get over the difficulty on error terms.

**Lemma 2.5.** *Let  $n \geq 3$ ,  $N \geq 1$  and  $u$  be a solution to (1.9) with  $F = 0$ . Then for any  $s \in [0, 1]$ ,  $\epsilon > 0$  and  $|\alpha| = N$ , we have*

$$\sum_{|\alpha|=N} \|\langle x \rangle^{-(1/2)-\epsilon} \partial_x^\alpha u\|_{L_t^2 \dot{H}^{s-1}} \lesssim \|u_0\|_{\dot{H}^{N+s-1} \cap \dot{H}^s} + \|u_1\|_{\dot{H}^{N+s-2} \cap \dot{H}^{s-1}}. \quad (2.12)$$

**Proof.** The estimate for  $s = 1$  follows directly from the KSS estimates (2.2) and Remark 2.1. Moreover, we have the following estimate

$$\begin{aligned} \|\langle x \rangle^{-(1/2)-\epsilon} u\|_{L_t^2 L_x^2} &= \|\langle x \rangle^{-(1/2)-\epsilon} P^{1/2} (P^{-1/2} u)\|_{L_t^2 L_x^2} \\ &\lesssim \|P^{-1/2} u_0\|_{\dot{H}^1} + \|P^{-1/2} u_1\|_{L_x^2} \lesssim \|u_0\|_{L_x^2} + \|u_1\|_{\dot{H}^{-1}}. \end{aligned} \quad (2.13)$$

For  $s = 0$ , first notice that since  $n \geq 3$ , we have Hardy's inequality

$$\|\langle x \rangle^{-2} x h\|_{L_x^2} \lesssim \|h\|_{\dot{H}^1},$$

and the duality gives

$$\|\langle x \rangle^{-2} x f\|_{\dot{H}^{-1}} \lesssim \|f\|_{L_x^2}.$$

Using the above estimate together with the KSS estimates and (2.13), we get

$$\begin{aligned} \|\langle x \rangle^{-(1/2)-\epsilon} \partial_x^\alpha u\|_{L_t^2 \dot{H}^{-1}} &\lesssim \|\langle x \rangle^{-(5/2)-\epsilon} x \partial_x^{\alpha-1} u\|_{L_t^2 \dot{H}^{-1}} + \|\langle x \rangle^{-(1/2)-\epsilon} \partial_x^{\alpha-1} u\|_{L_t^2 L_x^2} \\ &\lesssim \|\langle x \rangle^{-(1/2)-\epsilon} \partial_x^{\alpha-1} u\|_{L_t^2 L_x^2} \\ &\lesssim \|u_0\|_{\dot{H}^{N-1}} + \|u_1\|_{\dot{H}^{N-2} \cap \dot{H}^{-1}}. \end{aligned}$$

Now (2.12) follows from an interpolation between  $s = 0$  and  $s = 1$ .  $\square$

Next we give three lemmas that will be used to prove the second order part of the estimates (1.10) and (1.11).

**Lemma 2.6.** For  $0 < \mu \leq 3/2$  and  $k \geq 2$ , we have

$$\| \langle x \rangle^{-\mu} \tilde{\partial}_{j_1} \cdots \tilde{\partial}_{j_k} u \|_{L_x^2} \lesssim \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \| \langle x \rangle^{-\mu} \tilde{\partial} P^j u \|_{L_x^2} + \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \| \langle x \rangle^{-\mu} P^j u \|_{L_x^2}, \quad (2.14)$$

where  $[a]$  denotes the integer part of  $a$  ( $\max\{k \in \mathbb{Z}, k \leq a\}$ ).

**Proof.** This is just Lemma 4.8 in [2].  $\square$

**Lemma 2.7** (Fractional Leibniz rule). Let  $0 \leq s < n/2$ ,  $2 \leq p_i < \infty$  and  $1/2 = 1/p_i + 1/q_i$  ( $i = 1, 2$ ). Then

$$\| f g \|_{\dot{H}^s} \lesssim \| f \|_{L^{q_1}} \| g \|_{\dot{H}^{s, p_1}} + \| f \|_{\dot{H}^{s, p_2}} \| g \|_{L^{q_2}}.$$

Moreover, for any  $s \in (-n/2, 0) \cup (0, n/2)$ ,

$$\| f g \|_{\dot{H}^s} \lesssim \| f \|_{L^\infty \cap \dot{H}^{|s|, n/|s|}} \| g \|_{\dot{H}^s}.$$

**Proof.** The first inequality is well known, see, e.g., [12]. The second inequality with  $s \geq 0$  is an easy consequence of the first inequality together with Sobolev embedding. Then the result for negative  $s$  follows by duality.  $\square$

**Lemma 2.8.** For  $f \in \dot{H}^s(\mathbb{R}^n) \cap \dot{H}^{s+2}(\mathbb{R}^n)$  with  $n \geq 3$  and  $s \in [0, 1]$ , we have

$$\| \partial_x^2 f \|_{\dot{H}^s} \lesssim \| P f \|_{\dot{H}^s} + \| f \|_{\dot{H}^s}. \quad (2.15)$$

On the other hand,

$$\| P f \|_{\dot{H}^s} \lesssim \sum_{|\alpha| \leq 2} \| \partial_x^\alpha f \|_{\dot{H}^s}. \quad (2.16)$$

**Proof.** First, we give the proof for the estimate (2.16). When  $s = 0$ , noticing that  $P f = g^{ij} \partial_i \partial_j f + r_1 \partial_x f + r_2 f$ , we have

$$\| P f \|_{L_x^2} \lesssim \| \partial_x^2 f \|_{L_x^2} + \| \partial_x f \|_{L_x^2} + \| f \|_{L_x^2} \lesssim \| f \|_{\dot{H}^2 \cap L_x^2}.$$

When  $s = 1$ , recalling that  $\partial^j r_i = O(\langle x \rangle^{-\rho-i-j})$ , by Hardy's inequality,

$$\| \partial_x(r_2 f) \|_{L_x^2} \lesssim \| \partial_x(r_2) f \|_{L_x^2} + \| r_2 \partial_x f \|_{L_x^2} \lesssim \| \partial_x f \|_{L_x^2}.$$

Thus

$$\begin{aligned} \| P f \|_{\dot{H}_x^1} &= \| \partial_x P f \|_{L_x^2} \leq \| \partial_x(g^{ij} \partial_i \partial_j f) \|_{L_x^2} + \| \partial_x(r_1 \partial_x f) \|_{L_x^2} + \| \partial_x(r_2 f) \|_{L_x^2} \\ &\lesssim \| \partial_x^3 f \|_{L_x^2} + \| \partial_x f \|_{L_x^2} \\ &\lesssim \| f \|_{\dot{H}^3 \cap \dot{H}^1}. \end{aligned}$$

Our estimate (2.16) is obtained by an interpolation between the above two estimates on  $P f$ .

Now we turn to the proof of the estimate (2.15). First, when  $s = 0$ , by elliptic property of  $P$ , we have

$$\| \partial_x^2 f \|_{L_x^2} \lesssim \| P f \|_{L_x^2} + \| f \|_{L_x^2}. \quad (2.17)$$

Second, for  $s = 1$ , using (2.17),

$$\begin{aligned} \| \partial_x^3 f \|_{L_x^2} &\lesssim \| P \partial_x f \|_{L_x^2} + \| \partial_x f \|_{L_x^2} \\ &\lesssim \| [P, \partial_x] f \|_{L_x^2} + \| \partial_x P f \|_{L_x^2} + \| \partial_x f \|_{L_x^2} \\ &\lesssim \left\| \sum_{|\alpha| \leq 2} r_{3-|\alpha|} \partial_x^\alpha f \right\|_{L_x^2} + \| P f \|_{\dot{H}^1} + \| f \|_{\dot{H}^1} \\ &\lesssim \| P f \|_{\dot{H}^1} + \| f \|_{\dot{H}^1} + \| f \|_{\dot{H}^2} \\ &\lesssim \| P f \|_{\dot{H}^1} + \| f \|_{\dot{H}^1} + \epsilon \| f \|_{\dot{H}^3} + (1/\epsilon) \| f \|_{\dot{H}^1}, \quad \forall \epsilon > 0. \end{aligned}$$

Here we have used Hardy's inequality and the fact that  $\dot{H}^3 \cap \dot{H}^1 \subset \dot{H}^2$ . Now if we choose  $\epsilon > 0$  small enough and use (2.16) with  $s = 0$ , we have

$$\|\partial_x^2 f\|_{\dot{H}^1} \lesssim \|Pf\|_{\dot{H}^1} + \|f\|_{\dot{H}^1} \lesssim \|PP^{1/2}f\|_{L_x^2} + \|f\|_{\dot{H}^1} \lesssim \|P^{1/2}f\|_{\dot{H}^2} + \|P^{1/2}f\|_{L_x^2}. \quad (2.18)$$

On the basis of (2.17) and (2.18), by an interpolation for the operator  $\partial^2 P^{-1/2}$  and making use of Lemma 2.2, we have

$$\begin{aligned} \|\partial_x^2 f\|_{\dot{H}^s} &\lesssim \|P^{1/2}f\|_{\dot{H}^{1+s}} + \|P^{1/2}f\|_{\dot{H}^{s-1}} \\ &\lesssim \|P^{1/2}f\|_{\dot{H}^{1+s}} + \|P^{1/2+(s-1)/2}f\|_{L_x^2} \lesssim \|P^{1/2}f\|_{\dot{H}^{1+s}} + \|f\|_{\dot{H}^s}. \end{aligned} \quad (2.19)$$

We need only to deal with the term  $\|P^{1/2}f\|_{\dot{H}^{1+s}}$ . Noting that for  $s \in [0, 1]$ , we have

$$\|P^{-1/2}v\|_{\dot{H}^{1+s}} \lesssim \|v\|_{\dot{H}^s} + \|v\|_{\dot{H}^{-s}},$$

which is true for  $s = 0$  (see (2.3)) and  $s = 1$  (see (2.17)). Recalling that  $P - g^{ij}\partial_i\partial_j = r_1\partial_x + r_2$ , and by Leibniz rule (see Lemma 2.7), we have for any small  $0 < \epsilon < \rho$ ,

$$\begin{aligned} \|P^{1/2}f\|_{\dot{H}^{1+s}} &\lesssim \|Pf\|_{\dot{H}^s} + \|Pf\|_{\dot{H}^{-s}} \\ &\lesssim \|Pf\|_{\dot{H}^s} + \|f\|_{\dot{H}^{2-s}} + \|r_1\partial_x f\|_{\dot{H}^{-s}} + \|r_2f\|_{\dot{H}^{-s}} \\ &\lesssim \|Pf\|_{\dot{H}^s} + \|f\|_{\dot{H}^{2-s}} + \|f\|_{\dot{H}^{1+\epsilon}} \\ &\lesssim \|Pf\|_{\dot{H}^s} + \|f\|_{\dot{H}^s}^{\theta_1} \|f\|_{\dot{H}^{2+s}}^{1-\theta_1} + \|f\|_{\dot{H}^s}^{\theta_2} \|f\|_{\dot{H}^{2+s}}^{1-\theta_2}, \quad \text{where } \theta_i \in (0, 1) \\ &\lesssim \|Pf\|_{\dot{H}^s} + \|f\|_{\dot{H}^s}^{\theta_1} \|\partial_x^2 f\|_{\dot{H}^s}^{1-\theta_1} + \|f\|_{\dot{H}^s}^{\theta_2} \|\partial_x^2 f\|_{\dot{H}^s}^{1-\theta_2}, \end{aligned} \quad (2.20)$$

where we have used the fact that  $s \leq 1 + \epsilon$ ,  $2 - s < 2 + s$  (so that  $\theta_i > 0$ ) for  $s \in (0, 1]$ . Now our estimate (2.15) (for  $s > 0$ ) follows from (2.19) and (2.20).  $\square$

## 2.2. Proof of Theorem 1.3

Now we are ready to give the proof of Theorem 1.3. Recall that it has been proved in [17] that the result holds in the case with order 0 ( $|\alpha| = 0$ ) and  $\rho > 0$ . Specifically, by KSS estimates (2.2) and energy estimates, we have

$$\|u\|_{L_t^2 Y_{s,\epsilon}} + \|u\|_{L_t^\infty \dot{H}^s} + \|\partial_t u\|_{L_t^\infty \dot{H}^{s-1}} \lesssim \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}} \quad (2.21)$$

for the solution  $u$  to the homogeneous linear wave equation (1.9) and  $s \in [0, 1]$ .

Recall that Fang and Wang obtained the following Sobolev inequalities with angular regularity ((1.3) in [4])

$$\| |x|^{n/2-s} f(x) \|_{L_{|x|}^\infty L_\omega^{2+\eta}} \lesssim \| |x|^{n/2-s} f(x) \|_{L_{|x|}^\infty H_\omega^{s-1/2}} \lesssim \|f\|_{\dot{H}^s} \quad (2.22)$$

for  $s \in (1/2, n/2)$  and some  $\eta > 0$ . By Lemma 2.2, we have

$$\begin{aligned} \| |x|^{n/2-s} e^{itP^{1/2}} f(x) \|_{L_{t,|x|}^\infty L_\omega^{2+\eta}} &\lesssim \| e^{itP^{1/2}} f(x) \|_{L_t^\infty \dot{H}_x^s} \\ &\lesssim \| e^{itP^{1/2}} P^{s/2} f(x) \|_{L_t^\infty L_x^2} \lesssim \| P^{s/2} f \|_{L_x^2} \lesssim \|f\|_{\dot{H}^s} \end{aligned} \quad (2.23)$$

for  $s \in (1/2, 1]$ .

On the basis of KSS estimates, we can also obtain local energy decay estimates

$$\|\phi u\|_{L_t^2 \dot{H}^s} \lesssim \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}}$$

for  $\phi \in C_0^\infty$  and  $s \in [0, 1]$  (see Lemma 2.6 in [17]). Then for any  $p \geq 2$ ,

$$\|\phi u\|_{L_t^p \dot{H}^s} \lesssim \|\phi u\|_{L_t^2 \dot{H}^s} + \|\phi u\|_{L_t^\infty \dot{H}^s} \lesssim \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}}. \quad (2.24)$$

Now if we apply interpolation method, the estimates (1.10) and (1.11) with order 0 are direct consequences of (2.21), (2.23) and (2.24). Next we will prove these three estimates with order up to two.

**Proposition 2.9** (Generalized Morawetz estimates). *Let  $n \geq 3$ ,  $s \in [0, 1]$  and  $\rho > 2$ . Then for the solution  $u$  of Eq. (1.9) with  $F = 0$ , we have*

$$\sum_{|\alpha| \leq 2} \|Z^\alpha u\|_{L_t^2 Y_{s,\epsilon}} \lesssim \sum_{|\alpha| \leq 2} (\|Z^\alpha u_0\|_{\dot{H}^s} + \|Z^\alpha u_1\|_{\dot{H}^{s-1}}). \quad (2.25)$$

Moreover, if we assume only  $\rho > 1$  and  $s \in [0, 1]$ , the estimate still holds with  $|\alpha| \leq 1$ .



**Proof.** We first prove the estimate for  $Z^\alpha = \partial_x$ . Recall that for all  $-3/2 \leq \tilde{\mu} < \mu \leq 3/2$ , we have (Lemma 4.1 of [2])

$$\|\langle x \rangle^{-\mu} \tilde{\partial} u\|_{L_x^2} \lesssim \|\langle x \rangle^{-\tilde{\mu}} P^{1/2} u\|_{L_x^2}. \quad (2.26)$$

Also recall that  $\tilde{\partial} = \partial g^{-1}$ , a direct calculation induces

$$\sum_{|\alpha| \leq 1} \|\partial_x^\alpha u\|_{L_t^2 Y_{s,\epsilon}} \lesssim \sum_{|\alpha| \leq 1} \|\tilde{\partial}_x^\alpha u\|_{L_t^2 Y_{s,\epsilon}}.$$

Then for any  $\epsilon > 0$ , by (2.21),

$$\begin{aligned} \sum_{|\alpha| \leq 1} \|\partial_x^\alpha u\|_{L_t^2 Y_{s,\epsilon}} &\lesssim \sum_{|\alpha| \leq 1} \|\tilde{\partial}_x^\alpha u\|_{L_t^2 Y_{s,\epsilon}} \\ &\lesssim \sum_{j \leq 1} \|P^{j/2} u\|_{L_t^2 Y_{s,\epsilon/2}} \\ &\lesssim \sum_{j \leq 1} (\|P^{j/2} u_0\|_{\dot{H}^s} + \|P^{j/2} u_1\|_{\dot{H}^{s-1}}) \\ &\lesssim \sum_{|\alpha| \leq 1} \|\tilde{\partial}_x^\alpha u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}} + \|P^{(1+s-1)/2} u_1\|_{L_x^2} \\ &\lesssim \sum_{|\alpha| \leq 1} (\|\partial_x^\alpha u_0\|_{\dot{H}^s} + \|\partial_x^\alpha u_1\|_{\dot{H}^{s-1}}), \end{aligned}$$

where we have used the inequalities (2.3), (2.5) and Lemma 2.7 in the last two inequalities (note  $s \in [0, 1]$ ).

Next we check the case  $Z^\alpha = \Omega$ . Recall that by the interpolation of (2.8) and the duality of (2.8), we have

$$\|u\|_{L_t^2 Y_{s,\epsilon}} \leq \|F\|_{L_t^2 Y'_{1-s,\epsilon}}, \quad (2.27)$$

if  $u$  is a solution of (1.9) with vanishing initial data. Since  $[P, \Omega]u = \sum_{|\alpha| \leq 2} r_{2-|\alpha|} \partial_x^\alpha u$ , by using a combination of (6.7) in [17] and Lemma 2.3 for  $\Omega u$ , we have

$$\begin{aligned} \|\Omega u\|_{L_t^2 Y_{s,\epsilon}} &\lesssim \|\Omega u_0\|_{\dot{H}^s} + \|\Omega u_1\|_{\dot{H}^{s-1}} \\ &\quad + \sum_{|\alpha| \leq 1} \|\langle x \rangle^{3/2-s+\epsilon} r_{2-|\alpha|} \partial_x^\alpha u\|_{L_{t,x}^2} + \|r_0 \langle x \rangle^{1/2+\epsilon} \partial_x^2 u\|_{L_t^2 \dot{H}^{s-1}}. \end{aligned} \quad (2.28)$$

Now since  $\rho > 1$ , by (2.26) and Lemma 2.3,

$$\begin{aligned} \sum_{|\alpha| \leq 1} \|\langle x \rangle^{3/2-s+\epsilon} r_{2-|\alpha|} \partial_x^\alpha u\|_{L_{t,x}^2} &\lesssim \sum_{|\alpha| \leq 1} \|\langle x \rangle^{-1/2-s-\epsilon'} \partial_x^\alpha u\|_{L_{t,x}^2} \\ &\lesssim \sum_{|\alpha| \leq 1} \|\langle x \rangle^{-1/2-s-\epsilon'} \tilde{\partial}_x^\alpha u\|_{L_{t,x}^2} \\ &\lesssim \sum_{i \leq 1} \|\langle x \rangle^{-1/2-s-\epsilon'/2} P^{i/2} u\|_{L_{t,x}^2} \\ &\lesssim \sum_{i \leq 1} (\|P^{i/2} u_0\|_{\dot{H}^s} + \|P^{i/2} u_1\|_{\dot{H}^{s-1}}) \\ &\lesssim \sum_{|\alpha| \leq 1} \|\partial_x^\alpha u_0\|_{\dot{H}^s} + \|\partial_x^\alpha u_1\|_{\dot{H}^{s-1}} \end{aligned} \quad (2.29)$$

where in the last inequality we have used the inequalities (2.3), (2.5) and Lemma 2.7.

Let  $f(x) = r_0 \langle x \rangle^{1/2+\epsilon} = O(\langle x \rangle^{-\rho+1/2+\epsilon})$ . Then  $f'(x) = O(\langle x \rangle^{-\rho-1/2+\epsilon})$ . Since  $n \geq 3$ , by Hardy's inequality with duality, the KSS estimates (2.2) with Remark 2.1, and interpolation,

$$\begin{aligned} \|f \partial_x^2 u\|_{L_t^2 \dot{H}^{s-1}} &\leq \|\partial_x(f \partial_x u)\|_{L_t^2 \dot{H}^{s-1}} + \|f' \partial_x u\|_{L_t^2 \dot{H}^{s-1}} \\ &\lesssim \|f \partial_x u\|_{L_t^2 \dot{H}^s} + \|\langle x \rangle f' \partial_x u\|_{L_t^2 \dot{H}^s} \\ &\lesssim \sum_{|\alpha| \leq 1} \|\partial_x^\alpha u_0\|_{\dot{H}^s} + \sum_{|\alpha| \leq 1} \|\partial_x^\alpha u_1\|_{\dot{H}^{s-1}}. \end{aligned} \quad (2.30)$$

On the basis of (2.28), (2.29) and (2.30), we are done with  $Z^\alpha = \Omega$ . This completes the proof of the first order estimates under the condition  $\rho > 1$ .

For the second order part, we first consider the case  $Z^\alpha = \partial_x^2$ . Since  $s \in [0, 1)$ , we can always find  $\epsilon > 0$  such that  $1/2 + s + \epsilon \leq 3/2$ . By Lemma 2.6, the proof for  $Z^\alpha = \partial_x$ , Lemma 2.2 and Lemma 2.8, we have

$$\begin{aligned} \|\partial_x^2 u\|_{L_t^2 Y_{s,\epsilon}} &\lesssim \sum_{|\alpha| \leq 2} \|\tilde{\partial}_x^\alpha u\|_{L_t^2 Y_{s,\epsilon}} \\ &\lesssim \sum_{|\alpha| \leq 1} \|\tilde{\partial}_x^\alpha u\|_{L_t^2 Y_{s,\epsilon}} + \|Pu\|_{L_t^2 Y_{s,\epsilon}} \\ &\lesssim \sum_{|\alpha| \leq 1} (\|\partial_x^\alpha u_0\|_{\dot{H}^s} + \|\partial_x^\alpha u_1\|_{\dot{H}^{s-1}}) + \|Pu_0\|_{\dot{H}^s} + \|Pu_1\|_{\dot{H}^{s-1}} \\ &\approx \sum_{|\alpha| \leq 1} (\|\partial_x^\alpha u_0\|_{\dot{H}^s} + \|\partial_x^\alpha u_1\|_{\dot{H}^{s-1}}) + \|Pu_0\|_{\dot{H}^s} + \|P^{1/2}u_1\|_{\dot{H}^s} \\ &\lesssim \sum_{|\alpha| \leq 1} (\|\partial_x^\alpha u_0\|_{\dot{H}^s} + \|\partial_x^\alpha u_1\|_{\dot{H}^{s-1}}) + \sum_{|\alpha| \leq 2} \|\partial_x^\alpha u_0\|_{\dot{H}^s} + \sum_{|\alpha| \leq 1} \|\tilde{\partial}_x^\alpha u_1\|_{\dot{H}^s} \\ &\lesssim \sum_{|\alpha| \leq 2} (\|\partial_x^\alpha u_0\|_{\dot{H}^s} + \|\partial_x^\alpha u_1\|_{\dot{H}^{s-1}}), \end{aligned}$$

where the fractional Leibniz rule (Lemma 2.7) is used in the last inequality. Next, we consider the case  $Z^\alpha = \Omega^2$ . Since  $[P, \Omega^2]u = \sum_{|\alpha| \leq 3} (r_{2-|\alpha|} \partial_x^\alpha u)$ , and  $\Omega^2 u$  solves the wave equation with initial data  $(\Omega^2 u_0, \Omega^2 u_1)$  and nonlinear term  $[P, \Omega^2]u$ , by (2.27), Lemma 2.3, Lemma 2.5 and the higher order estimates we have proved

$$\begin{aligned} \|\Omega^2 u\|_{L_t^2 Y_{s,\epsilon}} &\lesssim \|\Omega^2 u_0\|_{\dot{H}^s} + \|\Omega^2 u_1\|_{\dot{H}^{s-1}} \\ &\quad + \sum_{|\alpha| \leq 2} \|\langle x \rangle^{3/2-s+\epsilon} r_{2-|\alpha|} \partial_x^\alpha u\|_{L_{t,x}^2} + \sum_{|\alpha|=3} \|\langle x \rangle^{1/2+\epsilon} r_{2-|\alpha|} \partial_x^\alpha u\|_{L_t^2 \dot{H}^{s-1}} \\ &\lesssim \|\Omega^2 u_0\|_{\dot{H}^s} + \|\Omega^2 u_1\|_{\dot{H}^{s-1}} + \sum_{|\alpha| \leq 2} \|\partial_x^\alpha u\|_{L_t^2 Y_{s,\epsilon}} + \sum_{|\alpha|=3} \|\langle x \rangle^{1/2+\epsilon} r_{-1} \partial_x^\alpha u\|_{L_t^2 \dot{H}^{s-1}} \\ &\lesssim \sum_{|\alpha| \leq 2} (\|Z^\alpha u_0\|_{\dot{H}^s} + \|Z^\alpha u_1\|_{\dot{H}^{s-1}}) + \sum_{|\alpha|=3} \|\langle x \rangle^{1/2+\epsilon} r_{-1} \partial_x^\alpha u\|_{L_t^2 \dot{H}^{s-1}} \\ &\lesssim \sum_{|\alpha| \leq 2} (\|Z^\alpha u_0\|_{\dot{H}^s} + \|Z^\alpha u_1\|_{\dot{H}^{s-1}}) + \|\langle x \rangle^{1+2\epsilon} r_{-1}\|_{L^\infty \cap \dot{W}^{1,n}} \|\langle x \rangle^{-1/2-\epsilon} \partial_x^3 u\|_{L_t^2 \dot{H}^{s-1}} \\ &\lesssim \sum_{|\alpha| \leq 2} (\|Z^\alpha u_0\|_{\dot{H}^s} + \|Z^\alpha u_1\|_{\dot{H}^{s-1}}) + \sum_{|\alpha| \leq 2} (\|\partial_x^\alpha u_0\|_{\dot{H}^s} + \|\partial_x^\alpha u_1\|_{\dot{H}^{s-1}}) \\ &\lesssim \sum_{|\alpha| \leq 2} (\|Z^\alpha u_0\|_{\dot{H}^s} + \|Z^\alpha u_1\|_{\dot{H}^{s-1}}) \end{aligned}$$

where we have used the fact that  $\rho > 2$ .

Since the commutator term  $[P, \partial \Omega]u = [P, \Omega \partial]u = \sum_{|\alpha| \leq 3} (r_{3-|\alpha|} \partial_x^\alpha u)$  corresponds to an even better case than what for  $\Omega^2$ , the proof proceeds in the same way. This completes the proof of the higher order estimates under the conditions  $\rho > 2$  and  $s \in [0, 1)$ .  $\square$

**Proposition 2.10** (Higher order energy estimates). *Let  $n \geq 3$ ,  $s \in [0, 1]$  and  $\rho > 2$ . Then for the solution  $u$  of Eq. (1.9) with  $F = 0$ , we have*

$$\sum_{|\alpha| \leq 2} \|Z^\alpha u(t, x)\|_{L_t^\infty \dot{H}^s} \lesssim \sum_{|\alpha| \leq 2} (\|Z^\alpha u_0\|_{\dot{H}^s} + \|Z^\alpha u_1\|_{\dot{H}^{s-1}}). \quad (2.31)$$

Moreover, if we assume only  $\rho > 1$ , the estimate still holds with  $|\alpha| \leq 1$ .

**Proof.** By Lemma 2.2 and elliptic regularity for  $P$ , we know

$$\|\partial_x u\|_{\dot{H}^1} \lesssim \|\partial_x^2 u\|_{L^2} \lesssim \|Pu\|_{L^2} + \|u\|_{L^2} \lesssim \|P^{1/2}u\|_{\dot{H}^1} + \|P^{1/2}u\|_{\dot{H}^{-1}}.$$

Interpolating this estimate with (2.3) with  $s = 1$ ,  $\|\partial_x u\|_{L^2} \simeq \|P^{1/2}u\|_{L^2}$ , we get that for  $s \in [0, 1]$ ,

$$\begin{aligned}\|\partial_x u\|_{\dot{H}^s} &\lesssim \|P^{1/2}u\|_{\dot{H}^s} + \|P^{1/2}u\|_{\dot{H}^{-s}} \\ &\lesssim \|P^{1/2}u\|_{\dot{H}^s} + \|u\|_{\dot{H}^{1-s}}.\end{aligned}\quad (2.32)$$

Thus by Lemma 2.2 we have for  $s \in [0, 1/2]$  (such that  $s \leq 1-s$  and  $\dot{H}^s \cap \dot{H}^{1+s} \subset \dot{H}^{1-s}$ ),

$$\begin{aligned}\sum_{|\alpha| \leq 1} \|\partial_x^\alpha u\|_{L_t^\infty \dot{H}^s} &\lesssim \sum_{j \leq 1} \|P^{j/2}u\|_{L_t^\infty \dot{H}^s} + \|u\|_{L_t^\infty \dot{H}^{1-s}} \\ &\lesssim \sum_{|\alpha| \leq 1} (\|\partial_x^\alpha u_0\|_{\dot{H}^s} + \|\partial_x^\alpha u_1\|_{\dot{H}^{s-1}}) + \|u_0\|_{\dot{H}^{1-s}} + \|u_1\|_{\dot{H}^{-s}} \\ &\lesssim \sum_{|\alpha| \leq 1} (\|\partial_x^\alpha u_0\|_{\dot{H}^s} + \|\partial_x^\alpha u_1\|_{\dot{H}^{s-1}}).\end{aligned}\quad (2.33)$$

Now we can deal with  $\Omega u$ . Noticing that

$$\tilde{\Omega}_{ij} f = g^{-1} \Omega_{ij} f + (x_i \partial_j g^{-1} - x_j \partial_i g^{-1}) f,$$

by the fractional Leibniz rule, we have

$$\|\tilde{\Omega} f\|_{\dot{H}^s} \lesssim \sum_{|\alpha| \leq 1} \|\Omega^\alpha f\|_{\dot{H}^s}, \quad |s| < n/2.$$

We have a similar relationship between  $\partial_x u$  and  $\tilde{\partial}_x u$ . By the Sobolev embedding, for any  $h \in L^n$ , we have

$$\begin{aligned}\|\langle x \rangle^{-1/2-\epsilon} h u\|_{\dot{H}^{s-1}} &\lesssim \|\langle x \rangle^{-1/2-\epsilon} h u\|_{L^{2n/(n+2(1-s))}} \\ &\lesssim \|h\|_{L^n} \|\langle x \rangle^{-1/2-\epsilon} u\|_{L^{2n/(n-2s)}} \\ &\lesssim \|\langle x \rangle^{-1/2-\epsilon} u\|_{\dot{H}^s}.\end{aligned}\quad (2.34)$$

Thus by the energy estimate, Lemmas 2.4, 2.7 and 2.5:

$$\begin{aligned}\|\tilde{\Omega} u\|_{L_t^\infty \dot{H}^s} &\lesssim \|\tilde{\Omega} u_0\|_{\dot{H}^s} + \|\tilde{\Omega} u_1\|_{\dot{H}^{s-1}} + \|\langle x \rangle^{1/2+\epsilon} [P, \tilde{\Omega}] u\|_{L_t^2 \dot{H}^{s-1}} \\ &\lesssim \|\tilde{\Omega} u_0\|_{\dot{H}^s} + \|\tilde{\Omega} u_1\|_{\dot{H}^{s-1}} + \sum_{1 \leq |\alpha| \leq 2} \|r_{2-|\alpha|} \langle x \rangle^{1/2+\epsilon} \tilde{\partial}_x^\alpha u\|_{L_t^2 \dot{H}^{s-1}} \\ &\lesssim \sum_{|\alpha| \leq 1} (\|\Omega^\alpha u_0\|_{\dot{H}^s} + \|\Omega^\alpha u_1\|_{\dot{H}^{s-1}}) \\ &\quad + \sum_{1 \leq |\alpha| \leq 2} \|r_{2-|\alpha|} \langle x \rangle^{1+2\epsilon} \|_{L^\infty \cap \dot{H}^{1-s, n/(1-s)}} \|\langle x \rangle^{-1/2-\epsilon} \tilde{\partial}_x^\alpha u\|_{L_t^2 \dot{H}^{s-1}} \\ &\lesssim \sum_{|\alpha| \leq 1} (\|\Omega^\alpha u_0\|_{\dot{H}^s} + \|\Omega^\alpha u_1\|_{\dot{H}^{s-1}}) + \sum_{1 \leq |\alpha| \leq 2} \|\langle x \rangle^{-1/2-\epsilon} \partial_x^\alpha u\|_{L_t^2 \dot{H}^{s-1}} \\ &\quad + \|\langle x \rangle^{-1/2-\epsilon} (\partial g^{-1}) u\|_{L_t^2 \dot{H}^{s-1}} + \|\langle x \rangle^{-1/2-\epsilon} [\partial(g^{-1} \partial g^{-1})] u\|_{L_t^2 \dot{H}^{s-1}} \\ &\lesssim \sum_{|\alpha| \leq 1} (\|Z^\alpha u_0\|_{\dot{H}^s} + \|Z^\alpha u_1\|_{\dot{H}^{s-1}}),\end{aligned}$$

where we have used the fact that  $\rho > 1$  and (2.34) with  $h = \partial g^{-1}$ ,  $\partial g^{-2}$  and  $h = \partial(g^{-1} \partial g^{-1})$  (the condition  $h \in L^n$  is satisfied since the condition (H1) on the metric  $g$ ). Noticing that  $\Omega u = g \tilde{\Omega} u - g(\Omega g^{-1})u$ , we hence have

$$\|\Omega u\|_{L_t^\infty \dot{H}^s} \lesssim \sum_{|\alpha| \leq 1} \|\tilde{\Omega} u\|_{L_t^\infty \dot{H}^s} \lesssim \sum_{|\alpha| \leq 1} (\|Z^\alpha u_0\|_{\dot{H}^s} + \|Z^\alpha u_1\|_{\dot{H}^{s-1}}). \quad (2.35)$$

On the basis of (2.33) and (2.35), we complete the proof of the energy estimates of order one, under the conditions  $s \in [0, 1/2]$  and  $\rho > 1$ .

For the part with second order derivatives, we need only to deal with  $\partial_x^2$  and  $\Omega^2$  as before.

By Lemma 2.8 and Lemma 2.2, we have

$$\begin{aligned}\|\partial_x^2 u\|_{L_t^\infty \dot{H}^s} &\lesssim \|Pu\|_{L_t^\infty \dot{H}^s} + \|u\|_{L_t^\infty \dot{H}^s} \\ &\lesssim \|Pu_0\|_{\dot{H}^s} + \|Pu_1\|_{\dot{H}^{s-1}} + \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}} \\ &\lesssim \sum_{|\alpha| \leq 2} (\|\partial_x^\alpha u_0\|_{\dot{H}^s} + \|\partial_x^\alpha u_1\|_{\dot{H}^{s-1}}).\end{aligned}\quad (2.36)$$

Here we remark that we can control  $\sum_{|\alpha|=1} \|\partial_x^\alpha u\|_{L_t^\infty \dot{H}^s}$  for  $s \in [0, 1]$  instead of the restriction  $s \in [0, 1/2]$  in (2.33), by (2.36) and (2.21), which enables us to relax the condition to  $s \in [0, 1]$  in the estimates of order one.

By Lemma 2.4, Lemma 2.5, and what we have gained in previous steps, if  $\rho > 2$ ,

$$\begin{aligned} \|\Omega^2 u\|_{L_t^\infty \dot{H}^s} &\lesssim \sum_{|\alpha| \leq 2} \|\tilde{\Omega}^\alpha u\|_{L_t^\infty \dot{H}^s} \\ &\lesssim \sum_{|\alpha| \leq 2} (\|\tilde{\Omega}^2 u_0\|_{\dot{H}^s} + \|\tilde{\Omega}^2 u_1\|_{\dot{H}^{s-1}}) + \sum_{1 \leq |\alpha| \leq 3} \|r_{2-|\alpha|} \langle x \rangle^{1/2+\epsilon} \tilde{\partial}_x^\alpha u\|_{L_t^2 \dot{H}^{s-1}} \\ &\lesssim \sum_{|\alpha| \leq 2} (\|\tilde{\Omega}^2 u_0\|_{\dot{H}^s} + \|\tilde{\Omega}^2 u_1\|_{\dot{H}^{s-1}}) + \sum_{1 \leq |\alpha| \leq 3} \|\langle x \rangle^{-1/2-\epsilon} \partial_x^\alpha u\|_{L_t^2 \dot{H}^{s-1}} \\ &\lesssim \sum_{|\alpha| \leq 2} (\|Z^\alpha u_0\|_{\dot{H}^s} + \|Z^\alpha u_1\|_{\dot{H}^{s-1}}). \end{aligned} \quad (2.37)$$

We are done with the second order estimates based on (2.36) and (2.37).  $\square$

**Proposition 2.11** (Sobolev inequality with angular smoothing). *Let  $u$  be a solution of (1.9) with  $F = 0$  and  $n \geq 3$ . Then for any  $s \in (1/2, 1]$  and  $\rho > 1$ , there exists a suitable  $\eta > 0$  so that we have:*

$$\sum_{|\alpha| \leq 1} \| |x|^{n/2-s} Z^\alpha u(t, x) \|_{L_{t,|x|}^\infty L_\omega^{2+\eta}} \lesssim \sum_{|\alpha| \leq 1} (\|Z^\alpha u_0\|_{\dot{H}^s} + \|Z^\alpha u_1\|_{\dot{H}^{s-1}}). \quad (2.38)$$

Furthermore, if we assume  $\rho > 2$ , then we have

$$\sum_{|\alpha| \leq 2} \| |x|^{n/2-s} Z^\alpha u(t, x) \|_{L_{t,|x|}^\infty L_\omega^{2+\eta}} \lesssim \sum_{|\alpha| \leq 2} (\|Z^\alpha u_0\|_{\dot{H}^s} + \|Z^\alpha u_1\|_{\dot{H}^{s-1}}). \quad (2.39)$$

**Proof.** This is a direct consequence of the energy estimates Proposition 2.10 and the inequality (2.22).  $\square$

**Proposition 2.12** (Local energy estimates). *Assume  $n \geq 3$ , let  $s \in [0, 1]$ ,  $p \geq 2$ ,  $k = 0, 1, 2$ ,  $\rho > k$  and  $u$  be a solution of (1.9) with  $F = 0$ . We have*

$$\sum_{|\alpha| \leq k} \|\phi Z^\alpha u\|_{L_t^p \dot{H}^s} \lesssim \sum_{|\alpha| \leq k} (\|Z^\alpha u_0\|_{\dot{H}^s} + \|Z^\alpha u_1\|_{\dot{H}^{s-1}}), \quad (2.40)$$

where  $\phi \in C_0^\infty(\mathbb{R}^n)$ .

**Proof.** The estimate with  $k = 0$  is just (2.24). For the higher order estimates with  $|\alpha| = k \geq 1$ , by the higher order KSS estimates (2.2),

$$\begin{aligned} \|\phi Z^\alpha u\|_{L_t^2 \dot{H}^1} &\lesssim \|\phi \partial_x Z^\alpha u\|_{L_{t,x}^2} + \|\phi' Z^\alpha u\|_{L_{t,x}^2} \\ &\lesssim \|\langle x \rangle^{-1/2-\epsilon} \partial_x Z^\alpha u\|_{L_{t,x}^2} + \|\langle x \rangle^{-3/2-\epsilon} Z^\alpha u\|_{L_{t,x}^2} \\ &\lesssim \sum_{|\alpha| \leq k} (\|Z^\alpha u_0\|_{\dot{H}^1} + \|Z^\alpha u_1\|_{L^2}). \end{aligned}$$

For  $s = 0$ , note that  $\phi \Omega = r_0 \partial_x$ ,

$$\begin{aligned} \|\phi Z^\alpha u\|_{L_{t,x}^2} &\lesssim \|\langle x \rangle^{-1/2-\epsilon} \partial_x Z^{\alpha-1} u\|_{L_{t,x}^2} \\ &\lesssim \sum_{|\alpha| \leq k-1} (\|Z^\alpha u_0\|_{\dot{H}^1} + \|Z^\alpha u_1\|_{L^2}) \\ &\lesssim \sum_{|\alpha| \leq k} (\|Z^\alpha u_0\|_{L^2} + \|Z^\alpha u_1\|_{\dot{H}^{-1}}). \end{aligned}$$

By interpolation between the above two estimates, we get (2.40) with  $p = 2$ . This will complete the proof if we combine it with the energy estimates in Proposition 2.10.  $\square$

**Proof of Theorem 1.3.** From the above four propositions, we have proved the higher order version of (2.21), (2.23) and (2.24), which gives us the required higher order estimates (1.10) and (1.11).  $\square$

### 3. Local in time Strichartz estimates

In this section, we give the proof of Theorem 1.4. The first lemma is concerned with the KSS estimates for the perturbed wave equation, obtained in Theorem 2.1 of [8] (see also Theorem 5.1 in [15]).

**Lemma 3.1.** *Let  $n \geq 3$ ,  $\square_h = \partial_t^2 - \Delta + h^{\alpha\beta}(t, x)\partial_\alpha\partial_\beta$ ,  $h^{\alpha\beta} = h^{\beta\alpha}$  and  $\sum |h^{\alpha\beta}| \leq 1/2$ . Then the solution to the equation  $\square_h u = F$  satisfies*

$$\begin{aligned} & (1+T)^{-2a} \left\| |x|^{-1/2+a} \left( |u'| + \frac{|u|}{|x|} \right) \right\|_{L^2([0,T] \times \mathbb{R}^n)}^2 + \left\| \langle x \rangle^{-1/2-\epsilon} \left( |u'| + \frac{|u|}{|x|} \right) \right\|_{L^2([0,T] \times \mathbb{R}^n)}^2 \\ & \lesssim \|u'(0, \cdot)\|_{L^2(\mathbb{R}^n)}^2 + \int_0^T \int \left( u' + \frac{u}{|x|} \right) \left( |F| + \left( |h'| + \frac{h}{|x|} \right) |u'| \right) dx dt \end{aligned} \quad (3.1)$$

for any  $\epsilon > 0$  and  $a \in (0, 1/2)$ .

On the basis of the KSS estimates for wave equations with variable coefficients and local energy decay (2.40), we can adapt the arguments in [17] to obtain the following KSS estimates for asymptotically Euclidean manifolds.

**Proposition 3.2.** *Assume that (H1) and (H2) hold with  $\rho > 1$ . Let  $N \geq 0$ ,  $0 < \mu < 1/2$ . Then the solution of (1.9) satisfies*

$$\begin{aligned} & \sum_{|\alpha| \leq N} (1+T)^{\mu-1/2} \left\| \langle x \rangle^{-\mu} \left( |(F^\alpha u)'| + \frac{|F^\alpha u|}{\langle x \rangle} \right) \right\|_{L_T^2 L_x^2} \\ & \lesssim \sum_{|\alpha| \leq N} \|(Z^\alpha u)'(0, \cdot)\|_{L_x^2} + \sum_{|\alpha| \leq N} \|F^\alpha F(s, \cdot)\|_{L_T^1 L_x^2}, \end{aligned} \quad (3.2)$$

where  $L_T^q L_x^r = L^q([0, T]; L^r(\mathbb{R}^n))$ .

As a consequence of this KSS estimate, similarly to the previous proof of Proposition 2.9, we can have the following estimates.

**Corollary 3.3.** *Assume that (H1) and (H2) hold with  $\rho > 2$ . Let  $0 < \mu \leq 1/2$  and*

$$A_\mu(T) = \begin{cases} (\log(2+T))^{-1/2}, & \mu = 1/2, \\ (1+T)^{\mu-1/2}, & 0 < \mu < 1/2. \end{cases}$$

We have

$$\|\langle x \rangle^{-\mu} e^{itP^{1/2}} f\|_{L_T^2 L_x^2} \lesssim A_\mu(T)^{-1} \|f\|_{L^2}. \quad (3.3)$$

Moreover, if  $0 < \mu < 1/2$ , for the solution  $u$  of Eq. (1.9) with  $F = 0$ , we have

$$\sum_{|\alpha| \leq 2} \|\langle x \rangle^{-\mu} Z^\alpha u\|_{L_T^2 L_x^2} \lesssim T^{1/2-\mu+\epsilon} \sum_{|\alpha| \leq 2} (\|Z^\alpha u_0\|_{L^2} + \|Z^\alpha u_1\|_{H^{-1}}). \quad (3.4)$$

And, if we assume  $\rho > 1$  instead of  $\rho > 2$ , we have the same estimates of first order ( $|\alpha| \leq 1$ ).

**Proof.** (3.3) is a direct consequence if we employ (3.2) with  $\alpha = 0$  for  $u' = \partial_t u$ . To obtain (3.4), we basically follow the argument as in Proposition 2.9 with some modifications. For the second order part, we first consider the case  $Z^\alpha = \partial_x^2$ . We claim that we have the following inequality

$$\|\langle x \rangle^{-\mu} \partial_x^2 u\|_{L_x^2} \leq \epsilon \|\langle x \rangle^{-\mu} \partial_x^2 u\|_{L_x^2} + C(\epsilon) \|\langle x \rangle^{-\mu} u\|_{L_x^2}. \quad (3.5)$$

By Lemma 2.6, Lemma 2.2 and Lemma 2.8, we have

$$\begin{aligned} A_\mu(T) \|\langle x \rangle^{-\mu} \partial_x^2 u\|_{L_T^2 L_x^2} & \lesssim A_\mu(T) \sum_{|\alpha| \leq 2} \|\langle x \rangle^{-\mu} \tilde{\partial}_x^\alpha u\|_{L_T^2 L_x^2} \\ & \lesssim A_\mu(T) \sum_{|\alpha| \leq 1} \|\langle x \rangle^{-\mu} \tilde{\partial}_x^\alpha u\|_{L_T^2 L_x^2} + A_\mu(T) \|\langle x \rangle^{-\mu} P u\|_{L_T^2 L_x^2} \end{aligned}$$

$$\begin{aligned}
&\lesssim A_\mu(T) \sum_{|\alpha| \leq 1} \|\langle x \rangle^{-\mu} \partial_x^\alpha u\|_{L_T^2 L_x^2} + A_\mu(T) \|\langle x \rangle^{-\mu} Pu\|_{L_T^2 L_x^2} \\
&\lesssim \epsilon A_\mu(T) \|\langle x \rangle^{-\mu} \partial_x^2 u\|_{L_T^2 L_x^2} + C(\epsilon) A_\mu(T) \|\langle x \rangle^{-\mu} u\|_{L_T^2 L_x^2} + A_\mu(T) \|\langle x \rangle^{-\mu} Pu\|_{L_T^2 L_x^2} \\
&\lesssim \epsilon A_\mu(T) \|\langle x \rangle^{-\mu} \partial_x^2 u\|_{L_T^2 L_x^2} + C(\epsilon) (\|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-1}}) + \|Pu_0\|_{L^2} + \|Pu_1\|_{\dot{H}^{-1}},
\end{aligned}$$

where we have used (3.3) and (3.5). Hence we have

$$\begin{aligned}
A_\mu(T) \|\langle x \rangle^{-\mu} \partial_x^2 u\|_{L_T^2 L_x^2} &\lesssim \|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-1}} + \|Pu_0\|_{L^2} + \|Pu_1\|_{\dot{H}^{-1}} \\
&\lesssim \|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-1}} + \|Pu_0\|_{L^2} + \|P^{1/2} u_1\|_{L^2} \\
&\lesssim \|u_1\|_{\dot{H}^{-1}} + \sum_{|\alpha| \leq 2} \|\partial_x^\alpha u_0\|_{L^2} + \|\tilde{\partial} u_1\|_{L^2} \\
&\lesssim \sum_{|\alpha| \leq 2} (\|\partial_x^\alpha u_0\|_{L^2} + \|\partial_x^\alpha u_1\|_{\dot{H}^{-1}}).
\end{aligned}$$

Now we are left with the norm for  $Z = \Omega, \Omega^2$ , but from the proof of Proposition 2.9, we know it suffices to prove the following estimates

$$\|\langle x \rangle^{-\mu} w\|_{L_{t,x}^2([0,T] \times \mathbb{R}^n)} \lesssim T^{1/2-\mu+\epsilon} \|\langle x \rangle^{1/2+\epsilon} F\|_{L_t^2 \dot{H}^{-1}([0,T] \times \mathbb{R}^n)}, \quad (3.6)$$

if  $w$  is the solution of (1.9) with vanishing initial data. Recall that we have proved in Lemma 2.3 that

$$\|\langle x \rangle^{-1/2-\epsilon} w\|_{L_{t,x}^2([0,T] \times \mathbb{R}^n)} \lesssim \|\langle x \rangle^{1/2+\epsilon} F\|_{L_t^2 \dot{H}^{-1}([0,T] \times \mathbb{R}^n)}. \quad (3.7)$$

Also if we restrict the time  $t$  in  $[0, T]$ , it is easy to verify that Lemma 2.4 still holds, i.e.

$$\|w\|_{L_t^2 L_x^2([0,T] \times \mathbb{R}^n)} \lesssim T^{1/2} \|w\|_{L_t^\infty L_x^2([0,T] \times \mathbb{R}^n)} \lesssim T^{1/2} \|\langle x \rangle^{1/2+\epsilon} F\|_{L_t^2 \dot{H}^{-1}([0,T] \times \mathbb{R}^n)}. \quad (3.8)$$

Now (3.6) just follows from the interpolation between (3.7) and (3.8). To conclude the proof of (3.4), it remains to prove the claim (3.5).

*Proof of (3.5).* This inequality is true for  $\mu = 0$ . For general  $\mu \geq 0$ , we apply the estimate for  $\mu = 0$  to  $v = \phi u$  with  $\phi = \psi(x/R)$ ,  $\psi \in C^\infty$ ,  $0 \leq \psi \leq 1$ ,  $\text{supp } \psi \subset \{1/4 < |x| < 2\}$ ,  $\psi = 1$  in  $B_1 \setminus B_{1/2}$  and  $R \geq 1$ . Because of  $\{x: \phi(x) = 1\} \subset \{|x| > R/4\}$  and  $\text{supp } \phi \subset \{R/4 < |x| < 2R\}$ , we get

$$\begin{aligned}
\|\langle x \rangle^{-\mu} \partial_x u\|_{L^2(\{x: \phi(x)=1\})} &= \|\langle x \rangle^{-\mu} \partial_x(\phi u)\|_{L^2(\{x: \phi(x)=1\})} \\
&\leq CR^{-\mu} \|\partial_x(\phi u)\|_{L^2(\mathbb{R}^n)} \\
&\leq CR^{-\mu} (\epsilon \|\partial_x^2(\phi u)\|_{L^2(\mathbb{R}^n)} + C(\epsilon) \|\phi u\|_{L^2(\mathbb{R}^n)}) \\
&\leq C(\epsilon \|\langle x \rangle^{-\mu} \partial_x^2(\phi u)\|_{L^2(\mathbb{R}^n)} + C(\epsilon) \|\langle x \rangle^{-\mu} \phi u\|_{L^2(\mathbb{R}^n)}) \\
&\leq C(\epsilon \|\langle x \rangle^{-\mu} \partial_x^2 u\|_{L^2(\text{supp } \phi)} + C\epsilon R^{-1} \|\langle x \rangle^{-\mu} \partial_x u\|_{L^2(\text{supp } \phi')} \\
&\quad + (C(\epsilon) + C\epsilon R^{-2}) \|\langle x \rangle^{-\mu} u\|_{L^2(\text{supp } \phi)}).
\end{aligned}$$

If we choose instead  $\psi = 1$  in  $B_1$  and 0 for  $|x| \geq 2$ , then

$$\begin{aligned}
\|\langle x \rangle^{-\mu} \partial_x u\|_{L^2(\{x: |x| \leq 1\})} &\leq C\epsilon \|\langle x \rangle^{-\mu} \partial_x^2 u\|_{L^2(\{x: |x| \leq 2\})} \\
&\quad + C\epsilon \|\langle x \rangle^{-\mu} \partial_x u\|_{L^2(\{x: |x| \leq 2\})} + (C(\epsilon) + C\epsilon) \|\langle x \rangle^{-\mu} u\|_{L^2(\{x: |x| \leq 2\})}.
\end{aligned}$$

Combining the above two inequalities, we see

$$\|\langle x \rangle^{-\mu} \partial_x u\|_{L^2(\mathbb{R}^n)} \leq C\epsilon \|\langle x \rangle^{-\mu} \partial_x^2 u\|_{L^2(\mathbb{R}^n)} + C\epsilon \|\langle x \rangle^{-\mu-1} \partial_x u\|_{L^2(\mathbb{R}^n)} + C(C(\epsilon) + \epsilon) \|\langle x \rangle^{-\mu} u\|_{L^2(\mathbb{R}^n)},$$

which implies (3.5), by choosing small enough  $\epsilon > 0$ .  $\square$

The next estimate is based on the endpoint trace lemma.

**Proposition 3.4.** Let  $\dot{B}_{pq}^s$  denote the homogeneous Besov space. Then we have

$$\| |x|^{(n-1)/2} e^{itP^{1/2}} f \|_{L_t^\infty L_r^\infty L_\omega^2} \lesssim \|f\|_{\dot{B}_{2,1}^{1/2}}. \quad (3.9)$$

**Proof.** Recall that we have the endpoint trace lemma (see (1.7) in [4]):

$$r^{(n-1)/2} \|f(r \cdot)\|_{L^2_\omega} \lesssim \|f\|_{\dot{B}^{1/2}_{2,1}}, \quad (3.10)$$

which gives that

$$\| |x|^{(n-1)/2} e^{itP^{1/2}} f \|_{L^\infty_r L^2_\omega} \lesssim \| e^{itP^{1/2}} f \|_{\dot{B}^{1/2}_{2,1}}. \quad (3.11)$$

On the other hand, by Lemma 2.2 we have

$$\| e^{itP^{1/2}} f \|_{\dot{H}^1} \lesssim \| P^{1/2} e^{itP^{1/2}} f \|_{L^2_x} \lesssim \| P^{1/2} f \|_{L^2_x} \lesssim \| f \|_{\dot{H}^1}.$$

Noticing that  $\|f\|_{\dot{B}^{s}_{2,2}} = \|f\|_{\dot{H}^s}$ , we can rewrite the above estimate as

$$\| e^{itP^{1/2}} f \|_{\dot{B}^{1}_{2,2}} \lesssim \| f \|_{\dot{B}^{1}_{2,2}}.$$

Interpolating this estimate with the energy estimate

$$\| e^{itP^{1/2}} f \|_{\dot{B}^0_{2,2}} \lesssim \| f \|_{\dot{B}^0_{2,2}}$$

gives

$$\| e^{itP^{1/2}} f \|_{\dot{B}^{1/2}_{2,1}} = \| e^{itP^{1/2}} f \|_{(\dot{B}^1_{2,2}, \dot{B}^0_{2,2})_{1/2,1}} \lesssim \| f \|_{(\dot{B}^1_{2,2}, \dot{B}^0_{2,2})_{1/2,1}} = \| f \|_{\dot{B}^{1/2}_{2,1}}, \quad (3.12)$$

where we have used the fact that (Theorem 6.4.5 in [1])

$$(\dot{B}^{s_0}_{pq_0}, \dot{B}^{s_1}_{pq_1})_{\theta,r} = \dot{B}^{s^*}_{pr}, \quad \text{if } s_0 \neq s_1, \quad 0 < \theta < 1, \quad r, q_0, q_1 \geq 1 \text{ and } s^* = (1-\theta)s_0 + \theta s_1.$$

Now our estimate (3.9) follows from (3.11) and (3.12).  $\square$

Now we are ready to obtain the local in time Strichartz estimates as follows.

**Proposition 3.5.** *Let  $2 \leq p < \infty$  and  $a \in (0, 1/p)$ . Then we have*

$$\| \langle x \rangle^{-a} |x|^{(n-1)(1/2-1/p)} e^{itP^{1/2}} f \|_{L^p_T L^p_r L^2_\omega} \lesssim (1+T)^{1/p-a} \| f \|_{\dot{H}^{1/2-1/p}}. \quad (3.13)$$

**Proof.** This estimate follows from the real interpolation between (3.3) and (3.9) with  $\theta = 2/p$  (for similar arguments, see, e.g., [6,21]).  $\square$

Finally we give the proof of Theorem 1.4.

**Proof of Theorem 1.4.** Since the estimates in Theorem 1.4 with order 0 are just obtained in Proposition 3.5, we are left with the higher order estimates. Similarly to the proof of Proposition 3.5, we need only to show the higher order estimates that correspond to (3.3) and (3.9).

The higher order estimates corresponding to (3.3) are known from Corollary 3.3. For the higher order estimates of (3.9), by (3.10) we have

$$\sum_{|\alpha| \leq 2} \| |x|^{(n-1)/2} Z^\alpha u(t, \cdot) \|_{L^\infty_r L^2_\omega} \lesssim \sum_{|\alpha| \leq 2} \| Z^\alpha u(t, \cdot) \|_{\dot{B}^{1/2}_{2,1}}. \quad (3.14)$$

On the other hand, from the energy estimates in Proposition 2.10, we have for any  $s \in [0, 1]$

$$\sum_{|\alpha| \leq 2} \| Z^\alpha u(t, \cdot) \|_{\dot{H}^s} \lesssim \sum_{|\alpha| \leq 2} (\| Z^\alpha u_0 \|_{\dot{H}^s} + \| Z^\alpha u_1 \|_{\dot{H}^{s-1}}).$$

Now the real interpolation between the above two estimates with  $s = 0$  and  $s = 1$  gives

$$\sum_{|\alpha| \leq 2} \| Z^\alpha u(t, \cdot) \|_{\dot{B}^{1/2}_{2,1}} \lesssim \sum_{|\alpha| \leq 2} (\| Z^\alpha u_0 \|_{\dot{B}^{1/2}_{2,1}} + \| Z^\alpha u_1 \|_{\dot{B}^{-1/2}_{2,1}}).$$

Combining this estimate with (3.14), we get the second order estimates of (3.9), which completes the proof of Theorem 1.4 for  $\rho > 2$ . When  $\rho > 1$ , we need only to use (2.22) instead of (3.10).  $\square$

#### 4. Strauss conjecture when $n = 3, 4$

In this section, we will prove the existence results in Theorem 1.1 and Theorem 1.2.

##### 4.1. Global results when $n = 3, 4$

In this subsection, we prove the Strauss conjecture stated in Theorem 1.1. The result when  $n = 3$  and  $\rho > 1$  has been proved in [17], under the additional assumption that  $g_{ij}$  is spherically symmetric. Since we have obtained the same estimates without this assumption, the existence result with a general metric follows from the same argument. Here we present the proof for  $n = 3, 4$  under the conditions  $\rho > 2$  and  $p > p_c$ , and we are following the argument as in [7].

We define  $X = X_{s,\epsilon,q}(\mathbb{R}^n)$  to be the space with norm defined by

$$\|h\|_{X_{s,\epsilon,q}} = \|h\|_{L^{qs}(|x| \leq 1)} + \| |x|^{n/2-(n+1)/q-s-\epsilon} h \|_{L^q_{|x|} L^{2+\eta}_{\omega}(|x| \geq 1)}, \quad (4.1)$$

where  $n(\frac{1}{2} - \frac{1}{q_s}) = s$ . Combining the Sobolev inequalities with angular regularity (2.22) with Sobolev embedding  $\dot{H}^s \subset L^{q_s}$ , we have the embedding

$$\dot{H}^s \subset X_{s,0,\infty}$$

for  $s \in (1/2, n/2)$  and some  $\eta > 0$ . By duality, we have (see Theorem 2.11 of [13])

$$X'_{1-s,0,\infty} \subset \dot{H}^{s-1} \quad \text{for } s \in ((2-n)/2, 1/2). \quad (4.2)$$

With these notations, Theorem 1.3 tells us that for the solution  $u$  to the linear wave equation  $\partial_t^2 u + Pu = 0$ , we have

$$\sum_{|\alpha| \leq 2} (\|Z^\alpha u\|_{L_t^\infty \dot{H}^s \cap L_t^p X_{s,\epsilon,p}} + \|\partial_t Z^\alpha u\|_{L_t^\infty \dot{H}^{s-1}}) \lesssim \sum_{|\alpha| \leq 2} (\|Z^\alpha u_0\|_{\dot{H}^s} + \|Z^\alpha u_1\|_{\dot{H}^{s-1}})$$

for  $s \in (1/2 - 1/p, 1)$ . By Duhamel's formula and (4.2), we see that for  $u$  solving the linear wave equation  $\partial_t^2 u + Pu = F$ , we have

$$\begin{aligned} & \sum_{|\alpha| \leq 2} (\|Z^\alpha u\|_{L_t^\infty \dot{H}^s \cap L_t^p X_{s,\epsilon,p}} + \|\partial_t Z^\alpha u\|_{L_t^\infty \dot{H}^{s-1}}) \\ & \lesssim \sum_{|\alpha| \leq 2} (\|Z^\alpha u_0\|_{\dot{H}^s} + \|Z^\alpha u_1\|_{\dot{H}^{s-1}} + \|Z^\alpha F\|_{L_t^1 \dot{H}^{s-1}}) \\ & \lesssim \sum_{|\alpha| \leq 2} (\|Z^\alpha u_0\|_{\dot{H}^s} + \|Z^\alpha u_1\|_{\dot{H}^{s-1}} + \|Z^\alpha F\|_{L_t^1 X'_{1-s,0,\infty}}) \end{aligned} \quad (4.3)$$

if  $\rho > 2$ ,  $p > 2$ ,  $s \in (1/2 - 1/p, 1/2)$ .

For the linear wave equation  $(\partial_t^2 - \Delta_g)u = F$ , using the observation (1.14), we have the same set of estimates.

Let us now see how we can use these estimates to prove Theorem 1.1. Considering the Cauchy data  $(u_0, u_1)$  satisfying the smallness condition (1.7), set  $u^{-1} \equiv 0$  and let  $u^{(0)}$  solve the Cauchy problem (1.2) with  $F = 0$ . We iteratively define  $u^{(k)}$ , for  $k \geq 1$ , by solving

$$\begin{cases} (\partial_t^2 - \Delta_g)u^{(k)}(t, x) = F_p(u^{(k-1)}(t, x)), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0, \cdot) = u_0, & \partial_t u(0, \cdot) = u_1. \end{cases}$$

Let  $s = s_c - p\epsilon/(p-1) = n/2 - 2/(p-1) - p\epsilon/(p-1)$ , our aim is to show that if the constant  $\delta > 0$  in (1.7) is small enough, then so is

$$M_k = \sum_{|\alpha| \leq 2} (\|Z^\alpha u^{(k)}\|_{L_t^\infty \dot{H}^s \cap L_t^p X_{s,\epsilon,p}} + \|\partial_t Z^\alpha u^{(k)}\|_{L_t^\infty \dot{H}^{s-1}})$$

for every  $k = 0, 1, 2, \dots$ . Notice that since  $p_c < p < 1 + 4/(n-1)$ , we can always choose  $\epsilon > 0$  small enough so that  $s \in (1/2 - 1/p, 1/2)$ . Note also that we have the identity

$$p(n/2 - (n+1)/p - s - \epsilon) = -(n/2 - (1-s)). \quad (4.4)$$

For  $k = 0$ , by (4.3) we have  $M_0 \leq C_0 \delta$ , with  $C_0$  a fixed constant. More generally, (4.3) implies that

$$\begin{aligned} M_k & \leq C_0 \delta + C_0 \sum_{|\alpha| \leq 2} (\| |x|^{-n/2+1-s} Z^\alpha F_p(u^{(k-1)}) \|_{L_t^1 L_{|x|}^1 L_{\omega}^2(\mathbb{R}_+ \times \{x: |x| \geq 1\})} \\ & \quad + \| Z^\alpha F_p(u^{(k-1)}) \|_{L_t^1 L_{x'}^{q'_1-s}(\mathbb{R}_+ \times \{x: |x| \leq 1\})}). \end{aligned} \quad (4.5)$$



Recall that our assumption (1.3) on the nonlinear term  $F_p$  implies that for small  $v$

$$\sum_{|\alpha| \leq 2} |Z^\alpha F_p(v)| \lesssim |v|^{p-1} \sum_{|\alpha| \leq 2} |Z^\alpha v| + |v|^{p-2} \sum_{|\alpha| \leq 1} |Z^\alpha v|^2. \quad (4.6)$$

Since the collection  $Z$  contains vectors spanning the tangent space to  $S^{n-1}$ , by Sobolev embedding we have

$$\|v(r \cdot)\|_{L^\infty_\omega} + \sum_{|\alpha| \leq 1} \|Z^\alpha v(r \cdot)\|_{L^4_\omega} \lesssim \sum_{|\alpha| \leq 2} \|Z^\alpha v(r \cdot)\|_{L^2_\omega}.$$

Consequently, for fixed  $t, r > 0$

$$\sum_{|\alpha| \leq 2} \|Z^\alpha F_p(u^{(k-1)}(t, r \cdot))\|_{L^2_\omega} \lesssim \sum_{|\alpha| \leq 2} \|Z^\alpha u^{(k-1)}(t, r \cdot)\|_{L^2_\omega}^p.$$

By (4.4), the first summand in the right-hand side of (4.5) is dominated by  $C_1 M_{k-1}^p$  for small  $u^{(k-1)}$ .

Since  $q'_{1-s} < 2 < q_s$ ,  $p > 2$  and  $n \leq 4$ , we can choose  $\eta > 0$  small enough such that  $p, q_s > 2 + \eta$  and so  $W^{2,2+\eta} \subset L^\infty$ ,  $H^1 \subset L^4$ . Thus, for each fixed  $t$ , we have

$$\begin{aligned} \sum_{|\alpha| \leq 2} \|Z^\alpha F_p(u^{(k-1)}(t, \cdot))\|_{L^{q'_{1-s}}(x: |x| \leq 1)} &\lesssim \sum_{|\alpha| \leq 2} \|u^{(k-1)}\|_{L^\infty(x: |x| \leq 1)}^{p-1} \|Z^\alpha u^{(k-1)}(t, \cdot)\|_{L^{q'_{1-s}}(x: |x| \leq 1)} \\ &\quad + \sum_{|\alpha| \leq 1} \|u^{(k-1)}\|_{L^\infty(x: |x| \leq 1)}^{p-2} \|Z^\alpha u^{(k-1)}(t, \cdot)\|_{L^{2q'_{1-s}}(x: |x| \leq 1)}^2 \\ &\lesssim \sum_{|\alpha| \leq 2} \|u^{(k-1)}\|_{W^{2,2+\eta}(x: |x| \leq 2)}^{p-1} \|Z^\alpha u^{(k-1)}(t, \cdot)\|_{L^{q_s}(x: |x| \leq 1)} \\ &\quad + \sum_{|\alpha| \leq 2} \|u^{(k-1)}\|_{W^{2,2+\eta}(x: |x| \leq 2)}^{p-2} \|Z^\alpha u^{(k-1)}(t, \cdot)\|_{L^2(x: |x| \leq 2)}^2 \\ &\lesssim \sum_{|\alpha| \leq 2} \|Z^\alpha u^{(k-1)}(t, \cdot)\|_{L^{q_s}(x: |x| \leq 1)}^p \\ &\quad + \sum_{|\beta| \leq 2} \| |x|^{n/2-(n+1)/p-s-\epsilon} Z^\beta u^{(k-1)}(t, \cdot) \|_{L^p_{|x|} L^{2+\eta}_\omega(|x| \geq 1)}^p. \end{aligned}$$

The second summand in the right-hand side of (4.5) is thus also dominated by  $C_1 M_{k-1}^p$ , and we conclude that  $M_k \leq C_0 \delta + 2C_0 C_1 M_{k-1}^p$ . Then

$$M_k \leq 2C_0 \delta, \quad k = 1, 2, 3, \dots, \quad (4.7)$$

for  $\delta > 0$  sufficiently small. Moreover, the smallness condition of (4.6) is verified for sufficiently small  $\delta > 0$ , since

$$\|u^{(k)}\|_{L^\infty_{t,x}} \lesssim M_k.$$

To finish the proof of Theorem 1.1 we need only to show that  $u^{(k)}$  converges to a solution of Eq. (1.2). For this it suffices to show that

$$A_k = \|u^{(k)} - u^{(k-1)}\|_{L^p_t X_{s,\epsilon,p}}$$

tends geometrically to zero as  $k \rightarrow \infty$ . Since  $|F_p(v) - F_p(w)| \lesssim |v - w|(|v|^{p-1} + |w|^{p-1})$ , the proof of (4.7) can be adapted to show that, for small  $\delta > 0$ , there is a uniform constant  $C$  so that

$$A_k \leq C A_{k-1} (M_{k-1} + M_{k-2})^{p-1},$$

which, by (4.7), implies that  $A_k \leq \frac{1}{2} A_{k-1}$  for small  $\delta$ . Since  $A_1$  is finite, the claim follows, which finishes the proof of Theorem 1.1.

#### 4.2. Local results when $n = 3$

In this subsection we prove Theorem 1.2. Let  $2 \leq p < p_c = 1 + \sqrt{2}$  and  $n = 3$ .

Define  $s = s_d = 1/2 - 1/p$ , and let  $a$  be the number such that

$$p[(n-1)(1/2 - 1/p) - a] = 1 - s - n/2,$$

i.e.,  $a = -1/p^2 - (n-1)/(2p) + (n-1)/2$ . Since  $2 \leq p < p_c$ , we have  $a \in (0, 1/p)$ . By the estimates (1.11), (1.12) and Duhamel's principle, we have for  $T \geq 1$

$$\begin{aligned} & \sum_{|\alpha| \leq 2} (\| |x|^{(n-1)(1/2-1/p)-a} Z^\alpha u \|_{L_t^p L_r^p L_\omega^2([0,T] \times \{|x|>1\})} + \| Z^\alpha u \|_{L_t^p L_x^{q_s}([0,T] \times \{|x|<1\})}) \\ & \lesssim T^{1/p-a+\epsilon} \sum_{|\alpha| \leq 2} (\| Z^\alpha u_0 \|_{\dot{H}^s} + \| Z^\alpha u_1 \|_{\dot{H}^{s-1}} + \| Z^\alpha F \|_{L_t^1 \dot{H}^{s-1}}) \\ & \lesssim T^{1/p-a+\epsilon} \sum_{|\alpha| \leq 2} (\| Z^\alpha u_0 \|_{\dot{H}^s} + \| Z^\alpha u_1 \|_{\dot{H}^{s-1}} + \| Z^\alpha F \|_{L_t^1 X_{1-s,0,\infty}'}). \end{aligned} \quad (4.8)$$

Now if we set

$$\begin{aligned} M_k &= \sum_{|\alpha| \leq 2} (\| Z^\alpha u^{(k)} \|_{L_t^\infty \dot{H}^s} + \| \partial_t Z^\alpha u^{(k)} \|_{L_t^\infty \dot{H}^{s-1}}) \\ &+ T^{a-1/p-\epsilon} \sum_{|\alpha| \leq 2} (\| |x|^{(-1/2-s)/p} Z^\alpha u \|_{L_t^p L_r^p L_\omega^2([0,T] \times \{|x|>1\})} + \| Z^\alpha u \|_{L_t^p L_x^{q_s}([0,T] \times \{|x|<1\})}), \end{aligned} \quad (4.9)$$

then on the basis of (1.11) and (4.8), we can use the iteration method (with  $\eta = 0$ ) as in Section 4.1 to get the existence result for  $2 \leq p < p_c$  and  $\rho > 2$  in Theorem 1.2.

Heuristically, the lifespan is given when we have

$$M_k \sim (T_\delta^{1/p-a+\epsilon} M_k)^p \sim \delta,$$

which yields that

$$T_\delta \sim \delta^{(p(p-1))/(p^2-2p-1)+\epsilon'}, \quad \forall \epsilon' > 0.$$

The case  $\rho > 1$  can be proved by the same argument in [17] combined with Theorem 1.4.

## Acknowledgment

The authors would like to thank the anonymous referee for a careful and thorough reading of the manuscript and a number of helpful comments. In particular, the authors are grateful for the suggestion that leads to an improvement of Theorem 1.2.

## References

- [1] J. Bergh, J. Löfström, Interpolation Spaces. An Introduction, Grundlehren Math. Wiss., vol. 223, Springer-Verlag, Berlin, New York, 1976, x+207 pp.
- [2] J.-F. Bony, D. Häfner, The semilinear wave equation on asymptotically Euclidean manifolds, *Comm. Partial Differential Equations* 35 (1) (2010) 23–67.
- [3] M. Christ, A. Kiselev, Maximal functions associated to filtrations, *J. Funct. Anal.* 179 (2001) 409–425.
- [4] D. Fang, C. Wang, Weighted Strichartz estimates with angular regularity and their applications, *Forum Math.*, doi:10.1515/FORM.2011.009, in press.
- [5] V. Georgiev, H. Lindblad, C.D. Sogge, Weighted Strichartz estimates and global existence for semilinear wave equations, *Amer. J. Math.* 119 (6) (1997) 1291–1319.
- [6] K. Hidano, Morawetz–Strichartz estimates for spherically symmetric solutions to wave equations and applications to semilinear Cauchy problems, *Differential Integral Equations* 20 (2007) 735–754.
- [7] K. Hidano, J. Metcalfe, H.F. Smith, C.D. Sogge, Y. Zhou, On abstract Strichartz estimates and the Strauss conjecture for nontrapping obstacles, *Trans. Amer. Math. Soc.* 362 (2010) 2789–2809.
- [8] K. Hidano, C. Wang, K. Yokoyama, On almost global existence and local well-posedness for some 3-D quasi-linear wave equations, arXiv:1004.3349.
- [9] J.-C. Jiang, C. Wang, X. Yu, Generalized and weighted Strichartz estimates, arXiv:1008.5397.
- [10] F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions, *Manuscripta Math.* 28 (1979) 235–265.
- [11] M. Keel, H. Smith, C.D. Sogge, Almost global existence for some semilinear wave equations, *J. Anal. Math.* 87 (2002) 265–279, dedicated to the memory of Thomas H. Wolff.
- [12] M. Kon, A. Gulisashvili, Exact smoothing properties of Schrödinger semigroups, *Amer. J. Math.* 118 (6) (1996) 1215–1248.
- [13] T.T. Li, Y. Zhou, A note on the life-span of classical solutions to nonlinear wave equations in four space dimensions, *Indiana Univ. Math. J.* 44 (1995) 1207–1248.
- [14] H. Lindblad, C.D. Sogge, Long-time existence for small amplitude semilinear wave equations, *Amer. J. Math.* 118 (5) (1996) 1047–1135.
- [15] J. Metcalfe, C. Sogge, Long-time existence of quasilinear wave equations exterior to star-shaped obstacles via energy methods, *SIAM J. Math. Anal.* 38 (1) (2006) 188–209.
- [16] T.C. Sideris, Nonexistence of global solutions to semilinear wave equations in high dimensions, *J. Differential Equations* 52 (1984) 378–406.
- [17] C.D. Sogge, C. Wang, Concerning the wave equation on asymptotically Euclidean manifolds, *J. Anal. Math.* 12 (1) (2010) 1–32.
- [18] C.D. Sogge, *Lectures on Nonlinear Wave Equations*, 2nd edition, International Press, Boston, MA, 2008.
- [19] D. Tataru, Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equation, *Trans. Amer. Math. Soc.* 353 (2) (2001) 795–807.
- [20] B. Yordanov, Q.S. Zhang, Finite time blow up for critical wave equations in high dimensions, *J. Funct. Anal.* 231 (2006) 361–374.
- [21] X. Yu, Generalized Strichartz estimates on perturbed wave equation and applications on Strauss conjecture, *Differential Integral Equations*, in press, arXiv:0905.0038.
- [22] Y. Zhou, Blow up of classical solutions to  $\square u = |u|^{1+\alpha}$  in three space dimensions, *J. Partial Differential Equations* 5 (3) (1992) 21–32.
- [23] Y. Zhou, Blow up of solutions to semilinear wave equations with critical exponent in high dimensions, *Chin. Ann. Math. Ser. B* 28 (2007) 205–212.
- [24] Y. Zhou, W. Han, Blow up of solutions to semilinear wave equations with variable coefficients and boundary, *J. Math. Anal. Appl.* 374 (2011) 585–601.