



Optimal lower bound estimates for the blow-up rate for the Zakharov system in a nonhomogeneous medium [☆]

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ABSTRACT

In this paper we obtain lower bound estimates for the blow-up rate of finite time blow-up solutions to the Cauchy problem for the Zakharov system in a nonhomogeneous medium in two space dimensions. By introducing suitable scale transformations of space and time, and the use of compactness arguments, we derive an optimal lower bound estimate in the energy space $H^2(\mathbf{R}^2) \times L^2(\mathbf{R}^2) \times H^1(\mathbf{R}^2)$ for the blow-up rate for t near the finite blow-up time T . Also we give an application to the virial identity for the Zakharov system under study.

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1. Introduction

In the study of laser plasma physics and soliton problems, various types of Zakharov equations play an important role (see Zakharov [21]). Here, we consider the Zakharov system in a nonhomogeneous medium in \mathbf{R}^2

$$i\Delta\varepsilon_t + \Delta^2\varepsilon = \nabla \cdot (n\nabla\varepsilon), \quad (1.1)$$

$$n_t = \Delta\phi, \quad (1.2)$$

$$\frac{1}{c_0^2}\phi_t = n + |\nabla\varepsilon|^2, \quad (1.3)$$

$$\varepsilon(0, x) = \varepsilon_0(x), \quad n(0, x) = n_0(x), \quad \phi(0, x) = \phi_0(x), \quad (1.4)$$

which describes the interaction of the electrostatic potential with the plasma density in the electrostatic limit (see Zakharov [21] and Zakharov, Mastryukov, Synakh [22]). Here, $c_0 > 0$, Δ is the Laplace operator on \mathbf{R}^2 , $\varepsilon : [0, T) \times \mathbf{R}^2 \rightarrow \mathbf{C}$, $n : [0, T) \times \mathbf{R}^2 \rightarrow \mathbf{R}$, $\phi : [0, T) \times \mathbf{R}^2 \rightarrow \mathbf{R}$. System (1.1)–(1.3) can be derived from the two-fluid Maxwell system and the vector Zakharov equations in a non-dimensional form (see Sulem and Sulem [19] for some rigorous derivations). When $c_0 = +\infty$, the Cauchy problem (1.1)–(1.4) formally reduces to the Cauchy problem of the fourth-order nonlinear Schrödinger equation

$$i\Delta\varepsilon_t + \Delta^2\varepsilon + \nabla \cdot (|\nabla\varepsilon|^2\nabla\varepsilon) = 0, \quad \varepsilon(0, x) = \varepsilon_0. \quad (1.5)$$

When $0 < c_0 < +\infty$, using the ideas of the papers [1,2,6,8,14,15,18], we get the local existence in time of solutions to the Cauchy problem (1.1)–(1.4) in spaces $H^2(\mathbf{R}^2) \times L^2(\mathbf{R}^2) \times H^1(\mathbf{R}^2)$: For all $(\varepsilon_0, n_0, \phi_0) \in H^3(\mathbf{R}^2) \times H^1(\mathbf{R}^2) \times H^2(\mathbf{R}^2)$, there

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is a unique solution $(\varepsilon, n, \phi)(t)$ in $H^3(\mathbf{R}^2) \times H^1(\mathbf{R}^2) \times H^2(\mathbf{R}^2)$ on $[0, T)$ with $T = +\infty$ or $T < +\infty$ and

$$|(\varepsilon, n, \phi)|_{H^2(\mathbf{R}^2) \times L^2(\mathbf{R}^2) \times H^1(\mathbf{R}^2)} \xrightarrow{t \rightarrow T} +\infty.$$

In addition, the following conservations of the longitudinal component and the energy hold

$$\int_{\mathbf{R}^2} |\nabla \varepsilon|^2 dx = \int_{\mathbf{R}^2} |\nabla \varepsilon_0|^2 dx, \quad (1.6)$$

$$\mathcal{H}(t) = \mathcal{H}(\varepsilon(t), n(t), \phi(t)) = \mathcal{H}(\varepsilon_0, n_0, \phi_0) = \mathcal{H}_0, \quad (1.7)$$

where

$$\mathcal{H}(\varepsilon, n, \phi) = \int_{\mathbf{R}^2} |\Delta \varepsilon|^2 dx + \int_{\mathbf{R}^2} n |\nabla \varepsilon|^2 dx + \frac{1}{2c_0^2} \int_{\mathbf{R}^2} |\nabla \phi|^2 dx + \frac{1}{2} \int_{\mathbf{R}^2} n^2 dx. \quad (1.8)$$

In this paper, we are devoted to obtaining an optimal lower bound estimate for the blow-up rate of finite time blow-up solutions to the Cauchy problem (1.1)–(1.4) with $0 < c_0 < +\infty$. Firstly, we recall some results on the study of the classical Zakharov system

$$\begin{aligned} iu_t + \Delta u &= nu, \\ n_t &= -\nabla \cdot v, \\ \frac{1}{c_0^2} v_t &= -\nabla n - \nabla |u|^2, \\ (u, n, v)(0, x) &= (u_0(x), n_0(x), v_0(x)). \end{aligned} \quad (1.9)$$

Local existence in time of solutions (u, n, v) for (1.9) has been studied by many authors (see Added and Added [1,2], Bourgain [3,4], Ginibre, Tsutsumi and Velo [6], Landman, Papanicolaou, Sulem et al. [10,16], Ozawa and Tsutsumi [15]). In addition, Merle in [11] proved a blow-up result for solutions with negative energy to the Cauchy problem (1.9) in two and three space dimensions. Merle in [12] obtained the lower bound estimates for the blow-up rate of finite time blow-up solution to the Cauchy problem (1.9) in two space dimensions. Gnanou and Merle in [7,8] studied the existence of self-similar blow-up solution, the concentration properties of blow-up solution and the instability result for (1.9) in two-dimensional space. Recently, Colin and Métivier in [5] proved the strong instability of Hadamard's type for a non-fully dispersive Zakharov system arising in the study of laser plasma interaction. Texier in [20] studied the validity of the Zakharov system which describes Langmuir turbulence. But for the study of the Zakharov system in a nonhomogeneous medium, few results are known except that Guo in [9] studied the existence and uniqueness of global smooth solutions for the Cauchy problem (1.1)–(1.4) in two dimensions by using the so-called continuity method and delicate a priori estimates. In addition, in the critical case ($N = 2$), from Gnanou and Merle [7,8], Nawa [13], there exists a family of blow-up solutions (ε, n, ϕ) of the Cauchy problem (1.1)–(1.4) satisfying that for $\omega > \omega_0$,

$$\nabla \varepsilon(t, x) = \frac{\omega}{T-t} e^{i[|x|^2/4(t-T) - \omega^2/(t-T)]} \mathbf{P}\left(\frac{x\omega}{T-t}\right), \quad (1.10)$$

$$n(t, x) = \left(\frac{\omega}{T-t}\right)^2 N\left(\frac{x\omega}{T-t}\right), \quad (1.11)$$

where $T > 0$ and $\mathbf{P}(x) = \mathbf{P}(|x|)$ is a vector-valued function of x , $N(x) = N(|x|)$,

$$\Delta P - P = NP, \quad (1.12)$$

$$\frac{1}{(c_0\omega)^2} (r^2 \partial_r^2 N + 6r \partial_r N + 6N) - \Delta N = \Delta P^2, \quad (1.13)$$

with $r = |x|$, $\Delta W = \partial_r^2 W + \frac{1}{r} \partial_r W$.

In this paper, we are interested in understanding the problem that how $|(\varepsilon, n, \phi)(t)|_{H^2(\mathbf{R}^2) \times L^2(\mathbf{R}^2) \times H^1(\mathbf{R}^2)}$ goes to infinity as $t \rightarrow T$ ($T < +\infty$) and establishing an optimal lower bound estimate for the blow-up rate of solutions to the Cauchy problem (1.1)–(1.4) in two space dimensions.

The main difficulty in the analysis of the Zakharov system (1.1)–(1.3) is the presence of the fourth-order derivative term for ε . For $c_0 = +\infty$, the Cauchy problem (1.1)–(1.4) reduces to the Cauchy problem of the fourth-order nonlinear Schrödinger equation (1.5). Unlike the second-order nonlinear Schrödinger equation, in this case, in order to establish a lower bound estimate, we must construct a different scaling transformation which is key to our arguments. For $0 < c_0 < +\infty$, we must also establish new scaling transformations for ε, n, ϕ to obtain a lower bound estimate. Especially, in order to establish an optimal lower bound estimate, we need to discuss non-vanishing properties and compactness properties of

the corresponding functions. These arguments are more difficult than the corresponding discussions of the second-order Zakharov system (1.9) to some extent. Moreover, to our knowledge, there are few results on the Zakharov system in a nonhomogeneous medium, especially for the optimal lower bound estimate for the blow-up rate.

At the end of this section, we give an outline of this paper. In Section 2, we give some preliminaries. In Section 3, we give an optimal lower bound estimate for blow-up rate. In Section 4, we give an application to the virial identity corresponding to the Cauchy problem (1.1)–(1.4).

2. Preliminaries

First we consider a finite time blow-up solution $(\varepsilon, n, \phi)(t, x)$ to the Cauchy problem for the Zakharov system in a nonhomogeneous medium (1.1)–(1.4). Here, in order to state our arguments more clearly, we rewrite (1.1)–(1.3) as

$$(Z1) \quad \begin{cases} i\Delta\varepsilon_t + \Delta^2\varepsilon = \nabla \cdot (n\nabla\varepsilon), & x \in \mathbf{R}^2, t > 0, \\ n_t = \Delta\phi, & x \in \mathbf{R}^2, t > 0, \end{cases} \quad (2.1)$$

$$(2.2)$$

$$\begin{cases} \frac{1}{c_0^2}\phi_t = n + |\nabla\varepsilon|^2, & x \in \mathbf{R}^2, t > 0, \end{cases} \quad (2.3)$$

and let T denote the blow-up time. By a direct calculation, we get

Proposition 2.1. *If $(\varepsilon, n, \phi)(t, x)$ is a solution of the system (2.1)–(2.3), then for any $\lambda > 0$ and $t < T$,*

$$(i) \quad (\varepsilon_1, n_1, \phi_1)(s_1, x) = \left(\varepsilon\left(t + \frac{s_1}{\lambda^2}, \frac{x}{\lambda}\right), \frac{1}{\lambda^2}n\left(t + \frac{s_1}{\lambda^2}, \frac{x}{\lambda}\right), \frac{1}{\lambda}\phi\left(t + \frac{s_1}{\lambda^2}, \frac{x}{\lambda}\right) \right) \quad (2.4)$$

satisfies

$$(Z2)_\lambda \quad \begin{cases} i\Delta\varepsilon_{1s_1} = -\Delta^2\varepsilon_1 + \nabla \cdot (n_1\nabla\varepsilon_1), \\ n_{1s_1} = \frac{1}{\lambda}\Delta\phi_1, \\ \frac{1}{c_0^2}\phi_{1s_1} = \frac{1}{\lambda}(n_1 + |\nabla\varepsilon_1|^2) \end{cases}$$

for $s_1 \in [0, \lambda^2(T - t)]$;

$$(ii) \quad (\varepsilon_2, n_2, \phi_2)(s_2, x) = \left(\varepsilon\left(t + \frac{s_2}{\lambda}, \frac{x}{\lambda}\right), \frac{1}{\lambda^2}n\left(t + \frac{s_2}{\lambda}, \frac{x}{\lambda}\right), \frac{1}{\lambda}\phi\left(t + \frac{s_2}{\lambda}, \frac{x}{\lambda}\right) \right) \quad (2.5)$$

satisfies

$$(Z3)_\lambda \quad \begin{cases} i\Delta\varepsilon_{2s_2} = -\lambda\Delta^2\varepsilon_2 + \lambda\nabla \cdot (n_2\nabla\varepsilon_2), \\ n_{2s_2} = \Delta\phi_2, \\ \frac{1}{c_0^2}\phi_{2s_2} = n_2 + |\nabla\varepsilon_2|^2 \end{cases}$$

for $s_2 \in [0, \lambda(T - t)]$.

In order to obtain some normalization properties, we choose λ such that

$$|(\varepsilon_1, n_1, \phi_1)(0, x)|_{H^2(\mathbf{R}^2) \times L^2(\mathbf{R}^2) \times H^1(\mathbf{R}^2)}^2 = \int_{\mathbf{R}^2} |\Delta\varepsilon_1(0, x)|^2 dx + \frac{1}{2} \int_{\mathbf{R}^2} n_1^2(0, x) dx + \frac{1}{2c_0^2} \int_{\mathbf{R}^2} |\nabla\phi_1(0, x)|^2 dx = 1 \quad (2.6)$$

and

$$|(\varepsilon_2, n_2, \phi_2)(0, x)|_{H^2(\mathbf{R}^2) \times L^2(\mathbf{R}^2) \times H^1(\mathbf{R}^2)}^2 = \int_{\mathbf{R}^2} |\Delta\varepsilon_2(0, x)|^2 dx + \frac{1}{2} \int_{\mathbf{R}^2} n_2^2(0, x) dx + \frac{1}{2c_0^2} \int_{\mathbf{R}^2} |\nabla\phi_2(0, x)|^2 dx = 1. \quad (2.7)$$

Since

$$\begin{aligned} \int_{\mathbf{R}^2} |\Delta\varepsilon_1(0, x)|^2 dx &= \int_{\mathbf{R}^2} \left| \Delta\left(\varepsilon\left(t, \frac{x}{\lambda}\right)\right) \right|^2 dx = \frac{1}{\lambda^2} \int_{\mathbf{R}^2} |\Delta\varepsilon(t, x)|^2 dx, \\ \int_{\mathbf{R}^2} n_1^2(0, x) dx &= \int_{\mathbf{R}^2} \frac{1}{\lambda^4} n^2\left(t, \frac{x}{\lambda}\right) dx = \frac{1}{\lambda^2} \int_{\mathbf{R}^2} n^2(t, x) dx, \end{aligned}$$

and

$$\int_{\mathbf{R}^2} |\nabla \phi_1(0, x)|^2 dx = \int_{\mathbf{R}^2} \left| \nabla \left(\frac{1}{\lambda} \phi \left(t, \frac{x}{\lambda} \right) \right) \right|^2 dx = \frac{1}{\lambda^2} \int_{\mathbf{R}^2} |\nabla \phi(t, x)|^2 dx,$$

from (2.6) and (2.7), if $(\varepsilon, n, \phi)(t)$ is a finite time blow-up solution to the Cauchy problem for (Z1), then we have

$$\lambda^2(t) = \int_{\mathbf{R}^2} |\Delta \varepsilon(t, x)|^2 dx + \frac{1}{2} \int_{\mathbf{R}^2} n^2(t, x) dx + \frac{1}{2c_0^2} \int_{\mathbf{R}^2} |\nabla \phi(t, x)|^2 dx, \quad (2.8)$$

and

$$\lambda(t) \rightarrow +\infty \quad \text{as } t \rightarrow T. \quad (2.9)$$

3. Optimal lower bound estimate for the blow-up rate

In this section, we give an optimal lower bound estimate for the blow-up rate of the solutions for the Cauchy problem for the Zakharov system (1.1)–(1.4) in order to understand how $|(\varepsilon, n, \phi)(t, x)|_{H^2(\mathbf{R}^2) \times L^2(\mathbf{R}^2) \times H^1(\mathbf{R}^2)}$ goes to infinity as $t \rightarrow T$.

Theorem 3.1. *Let $(\varepsilon, n, \phi)(t, x)$ be a finite time blow-up solution to the Cauchy problem (1.1)–(1.4) and T be the blow-up time.*

(i) *For some $c_1 > 0$ and $c_2 > 0$ (depending only on $|\nabla \varepsilon_0|_{L^2(\mathbf{R}^2)}$) and for t near T ,*

$$\begin{aligned} |(\varepsilon, n, \phi)(t, x)|_{H^2(\mathbf{R}^2) \times L^2(\mathbf{R}^2) \times H^1(\mathbf{R}^2)} &\geq \frac{c_1}{T-t}, \\ |\Delta \varepsilon(t, x)|_{L^2(\mathbf{R}^2)} &\geq \frac{c_2}{T-t}, \quad |n(t, x)|_{L^2(\mathbf{R}^2)} \geq \frac{c_2}{T-t}. \end{aligned}$$

(ii) *More precisely, for t near T ,*

$$\begin{aligned} |\Delta \varepsilon(t, x)|_{L^2(\mathbf{R}^2)} &\geq \frac{c_1}{(|\nabla \varepsilon_0|_{L^2(\mathbf{R}^2)}^2 - |Q|_{L^2(\mathbf{R}^2)}^2)^{\frac{1}{2}}} \frac{1}{T-t}, \\ |n(t, x)|_{L^2(\mathbf{R}^2)} &\geq \frac{c_1}{(|\nabla \varepsilon_0|_{L^2(\mathbf{R}^2)}^2 - |Q|_{L^2(\mathbf{R}^2)}^2)^{\frac{1}{2}}} \frac{1}{T-t}, \end{aligned}$$

where Q is the unique radial positive solution of

$$\Delta u + |u|^2 u = u. \quad (E)$$

Remark 3.1. We assume in Theorem 3.1 that t is near T . For a given solution, we can obtain a lower bound for $t \in [0, T)$ with constants c_1 and c_2 by a compactness discussion in time, where constants c_1 and c_2 depend on the solution and $|\nabla \varepsilon_0|_{L^2(\mathbf{R}^2)}$. Furthermore, if we consider the explicit blow-up solutions $(\varepsilon, n, \phi)(t)$ satisfying (1.10) and (1.11) as well as utilize the similar argument to that in [12], then we can prove that (i) and (ii) are both optimal.

In order to prove Theorem 3.1, we first give some propositions and lemmas.

Lemma 3.1. *For all $t \in [0, T)$, let*

$$(\tilde{\varepsilon}, \tilde{n}, \tilde{\phi})(s, x) = \left(\varepsilon \left(t + \frac{s}{\lambda(t)}, \frac{x}{\lambda(t)} \right), \frac{1}{\lambda^2(t)} n \left(t + \frac{s}{\lambda(t)}, \frac{x}{\lambda(t)} \right), \frac{1}{\lambda(t)} \phi \left(t + \frac{s}{\lambda(t)}, \frac{x}{\lambda(t)} \right) \right) \quad (3.1)$$

with

$$\lambda^2(t) = \int_{\mathbf{R}^2} |\Delta \varepsilon(t, x)|^2 dx + \frac{1}{2} \int_{\mathbf{R}^2} n^2(t, x) dx + \frac{1}{2c_0^2} \int_{\mathbf{R}^2} |\nabla \phi(t, x)|^2 dx. \quad (3.2)$$

Then $(\tilde{\varepsilon}, \tilde{n}, \tilde{\phi})(s, x)$ satisfies the following properties:

(1) $(\tilde{\varepsilon}, \tilde{n}, \tilde{\phi})(s, x)$ is defined for $s \in [0, \lambda(t)(T-t))$ and satisfies $(Z3)_\lambda$ on $[0, \lambda(t)(T-t))$.

$$(2) \quad \int_{\mathbb{R}^2} |\Delta \tilde{\varepsilon}(0, x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} \tilde{n}^2(0, x) dx + \frac{1}{2c_0^2} \int_{\mathbb{R}^2} |\nabla \tilde{\phi}(0, x)|^2 dx = 1 \quad (3.3)$$

and

$$\int_{\mathbb{R}^2} |\nabla \tilde{\varepsilon}(0, x)|^2 dx = \int_{\mathbb{R}^2} |\nabla \varepsilon(0, x)|^2 dx = \int_{\mathbb{R}^2} |\nabla \varepsilon_0(x)|^2 dx.$$

$$(3) \quad \lim_{s \rightarrow \lambda(t)(T-t)} \left(\int_{\mathbb{R}^2} |\Delta \tilde{\varepsilon}(s, x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} \tilde{n}^2(s, x) dx + \frac{1}{2c_0^2} \int_{\mathbb{R}^2} |\nabla \tilde{\phi}(s, x)|^2 dx \right) = +\infty. \quad (3.4)$$

$$(4) \quad \mathcal{H}(\tilde{\varepsilon}(s, x), \tilde{n}(s, x), \tilde{\phi}(s, x)) = \frac{1}{\lambda^2(t)} \mathcal{H}(t) = \frac{1}{\lambda^2(t)} \mathcal{H}_0. \quad (3.5)$$

Proof. By a direct calculation, we obtain the results of Lemma 3.1. \square

Lemma 3.2 (Sobolev's best constant estimate). (See [9,17].) Let $\varepsilon \in H^2(\mathbb{R}^2)$. Then

$$\frac{1}{2} \int_{\mathbb{R}^2} |\nabla \varepsilon|^4 dx \leq \left(\frac{\int_{\mathbb{R}^2} |\nabla \varepsilon|^2 dx}{\int_{\mathbb{R}^2} Q^2 dx} \right) \int_{\mathbb{R}^2} |\Delta \varepsilon|^2 dx,$$

where Q is the unique radial positive solution of

$$\Delta u + |u|^2 u = u. \quad (E)$$

Using the Hamiltonian structure of the Cauchy problem (1.1)–(1.4), (3.3) and (3.5), we obtain some rough estimates (Sobolev estimates) on $\tilde{\varepsilon}(0, x)$, $\tilde{n}(0, x)$ and $\tilde{\phi}(0, x)$.

Proposition 3.1. Let $\lambda(t)$ be defined by (3.2). Then there exist $\delta_1 > 0$, $c_1 > 0$ and $c_2 > 0$ such that for $t \in [T - \delta_1, T)$,

$$(1) \quad 0 < c_1 \leq \|\Delta \tilde{\varepsilon}(0, x)\|_{L^2(\mathbb{R}^2)} \leq c_2,$$

$$(2) \quad 0 < c_1 \leq \|\tilde{n}(0, x)\|_{L^2(\mathbb{R}^2)} \leq c_2, \quad 0 \leq \|\nabla \tilde{\phi}(0, x)\|_{L^2(\mathbb{R}^2)} \leq c_2,$$

where c_1 and c_2 depend only on $\|\nabla \varepsilon_0\|_{L^2(\mathbb{R}^2)}$.

Proof. (3.3) implies that

$$\|\Delta \tilde{\varepsilon}(0, x)\|_{L^2(\mathbb{R}^2)} \leq 1, \quad \|\tilde{n}(0, x)\|_{L^2(\mathbb{R}^2)} \leq \sqrt{2}, \quad \|\nabla \tilde{\phi}(0, x)\|_{L^2(\mathbb{R}^2)} \leq \sqrt{2}c_0. \quad (3.6)$$

Since $\lim_{t \rightarrow T} \lambda(t) = +\infty$, by (3.2) and (3.4), there is a $\delta_1 > 0$ such that for any $t \in [T - \delta_1, T)$,

$$\left| \frac{\mathcal{H}_0}{\lambda^2(t)} \right| \leq \frac{1}{\theta}, \quad (3.7)$$

where $\frac{1}{\theta} < 1$ is a constant. From (3.5) and (3.7) it follows that

$$\mathcal{H}(\tilde{\varepsilon}(0, x), \tilde{n}(0, x), \tilde{\phi}(0, x)) = \frac{\mathcal{H}(\varepsilon(t, x), n(t, x), \phi(t, x))}{\lambda^2(t)} \leq \frac{1}{\theta},$$

which is equivalent to

$$\int_{\mathbb{R}^2} |\Delta \tilde{\varepsilon}(0, x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} \tilde{n}^2(0, x) dx + \frac{1}{2c_0^2} \int_{\mathbb{R}^2} |\nabla \tilde{\phi}(0, x)|^2 dx \leq \frac{1}{\theta} - \int_{\mathbb{R}^2} \tilde{n}(0, x) |\nabla \tilde{\varepsilon}(0, x)|^2 dx. \quad (3.8)$$

Therefore, by Lemma 3.2, (3.3) and (3.8) we get

$$1 - \frac{1}{\theta} \leq \sqrt{2} \left(\frac{\int_{\mathbb{R}^2} |\nabla \tilde{\varepsilon}(0, x)|^2 dx}{\int_{\mathbb{R}^2} Q^2 dx} \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |\Delta \tilde{\varepsilon}(0, x)|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} \tilde{n}^2(0, x) dx \right)^{\frac{1}{2}},$$

which together with (3.3) yields that

$$1 - \frac{1}{\theta} \leq \sqrt{2} \frac{(\int_{\mathbb{R}^2} |\nabla \tilde{\varepsilon}(0, x)|^2 dx)^{\frac{1}{2}}}{(\int_{\mathbb{R}^2} Q^2 dx)^{\frac{1}{2}}} \left(\int_{\mathbb{R}^2} \tilde{n}^2(0, x) dx \right)^{\frac{1}{2}},$$

so there exists a c_1 such that $|\tilde{n}(0, x)|_{L^2(\mathbb{R}^2)} \geq c_1$. Similarly, we get $|\Delta \tilde{\varepsilon}(0, x)|_{L^2(\mathbb{R}^2)} \geq c_1$. Moreover, from (3.6) and taking $c_2 = \max\{\sqrt{2}, \sqrt{2}c_0\}$, we complete the proof of Proposition 3.1. \square

The following corollary follows from Proposition 3.1.

Corollary 3.1. *There are $\delta_1 > 0$, $c_3 > 0$ (depending only on $|\nabla \varepsilon_0|_{L^2(\mathbb{R}^2)}$) such that for any $t \in [T - \delta_1, T)$,*

- (1) $c_3 |n(t, x)|_{L^2(\mathbb{R}^2)} \leq |\Delta \varepsilon(t, x)|_{L^2(\mathbb{R}^2)} \leq \frac{1}{c_3} |n(t, x)|_{L^2(\mathbb{R}^2)},$
- (2) $|\nabla \phi(t, x)|_{L^2(\mathbb{R}^2)} \leq \frac{1}{c_3} |n(t, x)|_{L^2(\mathbb{R}^2)},$
- (3) $c_3 |n(t, x)|_{L^2(\mathbb{R}^2)} \leq |(\varepsilon, n, \phi)(t, x)|_{H^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)} \leq \frac{1}{c_3} |n(t, x)|_{L^2(\mathbb{R}^2)},$

where $|(\varepsilon, n, \phi)(t, x)|_{H^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)}$ is defined by

$$|(\varepsilon, n, \phi)(t, x)|_{H^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} |\Delta \varepsilon(t, x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} n^2(t, x) dx + \frac{1}{2c_0^2} \int_{\mathbb{R}^2} |\nabla \phi(t, x)|^2 dx. \quad (3.9a)$$

Proof. Since

$$\begin{aligned} |\Delta \tilde{\varepsilon}(0, x)|_{L^2(\mathbb{R}^2)} &= \frac{1}{\lambda(t)} |\Delta \varepsilon(t, x)|_{L^2(\mathbb{R}^2)}, & |\tilde{n}(0, x)|_{L^2(\mathbb{R}^2)} &= \frac{1}{\lambda(t)} |n(t, x)|_{L^2(\mathbb{R}^2)}, \\ |\nabla \tilde{\phi}(0, x)|_{L^2(\mathbb{R}^2)} &= \frac{1}{\lambda(t)} |\nabla \phi(t, x)|_{L^2(\mathbb{R}^2)}, \end{aligned}$$

Proposition 3.1 and (3.9a) yield that

$$\begin{aligned} c_1 &\leq \frac{|\tilde{n}(0, x)|_{L^2(\mathbb{R}^2)}}{|n(t, x)|_{L^2(\mathbb{R}^2)}} |\Delta \varepsilon(t, x)|_{L^2(\mathbb{R}^2)} \leq c_2 \\ \Rightarrow \quad \frac{c_1}{c_2} |n(t, x)|_{L^2(\mathbb{R}^2)} &\leq \frac{|\tilde{n}(0, x)|_{L^2(\mathbb{R}^2)}}{c_2} \cdot |\Delta \varepsilon(t, x)|_{L^2(\mathbb{R}^2)} \leq |\Delta \varepsilon(t, x)|_{L^2(\mathbb{R}^2)}, \end{aligned} \quad (3.9b)$$

$$|\Delta \varepsilon(t, x)|_{L^2(\mathbb{R}^2)} \leq c_2 \frac{|n(t, x)|_{L^2(\mathbb{R}^2)}}{|\tilde{n}(0, x)|_{L^2(\mathbb{R}^2)}} \leq \frac{c_2}{c_1} |n(t, x)|_{L^2(\mathbb{R}^2)}, \quad (3.9c)$$

$$\begin{aligned} 0 &\leq \frac{|\tilde{n}(0, x)|_{L^2(\mathbb{R}^2)}}{|n(t, x)|_{L^2(\mathbb{R}^2)}} |\nabla \phi(t, x)|_{L^2(\mathbb{R}^2)} \leq c_2 \\ \Rightarrow \quad 0 &\leq |\nabla \phi(t, x)|_{L^2(\mathbb{R}^2)} \leq \frac{c_2}{|\tilde{n}(0, x)|_{L^2(\mathbb{R}^2)}} |n(t, x)|_{L^2(\mathbb{R}^2)} \leq \frac{c_2}{c_1} |n(t, x)|_{L^2(\mathbb{R}^2)}. \end{aligned} \quad (3.10)$$

From (3.9b), (3.9c) and (3.10), we get

$$|(\varepsilon, n, \phi)(t, x)|_{H^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)}^2 \geq \frac{c_1^2}{c_2^2} \frac{2c_0^2}{3c_0^2 + 1} |n(t, x)|_{L^2(\mathbb{R}^2)}^2. \quad (3.11)$$

On the other hand,

$$|(\varepsilon, n, \phi)(t, x)|_{H^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)}^2 \leq \frac{3c_0^2 + 1}{2c_0^2} \frac{c_2^2}{c_1^2} |n(t, x)|_{L^2(\mathbb{R}^2)}^2. \quad (3.12)$$

Let $c_3^2 = \frac{c_1^2}{c_2^2} \frac{2c_0^2}{3c_0^2 + 1}$, by (3.9b)–(3.12) we establish the proof of Corollary 3.1. \square

For $(\varepsilon, n) \in H^2(\mathbf{R}^2) \times L^2(\mathbf{R}^2)$, we define

$$E(\varepsilon) = \int_{\mathbf{R}^2} |\Delta \varepsilon|^2 dx - \frac{1}{2} \int_{\mathbf{R}^2} |\nabla \varepsilon|^4 dx, \quad (3.13)$$

and

$$\mathcal{H}_1(\varepsilon, n) = E(\varepsilon) + \frac{1}{2} \int_{\mathbf{R}^2} (n + |\nabla \varepsilon|^2)^2 dx. \quad (3.14)$$

Let

$$\hat{\varepsilon}_m(x) = \tilde{\varepsilon}_m\left(\frac{x}{\lambda_m}\right), \quad \hat{n}_m(x) = \frac{1}{\lambda_m^2} \tilde{n}_m\left(\frac{x}{\lambda_m}\right), \quad \hat{\phi}_m(x) = \frac{1}{\lambda_m} \tilde{\phi}_m\left(\frac{x}{\lambda_m}\right), \quad (3.15)$$

where

$$\lambda_m^2 = \int_{\mathbf{R}^2} |\Delta \varepsilon(t_m, x)|^2 dx + \frac{1}{2} \int_{\mathbf{R}^2} n^2(t_m, x) dx + \frac{1}{2c_0^2} \int_{\mathbf{R}^2} |\nabla \phi(t_m, x)|^2 dx,$$

$$\tilde{\varepsilon}_m(x) = \varepsilon\left(t_m, \frac{x}{\lambda_m}\right), \quad \tilde{n}_m(x) = \frac{1}{\lambda_m^2} n\left(t_m, \frac{x}{\lambda_m}\right), \quad \tilde{\phi}_m(x) = \frac{1}{\lambda_m} \phi\left(t_m, \frac{x}{\lambda_m}\right).$$

By a direct calculation, we get

$$\int_{\mathbf{R}^2} |\nabla \hat{\varepsilon}_m(x)|^2 dx = \int_{\mathbf{R}^2} |\nabla \varepsilon_0|^2 dx,$$

$$\int_{\mathbf{R}^2} |\Delta \hat{\varepsilon}_m(x)|^2 dx + \frac{1}{2} \int_{\mathbf{R}^2} \hat{n}_m^2(x) dx + \frac{1}{2c_0^2} \int_{\mathbf{R}^2} |\nabla \hat{\phi}_m(x)|^2 dx = 1.$$

Thus, the following lemma is true.

Lemma 3.3. Assume that there is a sequence $(\hat{\varepsilon}_m(x), \hat{n}_m(x)) \in H^2(\mathbf{R}^2) \times L^2(\mathbf{R}^2)$ such that as $m \rightarrow +\infty$,

$$\int_{\mathbf{R}^2} |\nabla \hat{\varepsilon}_m|^2 dx \rightarrow c_1 > 0, \quad \int_{\mathbf{R}^2} \hat{n}_m |\nabla \hat{\varepsilon}_m|^2 dx \rightarrow -c_3 < 0,$$

$$\int_{\mathbf{R}^2} |\Delta \hat{\varepsilon}_m|^2 dx + \frac{1}{2} \int_{\mathbf{R}^2} |\hat{n}_m|^2 dx \rightarrow c_2 > 0.$$

Then there exist a constant $c_4 = c_4(c_1, c_2, c_3)$ and a sequence x_m such that

$$\int_{|x-x_m|<1} |\hat{n}_m| dx > c_4.$$

Proof. We prove this lemma by three steps.

Step 1. There exists some x_m such that for large m ,

$$\int_{C_m} (-\hat{n}_m |\nabla \hat{\varepsilon}_m|^2) dx \geq d \left(\int_{C_m} (|\Delta \hat{\varepsilon}_m|^2 + |\nabla \hat{\varepsilon}_m|^2 + \frac{1}{2} \hat{n}_m^2) dx \right), \quad (\text{A-1})$$

where C_m is the square of center x_m and $d = \frac{c_3}{2(c_1+c_2)} > 0$.

Proof. We prove this conclusion by contradiction. Assume that for any x_m ,

$$\int_{\mathbf{R}^2} (-\hat{n}_m |\nabla \hat{\varepsilon}_m|^2) dx < d \left(\int_{\mathbf{R}^2} (|\Delta \hat{\varepsilon}_m|^2 + |\nabla \hat{\varepsilon}_m|^2 + \frac{1}{2} \hat{n}_m^2) dx \right).$$

As $m \rightarrow +\infty$, we obtain

$$c_3 \leq d(c_1 + c_2) = \frac{c_3}{2},$$

which is a contradiction. So (A-1) is true. \square

Step 2. There exists $h > 0$ such that

$$\int_{C_m} (-\hat{n}_m |\nabla \hat{\varepsilon}_m|^2) dx \geq h \quad \text{and} \quad \int_{C_m} |\nabla \hat{\varepsilon}_m|^4 dx \geq h. \quad (\text{A-2})$$

Proof. From Sobolev embedding on C_m , it follows that there exists $s_0 > 0$ independent of m such that

$$\int_{C_m} (|\Delta \hat{\varepsilon}_m|^2 + |\nabla \hat{\varepsilon}_m|^2) dx \geq s_0 \left(\int_{C_m} |\nabla \hat{\varepsilon}_m|^4 dx \right)^{\frac{1}{2}},$$

which together with (A-1) yields that

$$ds_0 \left(\int_{C_m} |\nabla \hat{\varepsilon}_m|^4 dx \right)^{\frac{1}{2}} + \frac{d}{2} \int_{C_m} \hat{n}_m^2 dx \leq \left(\int_{C_m} \hat{n}_m^2 dx \right)^{\frac{1}{2}} \left(\int_{C_m} |\nabla \hat{\varepsilon}_m|^4 dx \right)^{\frac{1}{2}}. \quad (\text{A-3})$$

Since

$$ds_0 \left(\int_{C_m} |\nabla \hat{\varepsilon}_m|^4 dx \right)^{\frac{1}{2}} + \frac{d}{2} \int_{C_m} \hat{n}_m^2 dx \geq 2\sqrt{ds_0} \sqrt{\frac{d}{2}} \left(\int_{C_m} |\nabla \hat{\varepsilon}_m|^4 dx \right)^{\frac{1}{4}} \cdot \left(\int_{C_m} \hat{n}_m^2 dx \right)^{\frac{1}{2}},$$

by (A-3) we get

$$\left(\int_{C_m} |\nabla \hat{\varepsilon}_m|^4 dx \right)^{\frac{1}{4}} \geq \sqrt{2s_0} d \quad \text{and} \quad \int_{C_m} (-\hat{n}_m |\nabla \hat{\varepsilon}_m|^2) dx \geq 2s_0^2 d^3 > 0. \quad \square$$

Let $h = \min\{2s_0^2 d^3, 4s_0^2 d^4\}$, we get (A-2).

Step 3. We now conclude Lemma 3.3 by contradiction according to Step 1 and Step 2.

Assume that there exists a subsequence \hat{n}_m such that

$$\int_{C_m} |\hat{n}_m| dx \rightarrow 0 \quad \text{as } m \rightarrow +\infty. \quad (\text{A-4})$$

We can also assume that as $m \rightarrow +\infty$,

$$\hat{n}_m(x_m + \cdot) \rightharpoonup n^* \quad \text{in } L^2(\mathbf{R}^2), \quad \hat{\varepsilon}_m(x_m + \cdot) \rightharpoonup \varepsilon^* \quad \text{in } H^2(\mathbf{R}^2).$$

Then

$$\nabla \hat{\varepsilon}_m(x_m + \cdot) \rightarrow \nabla \varepsilon^* \quad \text{in } L_{\text{loc}}^4(\mathbf{R}^2), \quad \nabla \hat{\varepsilon}_m(x_m + \cdot) \rightarrow \nabla \varepsilon^* \quad \text{in } L_{\text{loc}}^2(\mathbf{R}^2). \quad (\text{A-5})$$

From (A-4) and (A-5), one has that as $m \rightarrow +\infty$, $\hat{n}_m(x_m + \cdot) \rightarrow 0$ in $L^2(C_0)$ and

$$\int_{C_m} \hat{n}_m |\nabla \hat{\varepsilon}_m|^2 dx = \int_{C_0} \hat{n}_m(x_m + x) |\nabla \hat{\varepsilon}_m(x_m + x)|^2 dx \rightarrow 0,$$

which is contradictory to (A-2). So far, the proof of Lemma 3.3 is completed. \square

Lemma 3.4. There exists $\beta_1 = \beta_1(|\nabla \varepsilon_0|_{L^2(\mathbf{R}^2)}) > 0$ such that the following property is true: Let $\tilde{\varepsilon}_m(x) \in H^2(\mathbf{R}^2)$, $\tilde{n}_m(x) \in L^2(\mathbf{R}^2)$, $\tilde{\phi}_m(x) \in H^1(\mathbf{R}^2)$ be sequences such that $|\nabla \tilde{\varepsilon}_m(x)|_{L^2(\mathbf{R}^2)}^2 = |\nabla \varepsilon_0(x)|_{L^2(\mathbf{R}^2)}^2$. Furthermore, assume that there exists $\delta_0 > 0$ such that $\forall R_0 > 0$, $\sup_y \int_{|y-x| < R_0} |\nabla \tilde{\varepsilon}_m(x)|^2 dx \leq |Q|_{L^2(\mathbf{R}^2)}^2 - \delta_0$ or $\sup_y \int_{|y-x| < R_0} |\tilde{n}_m| dx \leq \beta_1 - \delta_0$, where Q is the unique radial positive solution of

$$\Delta u + |u|^2 u = u. \quad (\text{E})$$

Then there exist constants $c_1 \geq 0$ and $c_2 > 0$ such that

$$\forall m, \quad -c_1 + c_2 \left(\int_{\mathbf{R}^2} |\Delta \tilde{\varepsilon}_m|^2 dx + \int_{\mathbf{R}^2} \tilde{n}_m^2 dx + \int_{\mathbf{R}^2} |\nabla \tilde{\phi}_m|^2 dx \right) \leq \mathcal{H}(\tilde{\varepsilon}_m, \tilde{n}_m, \tilde{\phi}_m).$$

Proof. We only need to substitute ∇u_k for $\Delta \tilde{\varepsilon}_m$ and v_k for $\nabla \tilde{\phi}_m$ in the proof of Proposition A.3 in [8]. \square

Now, let $(\varepsilon(t, x), n(t, x), \phi(t, x))$ and $(\tilde{\varepsilon}(0), \tilde{n}(0), \tilde{\phi}(0)) = (\tilde{\varepsilon}(0, x), \tilde{n}(0, x), \tilde{\phi}(0, x))$ be blow-up solutions for $t \in [0, T)$. Thus, we have the following non-vanishing properties for $\tilde{\varepsilon}(t, x), \tilde{n}(t, x)$ by (3.3) and (3.5).

Proposition 3.2 (Non-vanishing properties for $\tilde{\varepsilon}(0), \tilde{n}(0)$ as $t \rightarrow T$). Let $\lambda(t)$ be defined by (3.2). Then the following two conclusions are true:

(1) There are $R_1 > 0$ and $\beta_2 > 0$ (depending only on $|\nabla \varepsilon_0|_{L^2(\mathbf{R}^2)}$) such that for a sequence $x(t) \in \mathbf{R}^2$,

$$\liminf_{t \rightarrow T} |\nabla \tilde{\varepsilon}(0, x)|_{L^2(|x-x(t)| \leq R_1)} \geq \beta_2 > 0, \quad (3.16)$$

$$\liminf_{t \rightarrow T} |\tilde{n}(0, x)|_{L^2(|x-x(t)| \leq R_1)} \geq \beta_2 > 0. \quad (3.17)$$

(2) Let (ε_m, n_m) be a sequence satisfying the following inequalities:

$$|\nabla \varepsilon_m(0)|_{L^2(\mathbf{R}^2)} \leq |\nabla \varepsilon_0|_{L^2(\mathbf{R}^2)}, \quad (3.18)$$

$$c_1 \leq \int_{\mathbf{R}^2} |\Delta \varepsilon_m(0)|^2 dx \leq c_2, \quad (3.19)$$

$$c_1 \leq \int_{\mathbf{R}^2} |n_m(0)|^2 dx \leq c_2, \quad (3.20)$$

$$\limsup_{t \rightarrow T} \mathcal{H}(\varepsilon_m(0), n_m(0), 0) \leq 0. \quad (3.21)$$

Then there exist constants β_2 and R_1 depending only on $|\nabla \varepsilon_0|_{L^2(\mathbf{R}^2)}, c_1$ and c_2 such that for a sequence x_m ,

$$\lim_{m \rightarrow +\infty} |\nabla \varepsilon_m|_{L^2(|x-x_m| \leq R_1)} \geq \beta_2 > 0, \quad \lim_{m \rightarrow +\infty} |n_m|_{L^2(|x-x_m| \leq R_1)} \geq \beta_2 > 0.$$

Proof. (1) From (3.1), (3.2) and (3.5) we get $\mathcal{H}(\tilde{\varepsilon}(s, x), \tilde{n}(s, x), \tilde{\phi}(s, x)) = \frac{1}{\lambda^2(t)} \mathcal{H}_0$, where

$$\begin{aligned} \tilde{\varepsilon}(s, x) &= \varepsilon\left(t + \frac{s}{\lambda(t)}, \frac{x}{\lambda(t)}\right), \quad \tilde{n}(s, x) = \frac{1}{\lambda^2(t)} n\left(t + \frac{s}{\lambda(t)}, \frac{x}{\lambda(t)}\right), \\ \tilde{\phi}(s, x) &= \frac{1}{\lambda(t)} \phi\left(t + \frac{s}{\lambda(t)}, \frac{x}{\lambda(t)}\right). \end{aligned}$$

Let $\beta_1(|\nabla \varepsilon_0|_{L^2(\mathbf{R}^2)})$ be defined by Lemma 3.4. Assume that there are $\delta_0 > 0$ and a sequence $t_m \rightarrow T$ as $m \rightarrow +\infty$ such that

$$\liminf_{t \rightarrow T} \left(\sup_y \int_{|x-y| < R_0} |\nabla \tilde{\varepsilon}(0, x)|^2 dx \right) \leq |Q|_{L^2(\mathbf{R}^2)}^2 - \delta_0, \quad (3.22)$$

or

$$\liminf_{t \rightarrow T} \left(\sup_y \int_{|x-y| < R_0} |\tilde{n}(0, x)| dx \right) \leq \beta_1 - \delta_0. \quad (3.23)$$

Applying Lemma 3.4 with $(\tilde{\varepsilon}(0, x), \tilde{n}(0, x), \tilde{\phi}(0, x))$, we obtain

$$\int_{\mathbf{R}^2} |\Delta \tilde{\varepsilon}(0, x)|^2 dx + \int_{\mathbf{R}^2} |\tilde{n}(0, x)|^2 dx + \int_{\mathbf{R}^2} |\nabla \tilde{\phi}(0, x)|^2 dx \leq c \quad \text{and} \quad t \rightarrow T,$$

which is a contradiction. Thus, there are $x(t)$ and $R_1 > 0$ such that

$$\liminf_{t \rightarrow T} \left(\int_{|x-x(t)| \leq R_1} |\nabla \tilde{\varepsilon}(0, x)|^2 dx \right) \geq |Q|_{L^2(\mathbb{R}^2)}^2, \quad \liminf_{t \rightarrow T} \left(\int_{|x-x(t)| \leq R_1} |\tilde{n}(0, x)| dx \right) \geq \beta_1.$$

Using Hölder's inequality, we have

$$\beta_1 \leq \int_{|x-x(t)| \leq R_1} |\tilde{n}(0, x)| dx \leq R_1^{\frac{1}{2}} |\tilde{n}(0, x)|_{L^2(|x-x(t)| \leq R_1)}.$$

Taking $\beta_2^2 = \min\{|Q|_{L^2(\mathbb{R}^2)}^2, \beta_1^2 R_1^{-1}\}$, we complete the proof of (1) of Proposition 3.2.

(2) Let $t_m \rightarrow T$ as $m \rightarrow +\infty$,

$$\lambda_m^2 = \int_{\mathbb{R}^2} |\Delta \varepsilon(t_m, x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} |n(t_m, x)|^2 dx + \frac{1}{2c_0^2} \int_{\mathbb{R}^2} |\nabla \phi(t_m, x)|^2 dx$$

and

$$\hat{\varepsilon}_m(x) = \tilde{\varepsilon}_m\left(0, \frac{x}{\lambda_m}\right), \quad \hat{n}_m(x) = \frac{1}{\lambda_m^2} \tilde{n}_m\left(0, \frac{x}{\lambda_m}\right), \quad \hat{\phi}_m(x) = \frac{1}{\lambda_m} \tilde{\phi}_m\left(0, \frac{x}{\lambda_m}\right)$$

such that

$$\int_{\mathbb{R}^2} |\Delta \hat{\varepsilon}_m|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} |\hat{n}_m|^2 dx + \frac{1}{2c_0^2} \int_{\mathbb{R}^2} |\nabla \hat{\phi}_m|^2 dx = 1.$$

Thus,

$$\int_{\mathbb{R}^2} |\nabla \hat{\varepsilon}_m(x)|^2 dx = \int_{\mathbb{R}^2} |\nabla \tilde{\varepsilon}_m(0, x)|^2 dx, \quad \int_{\mathbb{R}^2} |\hat{n}_m(x)| dx = \int_{\mathbb{R}^2} |\tilde{n}_m(0, x)| dx.$$

Therefore, by the proof and conclusion of (1) of this proposition, there exist $\beta_1 > 0$ and $R_1 = R_1(|\nabla \varepsilon_0|_{L^2(\mathbb{R}^2)}^2, c_1, c_2)$ such that for a sequence x_m ,

$$\lim_{m \rightarrow +\infty} |\nabla \hat{\varepsilon}_m(x)|_{L^2(|x-x_m| \leq R_1)} \geq \beta_1 > 0, \quad \lim_{m \rightarrow +\infty} |\hat{n}_m(x)|_{L^2(|x-x_m| \leq R_1)} \geq \beta_1 > 0.$$

Let

$$\begin{cases} \tilde{\varepsilon}_m(0, x) = \varepsilon_m\left(t_m, \frac{x}{\lambda_m}\right) = \varepsilon_m\left(\frac{x}{\lambda_m}\right), \\ \tilde{n}_m(x) = \frac{1}{\lambda_m^2} n_m\left(t_m, \frac{x}{\lambda_m}\right) = \frac{1}{\lambda_m^2} n_m\left(\frac{x}{\lambda_m}\right), \\ \tilde{\phi}_m(x) = \frac{1}{\lambda_m} \phi_m\left(t_m, \frac{x}{\lambda_m}\right) = \frac{1}{\lambda_m} \phi_m\left(\frac{x}{\lambda_m}\right). \end{cases}$$

Since

$$|\nabla \hat{\varepsilon}_m(x)|_{L^2(|x-x_m| \leq R_1)}^2 = |\nabla \varepsilon_m(x)|_{L^2(|x-x_m| \leq R_1)}^2, \quad |\hat{n}_m(x)|_{L^1(|x-x_m| \leq R_1)} = |n_m(x)|_{L^1(|x-x_m| \leq R_1)},$$

by Hölder's inequality we have

$$\int_{|x-x_m| \leq R_1} |n_m(x)| dx \leq \left(\int_{|x-x_m| \leq R_1} 1 dx \right)^{\frac{1}{2}} \cdot \left(\int_{|x-x_m| \leq R_1} |n_m(x)|^2 dx \right)^{\frac{1}{2}},$$

we get

$$\lim_{m \rightarrow +\infty} |\nabla \varepsilon_m(x)|_{L^2(|x-x_m| \leq R_1)} \geq \beta_1 > 0, \quad \lim_{m \rightarrow +\infty} |n_m(x)|_{L^2(|x-x_m| \leq R_1)} \geq R_1^{-\frac{1}{2}} \beta_1 > 0.$$

Let $\beta_2 = \min\{\beta_1, R_1^{-\frac{1}{2}} \beta_1\}$, the above arguments complete the proof of (2). \square

Proposition 3.3 (Compactness for $\tilde{\varepsilon}(0, x)$). Let $t_m \rightarrow T$. There is a subsequence, also denoted by t_m , such that there are sequences $(x_m) \in \mathbb{R}^2$ and $(\varepsilon^*, n^*) \in H^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ for which as $m \rightarrow +\infty$,

$$\tilde{\varepsilon}(0, x_m +) \rightharpoonup \varepsilon^* \quad \text{in } H^2(\mathbb{R}^2), \quad \tilde{n}(0, x_m +) \rightharpoonup n^* \quad \text{in } L^2(\mathbb{R}^2). \quad (3.24)$$

Moreover, there are $\beta_2 > 0$ and $R_1 > 0$ (depending only on $|\nabla \varepsilon_0|_{L^2(\mathbb{R}^2)}$) such that

$$|\nabla \varepsilon^*|_{L^2(|x| \leq R_1)} \geq \beta_2 \quad \text{and} \quad \mathcal{H}(\varepsilon^*, n^*, 0) \leq 0. \quad (3.25)$$

Proof. Let β_2 be defined as in Proposition 3.2 and $(\varepsilon_m, n_m, \phi_m) \in H^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ satisfy

$$\int_{\mathbb{R}^2} |\nabla \varepsilon_m|^2 dx \leq \int_{\mathbb{R}^2} |\nabla \varepsilon_0|^2 dx, \quad \lim_{m \rightarrow +\infty} \mathcal{H}(\varepsilon_m, n_m, \phi_m) = 0, \quad (3.26)$$

$$\int_{\mathbb{R}^2} |\Delta \varepsilon_m|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} |n_m|^2 dx + \frac{1}{2c_0^2} \int_{\mathbb{R}^2} |\nabla \phi_m|^2 dx = 1. \quad (3.27)$$

From Proposition 3.1, it follows that there are $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \leq \int_{\mathbb{R}^2} |\Delta \varepsilon_m|^2 dx \leq c_2, \quad c_1 \leq \int_{\mathbb{R}^2} |n_m|^2 dx \leq c_2. \quad (3.28)$$

Let k_0 be the integer defined by

$$\int_{\mathbb{R}^2} |\nabla \varepsilon_m|^2 dx < (k_0 + 1)\beta_2^2 \quad \text{for } 1 \leq k_0 \leq \frac{|\nabla \varepsilon_0|_{L^2(\mathbb{R}^2)}^2}{\beta_2^2} - 1.$$

In the following, we prove the result of this proposition by induction on k_0 .

When $k_0 = 1$, from (3.26), (3.27), (3.28) and Proposition 3.2 we obtain the result of Proposition 3.3.

Now, we assume that the property is proven for k_0 and we prove it for $k_0 + 1$. Let (ε_m, n_m) be such that

$$\lim_{m \rightarrow +\infty} \mathcal{H}(\varepsilon_m, n_m, 0) \leq 0, \quad \int_{\mathbb{R}^2} |\nabla \varepsilon_m|^2 dx \leq (k_0 + 1)\beta_2^2, \quad c_1 \leq \int_{\mathbb{R}^2} |\Delta \varepsilon_m|^2 dx \leq c_2. \quad (3.29)$$

Thus from Proposition 3.2, we can assume that for $x_m \in \mathbb{R}^2$ and $R = R(c_1, c_2)$,

$$\int_{|x-x_m| \leq R} |\nabla \varepsilon_m|^2 dx \geq \beta_2^2 \quad (3.30)$$

and there is $(\hat{\varepsilon}, \hat{n}) \in H^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ such that

$$\varepsilon_m(+x_m) \rightharpoonup \hat{\varepsilon} \quad \text{in } H^2(\mathbb{R}^2), \quad n_m(+x_m) \rightharpoonup \hat{n} \quad \text{in } L^2(\mathbb{R}^2). \quad (3.31)$$

Extracting subsequences, we decompose

$$\varepsilon_m(+x_m) = \varepsilon_{m,1}(+x_m) + \varepsilon_{m,2}(+x_m), \quad n_m(+x_m) = n_{m,1}(+x_m) + n_{m,2}(+x_m),$$

where

$$\varepsilon_{m,1}(x) = n_{m,1}(x) = 0 \quad \text{for } |x| \leq \frac{R_m}{2},$$

$$\varepsilon_{m,2}(x) = n_{m,2}(x) = 0 \quad \text{for } |x| \geq R_m \text{ with } R_m \rightarrow +\infty,$$

$$\int_{\frac{R_m}{2} \leq |x| \leq R_m} (|\Delta \varepsilon_m|^2 + |n_m|^2 + |\nabla \varepsilon_m|^2) dx \rightarrow 0 \quad \text{as } m \rightarrow +\infty,$$

$$\int_{\mathbb{R}^2} (|\nabla \varepsilon_{m,1}|^2 + |\nabla \varepsilon_{m,2}|^2) dx - \int_{\mathbb{R}^2} |\nabla \varepsilon_m|^2 dx \rightarrow 0 \quad \text{as } m \rightarrow +\infty,$$

$$\int_{\mathbb{R}^2} (|\Delta \varepsilon_{m,1}|^2 + |\Delta \varepsilon_{m,2}|^2) dx - \int_{\mathbb{R}^2} |\Delta \varepsilon_m|^2 dx \rightarrow 0 \quad \text{as } m \rightarrow +\infty,$$

$$\int_{\mathbf{R}^2} (|n_{m,1}|^2 + |n_{m,2}|^2) dx - \int_{\mathbf{R}^2} |n_m|^2 dx \rightarrow 0 \quad \text{as } m \rightarrow +\infty,$$

$$\lim_{m \rightarrow +\infty} \mathcal{H}(\varepsilon_{m,1}, n_{m,1}, 0) + \lim_{m \rightarrow +\infty} \mathcal{H}(\varepsilon_{m,2}, n_{m,2}, 0) \leq 0.$$

Thus, we get $\int_{\mathbf{R}^2} |\nabla \varepsilon_{m,1}|^2 dx \rightarrow \int_{\mathbf{R}^2} |\nabla \hat{\varepsilon}|^2 dx$ as $m \rightarrow +\infty$. Especially, we have $\int_{\mathbf{R}^2} |\nabla \hat{\varepsilon}|^2 dx \geq \beta_2^2$. Therefore, from (3.29), for m large we get $\int_{\mathbf{R}^2} |\nabla \varepsilon_{m,2}|^2 dx < k_0 \beta_2^2$.

In the following, we continue to prove this proposition by considering two cases.

Case 1. $\mathcal{H}(\hat{\varepsilon}, \hat{n}, 0) \leq \lim_{m \rightarrow +\infty} \mathcal{H}(\varepsilon_{m,1}, n_{m,1}, 0) \leq 0$. In this case, by (3.29) and (3.30), taking $\varepsilon^* = \hat{\varepsilon}$ and $n^* = \hat{n}$, we get the conclusion of Proposition 3.3.

Case 2. $\mathcal{H}(\hat{\varepsilon}, \hat{n}, 0) > 0$. In this case, $p_1 = \lim_{m \rightarrow +\infty} \mathcal{H}(\varepsilon_{m,1}, n_{m,1}, 0) > 0$ and for m large, we have $\mathcal{H}(\varepsilon_{m,2}, n_{m,2}, 0) \leq -\frac{p_1}{2}$. Applying assumptions (3.28), (3.29) and (3.30), we obtain that there exists $y_m \in \mathbf{R}^2$ such that

$$\varepsilon_{m,2}(+y_m) \rightharpoonup \varepsilon^* \quad \text{in } H^2(\mathbf{R}^2), \quad n_{m,2}(+y_m) \rightharpoonup n^* \quad \text{in } L^2(\mathbf{R}^2)$$

with

$$\int_{\mathbf{R}^2} |\nabla \varepsilon^*|^2 dx \geq \beta_2^2 \quad \text{and} \quad \mathcal{H}(\varepsilon^*, n^*, 0) \leq 0. \quad (3.32)$$

Thus, we have as $m \rightarrow +\infty$,

$$\varepsilon_{m,2}(+x_m + y_m) \rightharpoonup \varepsilon^* \quad \text{in } H^2(\mathbf{R}^2), \quad n_{m,2}(+x_m + y_m) \rightharpoonup n^* \quad \text{in } L^2(\mathbf{R}^2),$$

by (3.32), the proof of Proposition 3.3 is completed. \square

Corollary 3.2 (Compactness for $\tilde{n}(0, x)$). *There is a $c_1^* > 0$ such that*

$$c_1^* \leq \|n^*\|_{L^2(\mathbf{R}^2)} \leq 1. \quad (3.33)$$

Proof. From Proposition 3.3, we have

$$\mathcal{H}(\varepsilon^*, n^*, 0) = \int_{\mathbf{R}^2} |\Delta \varepsilon^*|^2 dx + \int_{\mathbf{R}^2} n^* |\nabla \varepsilon^*|^2 dx + \frac{1}{2} \int_{\mathbf{R}^2} |n^*|^2 dx \leq 0. \quad (3.34)$$

Since $|\nabla \varepsilon^*|_{L^2(|x| \leq R_1)} \geq c_1 > 0$ (by (3.25)), (3.3), Proposition 3.3 and Sobolev imbedding yield that there exists a $c'_1 > 0$ such that

$$c'_1 \leq \left(\int_{\mathbf{R}^2} |\Delta \varepsilon^*|^2 dx \right)^{\frac{1}{2}} \leq 1. \quad (3.35)$$

Using (3.34) again, by Lemma 3.2 we have

$$\int_{\mathbf{R}^2} |\Delta \varepsilon^*|^2 dx \leq - \int_{\mathbf{R}^2} n^* |\nabla \varepsilon^*|^2 dx \leq \left(\int_{\mathbf{R}^2} |n^*|^2 dx \right)^{\frac{1}{2}} \left(\frac{2 \int_{\mathbf{R}^2} |\nabla \varepsilon^*|^2 dx}{\int_{\mathbf{R}^2} Q^2 dx} \cdot \int_{\mathbf{R}^2} |\Delta \varepsilon^*|^2 dx \right)^{\frac{1}{2}},$$

which together with Proposition 3.2, Proposition 3.3 and (3.35) concludes

$$c'_1 \leq \left(\int_{\mathbf{R}^2} |\Delta \varepsilon^*|^2 dx \right)^{\frac{1}{2}} \leq \sqrt{2} \left(\int_{\mathbf{R}^2} |n^*|^2 dx \right)^{\frac{1}{2}} \left[\left(\int_{\mathbf{R}^2} |\nabla \varepsilon_0|^2 dx \right)^{\frac{1}{2}} / \left(\int_{\mathbf{R}^2} Q^2 dx \right)^{\frac{1}{2}} \right].$$

Thus, we get

$$\left(\int_{\mathbf{R}^2} |n^*|^2 dx \right)^{\frac{1}{2}} \geq \frac{c'_1 (\int_{\mathbf{R}^2} Q^2 dx)^{\frac{1}{2}}}{\sqrt{2} (\int_{\mathbf{R}^2} |\nabla \varepsilon_0|^2 dx)^{\frac{1}{2}}}. \quad (3.36)$$

Furthermore, from (3.6), (3.34) and (3.35) it follows that

$$\frac{1}{2} \int_{\mathbf{R}^2} |n^*|^2 dx \leq -c_1'^2 + \left(\int_{\mathbf{R}^2} |n^*|^2 dx \right)^{\frac{1}{2}} \frac{\sqrt{2} (\int_{\mathbf{R}^2} |\nabla \varepsilon_0|^2 dx)^{\frac{1}{2}}}{(\int_{\mathbf{R}^2} Q^2 dx)^{\frac{1}{2}}}. \quad (3.37)$$

Choosing suitable ε_0 such that $\sqrt{2}c_1' \leq \frac{\sqrt{2}(\int_{\mathbf{R}^2} |\nabla \varepsilon_0|^2 dx)^{\frac{1}{2}}}{(\int_{\mathbf{R}^2} Q^2 dx)^{\frac{1}{2}}} \leq \frac{1}{2} + c_1'^2$, we get

$$\left(\int_{\mathbf{R}^2} |n^*|^2 dx \right)^{\frac{1}{2}} \leq 1. \quad (3.38)$$

On the other hand, since $\sqrt{2}c_1' < \frac{\sqrt{2}(\int_{\mathbf{R}^2} |\nabla \varepsilon_0|^2 dx)^{\frac{1}{2}}}{(\int_{\mathbf{R}^2} Q^2 dx)^{\frac{1}{2}}} \leq \frac{1}{2} + c_1'^2$, we get

$$\frac{c_1' (\int_{\mathbf{R}^2} Q^2 dx)^{\frac{1}{2}}}{\sqrt{2} (\int_{\mathbf{R}^2} |\nabla \varepsilon_0|^2 dx)^{\frac{1}{2}}} \leq \frac{\sqrt{2}}{2} < 1.$$

Let $\frac{c_1' (\int_{\mathbf{R}^2} Q^2 dx)^{\frac{1}{2}}}{\sqrt{2} (\int_{\mathbf{R}^2} |\nabla \varepsilon_0|^2 dx)^{\frac{1}{2}}} = c_1^*$. Thus, (3.36) and (3.38) conclude the proof of Corollary 3.2. \square

From the above results, we claim the following.

Theorem 3.2. Let

$$\lambda^2(t) = \int_{\mathbf{R}^2} |\Delta \varepsilon(t, x)|^2 dx + \frac{1}{2} \int_{\mathbf{R}^2} n^2(t, x) dx + \frac{1}{2c_0^2} \int_{\mathbf{R}^2} |\nabla \phi(t, x)|^2 dx$$

and

$$(\tilde{\varepsilon}(s, x), \tilde{n}(s, x), \tilde{\phi}(s, x)) = \left(\varepsilon(t + s/\lambda, x/\lambda), \frac{1}{\lambda^2} n(t + s/\lambda, x/\lambda), \frac{1}{\lambda} \phi(t + s/\lambda, x/\lambda) \right).$$

Then there are $\theta > 0$ and $A > 0$ (depending only on $|\nabla \varepsilon_0|_{L^2(\mathbf{R}^2)}$) such that for t near T ,

$$\forall s \in [0, \theta), \quad |(\tilde{\varepsilon}, \tilde{n}, \tilde{\phi})(s, x)|_{H^2(\mathbf{R}^2) \times L^2(\mathbf{R}^2) \times H^1(\mathbf{R}^2)} \leq A, \quad (3.39)$$

where

$$|(\tilde{\varepsilon}, \tilde{n}, \tilde{\phi})(s, x)|_{H^2(\mathbf{R}^2) \times L^2(\mathbf{R}^2) \times H^1(\mathbf{R}^2)}^2 = \int_{\mathbf{R}^2} |\Delta \varepsilon(s, x)|^2 dx + \frac{1}{2} \int_{\mathbf{R}^2} n^2(s, x) dx + \frac{1}{2c_0^2} \int_{\mathbf{R}^2} |\nabla \phi(s, x)|^2 dx.$$

Moreover, we can choose $\theta = \tilde{c}(|\nabla \varepsilon_0|_{L^2(\mathbf{R}^2)}^2 - |Q|_{L^2(\mathbf{R}^2)}^2)^{-\frac{1}{2}}$, where Q is the unique radial positive solution of (E).

Proof. From Proposition 2.1, it follows that $(\tilde{\varepsilon}(s, x), \tilde{n}(s, x), \tilde{\phi}(s, x))$ satisfies

$$\tilde{n}_s = \Delta \tilde{\phi}, \quad (3.40)$$

$$\frac{1}{c_0^2} \tilde{\phi}_s = \tilde{n} + |\nabla \tilde{\varepsilon}|^2. \quad (3.41)$$

Let $A > \max\{\sqrt{2}, \sqrt{2}c_0\}$ be fixed. We now consider $(\tilde{\varepsilon}(s, x), \tilde{n}(s, x), \tilde{\phi}(s, x))$ on $[0, \lambda(t)(T - t))$ for all $t > 0$. Since

$$\begin{aligned} \lambda^2(t) &= \int_{\mathbf{R}^2} |\Delta \varepsilon(t, x)|^2 dx + \frac{1}{2} \int_{\mathbf{R}^2} n^2(t, x) dx + \frac{1}{2c_0^2} \int_{\mathbf{R}^2} |\nabla \phi(t, x)|^2 dx \\ &= |(\varepsilon, n, \phi)(t, x)|_{H^2(\mathbf{R}^2) \times L^2(\mathbf{R}^2) \times H^1(\mathbf{R}^2)}^2, \\ |(\tilde{\varepsilon}, \tilde{n}, \tilde{\phi})(0, x)|_{H^2(\mathbf{R}^2) \times L^2(\mathbf{R}^2) \times H^1(\mathbf{R}^2)}^2 &= \frac{1}{\lambda^2(t)} \left(\int_{\mathbf{R}^2} |\Delta \varepsilon(t, x)|^2 dx + \frac{1}{2} \int_{\mathbf{R}^2} n^2(t, x) dx + \frac{1}{2c_0^2} \int_{\mathbf{R}^2} |\nabla \phi(t, x)|^2 dx \right) = 1, \end{aligned}$$

and

$$\lim_{s \rightarrow \lambda(t)(T-t)} |(\tilde{\varepsilon}, \tilde{n}, \tilde{\phi})(s, x)|_{H^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)} = +\infty, \quad (3.42)$$

by continuity there is a $\gamma(t) > 0$ such that

$$\forall s \in [0, \gamma(t)], \quad |(\tilde{\varepsilon}, \tilde{n}, \tilde{\phi})(s, x)|_{H^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)} \leq A \quad (3.43)$$

and

$$|(\tilde{\varepsilon}, \tilde{n}, \tilde{\phi})(\gamma(t), x)|_{H^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)} = A. \quad (3.44)$$

Thus, the following two conclusions hold.

Conclusion 1. There is a $\theta > 0$ (depending only on $|\nabla \varepsilon_0|_{L^2(\mathbb{R}^2)}$ and c' such that $\mathcal{H}_0 \leq c'$) such that for t near T ,

$$\gamma(t) \geq \theta. \quad (3.45)$$

Conclusion 2. For t near T ,

$$\forall s \in [0, \gamma(t)], \quad |\nabla \tilde{\phi}(s, x)|_{L^2(\mathbb{R}^2)} \leq A \tilde{c}_2 \left(\int_{\mathbb{R}^2} |\nabla \varepsilon_0|^2 dx - \int_{\mathbb{R}^2} Q^2 dx \right)^{\frac{1}{2}}. \quad (3.46)$$

Conclusion 1 and Conclusion 2 will be proven at the end of this section.

Therefore, Theorem 3.2 follows from (3.43)–(3.46). \square

Now, we begin to prove Theorem 3.1.

Proof of Theorem 3.1. (i) From Theorem 3.2, we have for $\theta = \theta(|\nabla \varepsilon_0|_{L^2(\mathbb{R}^2)})$ and $A > 0$, $\forall s \in [0, \theta]$,

$$|(\tilde{\varepsilon}, \tilde{n}, \tilde{\phi})(s, x)|_{H^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)} \leq A.$$

From (3.42), it follows that $\lambda(t)(T-t) \geq \theta$, that is, $\lambda(t) \geq \frac{\theta}{T-t}$, which together with (2.8) yields that

$$|(\varepsilon, n, \phi)(t, x)|_{H^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)} \geq \frac{c_1}{T-t} \quad (0 < c_1 \leq \theta). \quad (3.47)$$

Since

$$|\Delta \tilde{\varepsilon}(0, x)|_{L^2(\mathbb{R}^2)}^2 = \frac{1}{\lambda^2} \int_{\mathbb{R}^2} |\Delta \varepsilon|^2 dx, \quad \int_{\mathbb{R}^2} |\tilde{n}(0, x)|^2 dx = \frac{1}{\lambda^2} \int_{\mathbb{R}^2} |n(t, x)|^2 dx,$$

from Proposition 3.1 we get

$$\int_{\mathbb{R}^2} |\Delta \varepsilon|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} n^2 dx + \frac{1}{2c_0^2} \int_{\mathbb{R}^2} |\nabla \phi|^2 dx \geq \left(\frac{c_1}{T-t} \right)^2 \quad (3.48)$$

and

$$0 < c_1^2 \leq \frac{1}{\lambda^2} \int_{\mathbb{R}^2} |\Delta \varepsilon|^2 dx \leq c_2^2, \quad 0 < c_1^2 \leq \frac{1}{\lambda^2} \int_{\mathbb{R}^2} |n|^2 dx \leq c_2^2$$

which imply that

$$\int_{\mathbb{R}^2} |\Delta \varepsilon|^2 dx \geq \lambda^2 c_1^2 \geq \left(\frac{\theta}{T-t} \right)^2 c_1^2, \quad \int_{\mathbb{R}^2} |n|^2 dx \geq \lambda^2 c_1^2 \geq \left(\frac{\theta}{T-t} \right)^2 c_1^2. \quad (3.49)$$

Let $0 < c_2 \leq \theta c_1$, we get

$$|\Delta \varepsilon(t)|_{L^2(\mathbb{R}^2)} \geq \frac{c_2}{T-t}, \quad |n(t)|_{L^2(\mathbb{R}^2)} \geq \frac{c_2}{T-t}. \quad (3.50)$$

The proof of (i) of Theorem 3.1 follows from (3.48) and (3.50).

(ii) Choosing $\theta = \tilde{c}(|\nabla \varepsilon_0|_{L^2(\mathbf{R}^2)}^2 - |Q|_{L^2(\mathbf{R}^2)}^2)^{-\frac{1}{2}}$. By (3.49) we complete the proof of (ii) of Theorem 3.1. So far, we have completed the proof of Theorem 3.1. \square

At the end of this section, we prove Conclusion 1 and Conclusion 2 given in the proof of Theorem 3.2. Firstly, we give two propositions which are key to the proof of Conclusion 1 and Conclusion 2.

Proposition 3.4. *There exist $c_1 > 0$ and $c_2 > 0$ (independent of t and $A > \max\{\sqrt{2}, \sqrt{2}c_0\}$) such that the following results hold:*

(1) $\forall s \in [0, \gamma(t)]$,

$$|\Delta \tilde{\varepsilon}(s, x)|_{L^2(\mathbf{R}^2)} \leq Ac_2, \quad |\tilde{n}(s, x)|_{L^2(\mathbf{R}^2)} \leq Ac_2, \quad |\nabla \tilde{\phi}(s, x)|_{L^2(\mathbf{R}^2)} \leq Ac_2 c_0. \quad (3.51)$$

(2) Let $t_m \rightarrow T$. Then there is a subsequence, also denoted by t_m , such that for $x_m \in \mathbf{R}^2$ and $(\varepsilon^*, n^*) \in H^2(\mathbf{R}^2) \times L^2(\mathbf{R}^2)$,

$$\tilde{\varepsilon}(t_m, \gamma(t_m), x - x_m) = \tilde{\varepsilon}(\gamma(t_m), x - x_m) \rightharpoonup \varepsilon^* \quad \text{in } H^2(\mathbf{R}^2), \quad (3.52)$$

$$\tilde{n}(t_m, \gamma(t_m), x - x_m) = \tilde{n}(\gamma(t_m), x - x_m) \rightharpoonup n^* \quad \text{in } L^2(\mathbf{R}^2), \quad (3.53)$$

with

$$|\Delta \varepsilon^*|_{L^2(\mathbf{R}^2)} \geq Ac_1, \quad |n^*|_{L^2(\mathbf{R}^2)} \geq Ac_1. \quad (3.54)$$

Proof. Firstly, from (3.43), $\forall s \in [0, \gamma(t)]$, $|(\tilde{\varepsilon}, \tilde{n}, \tilde{\phi})(s, x)|_{H^2(\mathbf{R}^2) \times L^2(\mathbf{R}^2) \times H^1(\mathbf{R}^2)}^2 \leq A^2$, that is, $\int_{\mathbf{R}^2} |\Delta \tilde{\varepsilon}(s, x)|^2 dx + \frac{1}{2} \int_{\mathbf{R}^2} \tilde{n}^2(s, x) dx + \frac{1}{2c_0^2} \int_{\mathbf{R}^2} |\nabla \tilde{\phi}(s, x)|^2 dx \leq A^2$. Thus (1) holds. Secondly, since

$$(\tilde{\varepsilon}(s, x), \tilde{n}(s, x), \tilde{\phi}(s, x)) = \left(\varepsilon\left(t + \frac{s}{\lambda(t)}, \frac{x}{\lambda(t)}\right), \frac{1}{\lambda^2(t)} n\left(t + \frac{s}{\lambda(t)}, \frac{x}{\lambda(t)}\right), \frac{1}{\lambda(t)} \phi\left(t + \frac{s}{\lambda(t)}, \frac{x}{\lambda(t)}\right) \right),$$

by (3.2) we get

$$|(\tilde{\varepsilon}, \tilde{n}, \tilde{\phi})(s, x)|_{H^2(\mathbf{R}^2) \times L^2(\mathbf{R}^2) \times H^1(\mathbf{R}^2)}^2 = \left(\frac{\lambda(t + \frac{s}{\lambda(t)})}{\lambda(t)} \right)^2. \quad (3.55)$$

Thus, by the characterization of $\gamma(t)$ we obtain

$$\forall s \in [0, \gamma(t)], \quad \frac{\lambda(t + \frac{s}{\lambda(t)})}{\lambda(t)} \leq A \quad \text{and} \quad \frac{\lambda(t + \frac{\gamma(t)}{\lambda(t)})}{\lambda(t)} = A. \quad (3.56)$$

From (3.56), it follows that

$$\tilde{\varepsilon}(t, \gamma(t), x) = \varepsilon\left(t + \frac{\gamma(t)}{\lambda(t)}, \frac{x}{\lambda(t)}\right) = \tilde{\varepsilon}\left(t + \frac{\gamma(t)}{\lambda(t)}, 0, Ax\right), \quad (3.57)$$

and

$$\tilde{n}(t, \gamma(t), x) = \frac{1}{\lambda^2(t)} n\left(t + \frac{\gamma(t)}{\lambda(t)}, \frac{x}{\lambda(t)}\right) = A^2 \tilde{n}\left(t + \frac{\gamma(t)}{\lambda(t)}, 0, Ax\right). \quad (3.58)$$

Now, we consider a sequence $t_m \rightarrow T$. Then $t_m + \frac{\gamma(t_m)}{\lambda(t_m)} \rightarrow T$. From Proposition 3.3 and Corollary 3.2, there is a subsequence, also denoted by t_m such that there exist sequences $(x_m) \in \mathbf{R}^2$ and $(\varepsilon^*, n^*) \in H^2(\mathbf{R}^2) \times L^2(\mathbf{R}^2)$ such that

$$\tilde{\varepsilon}\left(t_m + \frac{\gamma(t_m)}{\lambda(t_m)}, 0, x + x_m\right) \rightharpoonup \varepsilon^* \quad \text{in } H^2(\mathbf{R}^2) \text{ as } m \rightarrow +\infty, \quad (3.59)$$

$$\tilde{n}\left(t_m + \frac{\gamma(t_m)}{\lambda(t_m)}, 0, x + x_m\right) \rightharpoonup n^* \quad \text{in } L^2(\mathbf{R}^2) \text{ as } m \rightarrow +\infty, \quad (3.60)$$

and there exists a $c_1 = c_1(|\nabla \varepsilon_0|_{L^2(\mathbf{R}^2)}) > 0$ such that

$$|\nabla \varepsilon^*|_{L^2(\mathbf{R}^2)} \geq c_1, \quad |n^*|_{L^2(\mathbf{R}^2)} \geq c_1. \quad (3.61)$$

Then, from (3.57)–(3.61) it follows that as $m \rightarrow +\infty$,

$$\tilde{\varepsilon}\left(t_m, \gamma(t_m), x + \frac{x_m}{A}\right) = \tilde{\varepsilon}\left(t_m + \frac{\gamma(t_m)}{\lambda(t_m)}, 0, Ax + x_m\right) \rightharpoonup \varepsilon^*(Ax) \quad \text{in } H^2(\mathbf{R}^2) \quad (3.62)$$

and

$$\tilde{n}\left(t_m, \gamma(t_m), x + \frac{x_m}{A}\right) = A^2 \tilde{n}\left(t_m + \frac{\gamma(t_m)}{\lambda(t_m)}, 0, Ax + x_m\right) \rightharpoonup A^2 n^*(Ax) \quad \text{in } L^2(\mathbf{R}^2), \quad (3.63)$$

with

$$|\Delta \varepsilon^*(Ax)|_{L^2(\mathbf{R}^2)} = \left(\int_{\mathbf{R}^2} |\Delta \varepsilon^*(Ax)|^2 dx \right)^{\frac{1}{2}} = A |\Delta \varepsilon^*|_{L^2(\mathbf{R}^2)} \geqslant Ac_1$$

and

$$|A^2 n^*(Ax)|_{L^2(\mathbf{R}^2)} = \left(\int_{\mathbf{R}^2} |A^2 n^*(Ax)|^2 dx \right)^{\frac{1}{2}} = A |n^*|_{L^2(\mathbf{R}^2)} \geqslant Ac_1,$$

which follow from Corollary 3.2 and its proof. Thus (2) holds.

So far, we have completed the proof of Proposition 3.4. \square

Let A satisfy

$$Ac_1 \geqslant 4 \quad (3.64)$$

so that $\tilde{n}(t, \gamma(t))$ can be different from $\tilde{n}(t, 0)$ ($|\tilde{n}(t, 0)|_{L^2(\mathbf{R}^2)} \leqslant \sqrt{2}$). Thus, we have

Proposition 3.5. *Let $t \rightarrow T$. Then there exists $c > 0$ such that*

$$\liminf_{t \rightarrow T} \int_0^{\gamma(t)} |\nabla \tilde{\phi}(s, x)|_{L^2(\mathbf{R}^2)} ds \geqslant c. \quad (3.65)$$

Proof. We prove this proposition by contradiction. Assume that there is a sequence $t_m \rightarrow T$ as $m \rightarrow +\infty$ such that

$$\int_0^{\gamma(t_m)} |\nabla \tilde{\phi}(s, x)|_{L^2(\mathbf{R}^2)} ds \xrightarrow{m \rightarrow +\infty} 0. \quad (3.66)$$

From (3.40), for all $\psi(x) \in \varphi^\infty$ with compact support we have

$$\int_{\mathbf{R}^2} \tilde{n}(t_m, \gamma(t_m), x) \psi dx - \int_{\mathbf{R}^2} \tilde{n}(t_m, 0, x) \psi dx = \int_0^{\gamma(t_m)} \int_{\mathbf{R}^2} \nabla \tilde{\phi}(s, x) \nabla \psi dx ds.$$

Thus,

$$\left| \int_{\mathbf{R}^2} \tilde{n}(t_m, \gamma(t_m), x) \psi dx - \int_{\mathbf{R}^2} \tilde{n}(t_m, 0, x) \psi dx \right| \leqslant \left(\int_0^{\gamma(t_m)} |\nabla \tilde{\phi}(s)|_{L^2(\mathbf{R}^2)} ds \right) |\nabla \psi|_{L^2(\mathbf{R}^2)}. \quad (3.67)$$

Extracting a sequence, denoted by t_m , by Proposition 3.4 we can assume that there are $x_m \in \mathbf{R}^2$, $n^* \in L^2(\mathbf{R}^2)$ such that

$$\tilde{n}(t_m, \gamma(t_m), x - x_m) \rightharpoonup n^* \quad \text{in } L^2(\mathbf{R}^2) \quad (3.68)$$

with $|n^*|_{L^2(\mathbf{R}^2)} \geqslant Ac_1$. Now we consider $\psi_0 \in \varphi^\infty$ with compact support such that

$$\left(\int_{\mathbf{R}^2} \psi_0^2 dx \right)^{\frac{1}{2}} = 1 \quad \text{and} \quad \int_{\mathbf{R}^2} n^* \psi_0 dx \geqslant \frac{1}{2} \left(\int_{\mathbf{R}^2} n^{*2} dx \right)^{\frac{1}{2}}. \quad (3.69)$$

From (3.66) and (3.67) it follows that

$$\left| \int_{\mathbb{R}^2} \tilde{n}(t_m, \gamma(t_m), x) \psi_0(x + x_m) dx - \int_{\mathbb{R}^2} \tilde{n}(t_m, 0, x) \psi_0(x + x_m) dx \right|$$

$$\leq \left(\int_0^{\gamma(t_m)} |\nabla \tilde{\phi}(s)|_{L^2(\mathbb{R}^2)} ds \right) |\nabla \psi_0|_{L^2(\mathbb{R}^2)} \xrightarrow{m \rightarrow +\infty} 0. \quad (3.70)$$

On the one hand, by (3.64), (3.68) and (3.69) we have

$$\int_{\mathbb{R}^2} \tilde{n}(t_m, \gamma(t_m), x) \psi_0(x + x_m) dx \geq \frac{1}{2} \left(\int_{\mathbb{R}^2} n^{*2} dx \right)^{\frac{1}{2}} \geq \frac{Ac_1}{2} \geq 2. \quad (3.71)$$

On the other hand,

$$\left| \int_{\mathbb{R}^2} \tilde{n}(t_m, 0, x) \psi_0 dx \right| \leq \left(\int_{\mathbb{R}^2} \tilde{n}^2(0, x) dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} \psi_0^2 dx \right)^{\frac{1}{2}} \leq \sqrt{2},$$

which contradicts (3.70)–(3.71).

From the above arguments, the proof of Proposition 3.5 is completed. \square

At the end of this section, we prove Conclusion 1 and Conclusion 2 mentioned in the proof of Theorem 3.2. Firstly, we prove Conclusion 1.

Proof of Conclusion 1. From Proposition 3.4, if

$$\int_0^{\gamma(t)} |\nabla \tilde{\phi}(s, x)|_{L^2(\mathbb{R}^2)} ds \leq \int_0^{\gamma(t)} Ac_2 c_0 ds = Ac_2 c_0 \gamma(t),$$

then Proposition 3.5 implies that for some $c > 0$, $\liminf_{t \rightarrow T} Ac_2 \gamma(t) \geq c$, that is, $\liminf_{t \rightarrow T} \gamma(t) > c$, which concludes Conclusion 1. \square

Finally, we prove Conclusion 2.

Proof of Conclusion 2. From (3.5) with $\lambda = \lambda(t)$, it follows that

$$\mathcal{H}(\tilde{\varepsilon}, \tilde{n}, \tilde{\phi}) = \int_{\mathbb{R}^2} |\Delta \tilde{\varepsilon}|^2 dx + \int_{\mathbb{R}^2} \tilde{n} |\nabla \tilde{\varepsilon}|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} |\tilde{n}|^2 dx + \frac{1}{2c_0^2} \int_{\mathbb{R}^2} |\nabla \tilde{\phi}|^2 dx = \frac{1}{\lambda^2(t)} \mathcal{H}_0,$$

or

$$\left(\int_{\mathbb{R}^2} |\Delta \tilde{\varepsilon}|^2 dx - \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \tilde{\varepsilon}|^4 dx \right) + \frac{1}{2} \int_{\mathbb{R}^2} (\tilde{n} + |\nabla \tilde{\varepsilon}|^2)^2 dx + \frac{1}{2c_0^2} \int_{\mathbb{R}^2} |\nabla \tilde{\phi}|^2 dx = \frac{1}{\lambda^2(t)} \mathcal{H}_0.$$

Thus, there exists $\mathcal{H}_0 \leq c$ such that

$$\left(\int_{\mathbb{R}^2} |\Delta \tilde{\varepsilon}|^2 dx - \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \tilde{\varepsilon}|^4 dx \right) + \frac{1}{2} \int_{\mathbb{R}^2} (\tilde{n} + |\nabla \tilde{\varepsilon}|^2)^2 dx + \frac{1}{2c_0^2} \int_{\mathbb{R}^2} |\nabla \tilde{\phi}|^2 dx \leq \frac{c}{\lambda^2}.$$

Therefore,

$$\int_{\mathbb{R}^2} |\Delta \tilde{\varepsilon}|^2 dx - \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \tilde{\varepsilon}|^4 dx \leq \frac{c}{\lambda^2} \quad (3.72)$$

and

$$\int_{\mathbb{R}^2} (\tilde{n} + |\nabla \tilde{\varepsilon}|^2)^2 dx + \frac{1}{c_0^2} \int_{\mathbb{R}^2} |\nabla \tilde{\phi}|^2 dx \leq \frac{2c}{\lambda^2} - 2 \left(\int_{\mathbb{R}^2} |\Delta \tilde{\varepsilon}|^2 dx - \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \tilde{\varepsilon}|^4 dx \right). \quad (3.73)$$

From Lemma 3.2 we have

$$\int_{\mathbb{R}^2} |\Delta \tilde{\varepsilon}|^2 dx - \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \tilde{\varepsilon}|^4 dx \geq \int_{\mathbb{R}^2} |\Delta \tilde{\varepsilon}|^2 dx \cdot \left(1 - \frac{\int_{\mathbb{R}^2} |\nabla \varepsilon_0|^2 dx}{\int_{\mathbb{R}^2} Q^2 dx}\right).$$

Consequently, from (3.73) and Proposition 3.4 we obtain

$$\int_{\mathbb{R}^2} (\tilde{n} + |\nabla \tilde{\varepsilon}|^2)^2 dx + \frac{1}{c_0^2} \int_{\mathbb{R}^2} |\nabla \tilde{\phi}|^2 dx \leq \frac{2c}{\lambda^2} + A\tilde{c}_2 \left(\int_{\mathbb{R}^2} |\nabla \varepsilon_0|^2 dx - \int_{\mathbb{R}^2} Q^2 dx \right).$$

Taking λ large enough ($t \rightarrow T$), we get Conclusion 2. \square

4. Application to the virial identity

In this section, we give an application of Theorem 3.1 to estimate some terms in the following identities. Let

$$F(t) = \frac{1}{4} \int_{\mathbb{R}^2} |x|^2 |\nabla \varepsilon|^2 dx - \int_0^t \frac{1}{c_0^2} \int_{\mathbb{R}^2} (x \cdot \nabla \phi) n dx ds. \quad (4.1)$$

Then, by a direct calculation we obtain

$$\frac{d}{dt} F(t) = \Im \int_{\mathbb{R}^2} (x \cdot \nabla \bar{\varepsilon}) \Delta \varepsilon dx - \frac{1}{c_0^2} \int_{\mathbb{R}^2} (x \cdot \nabla \phi) n dx, \quad (4.2)$$

$$\frac{d^2}{dt^2} F(t) = 2\mathcal{H}_0 - \frac{1}{c_0^2} \int_{\mathbb{R}^2} |\nabla \phi|^2 dx. \quad (4.3)$$

In the following, we are interested in showing that for any blow-up solution for the Cauchy problem (1.1)–(1.4), $\lim_{t \rightarrow T} F(t) = -\infty$. The approach we use here originates from Merle [11] and we show a blow-up result for the Cauchy problem (1.1)–(1.4) without using the positivity properties of $\int_{\mathbb{R}^2} |x|^2 |\nabla \varepsilon|^2 dx$. Now, the following results are true from (4.3) and the lower bound estimates obtained in the previous section.

Theorem 4.1. *Let (ε, n, ϕ) be a blow-up solution for the Cauchy problem (1.1)–(1.4). Then:*

(1) *There is a $c = c(|\nabla \varepsilon_0|_{L^2(\mathbb{R}^2)}) > 0$ such that for t near T ,*

$$\int_0^t |\nabla \phi(s, x)|_{L^2(\mathbb{R}^2)} ds \geq -c \log(T - t). \quad (4.4)$$

(2) *There is a $c = c(|\nabla \varepsilon_0|_{L^2(\mathbb{R}^2)}) > 0$ such that for t near T ,*

$$\int_0^t (T - \tau) |\nabla \phi(\tau, x)|_{L^2(\mathbb{R}^2)}^2 d\tau \geq -c \log(T - t). \quad (4.5)$$

Moreover, we have

$$\lim_{t \rightarrow T} \left(\frac{1}{4} \int_{\mathbb{R}^2} |x|^2 |\nabla \varepsilon|^2 dx - \frac{1}{c_0^2} \int_0^t \int_{\mathbb{R}^2} (x \cdot \nabla \phi) n dx ds \right) = -\infty. \quad (4.6)$$

Proof. (1) By using the notations in Section 3 and changing eventually the time and the value of A , from Proposition 3.1 and Proposition 3.5, we can assume that

$$1 < \lambda(0) < A, \quad \forall t > 0, \quad \int_0^{\gamma(t)} |\nabla \tilde{\phi}(s, x)|_{L^2(\mathbb{R}^2)} ds \geq c > 0. \quad (4.7)$$

Now, for $i = 1, \dots, +\infty$ we define

$$t_i = \min\{t \geq 0, \lambda(t) \geq A^i\}. \quad (4.8)$$

Thus, we get

$$t_1 < t_2 < t_3 < \dots < t_m < \dots < T, \quad t_m \rightarrow T \text{ as } m \rightarrow +\infty, \quad (4.9)$$

$$\lambda(t_i) = A^i \quad \text{and} \quad t_i + \frac{\gamma(t_i)}{\lambda(t_i)} = t_{i+1}. \quad (4.10)$$

By (4.7), (4.10) and the definition of $\tilde{\phi}(s, x)$, we get $\forall i$,

$$\int_0^{\gamma(t_i)} \left| \frac{1}{\lambda^2(t_i)} \nabla \phi \left(t + \frac{s}{\lambda(t_i)}, \frac{x}{\lambda(t_i)} \right) \right|_{L^2(\mathbb{R}^2)} ds = \int_{t_i}^{t_{i+1}} |\nabla \phi(t, x)|_{L^2(\mathbb{R}^2)} dt \geq c > 0. \quad (4.11)$$

Especially, $\forall i = 1, \dots, +\infty$, $\int_0^{t_i} |\nabla \phi(t, x)|_{L^2(\mathbb{R}^2)} dt \geq (i-1)c$ and for i large enough

$$\int_0^{t_i} |\nabla \phi(t, x)|_{L^2(\mathbb{R}^2)} dt \geq \frac{(1+i)c}{2} = \frac{c}{2 \log A} \log(\lambda(t_{i+1})).$$

In view of Theorem 3.2, for i large ($i \geq i_0$) we obtain $\lambda(t_{i+1}) \geq \frac{c}{T-t_{i+1}}$ and

$$\int_0^{t_i} |\nabla \phi(t, x)|_{L^2(\mathbb{R}^2)} dt \geq \frac{c}{2 \log A} \log(\lambda(t_{i+1})) \geq \frac{-c}{4 \log A} \log(T - t_{i+1}). \quad (4.12)$$

Now, for $t > t_{i_0}$, there is an $i \geq 1$ such that $t_i \leq t < t_{i+1}$ and then

$$\int_0^t |\nabla \phi(\tau, x)|_{L^2(\mathbb{R}^2)} d\tau \geq \int_0^{t_i} |\nabla \phi(\tau, x)|_{L^2(\mathbb{R}^2)} d\tau \geq \frac{-c}{4 \log A} \log(T - t),$$

which implies that for t near T , $\int_0^t |\nabla \phi(\tau, x)|_{L^2(\mathbb{R}^2)} d\tau \geq \frac{-c}{4 \log A} \log(T - t)$.

This completes the proof of (1) of Theorem 4.1.

In the following, we prove (2).

(2) For all $t > 0$,

$$\begin{aligned} \int_0^t |\nabla \phi(\tau, x)|_{L^2(\mathbb{R}^2)} d\tau &= \int_0^t \frac{(T - \tau)^{\frac{1}{2}}}{(T - \tau)^{\frac{1}{2}}} |\nabla \phi(\tau, x)|_{L^2(\mathbb{R}^2)} d\tau \\ &\leq \left(\int_0^t (T - \tau) |\nabla \phi(\tau, x)|_{L^2(\mathbb{R}^2)}^2 d\tau \right)^{\frac{1}{2}} (\log T - \log(T - t))^{\frac{1}{2}}. \end{aligned}$$

Thus, for t near T , from (4.4) we get

$$\frac{-c \log(T - t)}{[-\log(T - t)]^{\frac{1}{2}}} \leq \left(\int_0^t (T - \tau) |\nabla \phi(\tau, x)|_{L^2(\mathbb{R}^2)}^2 d\tau \right)^{\frac{1}{2}},$$

or equivalently,

$$-c \log(T - t) \leq \int_0^t (T - \tau) |\nabla \phi(\tau, x)|_{L^2(\mathbb{R}^2)}^2 d\tau,$$

which proves (4.5). By (4.1), (4.3) and Proposition 3.1, we have for $t > 0$ and $t < T$,

$$\begin{aligned}
 F(t) &= F(0) + F'(0)t + \int_0^t (t-\tau)F''(\tau) d\tau \\
 &\leq c - \frac{1}{c_0^2} \int_0^t (t-\tau) \|\nabla \phi(\tau, x)\|_{L^2(\mathbb{R}^2)}^2 d\tau.
 \end{aligned}$$

From (4.5), it follows that

$$\lim_{t \rightarrow T} \int_0^t (t-\tau) \|\nabla \phi(\tau, x)\|_{L^2(\mathbb{R}^2)}^2 d\tau = \lim_{t \rightarrow T} \int_0^T (T-\tau) \|\nabla \phi(\tau, x)\|_{L^2(\mathbb{R}^2)}^2 d\tau = +\infty,$$

which concludes (4.6). \square

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